Exactly Solvable LQC: New Insights on Some Old Questions

(Work with Abhay Ashtekar and Alejandro Corichi (*To appear*))

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Loops 2007, UNAM, Morelia

Motivation

- Extensive analytical and numerical methods in LQC: valuable insights on singularity resolution in symmetry reduced models. Big Bang replaced by Big Bounce. Non-local character of curvature leads to repulsive QG effects at Planck scale $\Rightarrow$ significant departures from classical dynamics. At low curvatures: LQC $\rightarrow$ GR

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  - What happens to the singularity resolution when "area gap goes to zero"? Does LQC approach WDW in this limit?
  - Does a continuum limit of LQC exist when area gap $\rightarrow 0$? Or is LQC a fundamentally discrete theory?

Outline

- A very short introduction of LQC
- Simplified LQC (SLQC) (contrast to Bojowald's model) and WDW in $\mathtt{b}$ representation
- Volume observable and its fluctuations (for arbitrary states)
- Comparison of SLQC and WDW
- Issues of limit

LQC: A Very Brief Introduction

- Gravitational + Matter Constraint (Massless Scalar)

$$-\gamma^{-2} \int_{\mathcal{V}} d^3x \ \varepsilon_{ijk} \frac{E^{ai} E^{bj}}{\sqrt{|\det E|}} F_{ab}^{k} \ + \ 8\pi G \frac{P_\phi^2}{2\sqrt{|\det E|}} \ = \ 0$$

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- Quantum constraint: $(\sin(\lambda \mathsf{b})\hat{A}\sin(\lambda \mathsf{b}) + KB(\nu)\hat{P}_\phi^2)\psi(\nu,\phi) = 0$

$\mathsf{b} := c/|p|^{1/2}$, $\{\mathsf{b},\nu\} = 2$, $\nu = V/(2\pi\gamma\ell_{\rm P}^2)$, $\lambda^2 =$ Area Gap.

$\hat{\mathsf{b}}\psi(\nu) = -2i\frac{\partial}{\partial\nu}\psi(\nu)$, $\exp(i\lambda b/2)\psi(\nu) = \psi(\nu+\lambda)$

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- For $\nu > 1$, $A(\nu) = 1$ and $B(\nu) \to 1/|\nu|$. States corresponding to a large classical universe at late times do not see $\nu \leq 1$.

Simplified LQC

- Quantum Constraint in $\mathtt{b}$ representation:

$$\Theta(\mathtt{b})\chi(\mathtt{b},\phi) \;=\; -12\pi G\,\frac{\sin(\lambda\mathtt{b})}{\lambda}\frac{\partial}{\partial\mathtt{b}}\,\frac{\sin(\lambda\mathtt{b})}{\lambda}\frac{\partial}{\partial\mathtt{b}}\,\chi(b,\phi) \;=\; -\,\partial_\phi^2\,\chi(\mathtt{b},\phi)$$

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- Introduce $x := (12\pi G)^{-1/2}\,\log(\tan(\lambda\mathtt{b}/2)) \Rightarrow \partial_\phi^2\,\chi = \partial_x^2\,\chi$
  General solution:

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- Physical states anti-symmetric in $\mathtt{b}$: $\chi(\mathtt{b},\phi) = -\chi(\mathtt{b}-\pi/2,\phi)$. Imposes relation between χ_+ and χ_-

Wheeler-DeWitt Theory

- Quantum constraint in $\mathtt{b}$ representation:

$$\Theta(\mathtt{b})\chi(\mathtt{b},\phi) \; = \; -12\pi G \; \mathtt{b}\frac{\partial}{\partial \mathtt{b}} \, \mathtt{b}\frac{\partial}{\partial \mathtt{b}}\chi(b,\phi) \; = \; -\partial_\phi^2 \chi(\mathtt{b},\phi)$$

- As in SLQC, we have an internal clock, Physical inner product and Dirac Obsevables.
- Introduce

$$y := (12\pi G)^{-1/2} \log\left(\mathtt{b}/2\mathtt{b}_o\right)$$

$\Rightarrow$

$$\partial_\phi^2 \chi(\phi,y) = \partial_y^2 \chi(\phi,y)$$

- General solution:

$$\chi = \chi_+(\phi+y) + \chi_-(\phi-y) := \chi_+(y_+) + \chi_-(y_-)$$

- Unlike SLQC, χ_+ (expanding) and χ_- (contracting) are disjoint.

Volume observable in WDW

$$
\begin{aligned}
(\chi, \hat{V}|_\phi\, \chi)_{\rm phy} &= 2\pi\gamma\ell_{\rm P}^2\, (\hat{\nu}\chi, \hat{\nu}\chi)_{\rm kin} \\
&= \frac{16\gamma\ell_{\rm P}^2}{\sqrt{12\pi G}\,\mathsf{b}_o^2} \int_{-\infty}^{\infty} dy_+ \left|\frac{d\chi_+}{dy_+}\right|^2 e^{\sqrt{12\pi G}(\phi - y_+)} \\
&= V_o\, e^{\sqrt{12\pi G}\phi} \,.
\end{aligned}
$$

- As $\phi \to -\infty$, $\langle \hat{V}|_\phi \rangle \to 0$. The backward evolution leads to the big bang singularity.
- Fluctuations:

$$(\chi, \hat{V}^2|_\phi\, \chi)_{\rm phy} = W_0\, e^{2\sqrt{12\pi G}\phi}$$

$$((\Delta V|_{\phi_0})/\langle \hat{V}|_{\phi_0}\rangle)^2 = (W_0/V_0)^2 - 1 \,.$$

Remains constant with evolution.

Volume observable in SLQC

$$
\begin{aligned}
(\chi, \hat{V}|_\phi\, \chi)_{\rm phy} \quad &= \quad \frac{8\gamma\ell_{\rm P}^2\lambda^2}{\sqrt{12\pi G}} \left[\int_{-\infty}^{\infty} dx_+ \, \left| \frac{d\chi_+}{dx_+} \right|^2 \cosh(\sqrt{12\pi G}(x_+ - \phi)) \right. \\
&\quad \left. + \int_{-\infty}^{\infty} dx_- \, \left| \frac{d\chi_-}{dx_-} \right|^2 \cosh(\sqrt{12\pi G}(-x_- + \phi)) \right] \\
&= \quad I_1\, e^{-\sqrt{12\pi G}\phi} + I_2\, e^{\sqrt{12\pi G}\phi}
\end{aligned}
$$

- There exists a minimum value of $\langle V|_{(\phi=\phi_B)}\rangle$ which occurs at

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\phi_B = (2\sqrt{12\pi G})^{-1} \, \log(I_1/I_2)
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- Relative dispersion bounded in time evolution. A single condition on the infinite dimensional space of initial data: $J_1 = \frac{I_1^2}{I_2^2} J_2$ implies symmetric fluctuations around bounce point and equality of asymptotic values.

Comparison between WDW and SLQC

- For a fixed value of λ select $\Psi_0(\mathtt{b})$. Initially, for $\phi = 0$

$$\langle \hat{V} \rangle_\lambda(\phi = 0) \;=\; \langle \hat{V} \rangle_{\mathrm{WDW}}(\phi = 0) \;=:\; V_0$$

Difference between expectation values:

$$\langle \hat{V} \rangle_{\mathrm{WDW}}(\phi) - \langle \hat{V} \rangle_\lambda(\phi) = (V_0 - I_2)\, e^{\sqrt{12\pi G}\,\phi} - I_1\, e^{-\sqrt{12\pi G}\,\phi}$$

Relative difference: bounded in future evolution

$$|\langle \hat{V} \rangle_{\mathrm{WDW}}(\phi) - \langle \hat{V} \rangle_\lambda(\phi)| / \langle \hat{V} \rangle_{\mathrm{WDW}}(\phi) \;\;\leq\;\; \delta := I_1/V_0 \text{ (very small)}$$

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- For any $N > 0$ (arbitrarily large) $\exists\, \phi$ such that

$$|\langle \hat{V} \rangle_{\mathrm{WDW}}(\phi) - \langle \hat{V} \rangle_\lambda(\phi)| \;\; > \;\; N$$

Fundamental discreteness of SLQC

- Start with an arbitrary λ_o, refine the area gap $(\lambda_o \to \lambda)$.
  For $\lambda < \lambda_o$, $\chi_i \in \mathcal{H}_{\lambda_o}$ under embedding $\chi_i \in \mathcal{H}_\lambda$.
  Under renormalization $\chi^\lambda := \sqrt{\lambda_o/\lambda}\,\chi^{\lambda_o}, \quad |\chi^\lambda|^2 = |\chi^{\lambda_o}|^2$.

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- Refinement in $\lambda \Rightarrow x_\lambda \neq x_{\lambda_o} \Rightarrow I_{1,2}(\lambda) \neq I_{1,2}(\lambda_o)$. $I_2(\lambda)$ is a monotonic decreasing function. As $\lambda \rightarrow 0$, $I_2(\lambda)$ grows.

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- Consequence: In the backward evolution of an expanding branch

$$\langle \hat{V} \rangle_{\lambda_o} - \langle \hat{V} \rangle_\lambda = (I_2(\lambda_o) - I_2(\lambda))e^{-\sqrt{12\pi G}\,\phi}$$

  which diverges as $\phi \to -\infty$.

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  $\phi_B = (2\sqrt{12\pi G})^{-1} \log(I_1/I_2) \quad \longrightarrow -\infty \quad \text{as} \quad \lambda \to 0.$

- Uniform limit does not exist. Contrast with results on Harmonic Oscillator (Corichi, Vukasinac, Zapata (07)).

Summary

- Without assuming semi-classicality at any level we presented analysis of arbitrary states in SLQC and WDW. Full analytical control to make predictions from the quantum theory.
- Bounce not restricted to states which are semi-classical at late times. There is a pre-big bang branch for a dense subspace of $\mathcal{H}_{\rm phy}$. Only one condition on the infinite dimensional space of initial data leads to symmetric fluctuations across the bounce point.
- SLQC and WDW approach GR at low curvatures. At large curvatures they depart significantly.
- In the backward evolution of the expanding branch for any given fixed time interval, SLQC and WDW agree to arbitrary accuracy by a choice of λ. However, for any given choice of λ, they diverge if one waits long enough.
- There is no limiting theory of SLQC when $\lambda \to 0$. Two different λ SLQC's depart in a similar way as they do from WDW. SLQC is a fundamentally discrete theory.