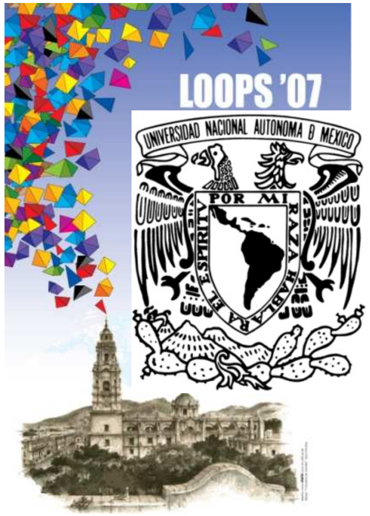
Entropic laws with bulk and boundary states



Danny Terno
Etera Livine

CENTRE FOR QUANTUM COMPUTER TECHNOLOGY
AUSTRALIAN RESEARCH COUNCIL CENTRE OF EXCELLENCE

MACQUARIE UNIVERSITY ~ SYDNEY

ECOLE NORMALE SUPERIEURE DE LYON

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Motivation

Black hole entropy,
holographic principle(s)

$$S \leq \frac{A}{4l_P^2}$$

Outline

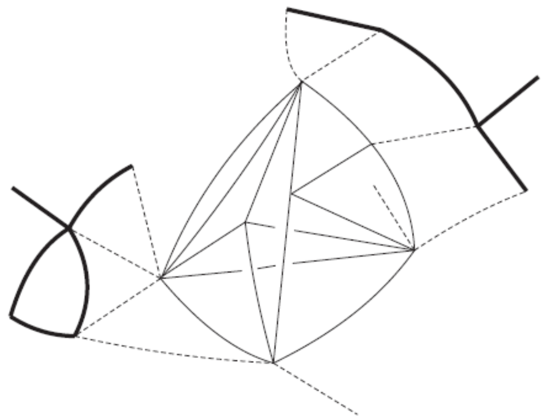
- Boundary entropy in LQG
- Bulk entropy
- Bulk entropy in BF theory

ERL & DRT, Nucl. Phys. B **741**, 131 (2006)
ERL & DRT, arXiv:gr-qc/0603008
ERL & DRT, arXiv:0706.0985 [gr-qc]
[soon ?]

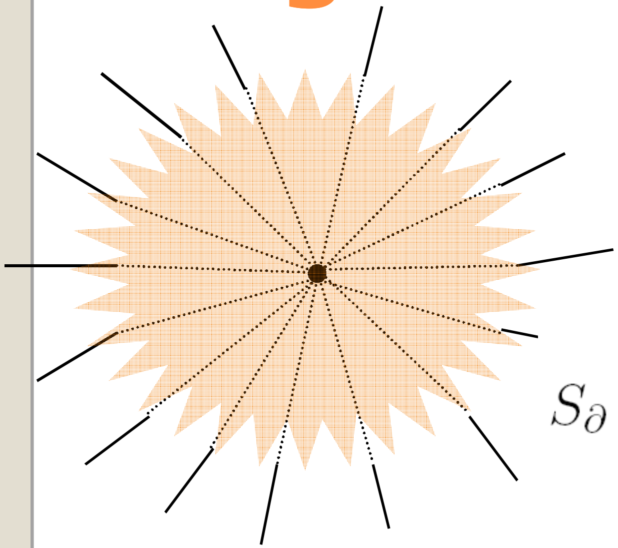
The model: *LQG*

- Count the degrees of freedom inside a region on a fixed graph before the Hamiltonian constraint is imposed
- To get finite entropy more gauge fixing is required
- Identify the scaling regimes [& see what is the relationship with the Hamiltonian constraint]

Assumption: boundary carry spin-½



Gauge-fixing: ideas



- "Black point"

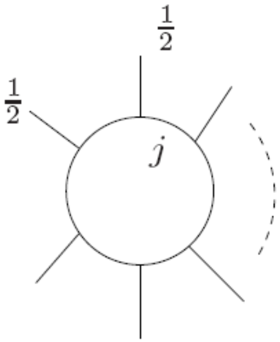
$2n$ spin-½ boundary edges
Total coarse graining/ no loops

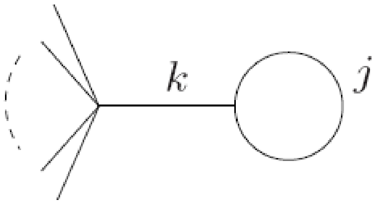
$$S_\partial \equiv \log N_\partial = \log \frac{1}{n+1}\binom{2n}{n} \underset{n\to\infty}{\sim} 2n\log 2 - \frac{3}{2}\log n$$

$$A = 2n a_{\frac{1}{2}} l_P^2 \Longrightarrow S_\partial \sim \frac{A}{l_P^2}\frac{\log 2}{a_{\frac{1}{2}}} - \frac{3}{2}\log A$$

- 1-loop

$2n$ spin-½ boundary edges
One loop inside

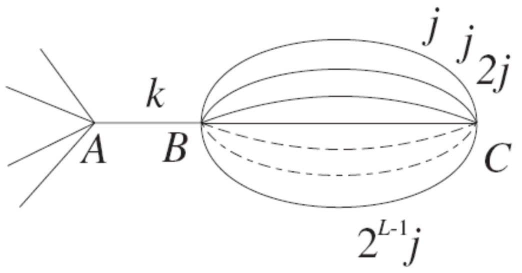




Entropy = infinity,
unless j is fixed

$$\dim \mathcal{H}_1 = \binom{2n}{n} \qquad S_1 \underset{n\to\infty}{\sim} 2n\log 2 - \frac{1}{2}\log n +$$

Gauge fixing: *L* loops



$$S_L \sim 2n\log 2 + \left(\frac{L}{2} - 1\right)\log n$$

$$S(L \gg n) \sim n\log L - (n + \tfrac{1}{2})\log n + n$$

If LQG is a holographic theory than more constraints are required!

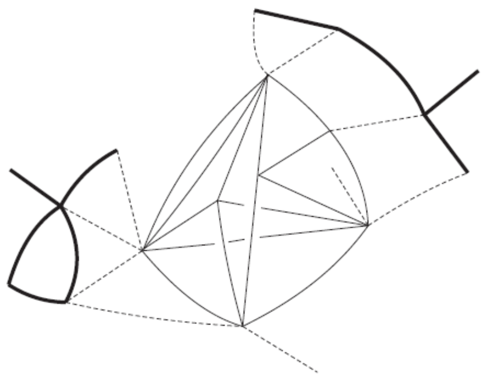
The model: *BF theory*

- Count the entanglement between inside and an outside regions on a fixed graph
- Regularized entanglement [von Neumann entropy of the reduced density matrix satisfies the area law
- Explicit calculations for SU(2) and $SU_q(2)$
- Renormalized entropy in the presence of holonomies

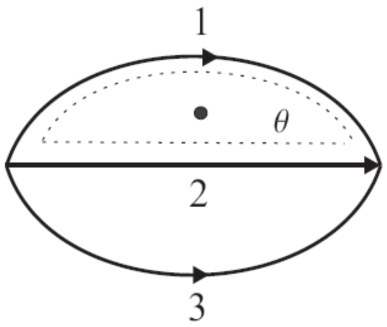
$$\mathcal{L} = \mathrm{tr}\,(E \wedge F)$$

$F = 0$ Flatness constraint

$d_A E = 0$ Gaussian constraint



Basic building blocks 1



$$\Psi(g_1, g_2, g_3) = \delta(g_1 g_3^{-1})\delta(g_2 g_3^{-1})$$

$$\Psi^{\mathrm{sym}}(g_1, g_2, g_3) = \delta(g_1 g_2^{-1})\delta(g_2 g_3^{-1})\delta(g_3 g_1^{-1})$$

Reduced density matrix

$$\begin{aligned}\rho(g_3, \tilde{g}_3) &= \int dg_1 dg_2 \delta(g_1 g_3^{-1})\delta(g_2 g_3^{-1})\delta(g_1 \tilde{g}_3^{-1})\delta(g_2 \tilde{g}_3^{-1}) \\ &= \delta(g_3 \tilde{g}_3^{-1})^2 = \delta(\mathbb{1})\delta(g_3 \tilde{g}_3^{-1})\end{aligned}$$

Regularization

$$\rho_J(g_3, \tilde{g}_3) = \frac{1}{N}\sum_{j_3=0}^{J}{}^{\frac{1}{2}} d_{j_3}\chi_{j_3}(g_3\tilde{g}_3^{-1}) = \frac{1}{N}\sum_{j_3=0}^{J}{}^{\frac{1}{2}} \sum_{n_3, m_3} d_{j_3} D^{(j_3)}(g_3)^{m_3}{}_{n_3} D^{(j_3)\dagger}(\tilde{g}_3)^{n_3}{}_{m_3}$$

Normalization constant

$$N = \Delta(J) := \sum_{j_{1,2}=0}^{J} {}^{\frac{1}{2}} d_j^2 = \frac{1}{3}(1+2J)(1+J)(3+4J) \sim 8J^3/3$$

Useful notation

$$\langle g|j,m,n\rangle = \sqrt{d_j} D^{(j)}(g)^m{}_n,$$
$$\langle j,m,n|j',m',n'\rangle = \delta_{jj'}\delta_{mm'}\delta_{nn'}$$

$$\rho_J = \frac{1}{\Delta(J)} \sum_{j_3=0}^{J} {}^{\frac{1}{2}} \sum_{m_3,n_3} |j_3 m_3 n_3\rangle\langle j_3 m_3 n_3|$$

Regularized entropy

$$E(\Psi|i:j,k) := S(\rho_J^i) = \log \Delta(J) \sim 3\log J + \log(8/3)$$

- Role of the cut-off

Basic building blocks 2

Non-zero holonomy around (13)

$$\Psi_\theta = \delta_\theta(g_1 g_3^{-1})\delta(g_2 g_3^{-1})$$

$$\left[\int dg\, \delta_\theta(g) f(g) = \frac{1}{4\pi}\int_{S^2} d^2\hat{u}\, f((\theta,\hat{u}))\right]$$

Reduced density matrix

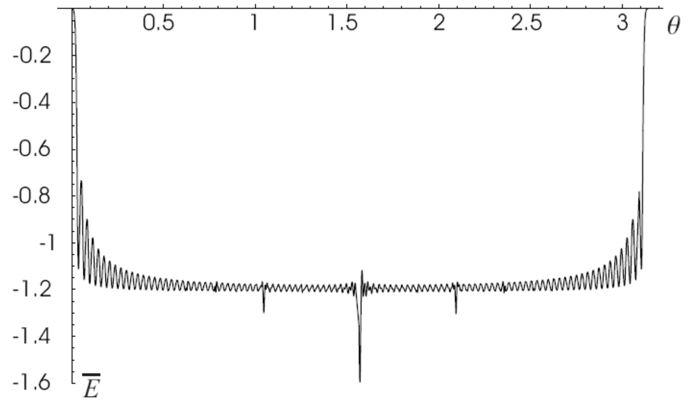
$$\rho^1_{\theta J} = \frac{1}{A(J,\theta)}\sum_j \sum_{mn} \frac{\chi_j^2(\theta)}{d_j^2} |jmn\rangle\langle jmn|$$

$$A(J,\theta) := \sum_j \chi_j^2(\theta) = \frac{(3+4J)\sin\theta - \sin(3+4J)\theta}{4\sin^3\theta}$$

Entropy

$$E(\Psi_J^\theta|1:2,3) = \log A - \frac{1}{A}\sum_{j_1} \chi_{j_1}^2(\theta) \log \frac{\chi_{j_1}^2(\theta)}{d_{j_1}^2}$$

$$E(\Psi_J^\theta|1:2,3) < E(\Psi_J|1:2,3)$$



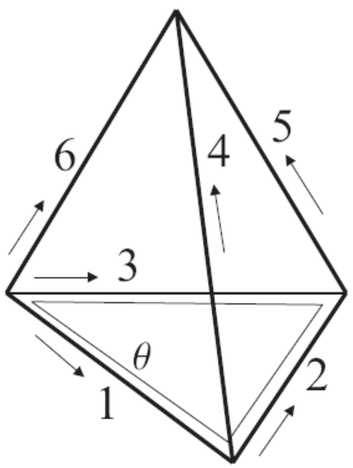
Regularized entropy

$$\bar{E}(\theta) := \lim_{J\to\infty} E(\Psi_J^\theta|1:2,3) - E(\Psi_J|1:2,3)$$

$$\bar{E}(\pi/2) = -2 + \log(3/2) \approx -1.5945, \qquad \bar{E} = -3 + \log 6 \approx -1.20824$$

$$\bar{E}(\pi/3) = -2 + \log 2 \approx -1.30436$$

A more complicated example

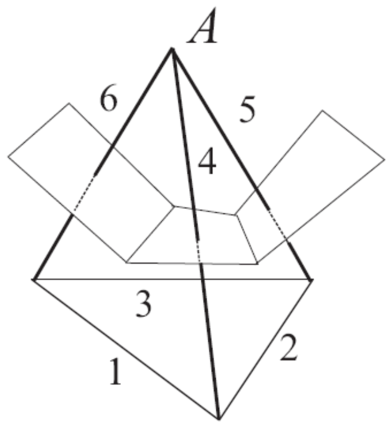


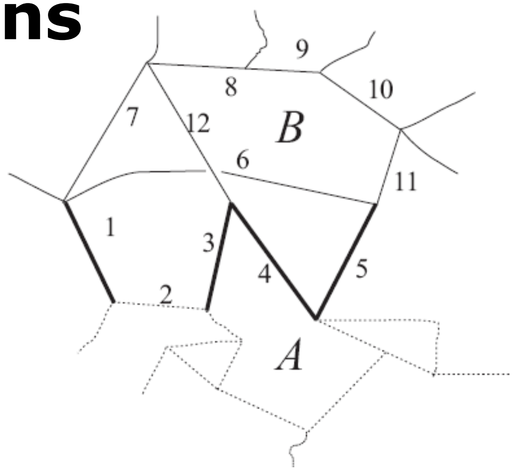
$$\bar{E}(\Phi_\theta|124:356)$$

$$= \lim_{J\to\infty} E(\Psi_J^\theta|1:2,3) - E(\Psi_J|1:2,3) =: \bar{E}(\theta)$$

General bounds

Basic definitions





Edges:

- internal g - - - - - - - - -
- boundary h ———
- external q ———

Loops: internal, boundary, frontier, external

Spaces: $\mathcal{H}_{\text{int}} = \mathcal{L}_2(G, dg)^{\otimes E_{\text{int}}}$ $\mathcal{H}_{\text{out}} = \mathcal{L}_2(G, dg)^{\otimes E_{\text{b}} + E_{\text{ext}}}$

Counting: $V = V_{\text{int}} + V_{\text{ext}}$ $E = E_{\text{int}} + E_{\text{b}} + E_{\text{ext}}$ $L = E - V + 1$

$L_{\text{int}} = E_{\text{int}} - V_{\text{int}} + 1$ $L_{\text{ext}} = E_{\text{ext}} - V_{\text{ext}} + 1$

$$L_{\text{b}} + L_{\text{f}} = L - L_{\text{int}} - L_{\text{ext}} = E_{\text{b}} - 1$$

Number of δ-functions: L

Reduced density operator:

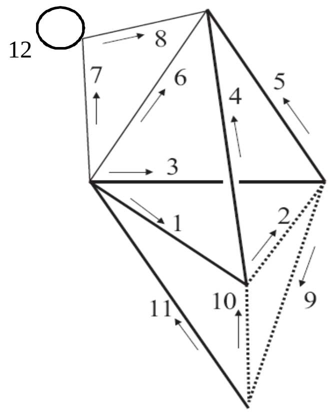
$$\rho^{\text{int}}(g,\tilde{g}) = \sigma^{\text{int}}(g,\tilde{g}) \int dq^{\otimes E_{\text{ext}}} dh^{\otimes E_{\text{b}}} \sigma^{\text{ext}}(q,\tilde{q}) \sigma^{\text{f}}(q,\tilde{q},h,\tilde{h}) \sigma^{\text{b}}(g,h,q,\tilde{g},\tilde{h},\tilde{q})$$

Important parameter: the minimal number of boundary loops that involve distinct internal edges $L_{\text{b}*}$

The boundary law

$$S(\rho^A) = L_{\text{b}*} \log \Delta$$

Example



$$S(\rho^A) = 2 \log \Delta$$

"Area law"

$$S(\rho^A) \leq (E_{\text{b}} - 1) \log \Delta$$

- LQG: the gauge-fixing is not enough. *If* some sort of a holographic bound is valid, then the Hamiltonian constraint reduces the graph complexity.

- BF theory: holographic, even with non-trivial holonomies.

Summary