Geometric Structures for a minimally modified CMPR action

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(Trabajo en proceso)

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- Geometric structures on GR formulations
- Effects on the geometric structures due to modified GR models

GR modified àla Krasnov

Krasnov introduces modifications to the Plebański action

$$S[A, B, \phi] = \int F_{AB} \wedge B^{AB} - \frac{1}{2}\phi_{ABCD} B^{AB} \wedge B^{CD}$$

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$$S[A,B,\phi] = \int F_{AB} \wedge B^{AB} - \frac{1}{2}\phi_{ABCD} B^{AB} \wedge B^{CD} - \frac{1}{2}\Phi(t_1, t_2) B^{AB} \wedge B_{AB}.$$

$F^{AB} = dA^{AB} + A^A{}_C \wedge A^C{}_B$ is the curvature of a $SL(2,C)$ connection A and $t_1 = \phi^{ABCD}\phi_{ABCD}$, $t_2 = \phi_{ABCD}\phi^{CD}{}_{FG}\phi^{ABFG}$ are the only two independent algebraic invariants.

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The former actions define a class of four-dimensional generally covariant theories propagating two degrees of freedom that reduces to vacuum (complex) GR with cosmological constant when $\Phi(t_1,t_2)\to\Lambda$.

CMPR action

It is possible to formulate GR as a constrained BF-type theory [1]

$$S[B, A, \Phi, \mu] = \int \left[B^{IJ} \wedge F_{IJ}(A) - \left(\frac{1}{2}\right) \Phi_{IJKL} B^{IJ} \wedge B^{KL} + \mu H(\Phi) \right],$$

where A^{IJ} is an $SO(3,1)$ valued connection,

$$H(\Phi) = a_1 \Phi_1 + a_2 \Phi_2,$$

and $\Phi_1 = \Phi^{IJ}{}_{IJ}$, $\Phi_2 = \epsilon^{IJKL} \Phi_{IJKL}$. (Lorentz indices are raised and lowered with Minkowski metric η^{IJ} and $*$ is the duality operator, $*^2 = \sigma$)

[1] Class. Quantum Grav. 18 (2001) L49

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For the associated constraints

$$\delta\Phi : \quad 2a_2 B^{MN} \wedge B_{MN} = \sigma a_1 B^{MN} \wedge {}^*B_{MN},$$

the general solution are

$$B^{IJ} = \alpha * (e^I \wedge e^J) + \beta (e^I \wedge e^J),$$

[1] Class. Quantum Grav. 18 (2001) L49

If we assume[2] $H(\Phi) = a_1 \Phi_1$, then

$$\mu = \frac{1}{12 a_1} B^{MN} \wedge B_{MN}, \quad (1)$$

$$B^{IJ} = *(e^I \wedge e^J) \pm \sqrt{-\sigma}\, e^I \wedge e^J, \quad (2)$$

where $\sigma = 1$ for Euclidean signature (-1 for Lorentzian)

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If we assume $H(\Phi) = a_2 \Phi_2$, the two-forms solution are

$$B_1^{IJ} = \kappa_1 * (e^I \wedge e^J), \tag{3}$$

$$B_2^{IJ} = \kappa_2 \ (e^I \wedge e^J), \tag{4}$$

and $\mu = \frac{\sigma}{12a_2} B^{MN} \wedge *B_{MN}$.

[2] see SIGMA 7(2011) 103

A minimally CMPR modified action

By analogy with *Class Quantum Grav. 27 (2010) 145011*, we modified the CMPR action by $\varepsilon\Phi_{IJKL}\Phi^{IJKL}$.

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$$S[\mathcal{B}, \omega, \phi, \mu] \to S[B + \varepsilon b, A + \varepsilon a, \Phi + \varepsilon \rho, \mu].$$

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From the equations of motion

$$\begin{aligned} \delta\mu &: \quad \phi_1 = 0, \\ \delta\phi &: \quad \mu = \left[\frac{1}{12\,\alpha_1}\right] \mathcal{B}^{MN} \wedge \mathcal{B}_{MN}, \\ & \quad\;\; \mathcal{B}^{MN} \wedge *\mathcal{B}_{MN} = -\varepsilon\,(\phi_2)\; \mathcal{B}^{MN} \wedge \mathcal{B}_{MN}. \end{aligned} \tag{5}$$

By using the expansion

$$\mathcal{B}^{IJ} = B^{IJ} + \varepsilon\, b^{IJ},$$

we realized that

$$B^{IJ} \;=\; *(e^I \wedge e^J) \pm \sqrt{-\sigma}(e^I \wedge e^J),$$

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By using the expansion

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we realized that

$$\begin{aligned} B^{IJ} &= *(e^I \wedge e^J) \pm \sqrt{-\sigma}(e^I \wedge e^J), \\ b^{IJ} &= A_1 \Phi_2\,(e^I \wedge e^J) \mp (1 + A_1)\, \left(\sigma\sqrt{-\sigma}\right)\, \Phi_2\; *(e^I \wedge e^J) \end{aligned}$$

solve equation (5)

The equation of motion for $\delta\omega$ and

$$\omega^{IJ} = A^{IJ} + \varepsilon a^{IJ},$$

leads us to

$$\begin{aligned} a^{IJ} &= a^{IJ}{}_{K}\, e^{K} \\ &= \left[\frac{\sigma}{4}\varepsilon^{IJ}{}_{LK} \pm \frac{\sigma\sqrt{-\sigma}}{2}(2A_1+1)\delta^{IJ}{}_{LK}\right]\left(\partial^{L}\Phi_2\right)\, e^{K}, \end{aligned}$$

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where $\delta^{IJ}{}_{KL} = \eta^I{}_{[K}\eta^J{}_{L]}$. This connection is not the Riemmanian one, then

$$\tau^I = \partial_K \Phi_2 \left[\frac{\sigma}{2} * (e^I \wedge e^K) \mp \frac{\sigma\sqrt{-\sigma}}{4}(2A_1+1)\,(e^I \wedge e^K)\right]$$

and

$$
\begin{aligned}
R_a^{IJ} &= \frac{1}{2}\Big[2\partial_M({a^{IJ}}_N) + 2{A^K}_{MN}{a^{IJ}}_K \\
&\quad + \frac{A(A+1)}{8}{\varepsilon^{IJ}}_{RS}\left[\partial^K{}_K\Phi_2\right]^{RS}{}_{MN} \\
&\quad \mp \frac{\sqrt{-\sigma}}{8}(2A+1)\left[\partial^K{}_K\Phi_2\right]^{IJ}{}_{MN}\Big](e^M \wedge e^N),
\end{aligned}
$$

where the ${A^{IJ}}_K$ are the Riemmannian connection components and

$$
\left[\partial^K{}_K\Phi_2\right]^{IJ}{}_{MN} = (\partial^K\Phi_2)(\partial_K\Phi_2){\varepsilon^{IJ}}_{MN} + 2(\partial_M\Phi_2)(\partial^K\Phi_2){\varepsilon^{IJ}}_{NK}
$$

Finally (from $\delta\mathcal{B}$)

$$\begin{aligned}\rho^{IJMN} &= (t_1)\delta^{IJMN} \\ &+\frac{\Phi_2}{2}\sigma\left[*\Phi^{MNIJ} \pm \sqrt{-\sigma}\,(1+2A_1)\Phi^{IJMN}\right] \\ &+\frac{\sigma}{4}\left[D_A a^{IJ}\right]_{KL}\left(\varepsilon^{KLMN} \mp 2\sqrt{-\sigma}\,\delta^{KLMN}\right)\end{aligned}$$

where

$$\begin{aligned}\left[D_A a^{IJ}\right]_{KL} &= \frac{\sigma}{4}\left\{\varepsilon^{IJ}{}_{R[L}\left[D^R\partial_{K]}\Phi_2\right]\right\} \\ &\pm\frac{\sigma\sqrt{-\sigma}}{2}(1+2A_1)\left\{\delta^{IJ}{}_{R[L}\left[D^R\partial_{K]}\Phi_2\right]\right\}\end{aligned}$$

and $\left[D^I\partial_J\Phi_2\right] = \partial^I\partial_J\Phi_2 + A^I{}_{KJ}\partial^K\Phi_2.$

Remarks

- The modifications proposed by Krasnov modify all the geometrical structures involved in the action.
- The Lagrange multiplier associated with the *cosmological function* is no longer the Weyl part of the Riemann curvature tensor.

Thanks

the ansatz

$$b_2^{IJ} = \mp 12\sqrt{-\sigma}\, A_2\, \Phi^{IJ}{}_{KL}\, \Sigma^{KL} + 12(A_2 - 1)\Phi^{IJ}{}_{KL} * \Sigma^{KL}, \qquad (6)$$

solves the equation of motion as well.

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solves the equation of motion as well.

From

$$\delta\omega : \quad d\mathcal{B}^{IJ} + \omega^I{}_K \wedge \mathcal{B}^{KJ} + \omega^J{}_K \wedge \mathcal{B}^{IK} = 0.$$

$$D_A b^{IJ} = \epsilon^{K[I}{}_{LM} a^{J]}{}_K \wedge \Sigma^{LM} \pm 2\sqrt{-\sigma} e^{[I} \wedge \tau^{J]} \qquad (7)$$

where

$$\tau^I = \frac{\sigma}{2}\partial_K \Phi_2 * \Sigma^{IK} \mp \frac{\sigma\sqrt{-\sigma}}{4}(2A_1 + 1)e^I \wedge d\Phi_2$$

Una *conexión*, ∇, en M, es una regla para calcular derivadas direccionales de los campos vectoriales sobre M. La conexión cumple con:

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Dada una conexión, ∇, se definen los *tensores* de *torsión* y *curvatura*, respectivamente como:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{10}$$

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{11}$$

Donde X, Y, Z son campos vectoriales en M