Spin foams, gauge–string duality and renormalization

Florian Conrady

Institute for Gravitational Physics and Geometry
Penn State University

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Outline of talk

1. Intro
2. Exact string representation of 3d SU(2) Yang–Mills theory
3. Renormalization in Yang–Mills theory
4. Summary

Gauge–string duality

Conjecture that gauge theory has an equivalent or effective description in terms of string–like degrees of freedom.

- String model of hadron scattering [Nambu, Nielsen, Susskind 1970]
- Strong–coupling expansion, flux lines [Wilson 1974]
- Large N limit ['t Hooft 1974]
- Continuum and lattice models of Nambu–Goto string
- Critical string theory [Polyakov 1981]
- AdS–CFT correspondence [Maldacena 1997]

Renormalization in Yang–Mills theory

Search for an effective low–energy description of gauge theories that incorporates non–perturbative effects.

- perturbative renormalization not sufficient
- non–perturbative effects due to flux lines, monopoles, vortices . . . ?

⇝ confinement and hadron spectrum of QCD?

⇝ mass generation for gauge bosons of electroweak theory? ['t Hooft 1998]

Part I: Exact string representation of 3d SU(2) YM theory

FC, Igor Khavkine, arXiv:0706.3423 [hep-th]

SU(2) lattice gauge theory in 3 dimensions

Partition function on a cubic lattice κ:

$$Z = \int \left(\prod_{e \subset \kappa} \mathrm{d} U_e \right) \exp\Big(-\beta\, \mathcal{S}(\{U_e\}) \Big)$$

Coupling:

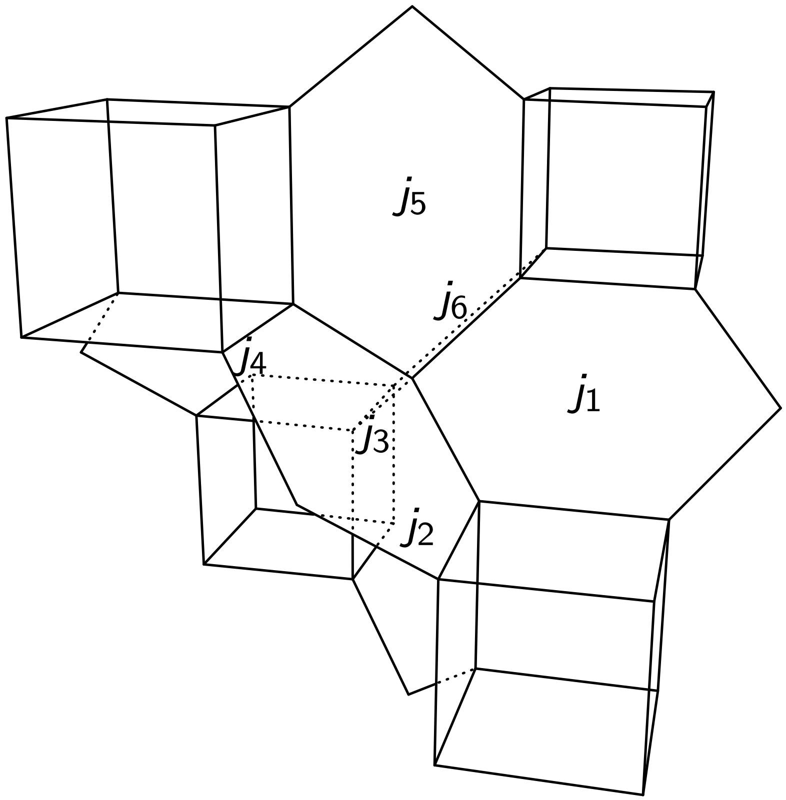
$$\beta = \frac{4}{a g^2} + \frac{1}{3}$$

The action is a sum of face/plaquette actions:

$$\mathcal{S} = \sum_{f \subset \kappa} \mathcal{S}_f$$

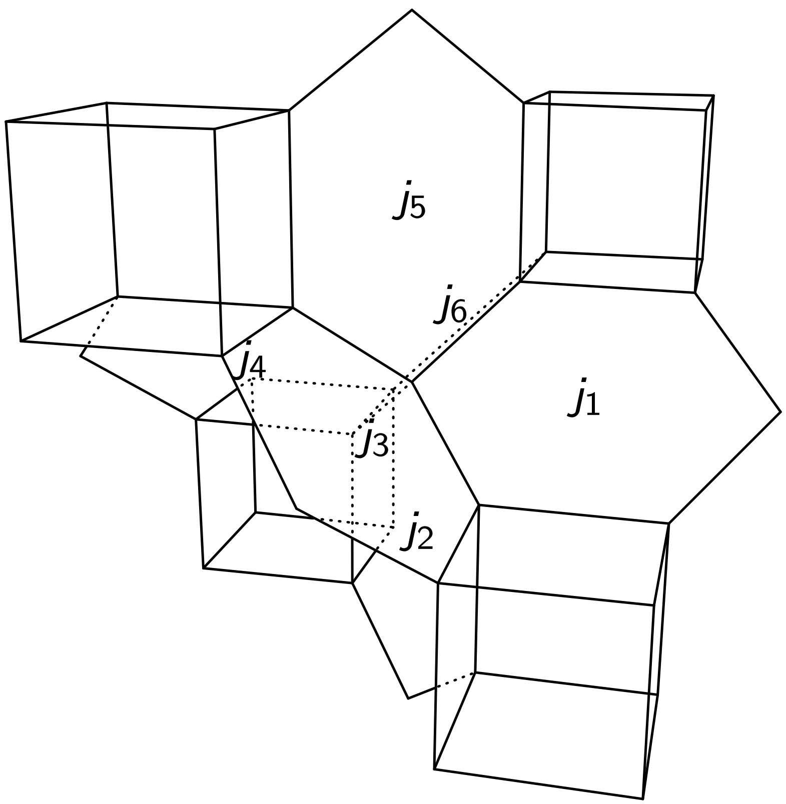
Spin foam representation

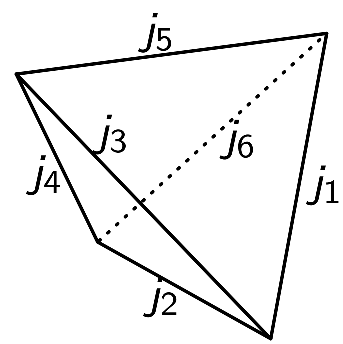
The amplitudes of the corresponding spin foam sum can be factorized into $6j$–symbols. Then, the partition function appears like a sum over spin foams on a modified lattice $\tilde{\kappa}$:



Every edge is shared by three faces.

Spin foam representation



$A_v =$ 

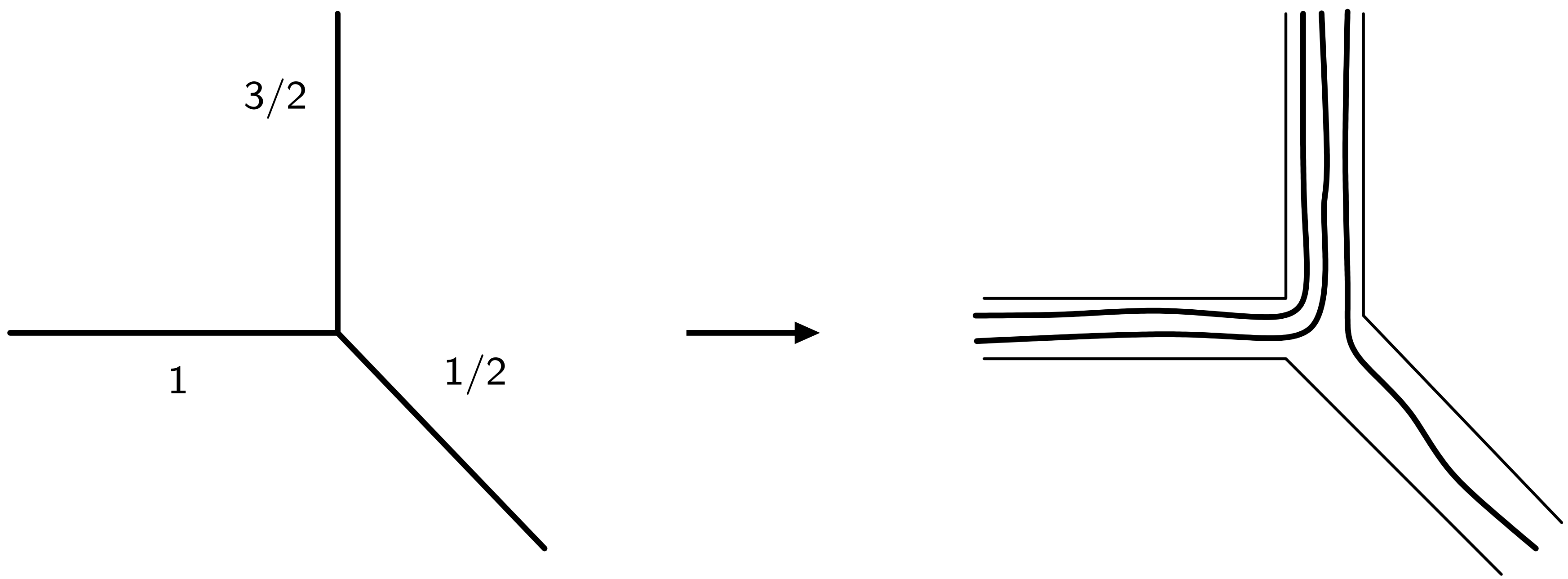
$= \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$

$$Z = \sum_{F \mid \partial F = \emptyset} \left(\prod_{f \subset \tilde{\kappa}} (2 j_f + 1) \right) \left(\prod_{v \subset \tilde{\kappa}} A_v \right) \left(\prod_{f \subset \kappa} (-1)^{2 j_f} \, \mathrm{e}^{-\frac{2}{\beta} j_f (j_f + 1)} \right)$$

Idea for string representation

For spin networks in boundary $\partial\tilde{\kappa}$:

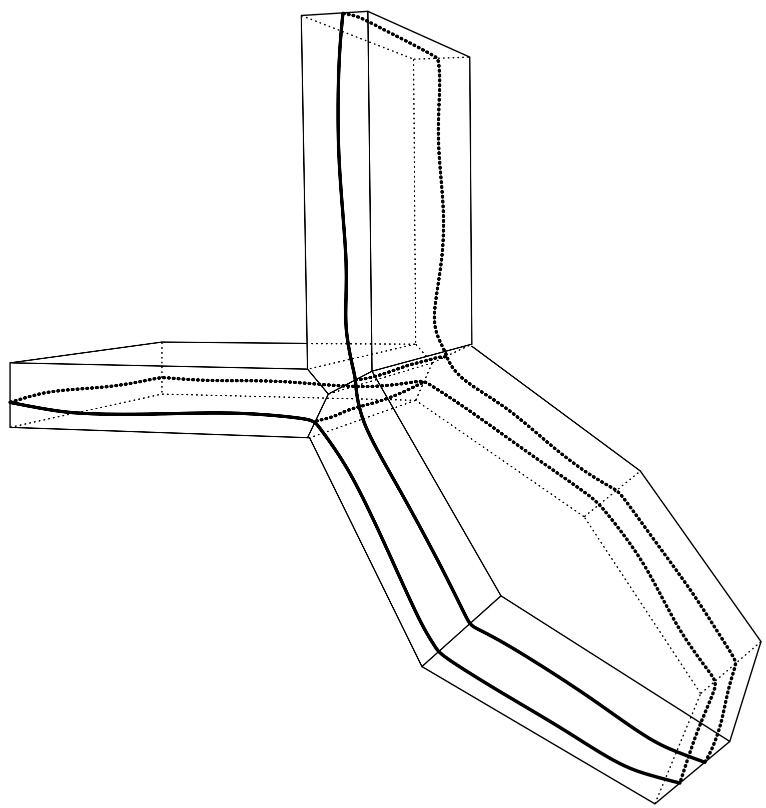
- Thickening (or framing) of 1–skeleton of boundary.
- Fill each thickened edge of spin j_e with $N_e = 2j_e$ lines without label.



Idea for string representation

For spin foams:

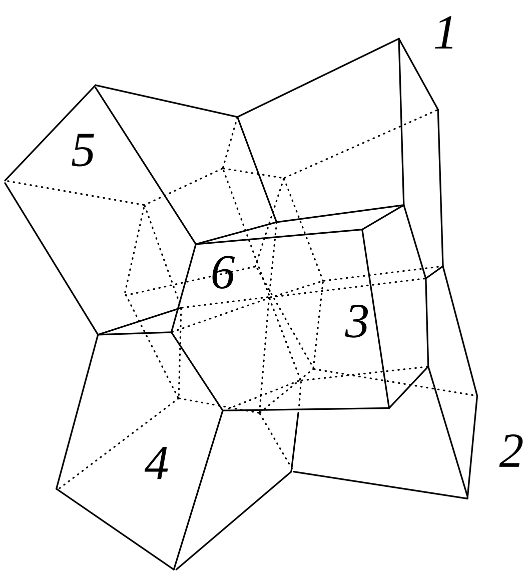
- Thickening (or framing) of 2–skeleton of lattice $\tilde{\kappa}$.
- Fill each thickened face of spin j_f with $N_f = 2j_f$ disks without label.

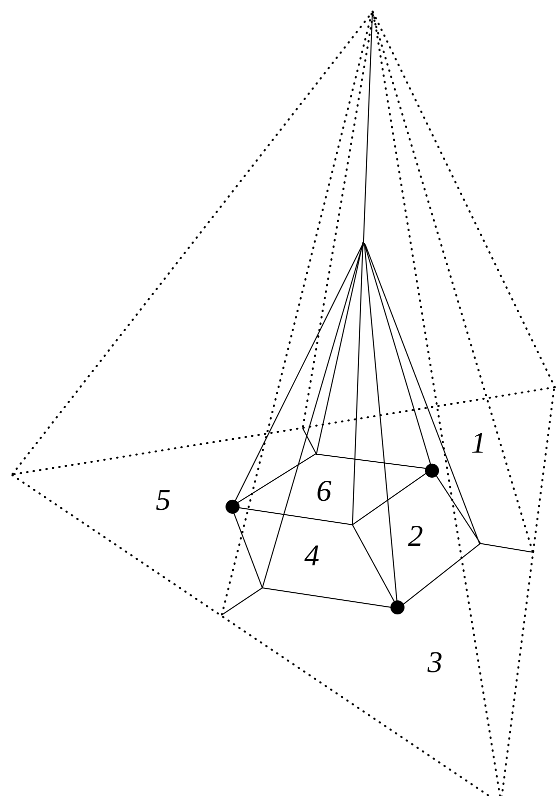


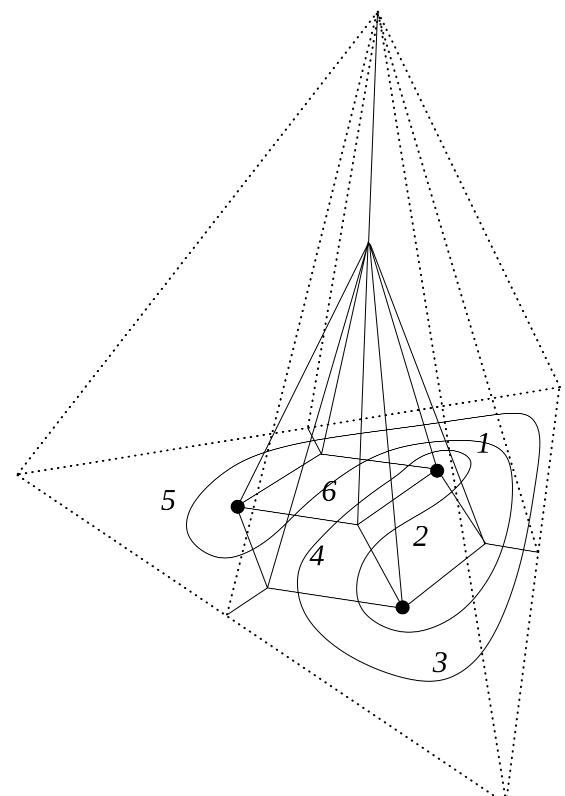
Glueing at vertices

Only difficult part:

Glueing of surfaces near vertices.







Bijection between worldsheets and spin foams

We restrict the surfaces such that

- thickened faces are only intersected transversely, and
- the intersections are disks.

We also take equivalence classes under homeomorphisms that preserve these conditions. Each equivalence class is called a worldsheet.

> We then prove that there is a bijection between the set of such worldsheets and the set of spin foams on $\tilde{\kappa}$.

FC, Igor Khavkine, arXiv:0706.3423 [hep-th]

String representation of partition function

Thus, we can translate spin foam sums into sums over worldsheets, and get an exact string representation of the lattice Yang–Mills theory:

$$Z = \sum_{w \mid \partial w = \emptyset} \left(\prod_{f \subset \tilde{\kappa}} (N_f + 1) \right) \left(\prod_{v \subset V} A_v(\{N_f/2\}) \right) \left(\prod_{f \subset \kappa} (-1)^{N_f} \, \mathrm{e}^{-\frac{1}{2\beta} N_f (N_f + 2)} \right)$$

Here, N_f is the number of sheets in each cell.

Difference to Nambu–Goto string

When a worldsheet does not run more than once through faces (i.e. when $N_f \leq 1$), the $6j$–symbols in the amplitude become trivial and the exponent in the amplitude is proportional to the area of the worldsheet. In these cases, the weighting resembles that of the Nambu–Goto string.

In general, however, a worldsheet intersects several times with the same cell, and then we have an interaction due to nonlinear dependences on N_f. That is, in addition to interactions by merging and splitting, there is an interaction of directly neighouring strings.

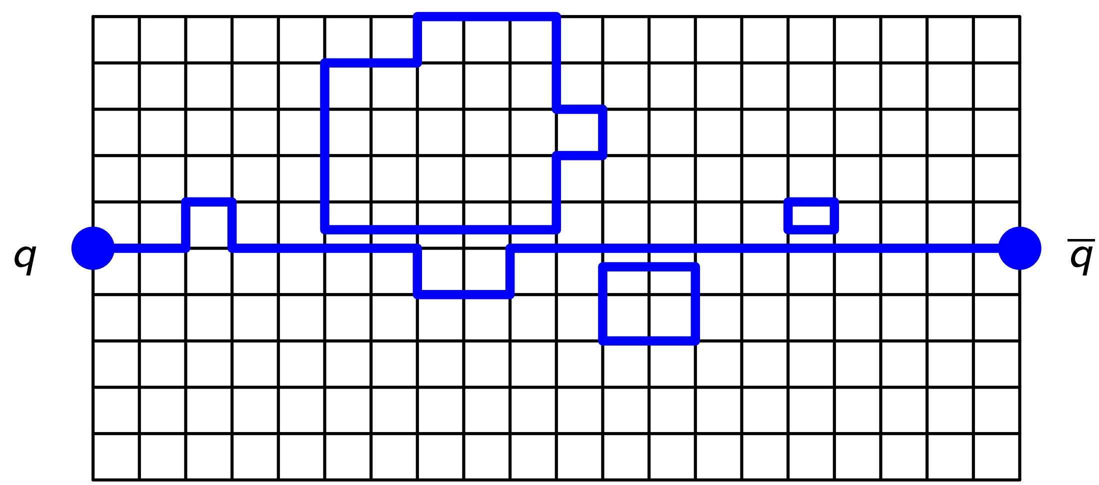
Part II: Renormalization in Yang–Mills theory

SU(2) lattice gauge theory in the continuum limit

Assume that we start from the SU(2) lattice gauge theory

$$\langle O \rangle = \int \left(\prod_{e \subset \kappa} \mathrm{d}U_e \right) O(\{U_e\}) \exp\Big(-\beta\, \mathcal{S}(\{U_e\}) \Big)$$

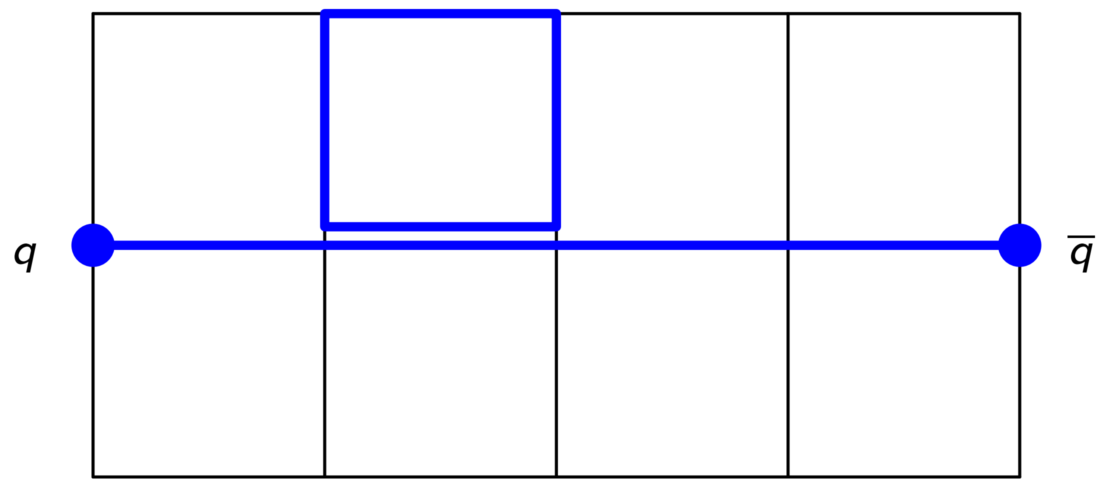
in the continuum limit. I.e. the physical length of the cutoff a is much smaller than the characteristic length scale L of the observable O.



Lattice gauge theory at strong coupling

To first approximation, the effective theory at scale L can be described by the same path integral with lattice cutoff $a' \simeq L$ and a larger coupling g':

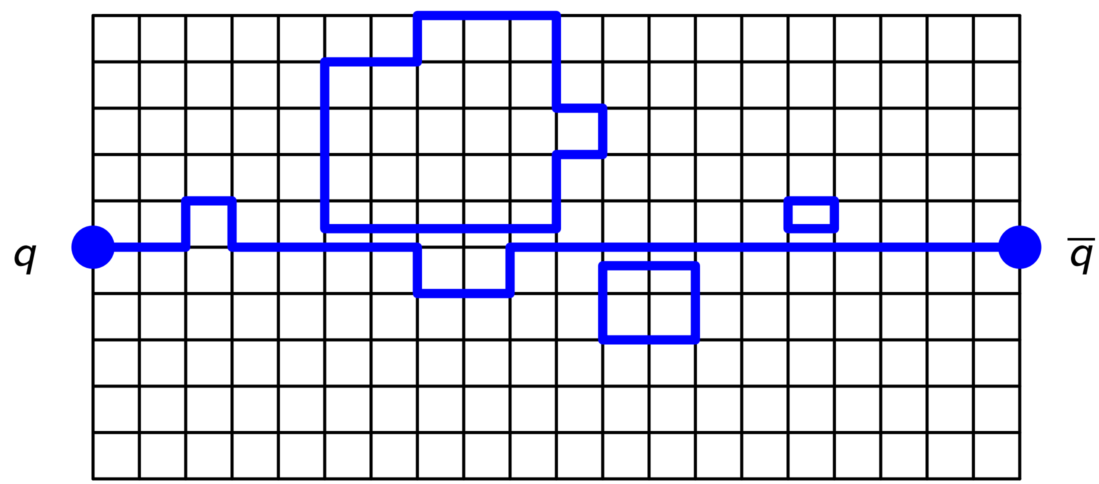
$$\langle O \rangle = \int \left(\prod_{e \subset \kappa'} \mathrm{d}U_e \right) \; O(\{U_e\}) \, \exp\Big(-\beta' \, \mathcal{S}(\{U_e\}) \Big)$$



Problem: This is not derived from the full theory, and the connection to the UV physics is missing.

Effective spin foam representation?

It would be preferable to start from the theory with small cutoff a, and integrate out the high energy fluctuations.

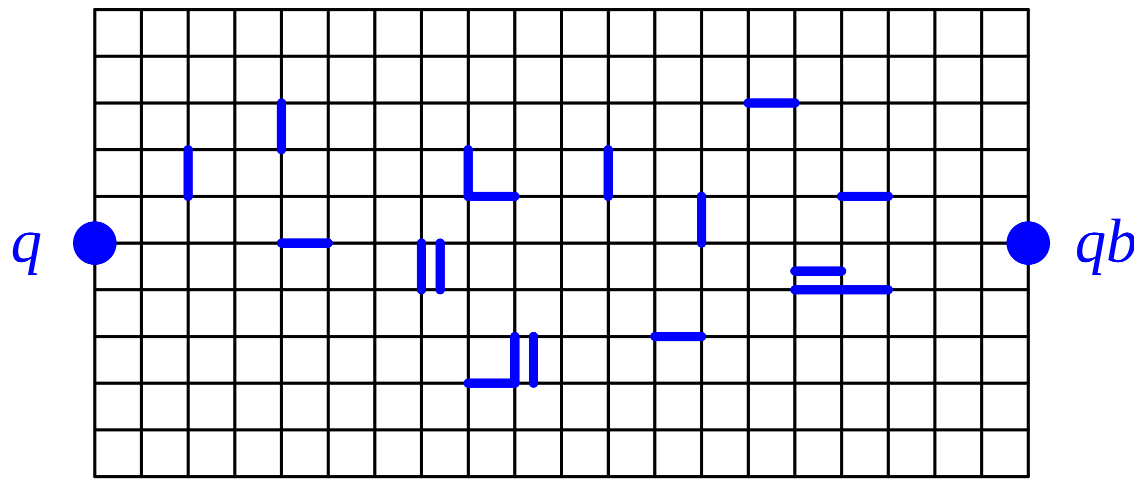


What would be the form of the effective spin foam representation?

First–order representation

When we expand the plaquette actions into characters, we obtain a first–order formulation with variables U_e and j_f.

$$Z = \int \left(\prod_e \mathrm{d}U_e\right) \sum_{\{j_f\}} \left(\prod_f (2j_f + 1)\, \chi_{j_f}(U_f)\, \mathrm{e}^{-\frac{2}{\beta} j_f (j_f+1)}\right)$$



Integrating out U_e leads to the spin foam representation.

BF Yang–Mills representation

Alternatively, we can use the Kirillov trace formula

$$\chi_j(U_f) = \frac{(2j+1)|\omega_f|/2}{4\pi\sin(|\omega_f|/2)} \int_{S^2} \mathrm{d}n\; \mathrm{e}^{\mathrm{i}\,(2j+1)\,n\cdot\omega_f/2}\,, \qquad U_f = \mathrm{e}^{\mathrm{i}\,\omega_f^a\sigma^a/2}\,,$$

and rewrite the path integral as

$$Z = \int \left(\prod_e \mathrm{d}U_e\right) \sum_{\{j_f\}} \left(\prod_f 2j_f\right) \int_{S^2} \left(\prod_f \mathrm{d}n_f\right)$$
$$\times \left(\prod_f \frac{|\omega_f|/2}{\sin(|\omega_f|/2)}\right) \exp\left[\sum_f \left(\mathrm{i}\, j_f\, n_f\cdot\omega_f - \frac{2}{\beta} j_f^2\right)\,.\right]$$

We approximated $2j_f + 1 \approx 2j_f$ and $j_f(j_f+1) \approx j_f^2$.

BF Yang–Mills representation

If we define vectors $b_f = j_f n_f$, the path integral looks like a BF Yang–Mills theory on the lattice:

$$Z = \int \left(\prod_e \mathrm{d}U_e\right) \int_{\mathbb{R}^3} \left(\prod_f \mathrm{d}^3 b_f \sum_{j\in\mathbb{Z}^+/2} \delta(|b_f| - j)\right)$$
$$\times \left(\prod_f \frac{|\omega_f|/2}{\sin(|\omega_f|/2)}\right) \exp\left[\sum_f \left(\mathrm{i}\, b_f \cdot \omega_f - \frac{2}{\beta}\, b_f^2\right)\right]$$

Since spins are discrete, the lengths of the b–vectors are constrained to be integers or half–integers.

Monopole–like excitations

By applying the Poisson summation formula, we can get rid of the discreteness constraint on the b–field, and get instead a $\mathbb{Z}$–valued 2–form m that is similar to monopoles of U(1):

$$Z = \int \left(\prod_e \mathrm{d}U_e\right) \int_{\mathbb{R}^3} \left(\prod_f \mathrm{d}^3 b_f\right) \sum_{\{m_f\}} \times \left(\prod_f \frac{|\omega_f|/2}{\sin(|\omega_f|/2)}\right) \exp\left[\sum_f \left(\mathrm{i}\, b_f \cdot \omega_f - \frac{2}{\beta}\, b_f^2 + 4\pi\mathrm{i}\, |b_f| m_f\right)\right]$$

Block spin transformation

A possible technique for renormalization are *block spin transformations*.

Let us introduce block variables

$$\overline{b}_{f'} = \frac{1}{N} \sum_{f \subset f'} b_f \equiv \langle b \rangle_{f'}$$

and

$$\overline{U}_{e'} = \prod_{e \subset e'} U_e$$

that live on a coarser lattice κ'.

Block spin transformation

Then, we can rewrite the path integral as

$$Z = \int \left(\prod_{e'} \mathrm{d}\overline{U}_{e'} \right) \int_{\mathbb{R}^3} \left(\prod_{f'} \mathrm{d}^3 \overline{b}_{f'} \right) \; \tilde{Z}(\{\overline{U}_{e'}\}, \{\overline{b}_{f'}\}) \,,$$

where

$$\begin{aligned} &\tilde{Z}(\{\overline{U}_{e'}\}, \{\overline{b}_{f'}\}) \\ &= \int \left(\prod_{e} \mathrm{d} U_e \right) \int_{\mathbb{R}^3} \left(\prod_{f} \mathrm{d}^3 b_f \right) \sum_{\{m_f\}} \delta\left(\overline{b}_{f'} - \langle b \rangle_{f'}\right) \, \delta\left(\overline{U}_{e'} - \prod_{e \subset e'} U_e \right) \\ &\times \left(\prod_f \frac{|\omega_f|/2}{\sin(|\omega_f|/2)} \right) \exp\left[\sum_f \left(\mathrm{i}\, b_f \cdot \omega_f - \frac{2}{\beta} b_f^2 + 4\pi \mathrm{i} \, |b_f| m_f \right) \right] . \end{aligned}$$

Perturbation theory to integrate out UV variables?

Since the theory is weakly coupled in the UV regime, one could try to integrate out the UV variables by perturbation theory.

In principle, we dispose of techniques to do perturbation theory with the above path integral.

We can borrow the gauge–fixing method from the work by Freidel and Louapre on 3d gravity.

Freidel, Louapre, Nucl.Phys. B662, 279, 2003

Zeroth order

If we just consider the partition function without block variables, the zeroth order of perturbation theory gives

$$Z = \int_{\mathbb{R}^3} \left(\prod_v \mathrm{d}^3 \varphi_v \right) \sum_{m_e} \times \exp \left[\sum_e \left(-\frac{2}{\beta} (\mathrm{d}\varphi_e)^2 + 4\pi \mathrm{i} \, |\mathrm{d}\varphi_e| \, m_e \right) \right] .$$

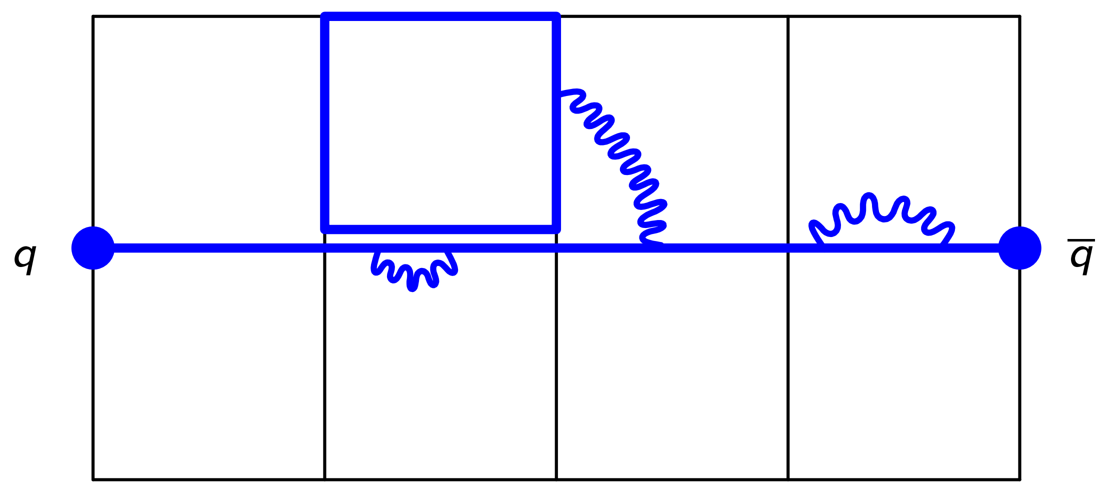
FC, hep-th/0610236

This expression is similar to the photon–monopole representation of U(1) lattice gauge theory. Could it be evaluated by similar methods?

Effective spin foam representation?

After the block spin transformation, one would integrate out the block connection $U_{e'}$ and obtain an effective spin foam representation.

An artists impression of the result:



Summary

- We showed that 3d SU(2) lattice Yang–Mills theory has an exact string representation:
  - conceptually interesting
  - surface picture might be useful for Monte Carlo simulations ($\rightarrow$ nonlocal moves)
- We sketched a possible approach to renormalization in SU(2) lattice Yang–Mills theory:
  - incorporation of non–perturbative effects ($\rightarrow$ monopoles and block variables as backgrounds)
  - computations feasible? right choice of block variables?
  - aims: confinement in QCD, mass generation in electroweak theory

References

- F.Conrady, I.Khavkine, An exact string representation of 3d SU(2) lattice Yang–Mills theory, arXiv:0706.3423 [hep-th].
- F.Conrady, Analytic derivation of dual gluons and monopoles from SU(2) lattice Yang-Mills theory
  - I. BF Yang-Mills representation, hep-th/0610236.
  - II. Spin foam representation, hep-th/0610237.
  - III. Plaquette representation, hep-th/0610238.
- M.Göpfert, G.Mack, Proof of confinement of static quarks in three-dimensional U(1) lattice gauge theory for all values of the coupling constant, Commun.Math.Phys. 82, 545, 1981.