Toward polymer quantum mechanics for fermionic systems.

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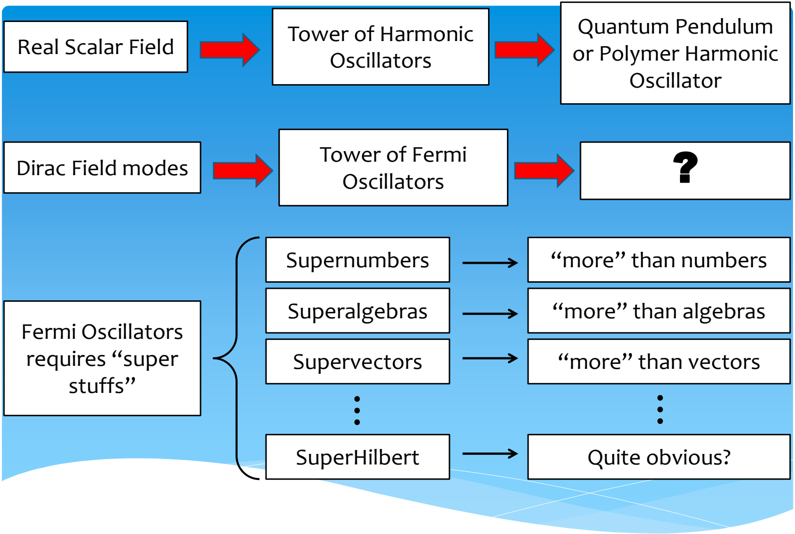
Motivation.


Real Scalar Field
Tower of Harmonic Oscillators
Quantum Pendulum or Polymer Harmonic Oscillator
Dirac Field modes
Tower of Fermi Oscillators
?
Fermi Oscillators requires "super stuffs"
Supernumbers
"more" than numbers
Superalgebras
"more" than algebras
Supervectors
"more" than vectors
SuperHilbert
Quite obvious?

Maps of contents

1. Super Hilbert Space ("Standard") ⇔ Super Clifford Algebra ⇒ Dynamics
2. Super Hilbert Space ("Standard") ⇔ Super Clifford Algebra ⇒ Super Weyl Algebra
3. "Polymer" Super Hilbert Space
4. Super Weyl Algebra ⇔ "Polymer" Super Hilbert Space

Dynamics?

0. Supernumbers "idea".

1. Take objects $\{\xi_j\}$, $j = 1, 2, 3, \ldots, n$ with the relation

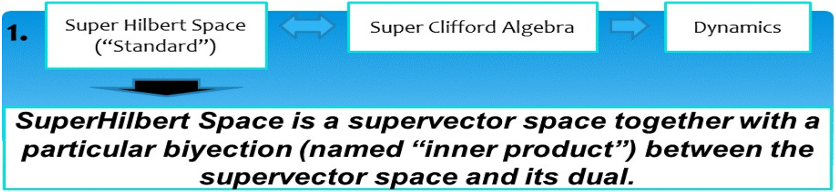
$$\xi_j \cdot \xi_k = -\xi_k \cdot \xi_j \quad \forall \quad j \neq k, \quad \text{and} \quad \xi_j \cdot \xi_j = 0.$$

2. Let us take the elements $1, \xi_j, \xi_j \cdot \xi_k, \ldots$ as generators of the algebra denoted by Λ_n.
3. Any element of Λ_n is of the form

$$z = z_0 + z_j \xi_j + \frac{1}{2} z_{jk} \xi_k \cdot \xi_j + \frac{1}{2} z_{ljk} \xi_k \cdot \xi_j \cdot \xi_l + \cdots +$$

where $z_0, z_j, z_{jk} \cdots \in \mathbb{C}$. Any $z_{a_1 a_2 \cdots a_m}$, with $m > 1$ is totally antisymmetric.

4. Take $n \to \infty$, then "$\Lambda_n \to \Lambda_\infty$".
5. A supernumber is an element $z \in \Lambda_\infty$.



- Supervector Space:
  $\mathcal{F} = \{\Psi : \mathbb{R}_a \to \Lambda_\infty | \Psi \text{ is super analytic}\}$. Any $\Psi \in \mathcal{F}$ may be written as $\Psi(x) = \Psi_0 + \Psi_1 x$.
- $(\Psi \cdot z)(x) := (\Psi_0 z) + (\Psi_1 z)x$.
- SuperDual: $\mathcal{F}^*$ with elements written as
  $$\omega(\cdot) = \int (\cdot) \left[\omega(x) \left(x + \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x} \right) \right] \mathrm{d}x.$$
- $\omega(x) \in \mathcal{F}'$ is a test function.
- "Inner"' product $f : \mathcal{F} \to \mathcal{F}^*; \Psi(x) \mapsto f_\Psi(\cdot)$.
- $f_\Psi(\Phi) := \int \Phi(x) \left[(\Psi^*)(x) \left(x + \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x} \right) \right] \mathrm{d}x = \Phi_0 \overline{\Psi}_0 + \Phi_1 \overline{\Psi}_1$.

2. Super Clifford Algebra.

- Clifford Algebra generators satisfy $[\widehat{x}_a, \widehat{x}_b]_+ = \hbar\delta_{ab}\widehat{1}$.
- Any element is: $a_0 + a_j\widehat{x}_j + b\widehat{x}_1\widehat{x}_2$, where $a_0, a_j, b \in \mathbb{C}$.
- SClifford Algebra replace $\mathbb{C} \to \Lambda_\infty$, now $a_0, a_j, b \in \Lambda_\infty$.
- Generators are "even" under supernumber multiplication.

$$\widehat{x}_1\Psi(x) = \frac{\sqrt{2\hbar}}{2}\Psi(x)\left(x + \frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}x}\right), \quad \widehat{x}_2\Psi(x) = -i\frac{\sqrt{2\hbar}}{2}\Psi(x)\left(x - \frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}x}\right)$$

- $\widehat{H} = \frac{\omega}{2i}(\widehat{x}_1\widehat{x}_2 - \widehat{x}_2\widehat{x}_1) \quad \to \quad \widehat{H}\Psi(x) = E\Psi(x)$
- $E = \pm\frac{\omega}{2}, \; \Psi_+(x) = x, \Psi_-(x) = 1.$

2. Super Weyl Algebra

- $\widehat{W}(\theta_1,\theta_2)\Psi(x) := e^{\theta_1\cdot\widehat{x}_1 + i\theta_2\cdot\widehat{x}_2} = \tilde{\Psi}_0 + \tilde{\Psi}_1 x, \quad \theta_1,\theta_2 \in \mathbb{R}_a.$
  $\tilde{\Psi}_0 = \left(1 + \frac{\hbar}{2}\theta_1\theta_2\right)\Psi_0 + \frac{\sqrt{2\hbar}}{2}\left(\theta_1 - \theta_2\right)\Psi_1,$
  $\tilde{\Psi}_1 = \left(1 - \frac{\hbar}{2}\theta_1\theta_2\right)\Psi_1 + \frac{\sqrt{2\hbar}}{2}\left(\theta_1 + \theta_2\right)\Psi_0.$
- $\widehat{W}(\tilde{\theta}_1,\tilde{\theta}_2)\cdot\widehat{W}(\theta_1,\theta_2) = e^{\frac{\hbar}{2}\left(\tilde{\theta}_1\theta_1 - \tilde{\theta}_2\theta_2\right)}\widehat{W}(\tilde{\theta}_1+\theta_1,\tilde{\theta}_2+\theta_2).$
- Any finite combination $\widehat{A} = \sum_j a_j \widehat{W}(\theta_{1j},\theta_{2j}),\ a_j \in \Lambda_\infty$
- $\widehat{x}_1\Psi(x) \to \frac{\partial}{\partial\theta_2}\widehat{W}(\theta_1,\theta_2)\Psi(x)|_{\theta_1=\theta_2=0}$
  $\widehat{x}_2\Psi(x) \to \frac{\partial}{\partial\theta_1}\widehat{W}(\theta_1,\theta_2)\Psi(x)|_{\theta_1=\theta_2=0}$
- The uniparametric groups elements are formed with:
  $$\widehat{U}(\theta_1) := \widehat{W}(\theta_1,\theta_1), \quad \widehat{V}(\theta_1) := \widehat{W}(\theta_1,-\theta_1)$$
- $\widehat{W}\Psi$ may be seen "pictorically" as
  $\widehat{W}(\theta_1,\theta_2)\Psi(x) = e^{\hbar\theta_1\theta_2}e^{\frac{\sqrt{2\hbar}}{2}x(\theta_1+\theta_2)}\Psi\left(x + \frac{\sqrt{2\hbar}}{2}(\theta_1-\theta_2)\right)$

3. "Polymer" Super Hilbert Space.

- Let us take any pair of sets $\{x_j\}$ and $\{\Psi_j\}$ each one having N elements such that $x_j \in \mathbb{R}_a$, and $\Psi_j \in \Lambda_\infty$.
- "Polymer" svector space is $\mathcal{F}_p = \{< \Psi| = \sum_j^N \Psi_j < x_j|\}$.
  - Sum is the same as in bosonic case.
  - $z\cdot < \Psi| := \sum_j^N z\Psi_{x_j} < x_j|, \qquad < \Psi| \cdot z := \sum_j^N \Psi_{x_j} < x_j| \cdot z$.
  - $(< \Psi|)^* = \sum_j \overline{\Psi}_j < x_j|$.
- "Polymer" sdual $\mathcal{F}_p^* = \{|\Psi >= \sum_j^N |x_j > \Psi_{x_j}\}$.
  - All the operations are implemented in the same manner as in $\mathcal{F}_p$.
  - $< x_j|x_k >:= \delta_{x_j,x_k}$ allows us to define $< \Phi|\omega >:= \omega(\Phi)$.
- The "inner" product f_p is just :

$$< \Phi|f_\Psi >= f_\Psi(\Phi) = \sum_j \Phi_j\overline{\Psi}_j.$$

$(\mathcal{F}_p, f_p)$ is our candidate for a "Polymer" Super Hilbert Space.

4. Representation of Super Weyl Algebra.

Representation on a Sdual polymer Ket.

$$\widehat{W}(\theta_1, \theta_2)|x_j >= e^{\hbar\theta_1\theta_2 + \frac{\sqrt{2\hbar}}{2} x_j(\theta_1+\theta_2)}|x_j + \frac{\sqrt{2\hbar}}{2}(\theta_1 - \theta_2) >$$

In terms of wave functions...

$$\Psi(x) = \sum_j \Psi_{x_j + \frac{\sqrt{2\hbar}}{2}(\theta_1 - \theta_2)} e^{\hbar\theta_1\theta_2 + \frac{\sqrt{2\hbar}}{2} x_j(\theta_1+\theta_2)} \delta_{x_j, x}$$

We say "Polymer", because as they are not super analytic then we can not recover SClifford Algebra by differentiation.

5. Next steps...

A hamiltonian candidate...

$$\widehat{H}_p \rightarrow \widehat{W}(\theta_1, \theta_2) + \widehat{W}(-\theta_1, -\theta_2) + 2$$

Something like the physical polymer SHilbert Space...

$$\mathcal{H}_{poly} \rightarrow \bigoplus_{x_0 \in [0, \frac{\sqrt{2\hbar}}{2}(\theta_1 - \theta_2))} \mathcal{H}_{poly}^{x_0}$$

Eigenvalues...

$$E_p \rightarrow \pm \frac{\hbar\omega}{2} + (\text{factor}) i \theta_1 \theta_2$$

Thanks!