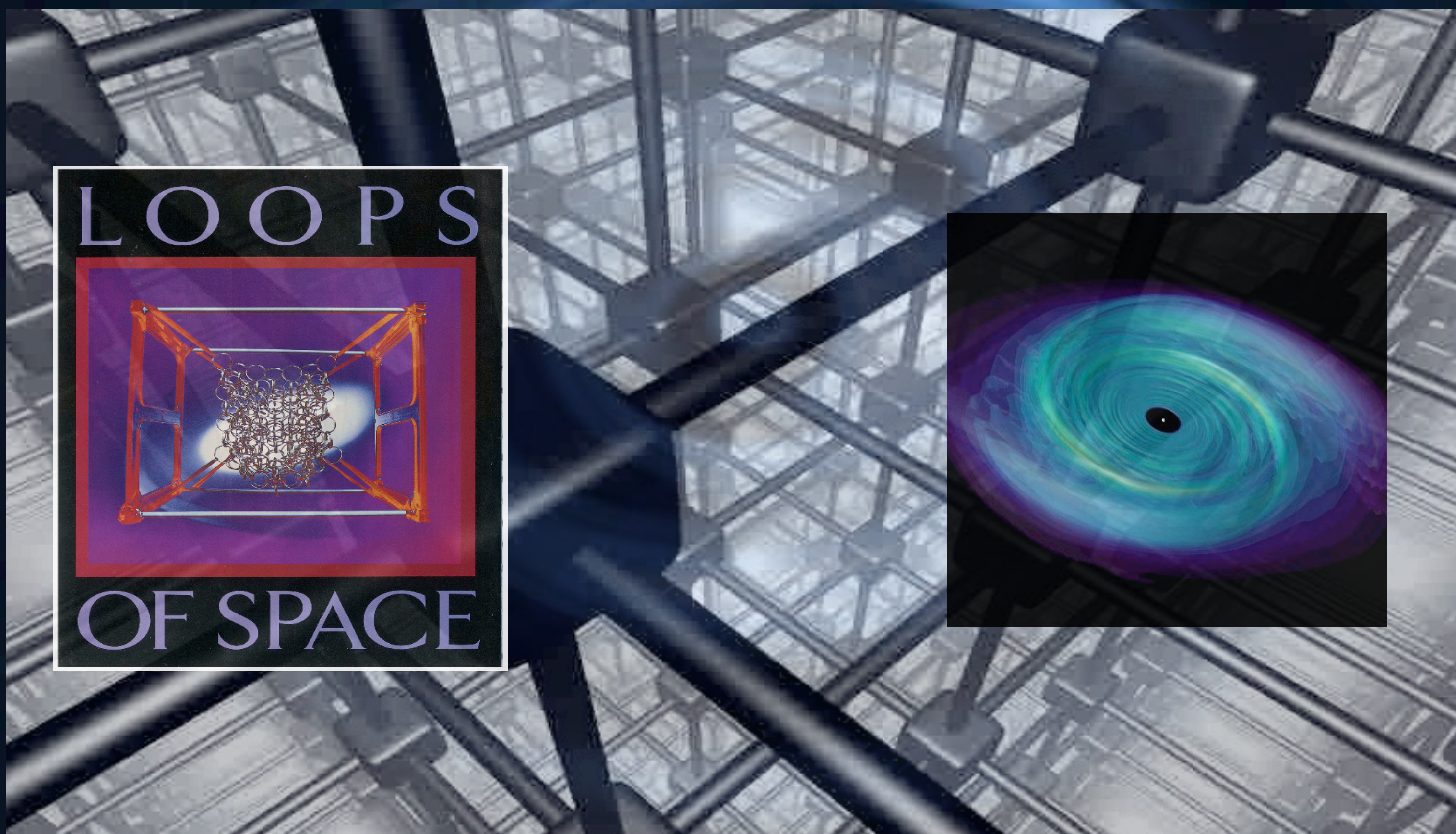
Loops07, Mexico 25-30 june 2007

Loop quantum black hole and evaporation



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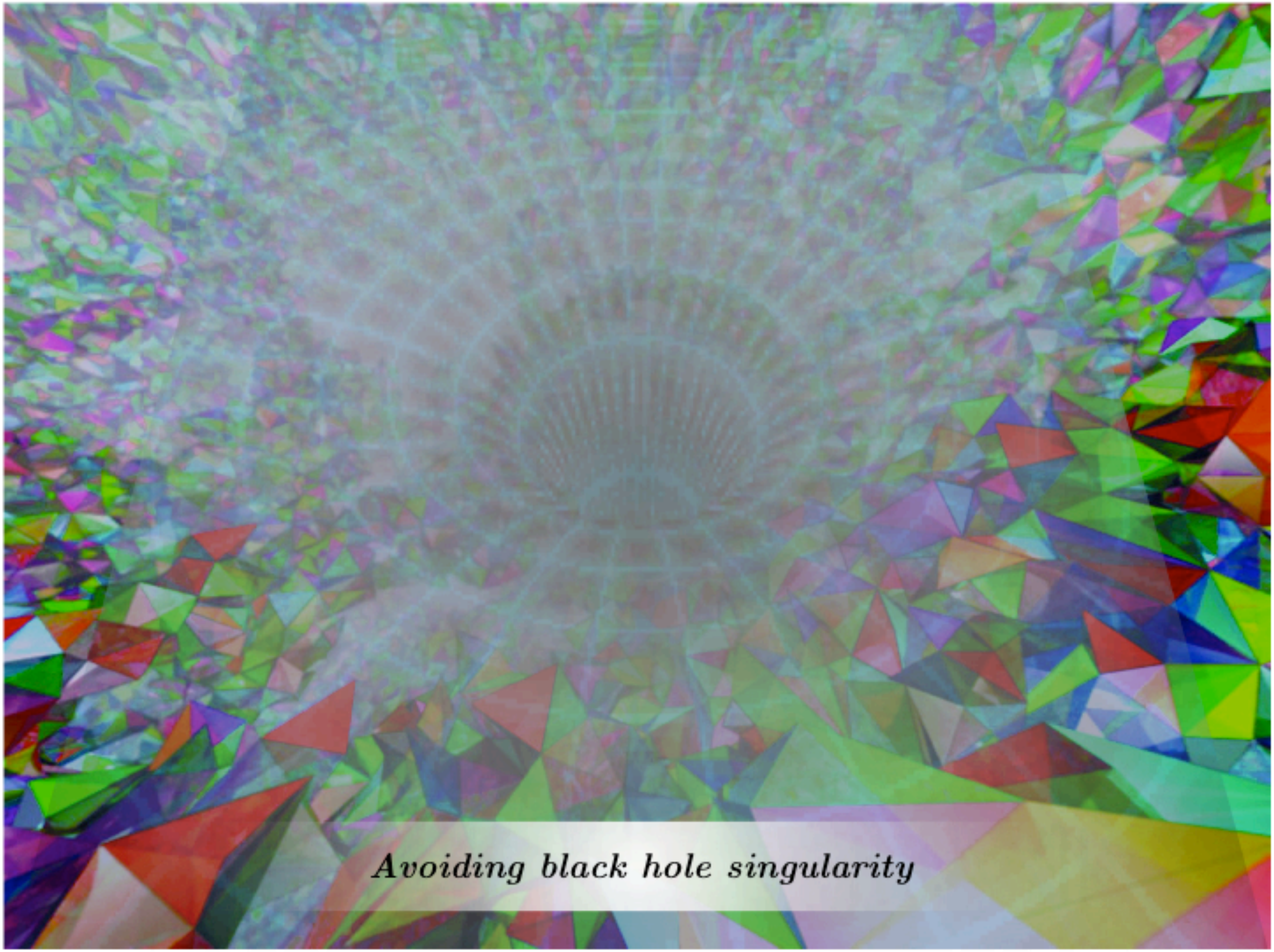
OUTLINE

Non singular black hole
from loop quantum gravity.

Semiclassical analysis and evaporation.


Avoiding black hole singularity

The Schwarzschild solution inside the horizon

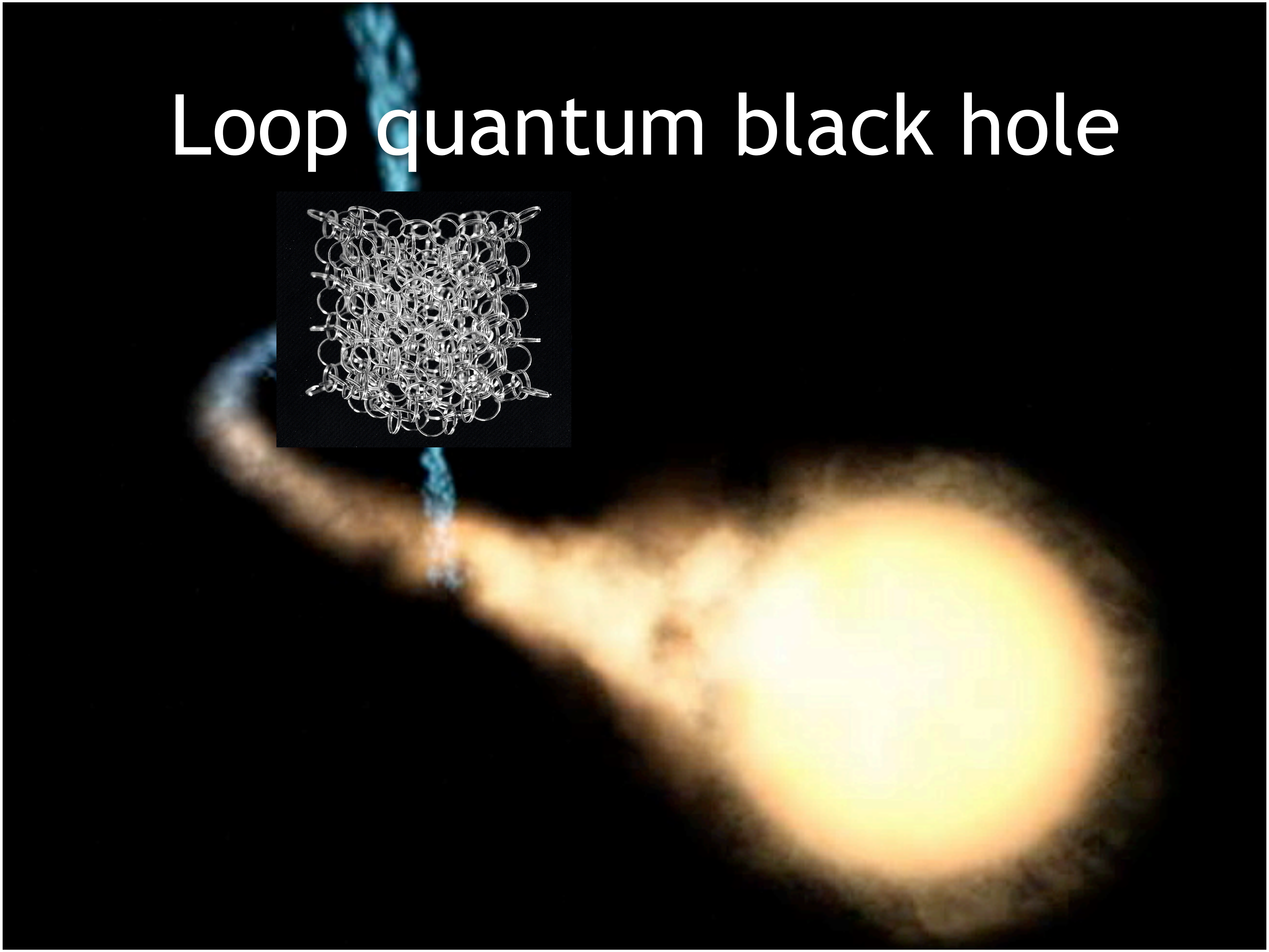
$$ds^2 = -\frac{dT^2}{\left(\frac{2MG_N}{T} - 1\right)} + \left(\frac{2MG_N}{T} - 1\right) dr^2 + T^2 (\sin^2\theta d\phi^2 + d\theta^2)$$

$$T \in]0, 2MG_N[\ , \ \ r \in]-\infty, +\infty[.$$

The Kantowski-Sachs space-time $(R \times R \times S^2)$:

$$ds^2 = -N^2(t)dt^2 + a^2(t)dr^2 + b^2(t)(\sin^2\theta d\phi^2 + d\theta^2)$$

Loop quantum black hole



Classical theory

Invariant 1-form connection $A_{[1]}$:

$$A_{[1]} = A_r(t)\,\tau_3\, dr + (A_1(t)\,\tau_1 + A_2(t)\tau_2)\, d\theta + (A_1(t)\,\tau_2 - A_2(t)\tau_1)\sin\theta\, d\phi + \tau_3\,\cos\theta\, d\phi$$

Invariant densitized triad :

$$E_{[1]} = E^r(t)\,\tau_3\,\sin\theta\,\frac{\partial}{\partial r} + (E^1(t)\,\tau_1 + E^2(t)\,\tau_2)\,\sin\theta\,\frac{\partial}{\partial\theta} + (E^1(t)\,\tau_2 - E^2(t)\tau_1)\frac{\partial}{\partial\phi}$$

Gauss constraint and Hamiltonian constrains :

$$G \sim A_1 E^2 - A_2 E^1$$

$$H_E = \frac{sgn[det(E_{[1]})]}{\sqrt{|E^r|[(E^1)^2 + (E^2)^2]}}\left[2A_r E^r(A_1 E^1 + A_2 E^2) + \left((A_1)^2 + (A_2)^2 - 1\right)[(E^1)^2 + (E^2)^2]\right]$$

For the Kantowski-Sachs space-time we fix the gauge $E^2 = E^1$ *and so* $A_2 = A_1$

The Hamiltonian constraint becomes : $H_E = \dfrac{sgn(E)}{\sqrt{|E|}|E^1|}\left[2AEA_1E^1 + (2(A_1)^2 - 1)(E^1)^2\right]$

Volume of the spatial section : $V = \displaystyle\int dr\, d\phi\, d\theta\,\sqrt{q} = 4\pi\sqrt{2}R\sqrt{|E|}|E^1|$

Background triad and co-triad : ${}^o e^a_I = diag(1, 1, \sin^{-1}\theta)$ ${}^o\omega^I_a = diag(1, 1, \sin\theta)$

$$q_{ab} = \begin{pmatrix} a^2(t) & 0 & 0 \\ 0 & b^2(t) & 0 \\ 0 & 0 & b^2(t)\,\sin^2\theta \end{pmatrix} = \begin{pmatrix} 2\frac{(E^1)^2}{|E|} & 0 & 0 \\ 0 & |E| & 0 \\ 0 & 0 & |E|\,\sin^2\theta \end{pmatrix}$$

Classical phase space

$$\textit{Canonical pairs} \;:\; (A, E) \;\; \textit{and} \;\; (A_1, E^1)$$

$$\textit{Symplectic structure}: \{A, E\} = \frac{\kappa}{l_P} \;\;,\;\; \{A_1, E^1\} = \frac{\kappa}{4 l_P}$$

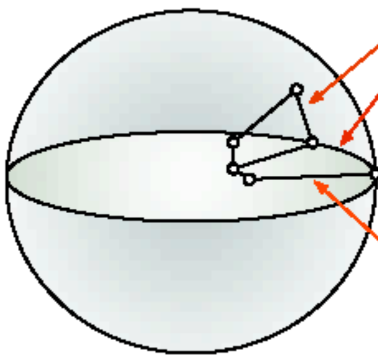
Holonomies

$$h_I = \exp\int A_I^i\,\tau_i\,d\lambda$$

$$h_1 = \exp\int A_1^i\tau_i d\lambda = \exp[A\mu_0 l_P\,\tau_3]$$

$$h_2 = \exp\int A_2^i\tau_i d\lambda = \exp[A_1\mu_0\,(\tau_2 + \tau_1)]$$

$$h_3 = \exp\int A_3^i\tau_i d\lambda = \exp[A_1\mu_0\,(\tau_2 - \tau_1)]$$



$$\textit{Curvature } F_{ab} \textit{ in terms of holonomies}: \quad F_{ab}^i\,\tau_i = {}^o\omega_a^I\,{}^o\omega_b^J\left[\frac{h_I h_J h_I^{-1} h_J^{-1} h_{[IJ]} - 1}{\epsilon(I)\epsilon(J)}\right]$$

Hamiltonian constraint

$$H^E(N) = \int_\Sigma d^3x\,N\frac{E_i^a E_j^b}{\sqrt{\det(E)}}\epsilon^{ij}{}_k F_{ab}^k, \qquad H^E(N) = \int_\Sigma dx^3\;\; N\;\; \epsilon^{abc}\delta_{ij}F_{ab}^i\left\{A_c^j,\,V\right\}.$$

$$H_E = -\frac{8\pi}{\mu_0^3}\sum_{IJK}\epsilon^{IJK}\,Tr\left[h_I h_J h_I^{-1} h_J^{-1} h_{[IJ]}\,h_K^{-1}\{h_K,\,V\}\right] \qquad \mu_0 = l_P/L_{Phys}$$

Quantum theory

$$Hilbert\ space\ :\ H_E \otimes H_{E^1} \sim L^2(R^2_{Bohr})$$

Basis in the Hilbert space:

$$|\mu_E, \mu_{E^1}\rangle \equiv |\mu_E\rangle \otimes |\mu_{E^1}\rangle \to \langle A|\mu_E\rangle \otimes \langle A_1|\mu_{E^1}\rangle = e^{\frac{i\mu_E l_P A}{2}} \otimes e^{\frac{i\mu_{E^1} A_1}{\sqrt{2}}} \qquad \langle \mu_E, \mu_{E^1}|\nu_E, \nu_{E^1}\rangle = \delta_{\mu_E,\nu_E}\,\delta_{\mu_{E^1},\nu_{E^1}}$$

Representation of the momentum operators:

$$\hat{E} \to -il_P \frac{d}{dA} \quad , \quad \hat{E}^1 \to -i\frac{l_P}{4}\frac{d}{dA_1}$$

$$\hat{E}|\mu_E, \mu_{E^1}\rangle = \frac{\mu_E\, l_P^2}{2}|\mu_E, \mu_{E^1}\rangle \quad , \quad \hat{E}^1|\mu_E, \mu_{E^1}\rangle = \frac{\mu_{E^1}\, l_P}{4\sqrt{2}}|\mu_E, \mu_{E^1}\rangle$$

Inverse volume operator

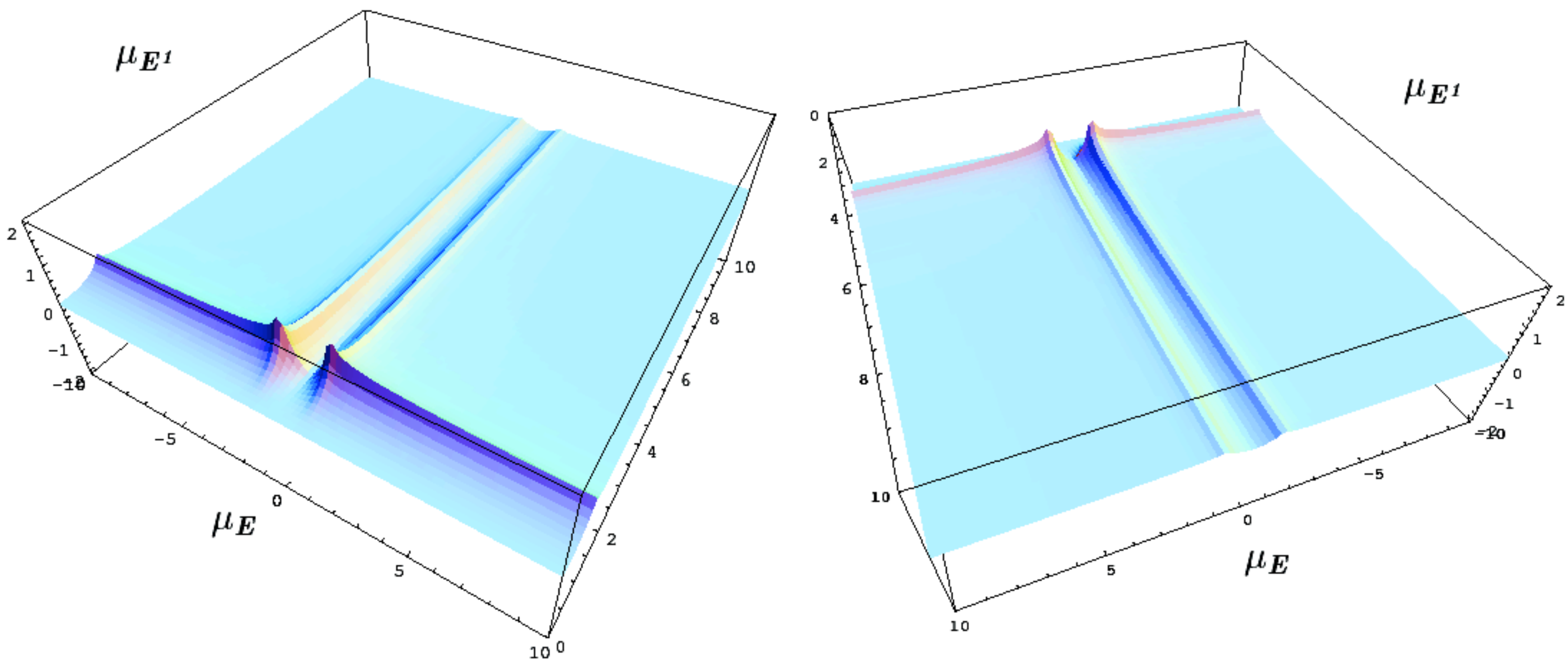
$$\frac{\widehat{sgn(E)}}{\sqrt{\bar{V}}} = \frac{512\, i}{3\, l_P^4 \mu_0^3}\,\epsilon_{ijk} \sum_{IJK} \epsilon^{IJK}\, Tr\left[\tau^i \hat{h}_I^{-1}[\hat{h}_I, \hat{\bar{V}}^{\frac{1}{2}}]\right]\, Tr\left[\tau^j \hat{h}_J^{-1}[\hat{h}_J, \hat{\bar{V}}^{\frac{1}{2}}]\right]\, Tr\left[\tau^k \hat{h}_K^{-1}[\hat{h}_K, \hat{\bar{V}}^{\frac{1}{2}}]\right]$$

Spectrum of $\widehat{\bar{V}}$ and $\widehat{1/\bar{V}}$

$$\hat{V}|\mu_E, \mu_{E^1}\rangle = \frac{4\pi\, l_P^3}{\sqrt{2}}\sqrt{|\mu_E|}\,|\mu_{E^1}||\mu_E, \mu_{E^1}\rangle$$

$$\frac{\widehat{sgn(E)}}{\sqrt{\bar{V}}}|\mu_E, \mu_{E^1}\rangle = \frac{8}{\sqrt{2}\, l_P \mu_0^3}\,|\mu_E|^{\frac{1}{2}}\,|\mu_{E^1}|^{\frac{1}{2}}\left(|\mu_E + \mu_0|^{\frac{1}{4}} - |\mu_E - \mu_0|^{\frac{1}{4}}\right)\left(|\mu_{E^1} + \mu_0|^{\frac{1}{2}} - |\mu_{E^1} - \mu_0|^{\frac{1}{2}}\right)^2 |\mu_E, \mu_{E^1}\rangle$$

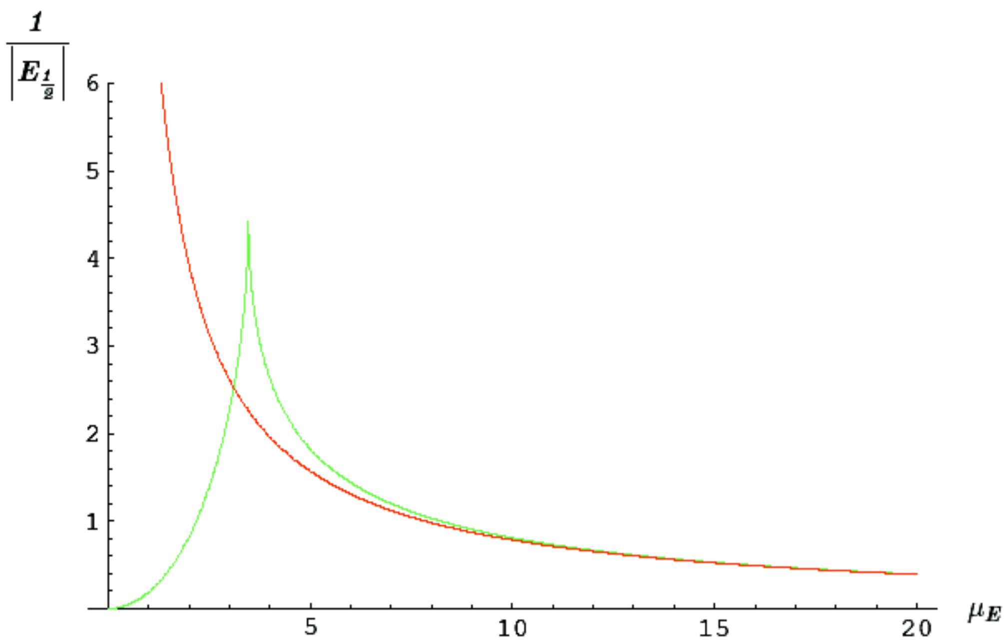
Plot of the inverse volume operator spectrum



Spatial curvature

$$^{(3)}R = \frac{2}{|E|} ,$$

$$\widehat{\frac{1}{|E_j|}}|\mu_E, \mu_{E_1}\rangle = \left(\frac{3}{\gamma^{\frac{1}{2}}\delta \ l_P j(j+1)(2j+1)} \sum_{k=-j}^{k=j} \left[k \left(\sqrt{|\mu_E|} - \sqrt{|\mu_E - 2k\delta|} \right) \right] \right)^2 |\mu_E, \mu_{E_1}\rangle .$$



Hamiltonian constraint

The solutions of the Hamiltonian constraint are in $C^\star$

dual of the dense subspace C of the kinematical space H_{kin}.

A generic element of this space is: $\langle\psi| = \sum_{\mu_E,\mu_{E^1}} \psi(\mu_E,\mu_{E^1})\langle\mu_E,\mu_{E^1}|.$

The constraint equation $\hat{H}_E|\psi\rangle = 0$ gives a relation for the coefficients $\psi(\mu_E,\nu_{E^1})$:

$$
\begin{aligned}
&-\alpha(\mu_E-2\mu_0,\mu_{E^1}-2\mu_0)\,\psi(\mu_E-2\mu_0,\mu_{E^1}-2\mu_0)+\alpha(\mu_E+2\mu_0,\mu_{E^1}-2\mu_0)\,\psi(\mu_E+2\mu_0,\mu_{E^1}-2\mu_0)\\
&+\alpha(\mu_E-2\mu_0,\mu_{E^1}+2\mu_0)\,\psi(\mu_E-2\mu_0,\mu_{E^1}+2\mu_0)-\alpha(\mu_E+2\mu_0,\mu_{E^1}+2\mu_0)\,\psi(\mu_E+2\mu_0,\mu_{E^1}+2\mu_0)\\
&+\frac{\sin(\mu_0^2/2)-\cos(\mu_0^2/2)}{2}\Big(\beta(\mu_E,\mu_{E^1}-4\mu_0)\,\psi(\mu_E,\mu_{E^1}-4\mu_0)-\beta(\mu_E,\mu_{E^1})\,\psi(\mu_E,\mu_{E^1})\\
&\qquad+\beta(\mu_E,\mu_{E^1}+4\mu_0)\,\psi(\mu_E,\mu_{E^1}+4\mu_0)\Big)\\
&-\sin(\mu_0^2/2)\Big(\beta(\mu_E,\mu_{E^1}-2\mu_0)\,\psi(\mu_E,\mu_{E^1}-2\mu_0)+\beta(\mu_E,\mu_{E^1}+2\mu_0)\,\psi(\mu_E,\mu_{E^1}+2\mu_0)\Big)=0
\end{aligned}
$$

$$
\alpha(\mu_E,\mu_{E^1}) \equiv |\mu_E|^{\frac{1}{2}}\left(|\mu_{E^1}+\mu_0|-|\mu_{E^1}-\mu_0|\right)
$$

$$
\beta(\mu_E,\mu_{E^1}) \equiv |\mu_{E^1}|\left(|\mu_E+\mu_0|^{\frac{1}{2}}-|\mu_E-\mu_0|^{\frac{1}{2}}\right)
$$

Semiclassical analysis

$$A = c\tau_3 dx + b\tau_2 d\theta - b\tau_1 \sin\theta d\phi + \tau_3 \cos\theta d\phi,$$

$$E = p_c \tau_3 \sin\theta \frac{\partial}{\partial x} + p_b \tau_2 \sin\theta \frac{\partial}{\partial \theta} - p_b \tau_1 \frac{\partial}{\partial \phi},$$

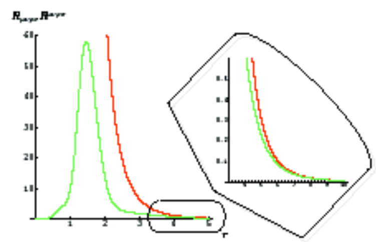
$$h_1 = \cos\frac{\delta c}{2} + 2\tau_3 \sin\frac{\delta c}{2}, \quad h_2 = \cos\frac{\delta b}{2} - 2\tau_1 \sin\frac{\delta b}{2}, \quad h_3 = \cos\frac{\delta b}{2} + 2\tau_2 \sin\frac{\delta b}{2}.$$

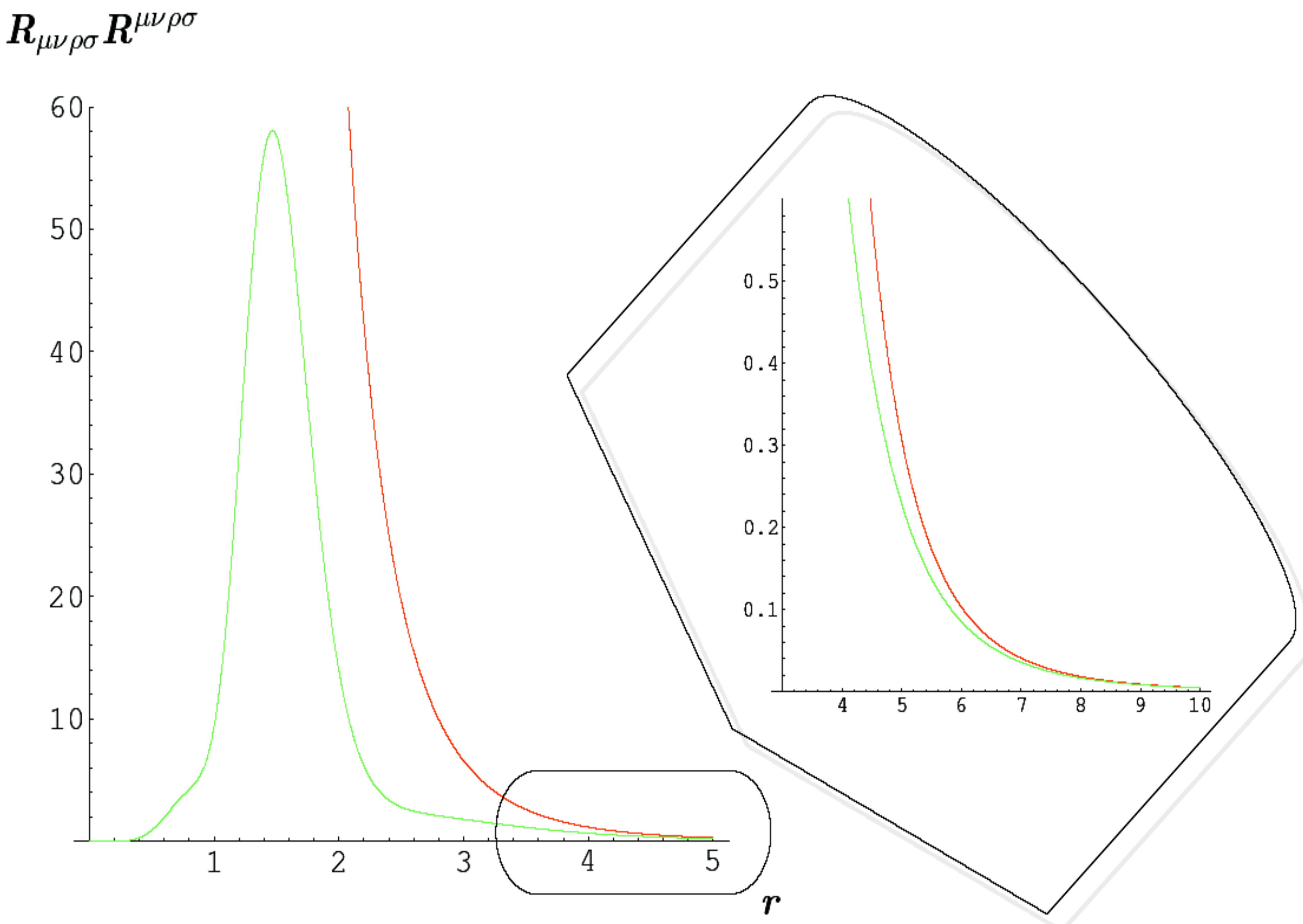
$$H^\delta = -\frac{2\hbar N}{\gamma^3 \delta^3 l_P^2} \mathrm{Tr}\left(\sum_{ijk} \epsilon^{ijk} h_i^{(\delta)} h_j^{(\delta)} h_i^{(\delta)-1} h_k^{(\delta)} \left\{ h_k^{(\delta)-1}, V \right\} + 2\gamma^2 \delta^2 \tau_3 h_1^{(\delta)} \left\{ h_1^{(\delta)-1}, V \right\} \right)$$

$$H^\delta = -\frac{1}{2\gamma G_N} \left\{ 2 \sin\delta c \;\; p_c + \left(\sin\delta b + \frac{\gamma^2 \delta^2}{\sin\delta b} \right) p_b \right\}.$$

$$N = \frac{\gamma \sqrt{|p_c|}\, sgn(p_c) \delta^2}{16\pi G_N \sin\delta b} \;\; \& \;\; \textbf{\textit{Hamilton e.m.}} \rightarrow g_{\mu\nu}.$$

Regular solution $\rightarrow$

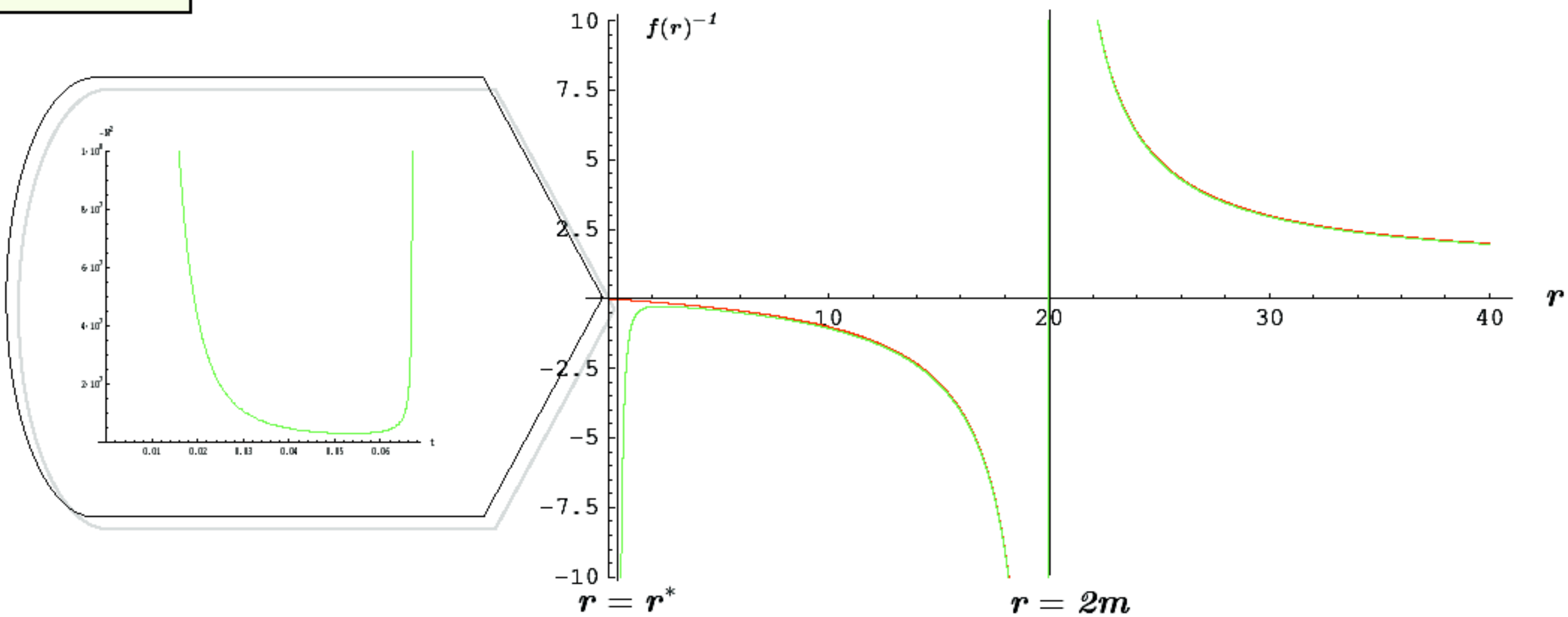




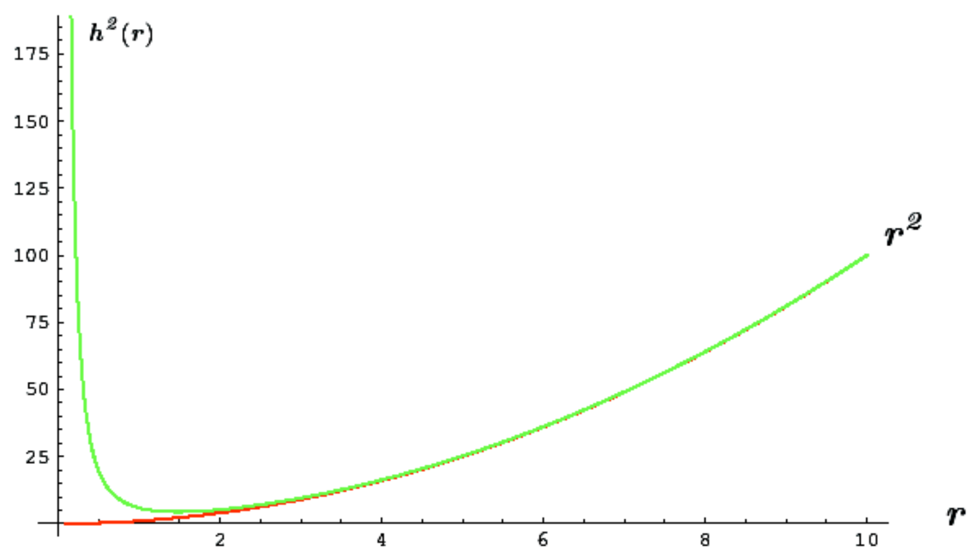
$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$
60
50
40
30
20
10
1
2
3
4
5
r
0.5
0.4
0.3
0.2
0.1
4
5
6
7
8
9
10

$$f(r)^{-1}$$

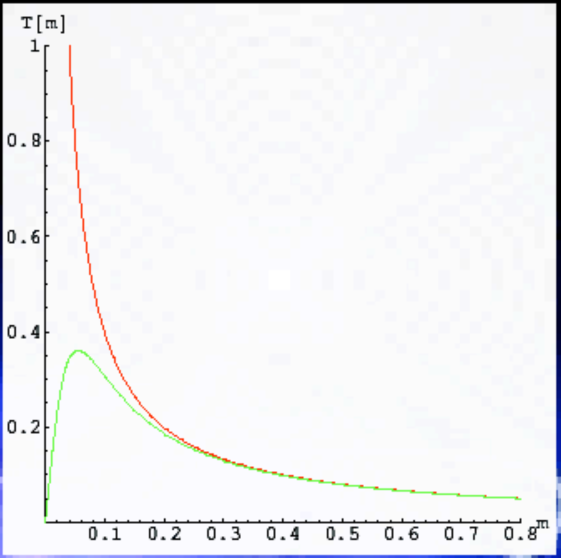
$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + h^2(r)(d\theta^2 + \sin\theta d\phi^2)$$



$$h^2(r)$$



$$Temperature: \quad T_{BH} = \frac{8m}{\pi(64m^2+\gamma^2\delta^2)}.$$



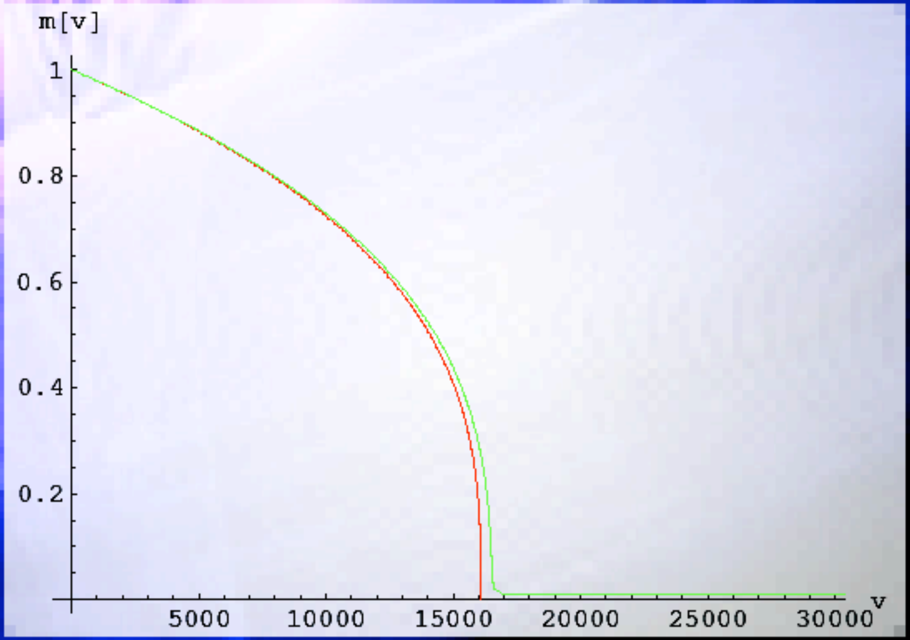
Entropy :

$$S = \frac{A}{4l_P^2} + \frac{\gamma^2}{16}\ln\left(\frac{A}{4l_P^2}\right) + \frac{\gamma^2}{16}\ln\left(1 - \frac{4\pi\gamma^2 l_P^2}{16A}\right) + \text{const.}$$

Evaporation process. Luminosity and mass decreasing.

$$L(m) = \frac{2^{16}m^6 + 2^{10}\gamma^2\delta^2 m^4}{60\pi(64m^2+\gamma^2\delta^2)^4},$$

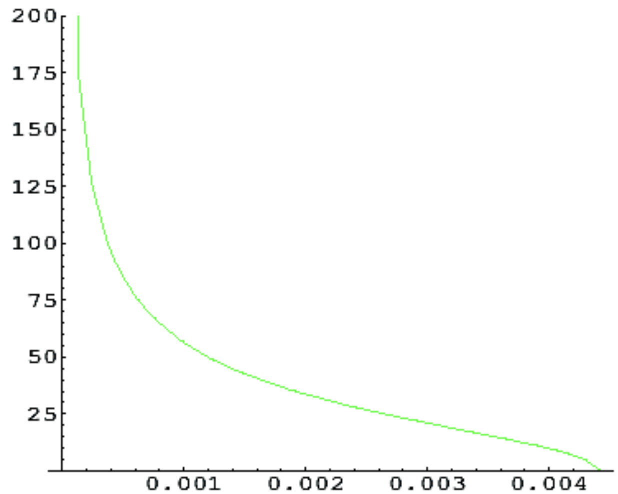
$$-\frac{dm(v)}{dv} = L[m(v)].$$

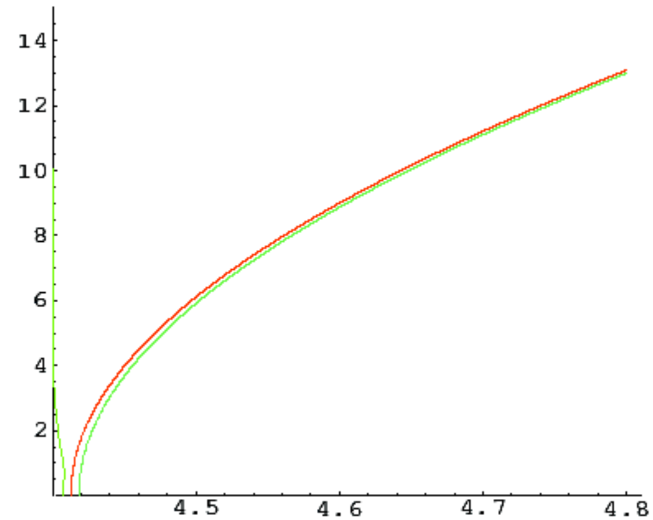


Inverse volume operator modifications

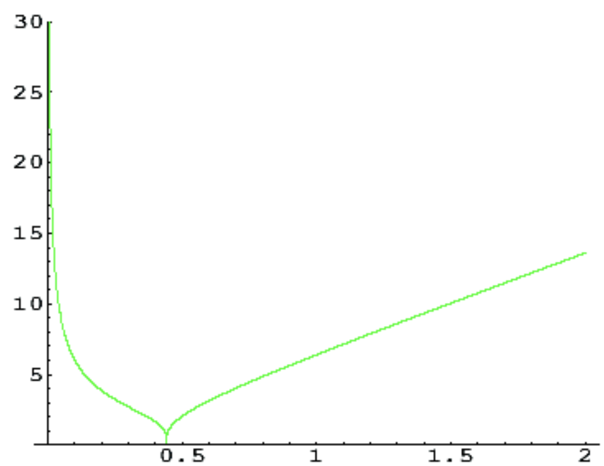
$$\dot{p}_c = -\frac{2\gamma\delta^2\sqrt{p_c - m^2\left(\sqrt{|p_c - \gamma\delta|} - \sqrt{|p_c - \gamma\delta|}\right)}}{\sqrt{|p_c - \gamma\delta|} - \sqrt{|p_c - \gamma\delta|}}$$

$$\boxed{m > \frac{1}{\sqrt{2}}}$$

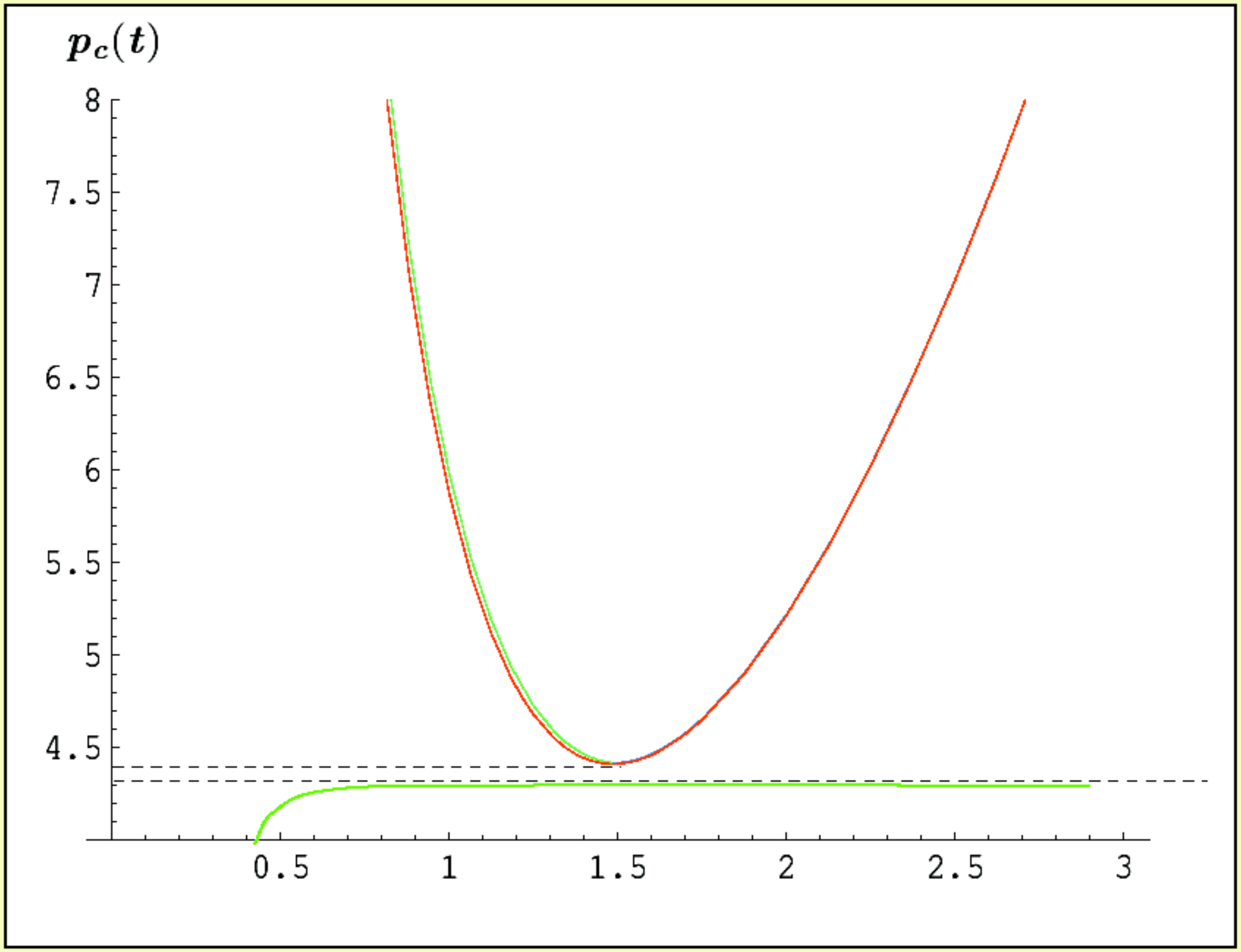




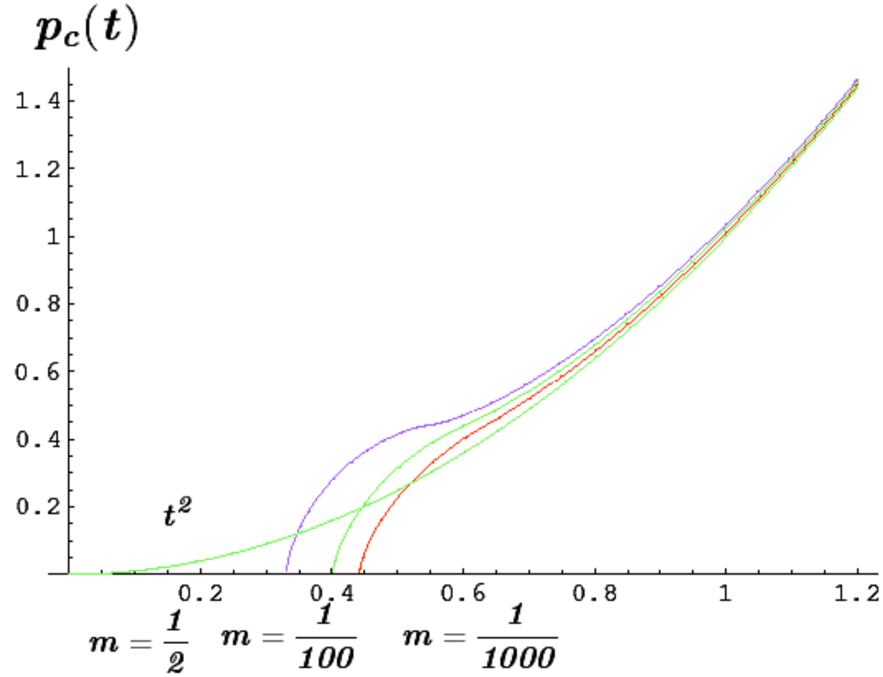
$$\boxed{m \leqslant \frac{1}{\sqrt{2}}}$$

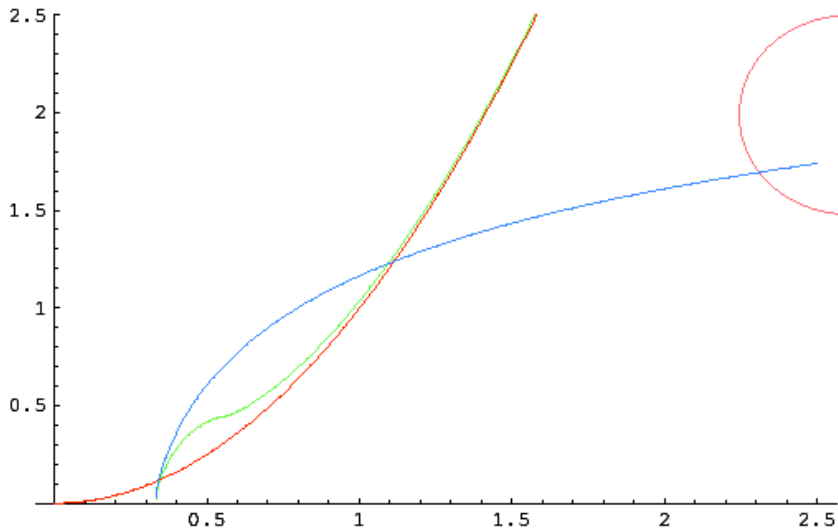


$m > \frac{1}{\sqrt{2}}$



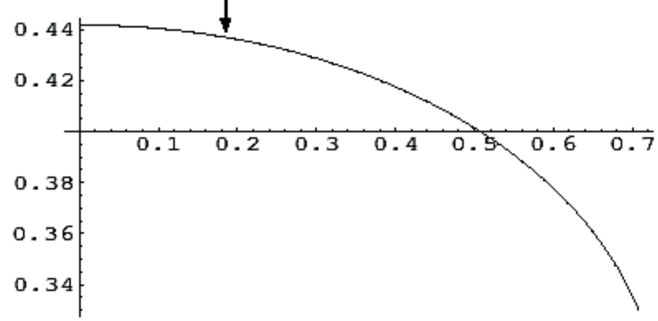
$$m \leqslant \frac{1}{\sqrt{2}}$$



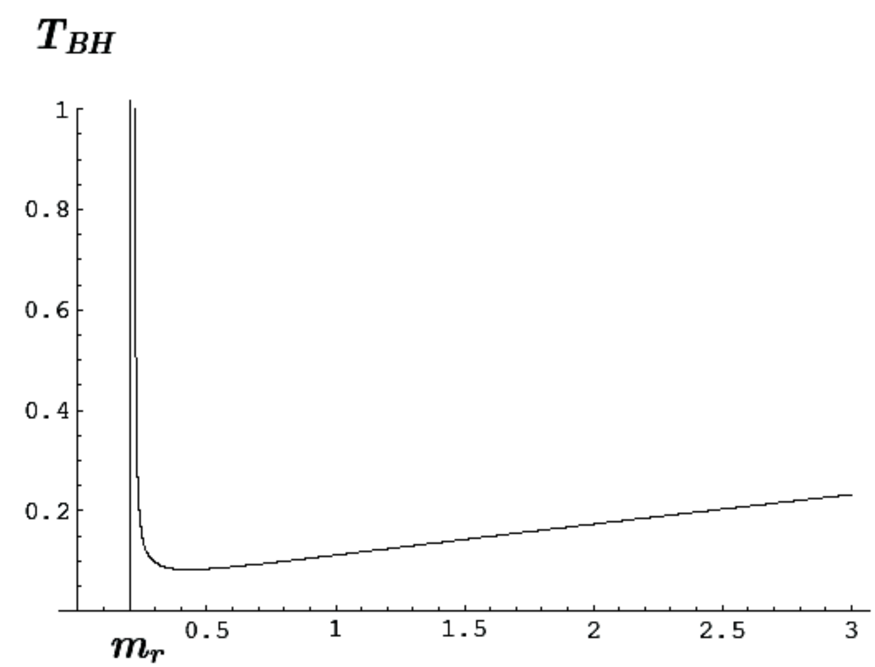


$$p_c(t) \sim \frac{4}{3}(\gamma\delta)^{\frac{1}{4}}\left(\log\left(\frac{t}{c(m)}\right)\right)^{\frac{2}{3}}$$

for $p_c \to 0$



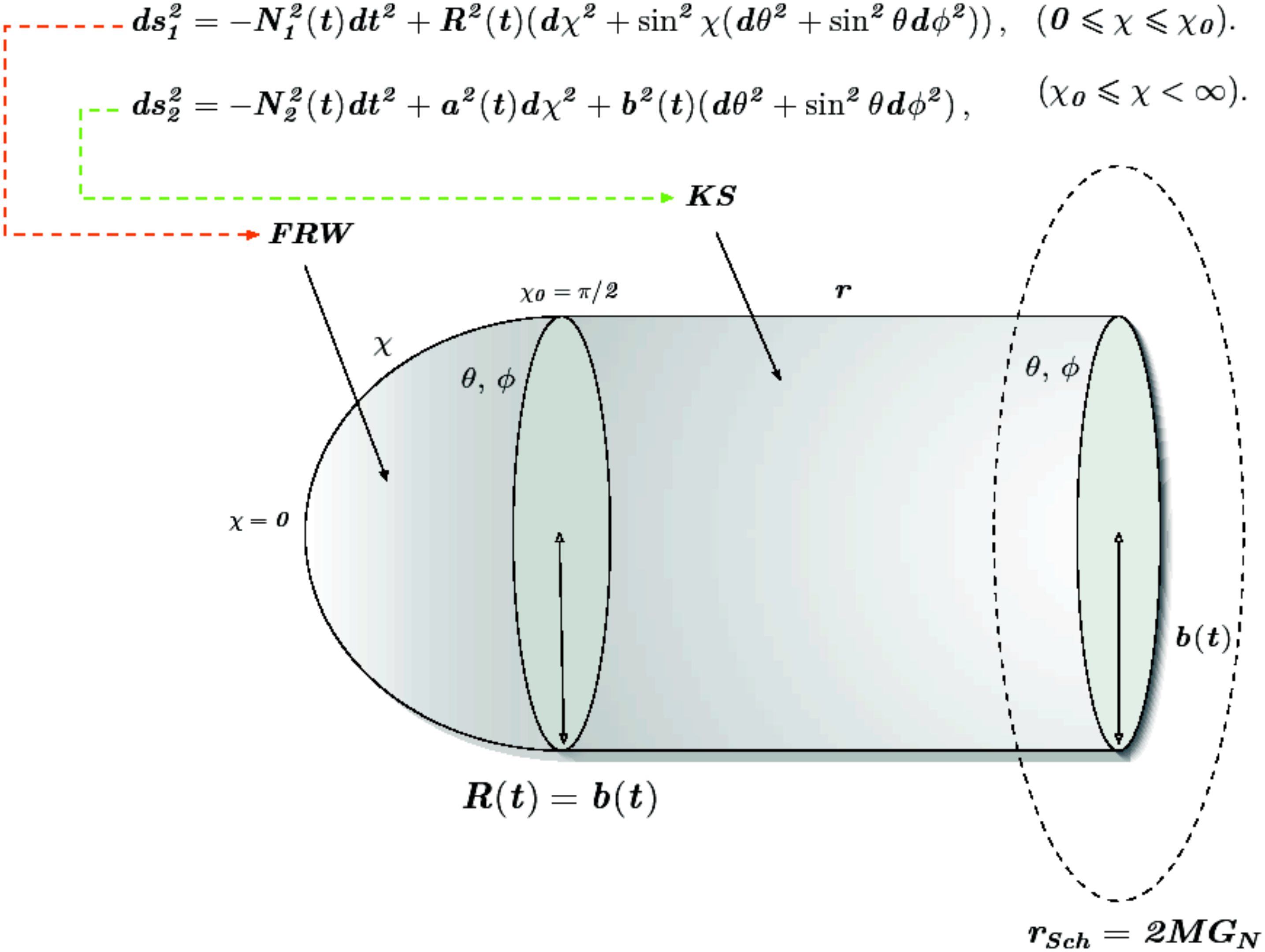
Temperature for $p_c \to 0$



Gravitational collapse

$$ds_1^2 = -N_1^2(t)dt^2 + R^2(t)(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2))\,, \quad (0 \leqslant \chi \leqslant \chi_0).$$

$$ds_2^2 = -N_2^2(t)dt^2 + a^2(t)d\chi^2 + b^2(t)(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (\chi_0 \leqslant \chi < \infty).$$



Classical theory

$$\boldsymbol{H_1} = -\frac{\kappa}{12\mathrm{V}_1}\frac{p_R^2}{R} - \frac{3\mathrm{V}_1}{\kappa}R + M_{\text{dust}} = 0,$$

$$\boldsymbol{H_2} = \frac{2\kappa}{\mathrm{V}_2}\left(-\frac{p_a p_b}{4b} + \frac{a\, p_a^2}{8b^2}\right) - \frac{\mathrm{V}_2}{\kappa}a = 0,$$

$$\boldsymbol{P_1} = \frac{p_R \sin^2\chi_0}{3\mathrm{V}_1} - \frac{p_a}{\mathrm{V}_2} = 0,$$

$$\boldsymbol{P_2} = R\sin\chi_0 - b = 0.$$

$$H^i\Big|_{\chi=\chi_0} = 0$$

Quantum theory

$$|\psi\rangle = \sum_{\mu,\mu_E,\mu_{E^1}} \psi_{\mu,\mu_E,\mu_{E^1}} |\mu,\mu_E,\mu_{E^1}\rangle. \quad \rightarrow \quad \begin{aligned} \hat{H}_1|\psi\rangle &= 0, \\ \hat{H}_2|\psi\rangle &= 0, \\ \hat{P}_1|\psi\rangle &= 0, \\ \hat{P}_2|\psi\rangle &= 0. \end{aligned} \quad \rightarrow \quad \psi_{\mu,\mu_E,\mu_{E^1}}.$$

CONCLUSIONS

The classical black hole singularity in $r = 0$ disappears from the quantum theory.

Classical divergent quantities are bounded in the quantum theory.

- ***Curvature invariant:*** $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \frac{48 M^2 G_N^2}{b(t)^6} \rightarrow \widehat{R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}} |\psi\rangle = \widehat{\frac{48 M^2 G_N^2}{b^6}} |\psi\rangle$

  is bounded for the Kantowski-Sachs model.

- ***The inverse volume operator $1/\sqrt{V}$ is bounded.***

The Hamiltonian constraint gives a difference equation for the coefficients of the physical states and we can evolve across the singularity.

... INSIDE ... ACROSS ... AND BEYOND ...

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Semiclassical analysis and evaporation

New regular black hole solution.

For $m > \frac{l_P}{\sqrt{2}}$ — *regular temperature,*
infinite evaporation time.

For $m \leqslant \frac{l_P}{\sqrt{2}}$ — *Hot remnant.*