UNITARITY AND UNIQUENESS OF THE FOCK QUANTIZATION OF THE GOWDY MODEL

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Motivation

- The extension of [L]QC to inhomogeneous spacetimes would validate present results and provide new predictions (CMB). Inhomogeneous quantum cosmological models would serve for comparison of techniques and interpretations.

- The linearly polarized Gowdy T^3 model is the simplest of all inhomogeneous cosmological systems.

- It can be formulated as 2+1 gravity coupled to a free massless scalar field with axial symmetry, an interesting system from the point of view of QFT in curved backgrounds.

- The model is non-stationary and infinite dimensional. Unitarity and uniqueness of the quantization are relevant issues.

The Gowdy T^3 model

- It describes inhomogeneous vacuum spacetimes with the spatial topology of T^3 and two commuting, axial, and hypersurface orthogonal Killing fields. Classical solutions have a cosmological singularity.

- After gauge fixing:

$$ds^2 = e^{\gamma(Q,p,\phi,P_\phi)} e^{-\phi/\sqrt{p}} \left(-dt^2 + d\theta^2\right) + e^{-\phi/\sqrt{p}} t^2 p^2 d\sigma^2 + e^{\phi/\sqrt{p}} d\delta^2 .$$

$\partial_a = \{\partial_\sigma, \partial_\delta\}$ are axial Killing vector fields. $\delta \in S^1, t > 0$.

$p = -\oint P_\gamma/(2\pi) > 0$ and $(Q, P = ln\, p)$ are canonically conjugate.

Deparametrization has been achieved by setting $det\left[(\partial_a)^i h_{ij} (\partial_b)^j\right] = t^2 p^2 .$

- There is some arbitrariness in the choice of: a) **time** and b) the **field** ϕ.

- There is a **homogeneous constraint**, which generates S^1-translations:

$$C_0 = \frac{1}{\sqrt{2\pi}} \oint P_\phi \phi' .$$

The Gowdy T^3 model (*old* description)

- The reduced dynamics is dictated by $H_r = \frac{1}{2t}\oint\left[P_\phi^2 + t^2(\phi')^2\right].$

- The field is subject to the wave equation $\ddot{\phi} + \frac{\dot{\phi}}{t} - \phi'' = 0.$

Smooth solutions have the form

$$\varphi(t,\theta) = \sum_{n=-\infty}^{\infty}\left[A_n f_n(t,\theta) + A^*_n f^*_n(t,\theta)\right]; \quad f_n(t,\theta) = \bar{f}_n(t)e^{in\theta},$$

$$\bar{f}_0(t) = \frac{1 - i\ln t}{\sqrt{4\pi}}, \qquad \bar{f}_n(t) = \frac{H_0(|n|t)}{\sqrt{8}} \quad n \neq 0.$$

We adopt a complex structure such that $\{A_n, A_n^*\}_{n\in\mathbb{Z}}$ are promoted to annihilation and creation operators.

- However this dynamics admits no unitary implementation, neither on the resulting Fock space, nor on the physical Hilbert space with $C_0 = 0$.

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New field parametrization

- For large n, our solutions behave as $f_n(t,\theta)=\frac{H_0(|n|t)\,e^{in\theta}}{\sqrt{8}}\approx\frac{e^{-i(|n|t-n\theta)}\,e^{i\pi/4}}{\sqrt{4\pi|n|t}}$.

By means of a **TIME DEPENDENT** canonical transformation, we introduce the new field parametrization:

$$\xi=\sqrt{t}\,\phi,\qquad P_\xi=\frac{1}{\sqrt{t}}\left(P_\phi+\frac{\phi}{2}\right).$$

- The reduced Hamiltonian becomes $H_r^{(\xi)}=\frac{1}{2}\oint\left[P_\xi^2+(\xi')^2+\frac{\xi^2}{4t^2}\right]$.

It is the Hamiltonian of a scalar field in a flat (2-dimensional) background subject to a time dependent potential that **tends to zero at large times**.

- The constraint that generates S^1-translations is $C_0=\frac{1}{\sqrt{2\pi}}\oint P_\xi\xi'$.

New field parametrization (description)

- We fix a reference time $t = t_0$ and expand the canonical fields (ξ, P_ξ) at t_0 *(Cauchy data)* in Fourier series. Next, we introduce "annihilation" and "creation" variables for $n \neq 0$ [*like in the "massless case"*] :

$$b_n = \frac{|n|\xi_{(n)} + iP_\xi^{(n)}}{\sqrt{2|n|}}, \quad b^*_{-n} = \frac{|n|\xi_{(n)} - iP_\xi^{(n)}}{\sqrt{2|n|}}.$$

We use coordinates $\{\boldsymbol{B_m}\}$; $\quad \boldsymbol{B_m} = (b_m, b^*_{-m}, b_{-m}, b^*_m)^{\boldsymbol{T}} \quad \forall m \in \mathbb{N}^+$.

- The S[1]-translations $T_\alpha : \theta \rightarrow \theta + \alpha \quad (\forall \alpha \in S^1)$ have a diagonal action:

$$T_\alpha(b_{\pm m}) = e^{\pm i m \alpha} b_{\pm m}, \quad T_\alpha(b^*_{\pm m}) = e^{-(\pm i) m \alpha} b^*_{\pm m}.$$

- The evolution is block diagonal $\boldsymbol{B}_m(t_1) = U_m(t_1, t_0)\, \boldsymbol{B}_m(t_0)$, with

$$U_m(t_1,t_0) = \begin{pmatrix} u_m(t_1,t_0) & \mathbf{0} \\ \mathbf{0} & u_m(t_1,t_0) \end{pmatrix}, \quad u_m(t_1,t_0) = \begin{pmatrix} \alpha_m(t_1,t_0) & \beta_m(t_1,t_0) \\ \beta^*_m(t_1,t_0) & \alpha^*_m(t_1,t_0) \end{pmatrix}.$$

New field parametrization (quantization)

- $U_m(t_1, t_0)$ is a Bogoliubov transformation. Defining $x_m^{(i)} = m t_i$,

$$\alpha_m(t_1, t_0) = c(x_m^{(1)}) c^*(x_m^{(0)}) - d(x_m^{(1)}) d^*(x_m^{(0)}), \quad \beta_m(t_1, t_0) = d(x_m^{(1)}) c(x_m^{(0)}) - c(x_m^{(1)}) d(x_m^{(0)}),$$

$$c(x_m^{(i)}) = \sqrt{\frac{\pi x_m^{(i)}}{8}} \left[\left(1 + \frac{i}{2 x_m^{(i)}} \right) H_0(x_m^{(i)}) - i H_1(x_m^{(i)}) \right], \qquad d(x_m^{(i)}) = \sqrt{\frac{\pi x_m^{(i)}}{2}} H_0^*(x_m^{(i)}) - c^*(x_m^{(i)}).$$

- We adopt the complex structure $J_0(\boldsymbol{B}_m) = (J_0)_m \boldsymbol{B}_m$, $(J_0)_m = diag(i, -i, i, -i)$.

We construct the one-particle Hilbert space determined by J_0, and with it the symmetric Fock space. The vacuum $|0\rangle$ satisfies $\hat{b}_{\pm m}|0\rangle = 0$.

- J_0 is invariant under the S^1-translations T_α.

One gets an invariant unitary implementation of the gauge group $(\hat{T}_\alpha |0\rangle = |0\rangle)$.

- Furthermore, one can check that $\sum_m |\beta_m(t_1, t_0)|^2 < \infty \quad \forall\, t_1, t_0 \neq 0$.

Therefore, the **evolution is unitarily implemented**.

Uniqueness of the quantization (for ξ)

- Are there other Fock representations with a unitary dynamics for (ξ, P_ξ)?

We consider compatible complex structures that are **invariant** under the group of **S¹-translations.**

- All such complex structure can be obtained as $J = K_J J_0 K_J^{-1}$ where K_J is a symplectic transformation with the block diagonal form in the $\{\boldsymbol{B}_m\}$ basis

$$(K_J)_m = \begin{pmatrix} (k_J)_m & \mathbf{0} \\ \mathbf{0} & (k_J)_m \end{pmatrix}, \quad (k_J)_m = \begin{pmatrix} \kappa_m & \lambda_m \\ \lambda_m^* & \kappa_m^* \end{pmatrix}, \quad \kappa_m > 0, \; |\kappa_m|^2 - |\lambda_m|^2 = 1 .$$

- The evolution $U(t_1, t_0)$ admits a unitary implementation $\forall\, t_1$ w.r.t. J iff so does $U^J(t_1, \tilde{t}_0) := K_J^{-1} U(t_1, \tilde{t}_0) K_J$ $\forall\, t_1, \tilde{t}_0$ w.r.t. J_0. We get the blocks

$$U_m^J(t_1, \tilde{t}_0) = \begin{pmatrix} u_m^J(t_1, \tilde{t}_0) & \mathbf{0} \\ \mathbf{0} & u_m^J(t_1, \tilde{t}_0) \end{pmatrix}, \quad u_m^J(t_1, \tilde{t}_0) = \begin{pmatrix} \alpha_m^J(t_1, \tilde{t}_0) & \beta_m^J(t_1, \tilde{t}_0) \\ \beta_m^{J*}(t_1, \tilde{t}_0) & \alpha_m^{J*}(t_1, \tilde{t}_0) \end{pmatrix},$$

$$\beta_m^J(t_1, \tilde{t}_0) = 2i\,\Im[\alpha_m(t_1, \tilde{t}_0)]\,\kappa_m^* \lambda_m + (\kappa_m^*)^2 \beta_m(t_1, \tilde{t}_0) - \lambda_m^2 \beta_m^*(t_1, \tilde{t}_0).$$

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Uniqueness of the quantization (for ξ)

- From the expression of $\beta_m^J(t_1,t_0)$ one gets that, $\forall a\geq 0,\ N\in\mathbb{N}^+$,

$$\sum_{m=1}^{N}\frac{2a|\lambda_m|^2}{1+a}\left\{1-(\Re[\alpha_m(t_1,\tilde{t}_0)])^2-a|\beta_m(t_1,\tilde{t}_0)|^2\right\}\leq\sum_{m=1}^{N}\left(2|\beta_m(t_1,\tilde{t}_0)|^2+|\beta_m^J(t_1,\tilde{t}_0)|^2\right).$$

The coefficients of $|\lambda_m|^2$ are **oscillating** functions of time.

- ✔ We use that (by assumption) the evolution is unitary w.r.t. both J_0 and J.
- ✔ We use that, for sufficiently large $\tilde{t}_0$, $|\alpha(\tilde{t}_0+\tau,\tilde{t}_0)-e^{im\tau}|<\epsilon\quad\forall\tau,\epsilon>0.$
- ✔ We average over τ on a suitable subset of $[0,\pi]$ (using Egorov's theorem).

- Then, one can show that the sequence $\left\{\sum_{m=1}^{N}|\lambda_m|^2,\ N\in\mathbb{N}^+\right\}$ is bounded.

Hence, the symplectic transformation K_J is implemented unitarily.

All compatible complex structures that are **invariant** under S^1-translations and allow a **unitary** implementation of the **dynamics** of (ξ, P_ξ) determine **unitarily equivalent Fock representations.**

Uniqueness (field parametrization)

- May another field parametrization lead to a non-equivalent Fock quantization with unitary dynamics and a natural unitary implementation of S^1-translations?

- We consider field parametrizations of the induced gauge-fixed metric which
  a) are local, b) are independent of the spatial coordinates,
  c) decouple the field, d) lead to a homogeneous Klein-Gordon equation.

Alternate field pairs are given by the time-dependent canonical transformations

$$\tilde{\xi} = f(t)\xi, \quad \tilde{P}_{\xi} = \frac{P_{\xi}}{f(t)} + g(t)\xi.$$

Absorbing a constant linear transformation, one can set $f(t_0)=1$, $g(t_0)=0$, so that the canonical pairs coincide at the reference time.

- The dynamics of $(\tilde{\xi}, \tilde{P}_{\xi})$ is unitary w.r.t. an **invariant** complex structure $J = K_J J_0 K_J^{-1}$ **iff** $\tilde{U}^J(t_1, t_0) := K_J^{-1} C(t_1) U(t_1, t_0) K_J$ is unitary w.r.t. J_0.
  $C(t_1)$ comes from the change of canonical pair at t_1.

Uniqueness (field parametrization)

- $C(t_1)$ has the 2x2 block-diagonal form:

$$C_m(t_1)=\frac{1}{2m}\begin{pmatrix} m f_+(t_1)+i g(t_1) & m f_-(t_1)+i g(t_1) \\ m f_-(t_1)-i g(t_1) & m f_+(t_1)-i g(t_1) \end{pmatrix}, \quad f_\pm(t_1)=f(t_1)\pm\frac{1}{f(t_1)}.$$

- One can prove that the β-coefficients of $\tilde{U}^J(t_1,t_0)$ do not go to zero $\forall\, t_1>0$ when $m\to\infty$ unless $f(t)$ is the **unit function**. This eliminates the *old* parametrization. Unitarity **selects** the canonical pairs $\left(\xi,\ \tilde{P}_\xi := P_\xi + g(t)\xi\right)$.

- From the expression of the β-coefficients of $\tilde{U}^J(t_1,t_0)$, one can also see that the dynamics of $(\xi,\tilde{P}_\xi)$ is unitary w.r.t. $J=K_J J_0 K_J^{-1}$ **iff** so is that of (ξ,P_ξ).

Therefore, there is a **unique S¹-translation invariant Fock representation** of the fields at t_0 such that the evolution of [all the pairs] $(\xi,\tilde{P}_\xi)$ is unitary.

- Asking that there exist a complex structure such that the vacuum is in the domain of the evolution generator (in the Schrödinger picture) fixes $g(t)=0$.

Conclusion

- We have introduced a new field parametrization for the Gowdy metric and completed it into a time dependent canonical transformation. In this way, we have redistributed the time dependence into an explicit and an implicit part, the latter being unitarily implementable in the quantum theory.

- There exists certain freedom in the deparametrization. Once a time gauge is chosen, the Fock representation is essentially **unique** if one demands a **unitary dynamics** and **invariance under S^1-translations**.

- The physics of the quantum cosmolocial model does not depend on the choice of **field parametrization** or **complex structure**.

- For any distinct (non-Fock) kind of quantization, there ought to exist a regime in which one would recover the Fock quantum theory.

- Our result shows that it is possible to attain uniqueness in standard quantum field theory even for non-stationary systems.

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