Cartan equations define a topological field theory

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Outline

- BF theory

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- General relativity as a constrained BF theory

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- Remarks and perspectives

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$$G_{IJ} := D_a \Pi^a{}_{IJ} \approx 0, \quad \Psi^a{}_{IJ} := \frac{1}{2} \varepsilon^{abc} F_{IJ\,bc} \approx 0$$

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- JMP **47** (2006) 022301, CQG **23** (2006) 2267

General relativity as a constrained BF theory

- Plebański (1977)

$$\Sigma^i = e^0 \wedge e^i + \frac{i}{2}\varepsilon^i{}_{jk} e^j \wedge e^k,$$

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- Einstein's equations:

$$D\Sigma^i := d\Sigma^i + \varepsilon^i{}_{jk} A^j \wedge \Sigma^k = 0,$$

$$\Sigma^i \wedge \Sigma^j = \frac{\delta^{ij}}{3}\left(\Sigma^k \wedge \Sigma_k\right),$$

$$F^i = C^i{}_j \Sigma^j, \quad \mathrm{Tr}\left(C^i{}_j\right) = C^i{}_i = 2i\lambda$$

General relativity as a constrained BF theory

- Action principle

$$S=\int\left[\Sigma_i\wedge F^i[A]-\frac{1}{2}C_{ij}\Sigma^i\wedge\Sigma^j-\rho\left(C^i{}_i-2i\lambda\right)\right]$$

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Cartan dynamics

- *Cartan's first structure equations*

$$de^I + \omega^I{}_J \wedge e^J = T^I, \tag{7}$$

Cartan's second structure equations

$$d\omega^I{}_J + \omega^I{}_K \wedge \omega^K{}_J \;=\; R^I{}_J, \tag{8}$$

the *first Bianchi identities*

$$dT^I + \omega^I{}_J \wedge T^J \;=\; R^I{}_J \wedge e^J, \tag{9}$$

and the *second Bianchi identities*

$$dR^I{}_J + \omega^I{}_K \wedge R^K{}_J - \omega^K{}_J \wedge R^I{}_K = 0. \tag{10}$$

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- Equations (7)-(10) are interpreted as equations of motion for a dynamical system that will be called *Cartan theory*, *Cartan dynamics* or *Cartan fields*. The solutions of (7)-(10) define, by construction, the space of solutions of the theory.

Cartan dynamics

- Action principle

$$S[e, \omega, T, R] =$$
$$c_2 \int \left[(e^I \wedge e^J) \wedge \left(d\omega_{IJ} + \omega_{IK} \wedge \omega^K{}_J \right) \right.$$
$$\left. -2T_I \wedge De^I + T_I \wedge T^I \right]$$
$$+ \int \left(c_1 R^{IJ} + c_3 {*R}^{IJ} \right) \wedge$$
$$\left(d\omega_{IJ} + \omega_{IK} \wedge \omega^K{}_J - \frac{1}{2} R_{IJ} \right) \qquad (11)$$

Cartan dynamics

- Equivalent action principle

$$S_3[e,\omega] = \int (c_2 L_{NY} + c_1 L_P + c_3 L_E)$$

$$L_{NY} = (e^I \wedge e^J) \wedge \left(d\omega_{IJ} + \omega_{IK} \wedge \omega^K{}_J\right)$$
$$- \eta_{IJ} De^I \wedge De^J$$

$$L_P = \frac{1}{2}\left(d\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}\right) \wedge$$
$$\left(d\omega_{IJ} + \omega_{IL} \wedge \omega^L{}_J\right)$$

$$L_E = \frac{1}{4}\varepsilon_{IJKL}\left(d\omega^{IJ} + \omega^I{}_M \wedge \omega^{MJ}\right) \wedge$$
$$\left(d\omega^{KL} + \omega^K{}_P \wedge \omega^{PL}\right)$$

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