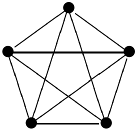
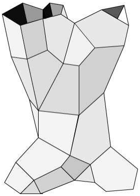
Computations involving spin networks, spin foams, quantum gravity and lattice gauge theory



Dan Christensen
University of Western Ontario
and many others

Loops '07
26 June 2007



Outline:

- Barrett-Crane model: behaviour, positivity, q-deformed version
- $10j$ symbol: asymptotics, graviton propagator
- Lattice gauge theory using spin foam methods

The Riemannian Barrett-Crane model

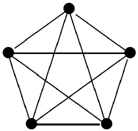
Let Δ be a triangulation of a closed 4-manifold. $\mathcal{F}$ = dual faces = triangles, $\mathcal{E}$ = dual edges = tets, $\mathcal{V}$ = dual vertices = 4-simplices.

A spin foam F is an assignment of a spin j_f to each dual face $f \in \mathcal{F}$.

The amplitude of F is

$$\mathcal{A}(F) := \left(\prod_{f\in\mathcal{F}} \mathcal{A}_f\right)\left(\prod_{e\in\mathcal{E}} \mathcal{A}_e\right)\left(\prod_{v\in\mathcal{V}} \mathcal{A}_v\right),$$

where

$\mathcal{A}_v$ =  = $10j$ symbol

and $\mathcal{A}_e$ and $\mathcal{A}_f$ are normalization factors that depend on the model.

Early computations

gr-qc/0202017, Baez-C-Halford-Tsang

Take Δ to be the simplest triangulation of the 4-sphere, as the boundary of the 5-simplex.

Using the Metropolis algorithm, we computed the expectation value of the average area of a triangle:

$$\langle O \rangle = \frac{\displaystyle\sum_F O(F)\mathcal{A}(F)}{\displaystyle\sum_F \mathcal{A}(F)} \qquad \text{where} \qquad O(F) = \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} \sqrt{j_f(j_f+1)}$$

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The results showed very strong dependence on the normalization factors:

- For the Perez-Rovelli model, spin zero dominanace.
- For the De Pietri-Freidel-Krasnov-Rovelli model, divergence.

Positivity

gr-qc/0110044, Baez-C

It was only after doing the above computations that it dawned on us that the amplitudes we were computing were always positive real numbers!

At first we suspected an error, but eventually we proved mathematically that this is correct.

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From a computational point of view, this was good news, because it meant that there was no sign problem in the Metropolis algorithm.

But conceptually it raised lots of questions as it meant that there was no interference in the path integral. This highlighted the interpretation of the path integral as a projection onto physical states.

q-deformed version

arXiv:0704.0278, C-Khavkine

The q-deformed Barrett-Crane model replaces the group SU(2) by the quantum group $\mathrm{SU}_q(2)$. When $q = \exp(i\pi/r)$ is a root of unity, this regularizes the theory by eliminating spins greater than $(r-2)/2$. As $r \to \infty$, $q \to 1$, the undeformed value.

Also, it has been suggested by Smolin that r is related to the cosmological constant:

$$\Lambda \sim 1/r.$$

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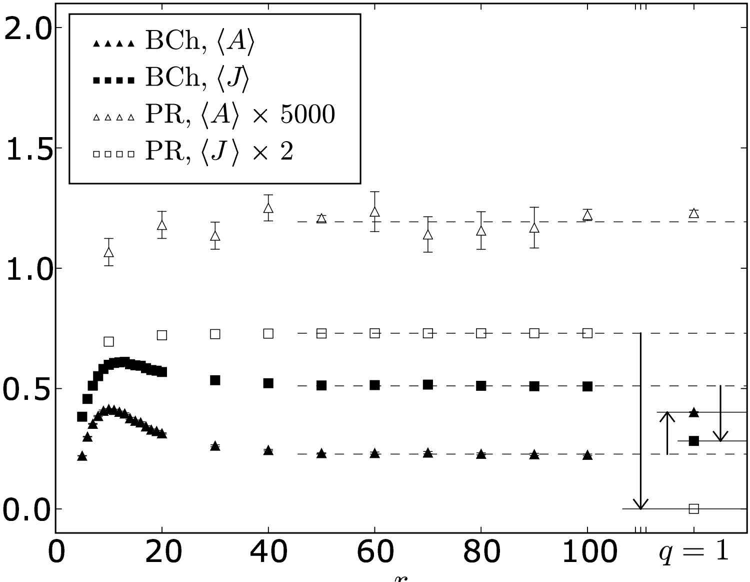
We have recently done computations of expectation values which greatly generalize earlier work:

- The deformation parameter q can be varied.
- The triangulation can be varied, and can be large.
- Several different observables have been used.

q-deformed results

arXiv:0704.0278, C-Khavkine



See Igor Khavkine's talk later today for details.

Asymptotics

gr-qc/9809032 Barrett-Williams

Since the $10j$ symbol is the key ingredient of the Barrett-Crane model, it has been well studied. It can be computed as an integral:

$$\int_{S^3}\int_{S^3}\int_{S^3}\int_{S^3}\int_{S^3}\prod_{1\le k<l\le 5} K_{j_{kl}}(\phi_{kl})\ dx_1\cdots dx_5,$$

where ϕ_{kl} is the angle between the unit vectors x_k and x_l, and

$$K_j(\phi) := \frac{\sin((2j+1)\phi)}{\sin(\phi)}.$$

The spins j_{kl} label the triangles of a 4-simplex, giving them each area $2j_{kl}+1$. The x_k can be thought of as normals to the 5 tetrahedra.

Barrett and Williams studied this integral for large spins. They showed that the stationary phase points correspond to 4-simplices with the prescribed triangle areas (up to scale) and that these points contribute according to the Regge action.

Degenerate points gr-qc/0208010, Baez-C-Egan; Barrett-Steele; Freidel-Louapre

We performed computations to verify that the $10j$ symbol behaved asymptotically like the Regge action, and found that this was false.

As the spins are scaled by a factor λ, the contribution from the stationary phase points goes like $\lambda^{-9/2}$. But we observed that the $10j$ symbol goes like λ^{-2}!

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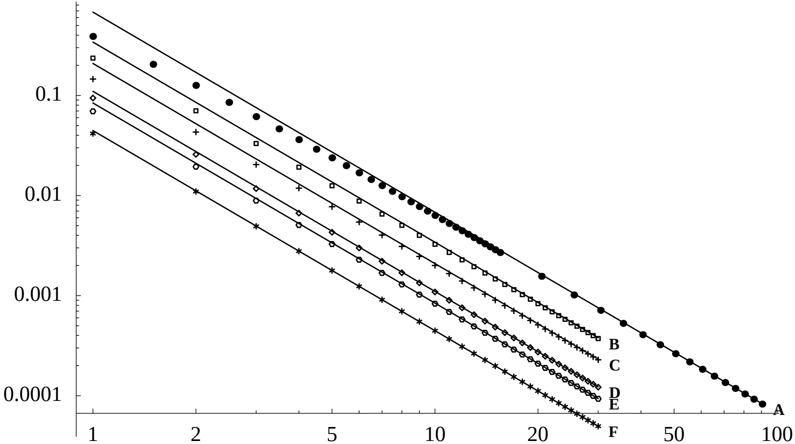
Further analytic study (by several independent groups) showed that this is due to contributions from degenerate 4-simplices, i.e. flat 4-simplices with zero volume. These were noticed but not studied by Barrett and Williams.

This has lead to new proposals for the vertex amplitude in quantum gravity.

Asymptotics

gr-qc/0208010, Baez-C-Egan



The points show the numerical evaluation of six different $10j$ symbols as the scale factor λ (x-axis) is varied. The lines show the asymptotic predictions using degenerate points.

Graviton Propagator

Rovelli, Bianchi, Modesto, Speziale, Livine, Willis, C, ...

Rovelli and others proposed a way to define 2-point functions in the Barrett-Crane model. The leading contribution is of the form

$$W_{ab} = \frac{\sum_{\{j_k\}} h(j_a)\, h(j_b)\, \Psi[j]\, \{10j\}}{\sum_{\{j_k\}} \Psi[j]\, \{10j\}}, \qquad h(j) = j(j+1) - j_0(j_0+1)$$

The sum is over ten spins labelling the triangles of a 4-simplex.
$h(j_a)h(j_b)$ is the field insertion. Ψ is a chosen boundary state.
$\{10j\}$ denotes the $10j$ symbol.

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More concisely:

$$W_{ab} = \frac{1}{\mathcal{N}} \sum_{\{j_k\}} h(j_a)\, h(j_b)\, \Psi[j]\, \{10j\}, \qquad h(j) = j(j+1) - j_0(j_0+1)$$

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Rovelli proposed a Gaussian boundary state:

$$\Psi[j] = \exp(-\frac{\alpha}{2j_0} \sum_k (j_k - j_0)^2 + i\Phi \sum_k j_k)$$

peaked around a regular 4-simplex, where $\alpha \in \mathbb{R}$ is a parameter.
Here j_0 determines the areas of the triangles of the regular 4-simplex, and $\Phi = \arccos(-1/4)$ is the dihedral angle.

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For large j_0, W_{ab} is expected to go as $1/j_0$, and Rovelli argued that this is indeed the case.

In numerical computations it was difficult to see this behaviour, at least in part because the computations were too difficult.

Livine-Speziale boundary state

Livine-Speziale, gr-qc/0608131

Livine and Speziale proposed a different boundary state:

$$\Psi[j] = \prod_k \psi(j_k), \quad \psi(j) = \frac{I_{|j-j_0|}(j_0/\alpha) - I_{j+j_0+1}(j_0/\alpha)}{\sqrt{I_0(2j_0/\alpha) - I_{2j_0+1}(2j_0/\alpha)}} \cos((2j+1)\Phi)$$

Here $I_n(z)$ is the modified Bessel function of the first kind.

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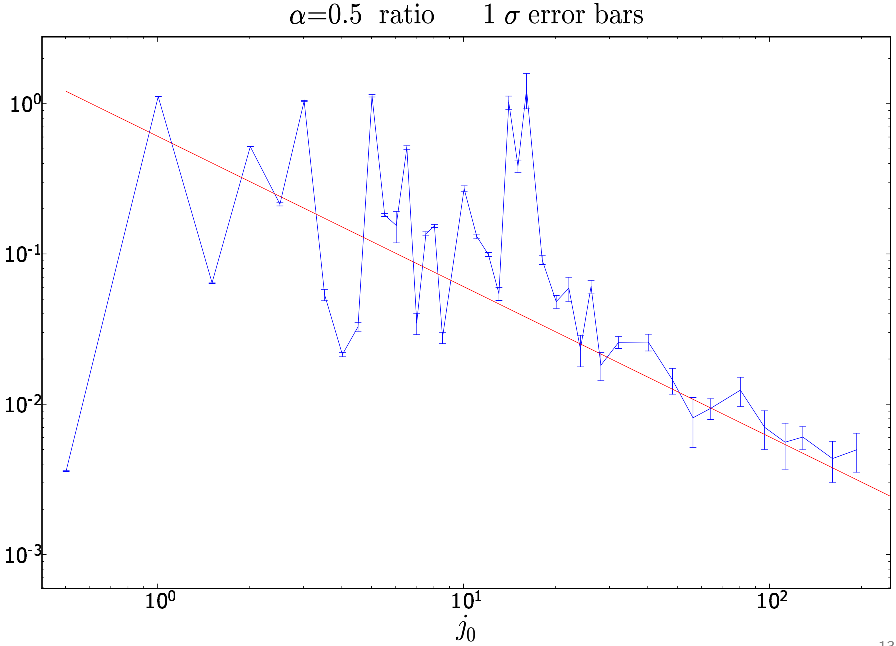
Computations using Livine-Speziale state

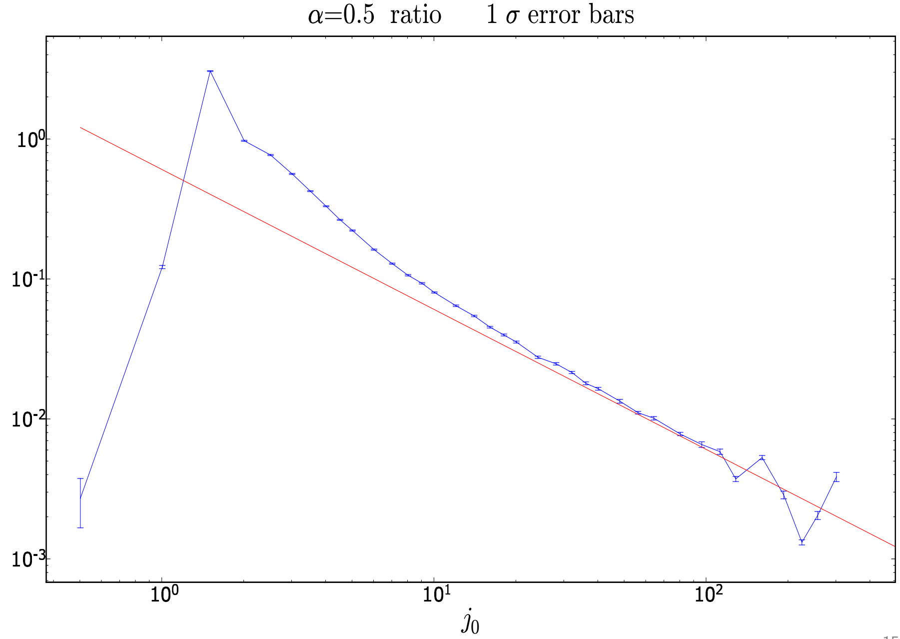


Breaking news

The above state implicitly had $\psi(j) = 0$ when $j - j_0$ is a half-integer. Speziale and I noticed that if you include all j, the Fourier transform is even simpler.

- Physically better boundary state.
- Numerical integration of Fourier transform easier.
- Graph is much cleaner:

Computations using new state



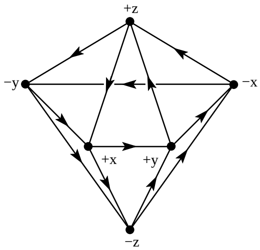
Lattice Gauge Theory

Conrady, Oeckl, Pfeiffer, ...

Many people have observed that spin foam methods can be used to provide a dual formulation of lattice gauge theory.

This is an exact duality. It replaces integrations over group variables labelling edges with summations over representation variables labelling edges and plaquettes (faces).

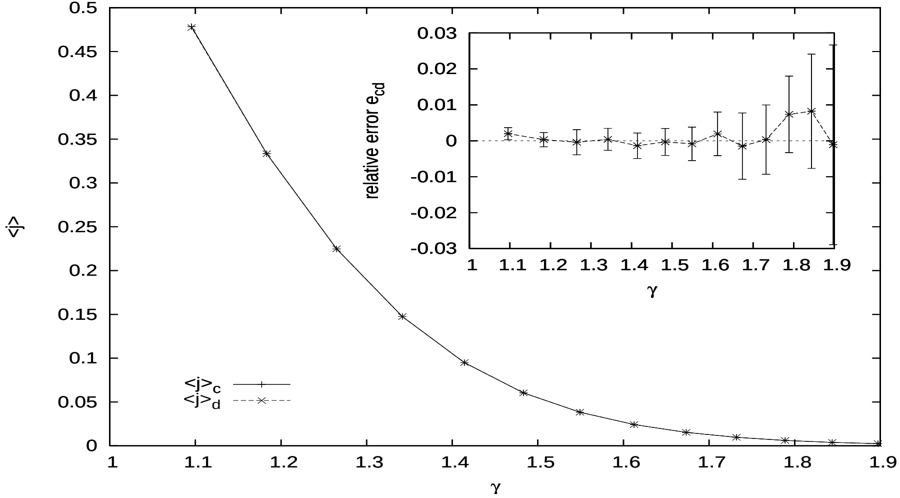
The terms in the summation involve evaluating complicated spin networks, such as the 18*j* symbol:



Computations

arXiv:0705.2629 **v2**, Cherrington-C-Khavkine

We found an efficient algorithm for the $18j$ symbol and performed computations for pure SU(2) Yang-Mills theory on a $D = 3$ cubic lattice. We get agreement with our conventional LGT computations:



See Wade Cherrington's talk, next, and Florian Conrady's talk, later today.

Conclusions

- Computation has repeatedly lead to new and often unexpected insights.
- These facts are often then derived analytically.
- The results of computation can help choose between existing models and can suggest new models.
- Computational techniques from one area (e.g. spin foams and spin networks) can be effective in another area (e.g. lattice gauge theory).

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