On the regularizability of the Ponzano-Regge model

Introduction

- The Ponzano-Regge model is a model for 3+0 dimensional quantum gravity with zero cosmological constant.

- In its formulation with observables defined in terms of group variables (Freidel and Louapre, 2005) there has been some question as to the regularizability of the observables

- Aims of talk
  1. to explain in terms of twisted cohomology whether an observable is well-defined or ill-defined
  2. to give a formula in terms of Reidemeister torsion for well-defined observables

Contents

- Definition of the observables

- Examples of well-defined and ill-defined observables

- Definition of twisted cohomology

- Result 1 - a cohomological criterion for the well-definition of an observable

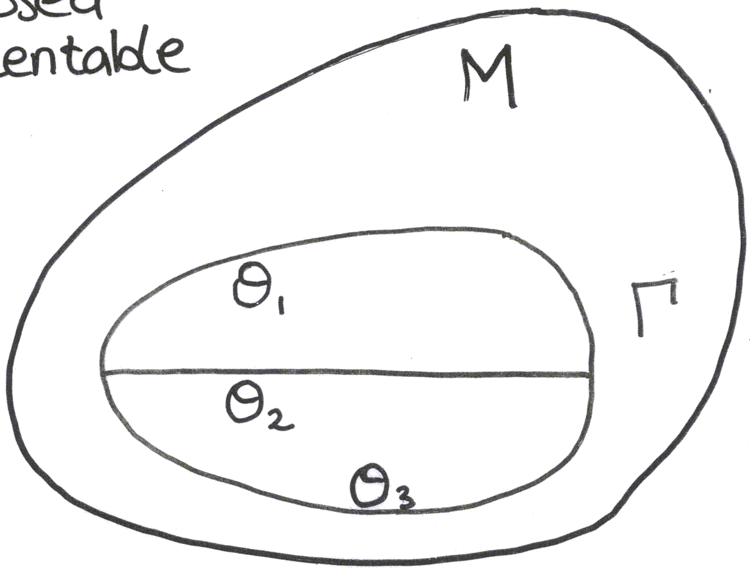
- Result 2 - a formula in terms of Reidemeister torsion for well-defined observables

- Examples

Definition of the observables $Z(M,\Gamma_\theta)$

M - 3 manifold
closed
orientable

Γ - graph
connected



$\theta_e \in [0,\pi]$

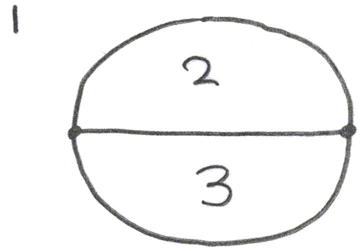
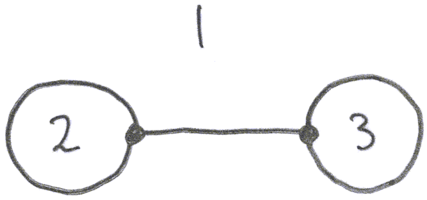
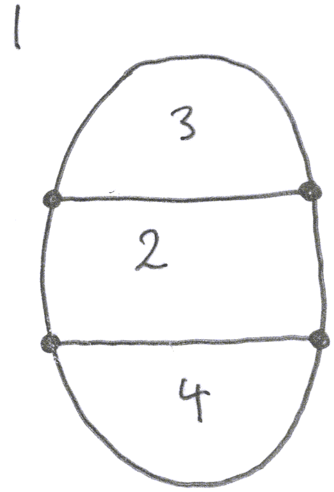
- Choose a triangulation Δ of M containing Γ
- Assign an elt. $g_{e*} \in SU(2)$ to each dual edge $e^* \in \Delta^*$
- Regularize : choose set of trees T in $\Delta - \Gamma$. Trees do not meet each other. Each tree meets graph at one vertex. T is maximal

$$Z(M,\Gamma_\theta) := \int \prod_{e^*\in\Delta^*} dg_{e*} \prod_{e\in\Gamma} \frac{\pi}{2}\,\delta(\theta_e - \theta(h_e)) \prod_{e\in\Delta-\Gamma-T} \delta(h_e)$$

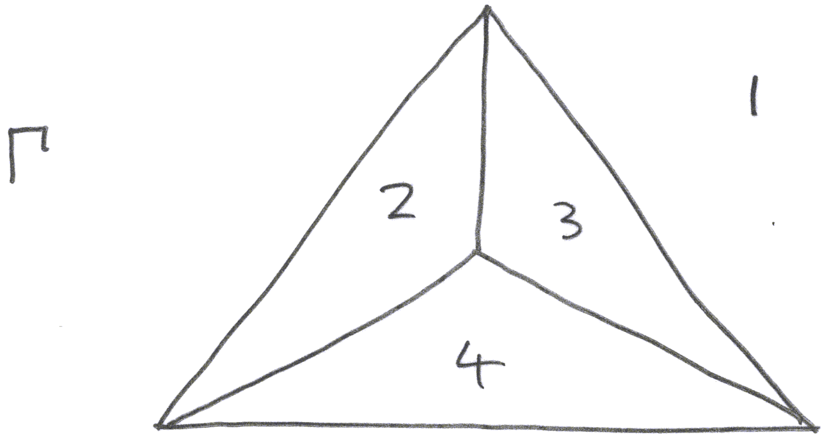
$\Rightarrow$ h_e is conjugate to $\begin{pmatrix} e^{i\theta_e} & 0 \\ 0 & e^{-i\theta_e} \end{pmatrix}$

h_e = holonomy around edge e

Some simple planar graphs in S^3

Γ	$Z(S^3,\Gamma)$
	$\sin^2\Theta$
	$\begin{cases} \frac{\pi}{4}\sin\theta_{12}\sin\theta_{13}\sin\theta_{23} & \text{if } (\theta_{12},\theta_{13},\theta_{14}) \text{ satisfy triangle inequalities} \\ 0 & \text{otherwise} \end{cases}$
	$\frac{\pi}{2}\sin^2\theta_{12}\sin^2\theta_{13}\,\delta(\theta_{11})$
	$\begin{cases} \frac{\pi^3}{2^5}\sin\theta_{13}\sin\theta_{14}\sin\theta_{23}\sin\theta_{24}\,\delta(\theta_{12}-\theta'_{12}) & \text{if } (\theta_{12},\theta_{13},\theta_{23}) \text{ and } (\theta_{12},\theta_{14},\theta_{23}) \text{ satisfy triangle inequalities} \\ 0 & \text{otherwise} \end{cases}$

The tetrahedron graph



Freidel and Louapre :

$$Z(S^3, \Gamma) = \frac{\pi^2}{2^5} \frac{\prod_{i<j} \sin\theta_{ij}}{\underbrace{\sqrt{\det[\cos\theta_{ij}]}}} \qquad i,j = 1, \ldots, 4$$

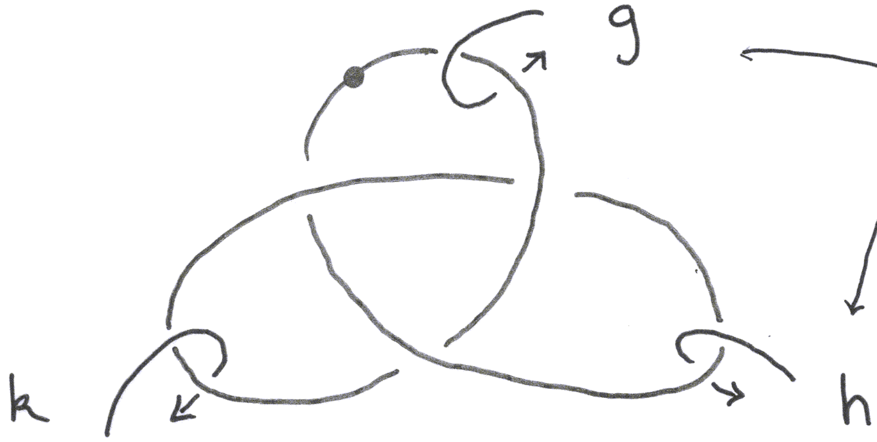
$\propto$ to volume of 4-simplex with vertices at $0, g_1, g_2, g_3, g_4 \in \mathbb{R}^4$

$\Rightarrow$ $Z(S^3, \Gamma)$ infinite for θ's defining a degenerate 5-simplex

- distributional interpretation

The trefoil knot

K



generators of $\pi_1(S^3 - K)$ in Wirtinger presentation

3-dimensional delta functions at the identity

$$Z(S^3, K) = \sin^2\theta \int_{\substack{g \in C_\theta \\ h, k \in SU(2)}} \overbrace{dg\, dh\, dk}^{8\text{ d.o.f.'s}}\; \delta(hgh^{-1}k^{-1})\, \delta(khk^{-1}g^{-1})$$

relations for $\pi_1(S^3 - K)$ in Wirtinger presentation

- Relations imposed by delta-functions have 2 branches of solutions. $Z = Z_1 + Z_2$
- 1st branch, $g = h = k$, exists $\forall \theta$, has symmetry orbit of dimension 2, so integral can be done:

$$Z_1(S^3, K) = \frac{\sin^2\theta}{|A_K(e^{2i\theta})|^2}$$

- 2nd branch, exists for $\theta \geq \pi/6$, has symmetry orbit of dimension 3, so integral makes no sense

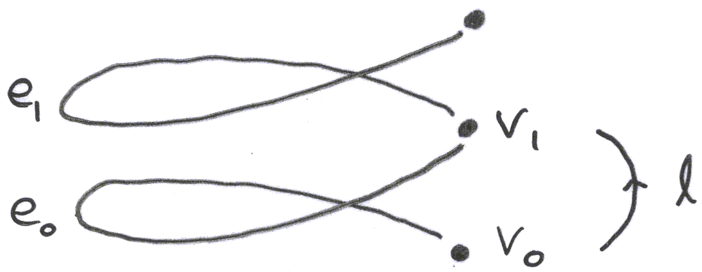
$$Z_2(S^3, K) \quad \text{ill-defined}$$

ρ-twisted cohomology groups of a cell complex W

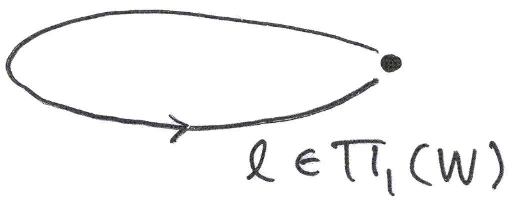
- $\rho \in \mathrm{Hom}(\pi_1(W), SU(2))$
- To define : the co-chain complex $C^*_\rho(W)$
- Then the homology groups of $C^*_\rho(W)$ are the ρ-twisted cohomology groups of W.

eg. the circle

$\tilde{W}$



W



- $\pi_1(W)$ acts on $\tilde{W}$
- $\pi_1(W)$ acts on $su(2)$: $\ell . a = \rho(\ell)\, a\, \rho(\ell)^{-1}$, $a \in su(2)$
- $C^*_\rho(W)$ is the set of homomorphisms from $C_*(\tilde{W})$ to $su(2)$ which are invariant under the action of $\pi_1(W)$

eg. $\phi \in C^0_\rho(W)$

$$\begin{aligned} \phi(v_1) &= \phi(\ell . v_0) \\ &= \ell . \phi(v_0) \\ &= \rho(\ell)\, \phi(v_0)\, \rho(\ell)^{-1} \end{aligned}$$

$$\boxed{\text{If } \exists \text{ a representation } \rho \text{ compatible with } \{\Theta_e\} \text{ for which } H^2_\rho(M-\Gamma) \neq 0 \text{ then } Z(M,\Gamma_\Theta) \text{ is ill-defined.}}$$

$$Z(M,\Gamma_\Theta) = \int \prod_{e^* \in \Delta^*} dg_{e^*} \prod_{e\in\Gamma} \frac{\pi}{2}\,\delta(\Theta_e - \Theta(h_e)) \underbrace{\prod_{e\in\Delta-\Gamma-T} \delta(h_e)}_{\delta(f)}$$

$$f : \underbrace{SU(2)^A}_{\#\text{edges in } \Delta^*} \longrightarrow \underbrace{SU(2)^B}_{\#\text{edges in } \Delta-\Gamma-T}$$

$$f(g_{e_1^*}, \dots, g_{e_A^*}) = (h_{e_1}, \dots, h_{e_B})$$

- $\delta(f)$ can only make sense if df is surjective
- But df is the boundary operator in the co-chain complex defining the twisted cohomology groups of the graph exterior
- So if $H^2_\rho(M-\Gamma) \neq 0$ then df cannot be surjective and so $\delta(f)$ makes no sense.

$Z(S^3, K)$ for K the trefoil and figure-eight knots

Twisted cohomology explanation for ill-definition above critical angle

Trefoil knot

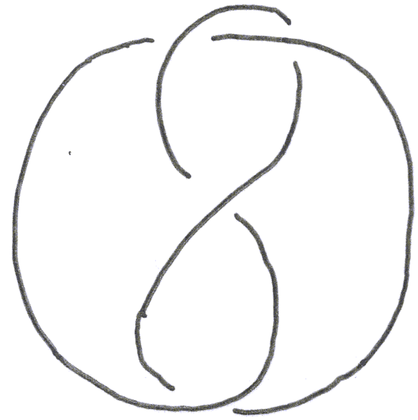


Figure-eight knot

$$Z(S^3; K) = \begin{cases} \dfrac{\sin^2\theta}{|A_K(e^{2i\theta})|^2} & \theta < \theta_c(K) \quad H^2_\rho(S^3 - K) = 0 \\ \text{ill-defined} & \theta > \theta_c(K) \quad H^2_\rho(S^3 - K) = \mathbb{R} \text{ for non-abelian branch of solutions } \rho \end{cases}$$

angle at which non-abelian representations of $\pi_1(S^3 - K)$ in $SU(2)$ begin

$$\theta_c(K) = \begin{cases} \pi/6 & K = \text{trefoil knot} \\ \pi/5 & K = \text{figure-eight knot} \end{cases}$$

If $H^2_\rho(M-\Gamma)=0$ $\forall$ representations ρ compatible with $\{\Theta_e\}$, then

$$Z = \int_A \alpha$$

A = flat connections / guage transformations

$$\alpha[h^1] = \text{const.}\ \mathrm{tor}(M-\Gamma,\rho)\prod_{e\in\Gamma}\delta(\Theta_e-\Theta(h_e))$$

h^1 = basis of $H^1 \simeq$ tangent space of A

$\mathrm{tor}(M-\Gamma,\rho)$ = Reidemeister torsion of $M-\Gamma$

- Reidemeister torsion is a **simple homotopy invariant** and a **homeomorphism invariant**.

- These invariance properties imply that $Z(M,\Gamma)$ is **regularization independent** (choice of trees T) and **triangulation independent** (where it is well-defined).

Reidemeister torsion formula for $Z(S^3, K_\theta)$, K a knot

- For a knot K, if $H^2_\rho(S^3 - K) = 0$

$$Z(S^3, K_\theta) = \text{const.} \left| \operatorname{tor}(S^3 - K, \rho_0) \right|$$

where

ρ_0 = abelian representation of $\pi_1(S^3 - K)$ s.t.

$\rho_0(x)$ is conjugate to $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ for all

Wirtinger generators x of $\pi_1(S^3 - K)$

- Simple homotopy invariance means we may calculate tor using the cell complex coming from the Wirtinger presentation of $\pi_1(S^3 - K)$. Then

$$Z(S^3, K_\theta) = \text{const.} \frac{\sin^2\theta}{|A_K(e^{2i\theta})|^2} \qquad \theta < \theta_c(K)$$

- This generalises our previous results for the trefoil and figure-eight knots

Graphs in general

- The Reidemeister torsion formula for a general graph gives a result which is in general a distribution in the Θ's.
- For a planar graph $H^2_\rho(M-\Gamma) = 0$. So $Z(M,\Gamma)$ is always well-defined for planar graphs
- The tetrahedron graph is thus well-defined. An infinite result is possible because the observable is a distribution in the Θ's.
- For non-planar graphs, $H^2_\rho(M-\Gamma)$ may be non-zero.