Discretization of Classical Mechanics

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- Review of Lagrangian Mechanics

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Lagrangian Mechanics

Q : n-dimensional Configuration manifold
L : $TQ \to \mathbb{R}$

$$S(q(t)) \equiv \int_a^b L\left(q^i(t), \frac{dq^i}{dt}(t)\right) dt$$

- Hamilton's Principle

$$\begin{aligned} dS(q(t)).\delta q(t) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \int_a^b L\left(q^i_\epsilon(t), \frac{dq^i_\epsilon}{dt}(t)\right) dt \\ &= \int_a^b \delta q^i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right) dt \\ &\quad + \frac{\partial L}{\partial \dot{q}^i}\delta q^i\Big|_a^b \end{aligned}$$

Lagrangian Mechanics

$$dS(q(t)).\delta q(t) = \int_a^b \delta q^i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right)dt + \frac{\partial L}{\partial \dot{q}^i}\delta q^i|_a^b$$

Boundary condition $\rightsquigarrow \delta q(a) = \delta q(b) = 0$

- Remove the boundary condition

  Lagrange 1-form

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i}\delta q^i$$

  Symplectic 2-form

$$-d\theta_L = \omega_L$$

$$F_t^* \omega_L = \omega_L$$

Discretization of Mechanics

Veselov Discretization,1988

Q: configuration space

Q×Q : discrete version of tangent bundle of configuration spce Q

tangent vector $\frac{(q_1 - q_0)}{\triangle t} \longleftrightarrow (q_0, q_1) \in Q \times Q$

$\Longrightarrow$ Discrete Lagrangian $L : Q \times Q = (q_0, q_1) \mapsto \mathbb{R}$

Action $S = \sum_{k=0}^{n} L(q_k, q_{k+1})$

Discretization of Mechanics

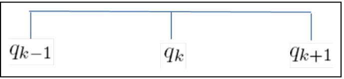
Veselov Discrete Euler-Lagrange equations, DEL

$$dS(q_0, ..., q_n).(\delta q_0, ..., \delta q_n) = \sum_{k=1}^{n-1} (\frac{\partial L}{\partial q_1}(q_k, q_{k-1}) + \frac{\partial L}{\partial q_0}(q_{k+1}, q_k))\delta q_k + \frac{\partial L}{\partial q_0}(q_1, q_0)\delta q_0 + \frac{\partial L}{\partial q_1}(q_n, q_{n-1})\delta q_n$$

$$\frac{\partial L(q_1, q_0)}{\partial q_1} + \frac{\partial L(q_2, q_1)}{\partial q_0} = 0$$

Descrete Flow F: $Q \times Q \longrightarrow Q \times Q$

$F(q_1, q_0) = (q_2, q_1)$ q_2 is found from DEL equations



Discretization of Mechanics

Lagrange 1-From

$$dS(q_0, ..., q_n).(\delta q_0, ..., \delta q_n)) = \sum_{k=1}^{n-1} (\frac{\partial L}{\partial q_1}(q_k, q_{k-1}) + \frac{\partial L}{\partial q_0}(q_{k+1}, q_k))\delta q_k$$
$$+ \frac{\partial L}{\partial q_0}(q_1, q_0)\delta q_0 + \frac{\partial L}{\partial q_1}(q_n, q_{n-1})\delta q_n$$

$$\theta_L^-(q_1, q_0).(\delta q_1, \delta q_0) \equiv \frac{\partial L}{\partial q_0}(q_1, q_0)\delta q_0$$

$$\theta_L^+(q_1, q_0).(\delta q_1, \delta q_0) \equiv \frac{\partial L}{\partial q_1}(q_1, q_0)\delta q_1$$

$$\Rightarrow \omega_L^+ = -d\theta_L^+$$

$$\omega_L^- = -d\theta_L^-$$

$$\theta_L^- + \theta_L^+ = dL \quad \rightarrow d\theta_L^- + d\theta_L^+ = 0$$

Discretization of Mechanics

Modified Veselov Discretization

$\Longrightarrow$ Discrete Lagrangian

$$L^- : Q^- \times Q = (q_k^-, q_k) \mapsto \mathbb{R}$$

$$L^+ : Q \times Q^+ = (q_k, q_k^+) \mapsto \mathbb{R}$$

Action $S = \sum_{k=1}^{n}(L^-(q_k, q_k^-) + L^+(q_k, q_k^+))$

Discretization of Mechanics

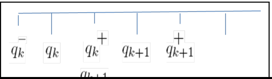
Modified Veselov Discrete Euler-Lagrange equations, DEL

$$dS(q_1, \ldots, q_n, q_1^+, \ldots, q_n^+).(\delta q_1, \ldots, \delta q_n^+) = \frac{\partial L_1^-}{\partial q_1^-}\delta q_1^- +$$

$$\sum_{i=1}^{n-1}\left(\frac{\partial L_i^+}{\partial q_i^+} + \frac{\partial L_{i+1}^-}{\partial q_i^+}\right)\delta q_i^+ + \sum_{i=2}^{n-1}\left(\frac{\partial L_i^+}{\partial q_i} + \frac{\partial L_i^-}{\partial q_i}\right)\delta q_i + \frac{\partial L_n^+}{\partial q_n^+}\delta q_n^+$$

$$\frac{\partial L_i^+(q_i, q_i^+)}{\partial q_i^+} + \frac{\partial L_{i+1}^-(q_{i+1}^-, q_{i+1})}{\partial q_i^+} = 0 \qquad (q_i^+ = q_{i+1}^-)$$

$$\frac{\partial L_i^+(q_i^-, q_i)}{\partial q_i} + \frac{\partial L_i^-(q_i, q_i^+)}{\partial q_i} = 0$$



Discretization of Mechanics

Modified Lagrange 1-From

$$dS(q_1,...,q_n,q_1^+,...,q_n^+).(\delta q_1,...,\delta q_n^+) = \frac{\partial L_1^-}{\partial q_1^-}\delta q_1^- +$$

$$\sum_{i=1}^{n-1}\left(\frac{\partial L_i^+}{\partial q_i^+} + \frac{\partial L_{i+1}^-}{\partial q_i^+}\right)\delta q_i^+ + \sum_{i=2}^{n-1}\left(\frac{\partial L_i^+}{\partial q_i} + \frac{\partial L_i^-}{\partial q_i}\right)\delta q_i + \frac{\partial L_n^+}{\partial q_n^+}\delta q_n^+$$

$$\theta_L^- = \frac{\partial L_1^-}{\partial q_1^-}\delta q_1^-$$

$$\theta_L^+ = \frac{\partial L_n^+}{\partial q_n^+}\delta q_n^+$$

In this model Lagrange 1-form doesn't depend on any Potential

Example 1,Particle in line

$$S_v = \frac{m_v}{2}(q_{n+1} - q_n)^2 - V_v(q_n) - V_v(q_{n+1})$$

$$S_m = \frac{m_m}{2}(q_n^+ - q_n)^2 - V_m(q_n) + \frac{m_m}{2}(q_{n+1} - q_{n+1}^-)^2 - V_m(q_{n+1})$$

$$\frac{\partial L_i^+}{\partial q_i^+} + \frac{\partial L_{i+1}^-}{\partial q_i^+} = 0 \Rightarrow q_{n+1} - q_{n+1}^- = q_{n+1} - q_n^+ = q_n^+ - q_n$$

$$\frac{\partial L_i^+}{\partial q_i} + \frac{\partial L_i^-}{\partial q_i} = 0$$

$$\rightsquigarrow q_{n+1} - q_n = q_{n+1} - q_n^+ + q_n^+ - q_n = 2(q_{n+1} - q_n^+) = 2(q_n^+ - q_n)$$

Example 1,Particle in line

$$\rightarrow S_m = \frac{m_m}{4}(q_{n+1} - q_n)^2 - V_m(q_n) - V_m(q_{n+1})$$
$$= \frac{m_v}{2}(q_{n+1} - q_n)^2 - V_v(q_n) - V_v(q_{n+1}) = S_v$$
$$[m_m = 2m_v, V_m(q_n) = V_v(q_n)]$$

- Our variation of framework and that proposed by Veselov et al lead to the same space of solutions and that the induced geometric structures in the space of solutions agree.

Example 2,Particle in circle

Example 2,Particle in circle

$$S \equiv \sum_{k=1}^{n} L^-(\varphi_k, \varphi_k^-) + L^+(\varphi_k^+, \varphi_k)$$

$$L_n^- = \frac{m_m}{2}(\varphi_n - \varphi_n^- + 2\pi z_n^-)^2 - V_m(\varphi_n)$$

$$L_n^+ = \frac{m_m}{2}(\varphi_n^+ - \varphi_n + 2\pi z_n^+)^2 - V_m(\varphi_n)$$

Example 2,Particle in circle

- DEL equations

$$\frac{\partial L_i^+}{\partial \varphi_i} + \frac{\partial L_i^-}{\partial \varphi_i} = 0$$

$$\frac{\partial L_i^+}{\partial \varphi_i} + \frac{\partial L_i^-}{\partial \varphi_i} = m_m \left(\varphi_i - (\varphi_i^+ + 2\pi z_i^+) + \varphi_i - (\varphi_i^- - 2\pi z_i^-)\right) - 2\left(\frac{\partial V_m}{\partial \varphi_i}\right) = 0$$

$$\frac{\partial L_i^+}{\partial \varphi_i^+} + \frac{\partial L_{i+1}^-}{\partial \varphi_i^+} = 0$$

$$\frac{\partial L_i^+}{\partial \varphi_i^+} + \frac{\partial L_{i+1}^-}{\partial \varphi_i^+} = \varphi_i^+ - \varphi_i + 2\pi z_i^+ + \varphi_i^+ - (2\pi z_{i+1}^- + \varphi_{i+1}) = 0$$

Example 2,Particle in circle

$$\theta_n^- = (m_m(\varphi_n^- - (\varphi_n + 2\pi z_n^-)))d\varphi_n^- + (m_m(\varphi_n + 2\pi z_n - \varphi_n^-) - \frac{\partial V_m(\varphi_n)}{\partial \varphi_n})d\varphi_n$$

$$\theta_n^+ = (m_m(\varphi_n^+ - \varphi_n + 2\pi z_n^+)))d\varphi_n^+ + (m_m(\varphi_n - 2\pi z_n^+ - \varphi_n^+) - \frac{\partial V_m(\varphi_n)}{\partial \varphi_n})d\varphi_n$$

- **Lagrange 1-Form**

  $$\theta = m_m\left(\varphi_1^- - (\varphi_1 + 2\pi z_1^-)\right)d\varphi_1^- + m_m\left(\varphi_n^+ - \varphi_n + 2\pi z_n^+\right)d\varphi_n^+$$

  NO POTENTIAL

Example 2,Particle in circle

$$\theta_n^- = (m_m(\varphi_n^- - (\varphi_n + 2\pi z_n^-)))d\varphi_n^- + (m_m(\varphi_n + 2\pi z_n - \varphi_n^-) - \frac{\partial V_m(\varphi_n)}{\partial \varphi_n})d\varphi_n$$

$$\theta_n^+ = (m_m(\varphi_n^+ - \varphi_n + 2\pi z_n^+)))d\varphi_n^+ + (m_m(\varphi_n - 2\pi z_n^+ - \varphi_n^+) - \frac{\partial V_m(\varphi_n)}{\partial \varphi_n})d\varphi_n$$

- **Lagrange 1-Form**

  $$\theta = m_m\left(\varphi_1^- - (\varphi_1 + 2\pi z_1^-)\right) d\varphi_1^- + m_m\left(\varphi_n^+ - \varphi_n + 2\pi z_n^+\right) d\varphi_n^+$$

  NO POTENTIAL
- **Symplectic Form**

  $$\omega = \frac{\partial^2 L_1^-}{\partial \varphi_1 \partial \varphi_1^-}|_{initialdata} d\varphi_1 \wedge d\varphi_1^- = -\frac{\partial^2 L_n^+}{\partial \varphi_n \partial \varphi_n^+}|_{Finaldata} d\varphi_n \wedge d\varphi_n^+$$

Review of Reisenbergers model

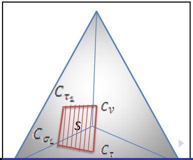
Plebanski Action

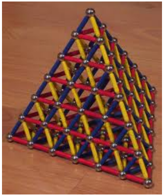
$$I_p(\Sigma, A) = \int \Sigma_i \wedge F^i - \frac{1}{2}\Phi^{ij}\Sigma_i \wedge \Sigma_j$$

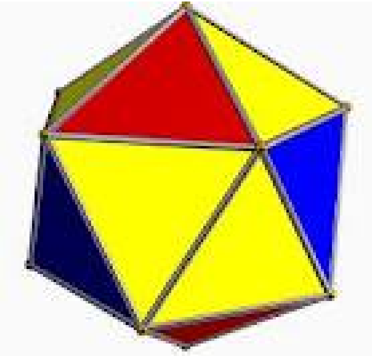
Reisenberger Discrete Action

$$I_R(e, h, k, \phi) = \sum_{\nu<\Delta}\Big(\sum_{s<\nu} e_{si}\theta^i_s - \frac{1}{60}\phi^{ij}_{\nu}\sum_{s,\bar{s}<\nu} e_{si}e_{\bar{s}j}sgn(s, \bar{s})\Big)$$

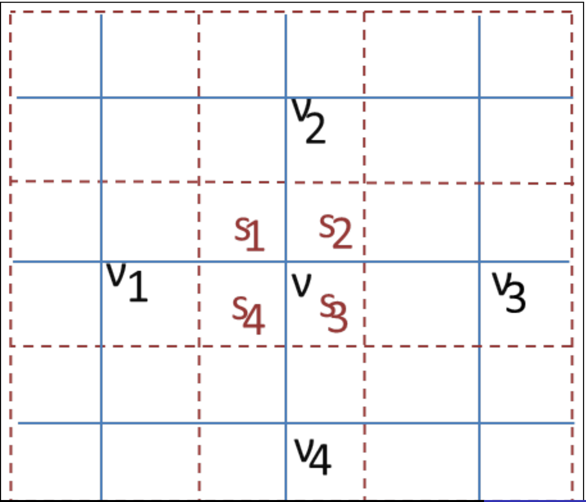
$$\theta^i_s = tr[J^i g_{\partial s}]$$





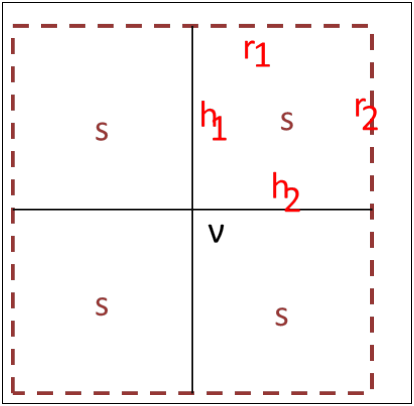


Reisenbergers model in 2 Dimesion



Reisenbergers model in 2 Dimesion

- Single ν



Example:SO(2) Gauge Theory for BF

- Action

$$S_\nu(\Phi, e) = \sum_{s<\nu} [e_\nu(\Phi_{s_i} + 2\pi z_{s_i})] = \sum_{i=1}^{4} L_\nu^{s_i} \qquad \Phi_s = h_{l_1} + h_{l_2} + k_{r_1} + k_{r_2}$$

Example:SO(2) Gauge Theory for BF

- Action

$$S_\nu(\Phi, e) = \sum_{s<\nu} [e_\nu(\Phi_{s_i} + 2\pi z_{s_i})] = \sum_{i=1}^{4} L_\nu^{s_i} \qquad \Phi_s = h_{l_1} + h_{l_2} + k_{r_1} + k_{r_2}$$

- DEL equations for single ν

$$\frac{\partial L_\nu^{s_i}}{\partial h_{l_2}} + \frac{\partial L_\nu^{s_{i+1}}}{\partial h_{l_1}} = 0 \qquad \Rightarrow e_\nu - e_\nu = 0$$

$$\frac{\partial L_\nu^{s_i}}{\partial h_{l_1}} + \frac{\partial L_\nu^{s_{i-1}}}{\partial h_{l_2}} = 0 \qquad \Rightarrow e_\nu - e_\nu = 0$$

$$\sum_{i=1}^{4} \frac{\partial L_\nu^{s_i}}{\partial e_\nu} = 0 \qquad \Rightarrow \sum_{s<\nu} \Phi_{s_i} + 2\pi z_{s_i} = 0$$

Example:SO(2) Gauge Theory for BF

- Lagrange 1-Form

$$\theta = \sum_{i=1}^{4} (e_\nu dk^i_{r_1} + e_\nu dk^i_{r_2})$$

Example:SO(2) Gauge Theory for BF

- Lagrange 1-Form

$$\theta = \sum_{i=1}^{4} (e_\nu dk^i_{r_1} + e_\nu dk^i_{r_2})$$

- Lagrange 2-Form

$$\omega = \sum_{i=1}^{4} (de_\nu \wedge dk^i_{r_1} + de_\nu \wedge dk^i_{r_2}) = de_\nu \wedge \sum_{i=1}^{4} (dk^i_{r_1} + dk^i_{r_2})$$

Example:SO(2) Gauge Theory for BF

Equation of motion: $\sum_{s<\nu} \Phi_{s_i} + 2\pi z_{s_i} = 0$

$$\Rightarrow \sum_{i=1}^{4} k_r^i + 2\pi z_{s_i} = 0$$

Also for ω we have

$$\omega = de_\nu \wedge \sum_{i=1}^{4} (dk_{r_1}^i + dk_{r_2}^i) = de_\nu \wedge \sum_{i=1}^{4} d(k_{r_1}^i + k_{r_2}^i) = de_\nu \wedge d(k_r + 2\pi z_{s_i})$$

$$\Rightarrow \omega = 0$$

Example:SO(2) Gauge Theory for 2-D Gravity

- Action

$$S_\nu(\Phi, e) = \sum_{s<\nu} [e_\nu(\Phi_{s_i} + 2\pi z_{s_i}) - \lambda_\nu(e^2 - 1)] = \sum_{i=1}^{4} L_\nu^{s_i}$$

$$\Phi_s = h_{l_1} + h_{l_2} + k_{r_1} + k_{r_2}$$

Example:SO(2) Gauge Theory for 2-D Gravity

- DEL equation for single ν

$$\frac{\partial L_\nu^{s_i}}{\partial h_{l_2}} + \frac{\partial L_\nu^{s_{i+1}}}{\partial h_{l_1}} = 0 \qquad \Rightarrow e_\nu - e_\nu = 0$$

$$\frac{\partial L_\nu^{s_i}}{\partial h_{l_1}} + \frac{\partial L_\nu^{s_{i-1}}}{\partial h_{l_2}} = 0 \qquad \Rightarrow e_\nu - e_\nu = 0$$

$$\sum_{i=1}^{4} \frac{\partial L_\nu^{s_i}}{\partial e_\nu} = 0 \qquad \Rightarrow \sum_{s<\nu} [\Phi_{s_i} + 2\pi z_{s_i}] - 4 \times 2\lambda e_\nu = 0$$

$$\sum_{i=1}^{4} \frac{\partial L_\nu^{s_i}}{\partial \lambda_\nu} = 0 \qquad \Rightarrow e^2 - 1 = 0 \qquad \Rightarrow de = 0$$

- Lagrange 1-Form

$$\theta = \sum_{i=1}^{4} (e_\nu dk^i_{r_1} + e_\nu dk^i_{r_2})$$

- Lagrange 1-Form

$$\theta = \sum_{i=1}^{4} (e_\nu dk^i_{r_1} + e_\nu dk^i_{r_2})$$

- Lagrange 2-Form

$$\omega = -d\theta$$

$$\omega = \sum_{i=1}^{4} de_\nu \wedge (dk^i_{r_1} + dk^i_{r_2})$$

DEL$\rightsquigarrow$ $de = 0 \Rightarrow \omega = 0$

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