Spin foam vertex and loop gravity

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Plan

- Introduction and state of art
  - $\hookrightarrow$ the BC model and its problems
  - $\hookrightarrow$ SO(4) Plebanski action

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  - $\hookrightarrow$ constraints
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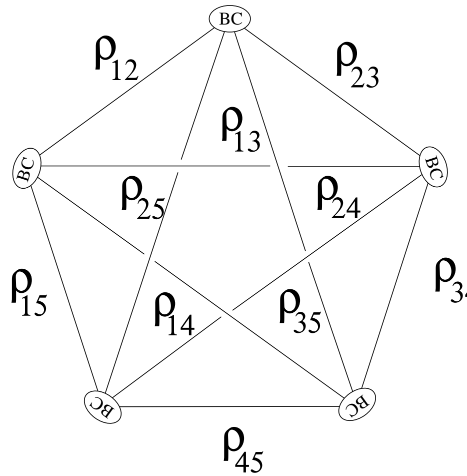
- Conclusion

State of art and the BC model



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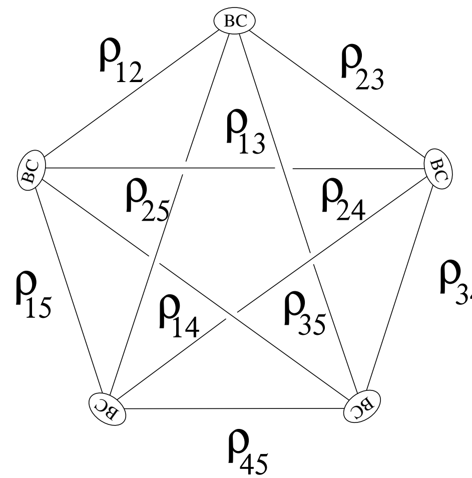

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Quantization of the classical tetrahedron imposing a set of constraints à là Dirac: [Baez,Barrett '99]

$$\hat{C}|\psi\rangle = 0$$

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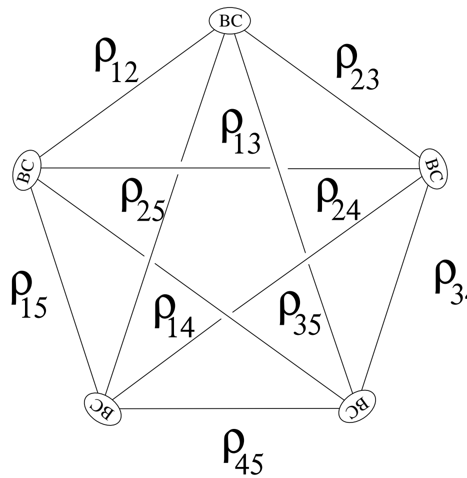

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unique solution of the constraints [Reisenberger '99]

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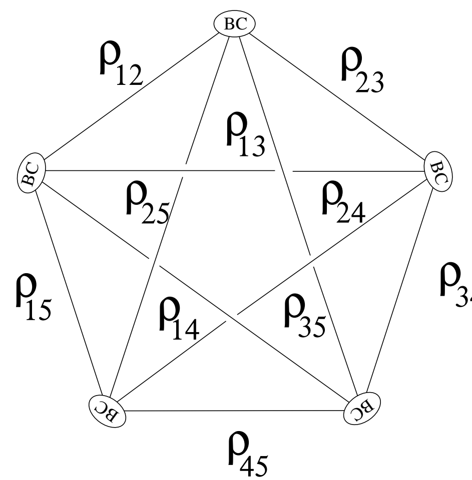

[Barrett,Crane '97]
Quantization of the classical tetrahedron imposing a set of constraints à là Dirac: [Baez,Barrett '99]

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Boundary states do not agree with LQG states and problems in the calculation of non diagonal components of the propagator. [Alesci,Rovelli to appear] and talk by Carlo Rovelli.

Plebanski action and constraints

$$S_P[B,\omega] = \int B^{IJ} \wedge F_{IJ}(\omega) + \phi_{IJKL} B^{IJ} \wedge B^{KL}$$

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Two sectors of solutions: [Reisenberger '98;De Pietri,Freidel '99;Perez '02]

$$\begin{aligned} B &= (\pm)\, e \wedge e \\ B &= (\pm) \star (e \wedge e) \end{aligned}$$

Regge discretization

$\hookrightarrow$ Simplicial discretization of a 4-dimensional space time: 4-simplices (v), tetrahedra (t) and triangles (f)

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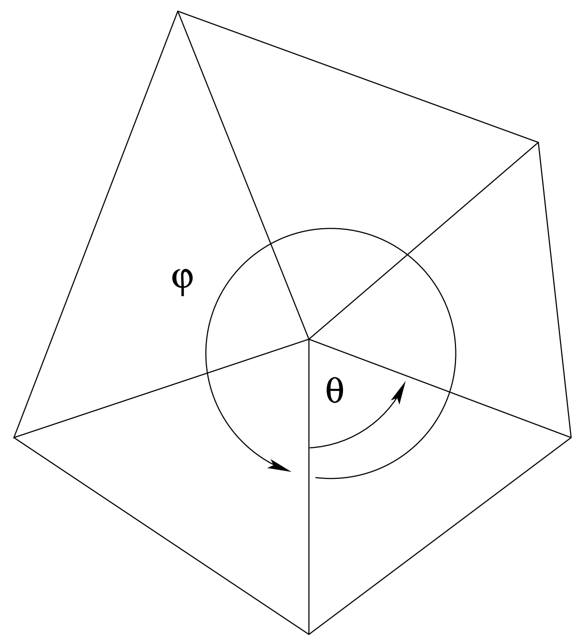
In 2d:



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↪ Simplicial discretization of a 4-dimensional space time: 4-simplices (v), tetrahedra (t) and triangles (f)

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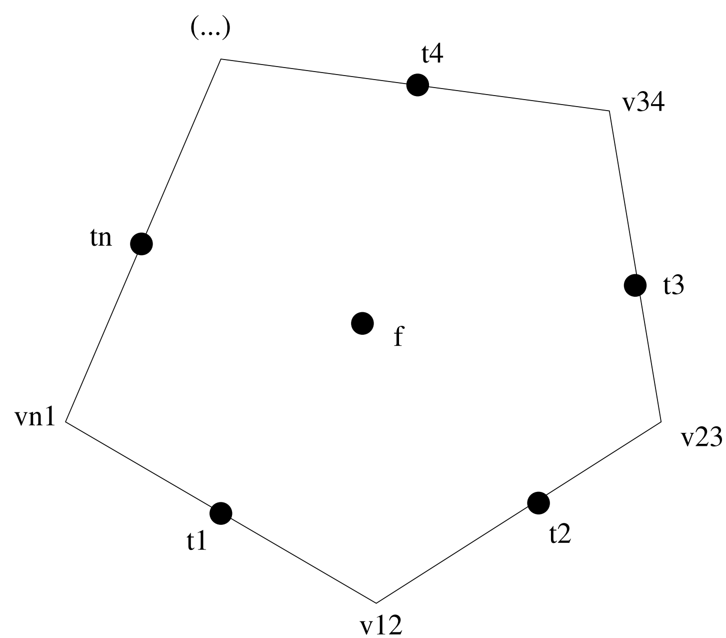
In our case:



Discrete variables

To each t: $e(t) = e(t)^I_a v_I dx^a$

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To each $f = t \cap t'$: $e(t') = U_{t't} \, e(t)$

and :$B(t)^{IJ}_{ab} = \epsilon^{IJ}{}_{KL} e(t)^K_a e(t)^L_b \rightarrow B_f(t) = \int_f \, B(t) \qquad \left(B(t') = U_{t't} B(t) U_{tt'}\right)$

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Flatness implies: $U_{t_1 t_n} = V_{t_1 v_{12}} ... V_{v_{(n-1)n} t_n}$

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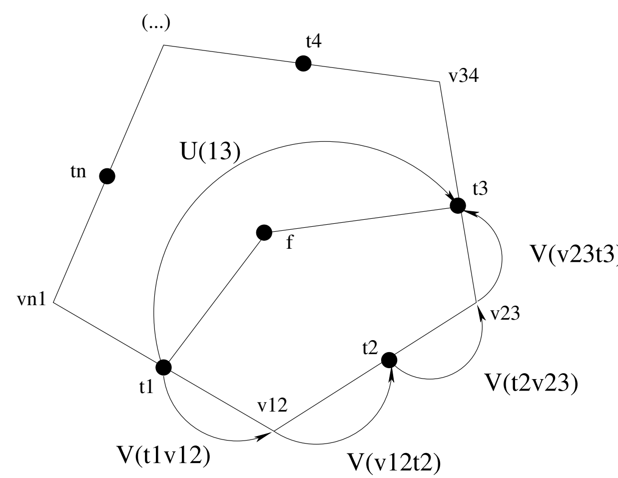

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Curvature is given by the holonomy around a link: $U_f(t_1) := V_{t_1 v_{12}} ... V_{v_{n1} t_1}$

Discrete constraints

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$\hookrightarrow$ the solutions to these constraints correspond to the two sectors of Plebanski theory.

Boundary variables

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faces (f) $\longrightarrow$ links (l)

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$\hookrightarrow$ phase space of a SO(4) Yang-Mills lattice gauge theory (!!)

Dynamics I: bulk

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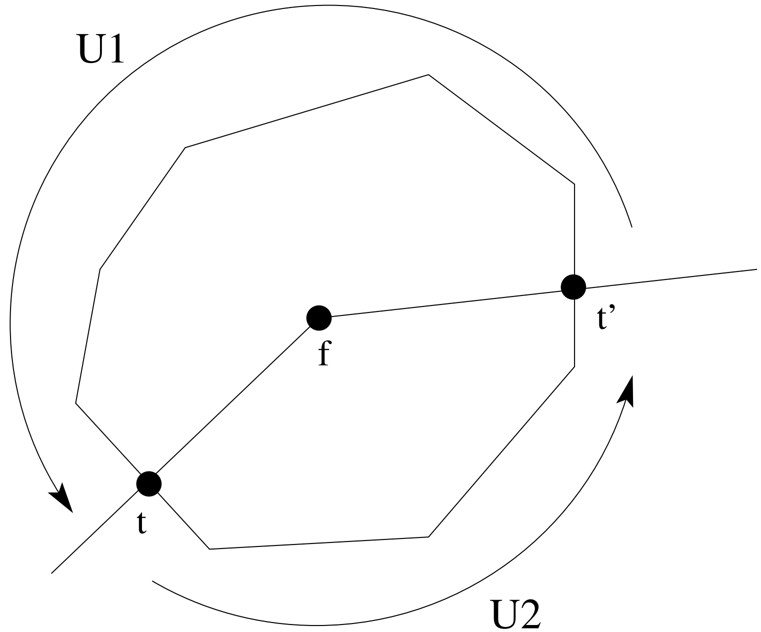
$\hookrightarrow$ clear relation with the Regge action (!!):

$$S_{Regge} = \sum_f A_f \ \sin \epsilon_f$$

Dynamics II: boundary



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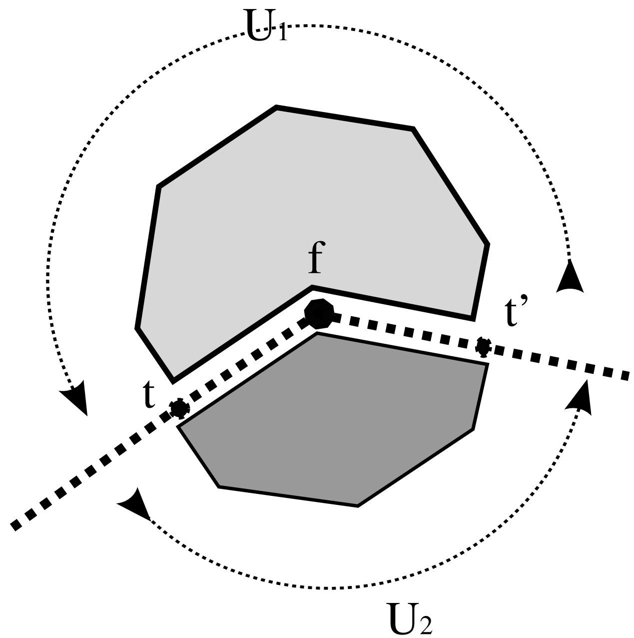

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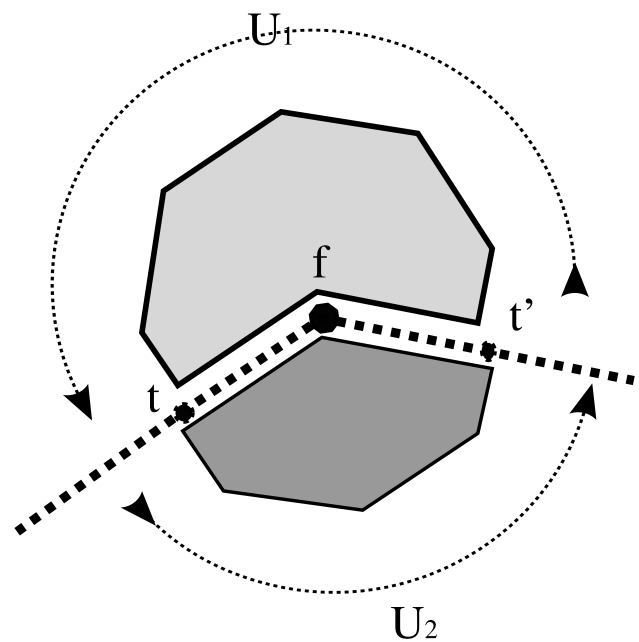

$$S_{\partial\Delta} = \sum_{f\in\partial\Delta} Tr[B_f(t)U_{tt'}]$$

Summary of the classical theory

$$S_{LGR} = \sum_{f \in int\Delta} Tr[B_f(t)U_f(t)] + \sum_{f \in \partial\Delta} Tr[B_f(t)U_{tt'}]$$

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$$\sum_{f \in t} B_f(t) = 0,$$
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Boundary variables : $\{(B_l, U_l)\}$

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Quantization: constraints

$$\psi_{j_l^+ j_l^- i_n^+ i_n^-}(g_l^+, g_l^-) = \left(\bigotimes_l D^{(j_l^+)}(g_l^+) \cdot \bigotimes_n i_n^+\right) \left(\bigotimes_l D^{(j_l^-)}(g_l^-) \cdot \bigotimes_n i_n^-\right)$$

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Motivation: Gupta-Bleur, doubled system.

Solution

A state $|\psi\rangle$ in $\mathcal{H}_e \; := \; \text{Inv}\left(\mathcal{H}_{(j_1,j_1)} \otimes ... \otimes \mathcal{H}_{(j_4,j_4)}\right)$ can be written (in a given pairing) as:

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Answer: Yes (!!). It is given by a suitable projection

$\mathcal{H}_e \longrightarrow$ Inv $\left(\mathcal{H}_{2j_1} \otimes ... \otimes \mathcal{H}_{2j_4}\right)$

Projection

Spinor notation for $I \in \mathcal{H}_e$:

$$I^{(A_1 \ldots A_{2j_1})(A'_1 \ldots A'_{2j_1}) \ \cdots \ (D_1 \ldots D_{2j_4})(D'_1 \ldots D'_{2j_4})} =: I^{\mathcal{A}\mathcal{A}' \ \cdots \ \mathcal{D}\mathcal{D}'} = i_+^{\mathcal{A} \ \cdots \ \mathcal{D}} \ i_-^{\mathcal{A}' \ \cdots \ \mathcal{D}'}$$

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The projection of the complete hilbert space of $SO(4)$ s-nets can be defined and it matches the hilbert space of $SO(3)$ s-nets (!)

Embedding

To every projection there is an embedding. The image of a $SO(3)$ intertwiner i, $f(i)$ can be expanded in the $SO(4)$ basis:

$$f(i) = \sum_{i_+, i_-} f^i_{i_+ i_-} \, |i_+, i_-\rangle$$

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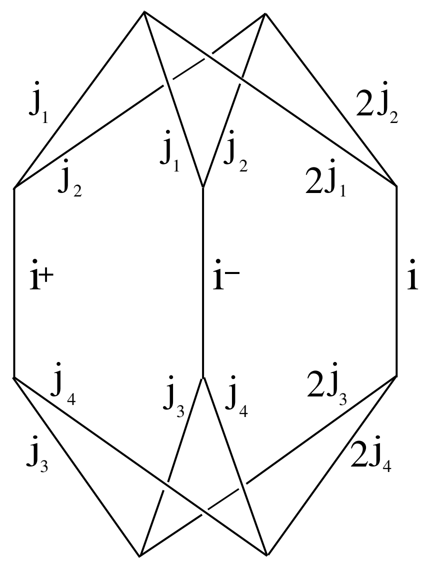
Its components can be seen as the evaluation of a $15j$ symbol:



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The image of the embedding $f\left[\text{Inv}\left(\mathcal{H}_{2j_1} \otimes ... \otimes \mathcal{H}_{2j_4}\right)\right] =: \mathcal{K}_e$ can be interpreted as a subspace of $\mathcal{H}_e$ obtained by imposing a geometrical constraint. Attention: It depends on a different choice of quantization: $B_f \to \star J_f$.

Dynamics I: setup

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The quantum amplitude for a given boundary state $|\psi\rangle$ is given by

[Oeckl'03]:

$$A(\psi) := \int dU_{ab} \; A(U_{ab}) \; \psi(U_{ab})$$

Dynamics II: vertex

For our model the vertex amplitude is given by:

$$A(\{j_{ab}\},\{i^a\}) := A(\psi_{\{j_{ab}\},\{i^a\}}) = \sum_{i^a_+ i^a_-} 15j\left(\left(\frac{j_{ab}}{2},\frac{j_{ab}}{2}\right);(i^a_+,i^a_-)\right)\; f^{i^a}_{i^a_+ i^a_-}$$

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The final amplitude is obtained by gluing 4-simplices:

$$Z = \sum_{j_f,i_e}\prod_f (\dim \tfrac{j_f}{2})^2 \prod_v A(j_f,i_e)$$

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- Geometrical interpretation forces quantization with the dual SO(4) generators
- Next step: semiclassical analysis and propagator.

Doubled system

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Solution: impose half of them (!) : $\hat{z}|\psi\rangle = 0 \Rightarrow \langle\phi|\hat{\bar{z}}|\psi\rangle = 0$

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$$f \ : \ \left(\bigotimes_l D^{(j_l)}(g_l)\right) \cdot (\bigotimes_n i_n) \longmapsto$$

$$\int_{SO(4)^N} \prod_e dV_e \left(\bigotimes_l D^{(\frac{j_l}{2},\frac{j_l}{2})}\left(V_{s(l)}(g_l^+, g_l^-)V_{t(l)}^{-1}\right)\right) \cdot (\bigotimes_n e(i)_n)$$

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This constraint selects the highest $SU(2)$ irreducible.