THE GROMOV-LAWSON-ROSENBERG CONJECTURE FOR THE GROUP $\mathbb{Z}/4 \times \mathbb{Z}/4$

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1. Introduction

The study of metrics of positive scalar curvature on spin manifolds has been traditionally related to the spectral properties of the Dirac operator.

The fact that the vanishing of polynomials in Pontrjagyin classes, or the non-existence of harmonic spinors is a necessary condition for the existence of a metric of positive scalar curvature is known since Lichnerovicz [24] and Bochner-Yano [41].

The emergence of the Atiyah-Singer Index Theorem and its cohomological formulas in preliminary form led to Hitchin [20] to formulate the vanishing of the mod 2 index of the dirac operator, which finally led to formulations in terms of real K-theory and Clifford-linear operators.

Given a smooth compact spin manifold M of even dimension and a choice of a spin structure, there exists a polynomial in Pontryagin classes $\hat{A}$, such that the index of the Dirac operator of M with respect to the spin structure satisfies the equality

$$\hat{A}(M) = \mathrm{Index}(D).$$

Hitchin constructed an invariant taking values on the coefficients of real K-Theory, viewed as the K-theory of modules over a Clifford algebra.

In symbols,

$$\alpha(M) \in KO_n.$$

Denoting the fundamental group of the spin manifold M by π, the (spin) bordism invariance of the index has as consequence that Hitchin's construction defines a natural transformation of homology theories between spin bordism and connective real K-homology

$$D(M) : \Omega_n^{\mathrm{Spin}}(B\pi) \to ko_n(B\pi).$$

Gromov and Lawson [17] established the fact that the vanishing of the KO- Pontryagin nubers of Anderson-Brown-Peterson [2] is a necesary condition for the existence of a metric of positive scalar curvature on a spin manifold. Trough the surgery theorem in loc.cit [17], and [18], they established that the question of whether a spin manifold of positive scalar curvature of dimension greater or equal than 5 is a problem of the spin bordism class of the given spin manifold.

They also conjectured that the vanishing of the α-invariant described above is sufficient for the manifold with fundamental group π to admit a metric of positive scalar curvature.

Rosenberg in [33] elaborated on these results to construct a refinement of the α-invariant and he formulated the following Conjecture which will be the main topic of this note.

Date: August 21, 2024.

Conjecture 1.1 (Gromov-Lawson-Rosenberg). *Let M be a smooth, compact spin manifold of dimension $n \geq 5$, and fundamental group π. Then, M admits a metric of positive scalar curvature if and only if the invariant* $\mathrm{Ind}(M) = A \circ \mathrm{per} \circ D$ *taking values on the real K theory of the reduced real group C^*- algebra $KO_n(C_r^*(\pi))$ vanishes.*

The invariant $\mathrm{ind}(M)$ is defined as follows. The first map in the composition is the map D, which associates to the cycle for spin bordism (M, f), the image of the ko-theoretical fundamental class $f_*(D(M))$. The map $f : M \to B\pi$ is the classifying map for the fundamental group, and the periodicity map per from connective to periodic real K- theory followed by the Baum-Connes assembly map A. In symbols

$$\Omega_n^{\mathrm{Spin}}(B\pi) \overset{D}{\to} ko_n(B\pi) \overset{\mathrm{per}}{\to} KO_n(B\pi) \overset{A}{\to} KO_n(C_r^*(\pi)).$$

S. Stolz in [36] proved the conjecture in the simply-connected case, using his previous result in [37] which contains an identification trough computations with the Adams spectral sequence of the kernel of the α-invariant. In slightly more detail, the kernel of the α-invariant is generated by manifolds which are fiber bundles with fiber $\mathbb{H}P^2$ and structural group $PSp(3)$.

The conjecture has been proved for several groups, including groups with periodic cohomology [6], the semidihedral group of order 16 [25], and several torsionfree groups for which the Baum-Connes assembly map is injective. This includes notably Fuchsian groups [11], surface groups, and free groups. On the other hand, there exists a reduced number of infinite groups containing torsion [21], [10] for which the conjecture is known to hold as a consequence of computations of connective ko-homology.

The conjecture is known to be false for the group $\mathbb{Z}^3 \times \mathbb{Z}/4$ [34], and several torsionfree groups [14] but the question whether the conjecture is valid for finite groups remains open.

We will prove in this text the following result

Theorem 1.2 (Main Result). *The Gromov-Lawson-Rosenberg Conjecture is true for the group $\mathbb{Z}/4 \times \mathbb{Z}/4$.*

Theorem 1.2 includes substancial previous work of the Ph. D. theses of Christian Siegemeyer [35] and Raphael Reinauer [31], defended at the University of Münster under the supervision of Michael Joachim.

The method we employ will consist of first establishing in 3 the structure of the integral and mod 2 cohomology of $\mathbb{Z}/4 \times \mathbb{Z}/4$ as a module over the Steenrod Algebra and the subalgebras $\mathcal{A}_1$ and $E(1)$. Ingredients here are, besides from the actual cohomology computation, the splitting theorem 3.2.

Moreover, we will produce a minimal resolution of all relevant modules over the Steenrod algebra.

In section 2 we use the previous information as input for the determination of the connective ko-Theory groups of the group $\mathbb{Z}/4 \times \mathbb{Z}/4$ by the Adams spectral sequence.

We need to determine differentials of the Adams Spectral sequence; we will do this in section 6. We do so by comparing to the Atiyah-Hirzebuch spectral sequence in the η-c-r exact sequence 5.1, and obtain differentials in Adams degree up to four.

We will prove that there are no higher differentials, and after analyzing hidden Adams extensions, the computation of the connective ko-theory of the classifying space for $\mathbb{Z}/4 \times \mathbb{Z}/4$ is achieved.

In the final section, 6, we guarantee that the subgroup of spin bordism classes of manifolds with positive scalar curvature exhausts the kernel of the α-invariant by constructing explicitely the manifolds representing the kernel of the alpha invariant.

We use two distinct methods for the even and odd dimensional case. In the even dimensional case, we introduce homological considerations, and the odd dimensional argument follows closely previous constructions using η-invariants.

1.1. **Aknowledgements.** The first author thanks suport of DGAPA-UNAM grant IN101423. The first two authors aknowledge support of CONAHCYT Grant CF-2019 217392. The first and third author thank Michael Joachim for the generous introduction to the topic, and several discussions along the years.

2. The Adams Spectral sequence and the Atiyah-Hirzebruch Spectral Sequences

We will introduce now the mod 2 -Adams spectral sequence.

Definition 2.1. The mod 2 Steenrod algebra $\mathcal{A}$ is the $\mathbb{F}_2$- algebra of stable cohomology operations in mod 2 cohomology. It can be defined in terms of the following axioms.

(i) The algebra $\mathcal{A}$ is generated by elements
$$\{Sq^i \mid i \in \mathbb{N} \cup \{0\}\}$$
called Steenrod squares.

(ii) $Sq^0 = 1$.

(iii) (Adem relation) $Sq^a \circ Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-a}{a-2c} Sq^{a+b-c} Sq^c$.

Recall the following well-known cohomological properties of Steenrod squares. The multiplicative structure refers to the usual cup product in ordinary cohomology. See [28], [1] for more details.

- The elements Sq^i correspond to cohomology operations
$$Sq^i : H^*(\quad) \to H^{*+i}(\quad).$$
Moreover, this defines a structure of graded module over the mod 2-Steenrod algebra on the cohomology of a fixed space $H^*(X)$.
- Sq^1 is the mod 2-Bockstein homomorphism.
- If $x \in H^*(X)$, and $i > deg(x)$, then $Sq^i(x) = 0$.
- If x is of cohomological degree n, then $Sq^n(x) = x^2$.
- For the connecting homomorphism for the lang exact sequence in cohomology δ^*, the equality $Sq^i \circ \delta^* = \delta^* \circ Sq^i$ holds.
- The cartan formula holds: $Sq^n(xy) = \sum_{i+j=n} Sq^i(x)Sq^j(y)$.

The following subalgebras of the mod 2 Steenrod algebra will be important for the determination of the E_2 terms of the Adams spectral sequences.

Definition 2.2. We introduce the following subalgebras of the mod 2 Steenrod algebra.

- The subalgebra $\mathcal{A}_1$ is defined as the subalgebra generated by Sq^1 and Sq^2. In terms of generators and relations , it is the quotient
$$\langle Sq^1, Sq^2 \mid Sq^1 Sq^2 Sq^1 = Sq^2 Sq^2 \rangle.$$
- The subalgebra $E(1)$ is the exterior algebra generated by $Q_0 = Sq^1$ and $Q_1 = Q_0 Sq^2 - Sq^2 Q_0$.

We will consider (graded) modules over the algebras $\mathcal{A}$, $\mathcal{A}_1$, and $E(1)$. We recall briefly the relevant definitions, which appear for instance in [40], [26].

Let Γ be an algebra over a field $\mathbb{K}$ with augmentation
$$\epsilon : \mathbb{K} \to \Gamma,$$

and unit

$$\eta : \Gamma \to \mathbb{K}.$$

Definition 2.3. Given two graded modules M and N over Γ, a $\mathbb{K}$- homomorphism $f : M \to N$ is of degree t, where t is a natural number if it satisfies $f(M_q) \subset N_{q+t}$. Denote the $\mathbb{K}$- vector space of homomorphisms of degree t between graded modules M, and N as

$$\hom^t(M, N).$$

Definition 2.4. The suspension functor $\Sigma : \Gamma - \mathsf{MODULES} \to \Gamma - \mathsf{MODULES}$ is defined on a graded Γ- module M as

$$\Sigma M_n = M_{n+1}.$$

The iterated suspension functor Σ^n is defined inductively as

$$\Sigma^0 = 1,\ \Sigma^{k=1} = \Sigma \circ \Sigma^{k-1}.$$

Definition 2.5. Let M, and N be a pair of graded Γ-modules, and let $P_*(M)$ be a projective resolution of the Γ-module M, $s \geq 0$, and $t \in \mathbb{Z}$. The Ext- groups $\mathrm{Ext}_\Gamma^{s,t}$ are defined as

$$Ext_\Gamma^{s,t}(M, N) = H^s(\hom^t(P_*(M)), N).$$

As it is usual in homological algebra, the definition does not depend on the particular resolution.

Recall the existence of the Yoneda product

$$Ext_\Gamma^{s,t}(L, M) \otimes Ext_\Gamma^{s',t'}(M, N) \longrightarrow Ext_\Gamma^{s+s',t+t'}(L, N).$$

The following result concerns the construction and convergence of the Adams spectral sequence, and it is proved in [1], chapter 15 in page 316.

See [1], chapter 15 and [29], Chapter 2 for proofs of the following result.

Theorem 2.6. *Let X, and Y be connective CW-spectra for which the* mod 2*-homology of X if is finitely generated in every degree, and for which Y is finite. Then, there exists a spectral sequence with E_2 term*

$$\mathrm{Ext}_{\mathcal{A}}^{s,t}(H^*(X), H^*(Y)).$$

It converges to the 2-adical completion of the stable homotopy groups of classes of maps between X and Y.

$$\pi_{s-t}[X, Y]_{\hat{2}}.$$

Remark 2.7. For further reference, let us unravel the gradings of differentials for the classical Adams spectral sequence. The differentials d_r have the Adams grading,

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t+r-1}.$$

Denote by ko the connective real K- theory spectrum.

Similarly, denote by ku the connective complex K- theory spectrum.

The following result, atributed to Stong [38] simplifies substantially the E_2 terms of the Adams spectral sequences converging to the real and complex connective K-homology groups.

Theorem 2.8. *The following $\mathbb{F}_2$-algebras are isomorphic.*

(i) $H^*(ko, \mathbb{F}_2)$ *and* $\mathcal{A} \otimes_{\mathcal{A}_1} \mathbb{F}_2$.
(ii) $H^*(ku, \mathbb{F}_2)$ *and* $\mathcal{A} \otimes_{E(1)} \mathbb{F}_2$.
(iii) $H^*(H\mathbb{Z})$ *and* $E(0) = \mathcal{A} \otimes_{\mathcal{A}_1} \mathcal{A}_1/\langle Sq^1\rangle$.

The existence of the Adams spectral sequence together with the previous theorem has as consequence the following corollary.

Corollary 2.9. *Let G be a finite 2-group.*

- *The E_2 term of the Adams spectral sequence converging to $ko_*(BG)$ can be identified as*
$$\mathrm{Ext}^{s,t}_{\mathcal{A}_1}(H^*(BG), \mathbb{F}_2).$$
It converges to $ko_{t-s}(BG)$.
- *The E_2 term of the Adams spectral sequence converging to $ku_*(BG)$ can be identified as*
$$\mathrm{Ext}^{s,t}_{E(1)}(H^*(BG), \mathbb{F}_2).$$
It converges to $ku_{t-s}(BG)$.

Section 3 will discuss explicit resolutions of the group cohomology as a module over the Algebras $\mathcal{A}_1$, and $E(1)$.

We introduce now notation which will allow us to understand periodicity phenomenae on the E_2-term of the Adams spectral sequence and the ko, respectively ku-homology of finite groups.

Recall that the homotopy groups of real connective K-Theory are as follows
$$\pi_i(ko) = \begin{cases} \mathbb{Z}/2 & i \overset{4}{\equiv} 1, 2 \\ \mathbb{Z} & i \overset{4}{\equiv} 0 \\ 0 & \text{else.} \end{cases}$$

As a graded algebra, the coefficients
$$\bigoplus_* \pi_*(ko)$$
are the truncated algebra
$$\mathbb{Z}[\eta, \alpha, \beta]/\eta^3, 2\eta, \alpha\beta, \alpha^2 - 4\beta,$$
where η is of degree 1, ω is of degree 4, and μ is of degree 8.

Lemma 2.10. *The Adams spectral sequence converging to the coefficients of $ko_{\hat{2}}$ has as E_2 term*
$$\frac{\mathbb{F}_2[h_0, h_1, a, b]}{h_0 h_1, h_1^3, h_1 a, a^2 - h_0 b.}$$
The element h_0 has degree $(1, 1)$, h_1 has degree $(1, 2)$, a has degree $(3, 7)$, and b has degree $(4, 12)$. It collapses and it is depicted in picture 2.1.

Lemma 2.11. *The Adams spectral sequence converging to the coefficients of $ku_{\hat{2}}$ has as E_2 term*
$$\mathbb{F}_2[h_0, v],$$
with h_0 of degree $(1, 1)$, and v of degree $(1, 3)$. It collapses and it is depicted in picture 2.2.

Besides from the Yoneda product, we will need the cap product structure for the spectral sequences converging to ko and ku.

Theorem 2.12. *Let E be a connected ring spectrum, and let X be the suspension spectrum of a pointed space. Then, there exists a cap pairing*
$$\cap : E^s(X) \otimes E_{s+t}(X) \to E_t(X)$$

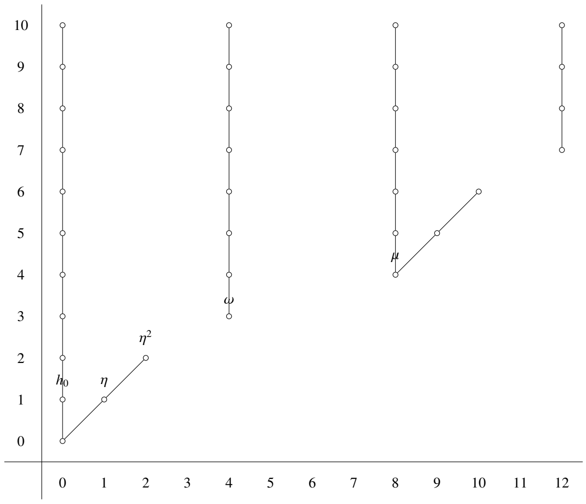


FIGURE 2.1. Adams spectral sequence converging to $\pi_*(ko_{\hat{2}})$.

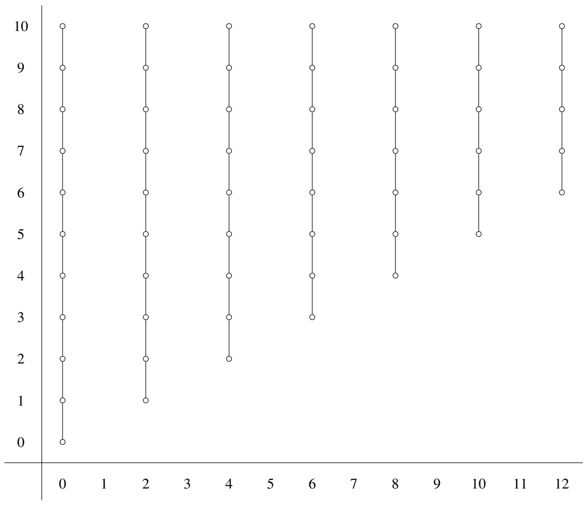


FIGURE 2.2. Adams spectral sequence converging to $\pi_*(ku_{\hat{2}})$.

Proof. The pairing is given by assigning to representatives

$$\alpha : \Sigma^{\infty-s}(X) \to E \in E^s(X),$$

and

$$\beta : S^{s+t} \to E \wedge X \in E_{s+t}(X)$$

the composition

$$S^t \xrightarrow{\beta} S^{-s} \wedge E \wedge X \overset{(\tau\wedge 1)\circ(1\wedge\Delta)}{\to} E \wedge S^{-s} \wedge X \wedge X \overset{1\wedge\alpha\wedge 1}{\to} E \wedge E \wedge X \overset{\mu\wedge 1}{\to} X.$$

□

Remark 2.13 (Cap pairing for the Adams Hirzebruch spectral sequence)**.** We will need both the cap pairing and the external smash product for the Adams spectral sequence converging to ko and ku. In a second stage, we will need these algebraic structures for the Ext- terms obtained for modules over $\mathcal{A}_1$ and $E(1)$. The following result has as objective the external spash product.

Theorem 2.14. *Let E be a connected ring spectrum such that $H_*(E)$ is of finite type, and let X be the suspension spectra of a finite CW complex. Then, there exists a pairing*

$$\hat{E}_r^{s,t} \otimes E_r^{u,v} \to E_r^{u+s,v-t}$$

of the Adams spectral sequences $\hat{E}_r^{s,t} \Rightarrow E^{s+t}(X)$, where

$$\hat{E}_r^{s,t} = \operatorname{Ext}_{\mathcal{A}}^{s,-t}(H^*(E), H^*(X)),$$

and

$$E_r^{u,v} \Rightarrow E_{u+v}(X),$$

for

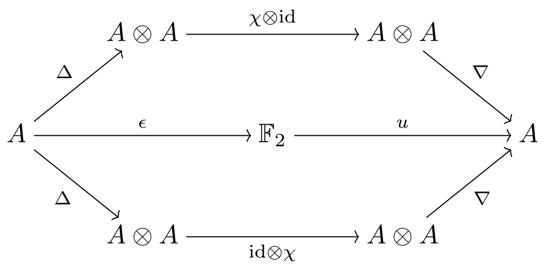
$$E_r^{u,v} = \operatorname{Ext}_{\mathcal{A}_1}^{u,v}(H^*(E) \otimes H^*(X), \mathbb{F}_2).$$

This product is compatible with the cap product pairing of definition 2.12.

Since we will be dealing with the Hopf subalgebras $E(1)$, and $\mathcal{A}_1$ of the Steenrod algebra, we will need the following results and definitions.

Definition 2.15. A Hopf algebra over $\mathbb{F}_2$ is a tuple $(A, \nabla, u, \Delta, \epsilon, S)$ consisting of

(i) A connected and graded $\mathbb{F}_2$ algebra (A, ∇, u).
(ii) A connected and graded $\mathbb{F}_2$-coalgebra (A, Δ, ϵ).
(iii) An $\mathbb{F}_2$- linear map $\chi : A \to A$, called conjugation such that the algebra structure is compatible with the coalgebra structure: ∇ and u are coalgebra homomorphisms, Δ and ϵ are algebra homomorphisms, and the following diagram commutes:



Definition 2.16. Let A be a Hopf algebra over $\mathbb{F}_2$.

A Hopf ideal is an $\mathbb{F}_2$-algebra which is an ideal when restricting to the algebra structure of A such that

$$\Delta(I) \subset I \otimes A + A \otimes I.$$

(i) The subset $I(A) := \ker(\epsilon)$ is an ideal, called the augmentation ideal.
(ii) A subalgebra $B \subset A$ is a Hopf subalgebra if it is a subcoalgebra and $\chi(B) \subset B$.
(iii) A subalgebra B is called normal if

$$AI(B) = I(B)A,$$

where $I(B)$ is the aumentation ideal of B.
(iv) An $\mathbb{F}_2$-vector space together with a structure of a left A-module is called a left module of A.

Remark 2.17. Given a Hopf algebra A, the coaction defines a natural A-module structure on a tensor product $M \otimes N$ of left A- modules by

$$A \otimes (M \otimes N) \overset{\nabla\otimes 1\otimes 1}{\to} A \otimes A \otimes M \otimes N \overset{1\otimes\tau\otimes 1}{\to} A \otimes M \otimes A \otimes N \overset{\lambda_M\otimes\lambda_N}{\to} M \otimes N,$$

where τ interchanges the factors A and M, and λ_N, λ_M denote the homomorphisms given by the module structures of M and N.

The following lemma is proved in [38], [27].

Lemma 2.18. *The Steenrod Algebra $\mathcal{A}$ is a Hopf algebra. Both $\mathcal{A}_1$ and $E(1)$ are normal subalgebras of the Steenrod Algebra $\mathcal{A}$.*

Lemma 2.19. *For a normal subalgebra $B \subset A$, $J := AI(B)$ is a 2-sided ideal, and*

$$A//B := A/J$$

inherits the structure of a Hopf algebra. There exists an isomorphism of Hopf algebras

$$A//B \cong A \otimes_B \mathbb{F}_2.$$

Theorem 2.20. *Let A be a Hopf algebra, and let B be a subalgebra. For an A-module M with module structure*

$$\lambda : A \otimes M \to M,$$

we denote the underlying B-module M by $\mid M \mid$. Then

- *There is an isomorphism of A-modules*

$$A \otimes_B \mid M \mid \cong A//B \otimes M. \tag{2.21}$$

  where we denote by $\otimes$ the tensor product over $\mathbb{F}_2$. The algebra A acts on the left hand side only on the A-factor.
- *On the right hand side of 2.21, the module structure over the algebra A is defined by*

$$\begin{aligned} A \otimes A//B \otimes M &\to A//B \otimes M \\ a \otimes c \otimes m &\mapsto \sum_i (a_i^{'} c) \otimes (a_i^{''} m), \end{aligned}$$

  where

$$\Delta(a) = \sum_i a_i^{'} \otimes a_i^{''}.$$

Theorem 2.20 has the following consequence for the subalgebras $\mathcal{A}_1$, and $E(1)$ [38].

Corollary 2.22. *Let X be a spectrum. Then, there exist isomorphisms of $\mathcal{A}$-modules*

- $\theta_{\mathcal{A}_1} : \mathcal{A} \otimes_{\mathcal{A}_1} H^*(X;\mathbb{F}_2) \to (\mathcal{A} \otimes_{\mathcal{A}_1} \mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2)$.
- $\theta_{E(1)} : \mathcal{A} \otimes_{E(1)} H^*(X;\mathbb{F}_2) \to (\mathcal{A} \otimes_{E(1)} \mathbb{F}_2) \otimes H^*(X;\mathbb{F}_2)$
- *The isomorphisms θ induce isomorphisms*

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(H^*(X);\mathbb{F}_2) \cong \mathrm{Ext}^{*,*}_{\mathcal{A}}(H^*(ko) \otimes H^*(X), \mathbb{F}_2).$$

$$\mathrm{Ext}^{*,*}_{E(1)}(H^*(X);\mathbb{F}_2) \cong \mathrm{Ext}^{*,*}_{\mathcal{A}}(H^*(ku) \otimes H^*(X), \mathbb{F}_2)$$

Lemma 2.23. *Let X and Y be connected spectra such that $H_*(X)$ and $H^*(Y)$ are of finite type. Let $P_* \to H^*(X)$ and $Q_* \to H^*(Y)$ be free $\mathcal{A}$- resolutions. Then $P_* \otimes Q_* \to H^*(X \wedge Y)$ is a free $\mathcal{A}$- resolution. The pairing*

$$\hom_{\mathcal{A}}(H^*(X), \mathbb{F}_2) \otimes (\hom_{\mathcal{A}}(H^*(Y), \mathbb{F}_2)) \to \hom_{\mathcal{A}}(P_* \otimes Q_*, \mathbb{F}_2)$$

given by

$$(P_s \to \Sigma^t \mathbb{F}_2) \otimes (Q_u \to \Sigma^v \mathbb{F}_2) \mapsto (P_s \otimes Q_u \to \Sigma^{t+v} \mathbb{F}_2).$$

induces a pairing of Ext- *groups*

$$\mathrm{Ext}^{s,t}(H^*(X), \mathbb{F}_2) \otimes \mathrm{Ext}^{u,v}(H^*(Y), \mathbb{F}_2).$$

Theorem 2.24. *Let $\mathcal{A}$ be the Steenrod algebra, and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Let X and Y be connected spectra of finite type, and let E be a ring spectrum with $H^*(E) = \mathcal{A}//\mathcal{B}$ and multiplication $\mu : E \wedge E \to E$. Then, the smash pairing of* Ext *groups*

$$\begin{array}{c} \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(E) \otimes H^*(X), \mathbb{F}_2) \otimes \mathrm{Ext}^{u,v}_{\mathcal{A}}(H^*(E) \otimes H^*(Y), \mathbb{F}_2) \\ \Big\downarrow \\ \mathrm{Ext}^{s+u,t+v}(H^*(E) \otimes H^*(E) \otimes H^*(X) \otimes H^*(Y), \mathbb{F}_2) \\ \Big\downarrow \\ \mathrm{Ext}^{s+u,t+v}(H^*(E) \otimes H^*(X) \otimes H^*(Y), \mathbb{F}_2) \end{array}$$

is compatible via the change of rings isomorphism with the cap product pairing of Ext *groups for $\mathcal{B}$-modules*

$$\mathrm{Ext}^{s,t}(H^*(X), \mathbb{F}_2) \otimes \mathrm{Ext}^{u,v}(H^*(Y), \mathbb{F}_2) \to \mathrm{Ext}^{s+u,t+v}(H^*(X) \otimes H^*(Y), \mathbb{F}_2).$$

Here, the first morphism denotes the tensor product on the Ext *groups introduced in 2.23. The second morphism is induced by μ.*

We will use the smash product structure of the Adams spectral sequence to define a cap product.

Let X be the suspension spectrum of a pointed CW-complex. Denote the Spanier-Whitehead dual of X by $D(X)$. Consider the duality morphism

$$u : S \to D(X) \wedge X.$$

The functional spectrum is denoted by

$$F(X, E).$$

Notice the weak equivalence

$$D(X) \simeq F(X, S).$$

A general reference for Spanier Whitehead duality in the category of spectra is [30]. We will need the following two results for the construction of the cap structure.

Lemma 2.25. *The duality morphism induces*

- *An $\mathcal{A}$-linear pairing*

$$u^* : H^*(DX) \otimes_{\mathbb{F}_2} H^*(X) \to \mathbb{F}_2.$$

- *An isomorphism of $\mathcal{A}$-algebras*

$$H^*(DX) \cong \hom_{\mathbb{F}_2}(H^*(X), \mathbb{F}_2).$$

Proposition 2.26. *Let E be a spectrum and let X be the suspension spectrum of a finite CW complex. Then there is a natural isomorphism*

$$\hom_{\mathcal{A}}(H^*(E \wedge D(X)), \mathbb{F}_2) \cong \hom_{\mathcal{A}}(H^*(E), H^*(X)).$$

Proof. Let $\{e_\alpha\}$ be an $\mathbb{F}_2$- basis of $H^*(X)$, and let $\{e^\alpha\}$ be the corresponding dual basis of $H^*(D(X))$ with respect to the nondegenerate pairing $u^* : H^*(D(X)) \otimes_{\mathbb{F}_2} H^*(X) \to \mathbb{F}_2$. Furthermore, let $\{f_\beta\}$ be a basis of $H^*(E)$. We will define maps

$$\hom_{\mathcal{A}}(H^*(E \wedge D(X)), \mathbb{F}_2)) \underset{\psi}{\overset{\phi}{\rightleftarrows}} \hom_{\mathcal{A}}(H^*(E), H^*(X)).$$

For $T \in \hom_{\mathcal{A}}(H^*(E \wedge D(X)), \mathbb{F}_2)$, we define $\phi(T)$ to be the composite

$$f_\beta \mapsto \sum_\alpha f_\beta \otimes e^\alpha \otimes e_\alpha \overset{T \otimes 1}{\mapsto} \sum_\alpha T^\alpha_\beta(e_\alpha),$$

where $T^{\alpha}_{\beta} := T(f_\beta \otimes e^\alpha)$.

For an element

$$S \in \hom_{\mathcal{A}}(H^*(E), H^*(X)),$$

we define $\psi(S)$ to be the composite

$$f_\beta \otimes e^\alpha \overset{S\otimes 1}{\mapsto} \sum_{\alpha'} S^{\alpha'}_{\beta} e_{\alpha'} \otimes e^\alpha \overset{u^*}{\mapsto} S^{\alpha}_{\beta},$$

where $S^{\alpha}_{\beta} = u^* \circ (S \otimes 1)(f_\beta \otimes e_\alpha)$.

□

Theorem 2.27. *Let X be the suspension spectrum of a finite pointed CW-complex. Let E be a connected spectrum such that $H_*(E)$ is of finite type. Then, the Adams spectral sequence*

$$\mathrm{Ext}^{s,t}(H^*(F(X,E)), \mathbb{F}_2) \Rightarrow [S, F(X,E)]_{\hat{2}} \cong [X,E]_{\hat{2}}.$$

Is naturally isomorphic to the Adams spectral sequence

$$\mathrm{Ext}^{s,t}(H^*(E), H^*(X)) \Rightarrow [X,E]_{\hat{2}}.$$

Proof. Given an Adams resolution of E

$$\begin{array}{ccccccccc} \dots \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \overset{=}{\longrightarrow} & E, \\ & \downarrow & & \downarrow & & \downarrow & & \\ & K_2 & & K_1 & & K_0 & & \end{array}$$

we smash with $D(X)$ to obtain an Adams resolution for $E \wedge D(X)$.

$$\begin{array}{ccccccccc} \dots \longrightarrow & F_2 \wedge D(X) & \longrightarrow & F_1 \wedge D(X) & \longrightarrow & F_0 \wedge D(X) & \overset{=}{\longrightarrow} & E \wedge D(X). \\ & \downarrow & & \downarrow & & \downarrow & & \\ & K_2 \wedge D(X) & & K_1 \wedge D(X) & & K_0 \wedge D(X) & & \end{array}$$

Due to proposition 2.26 we obtain an isomorphism

$$\hom^t_{\mathcal{A}}(H^*(\Sigma^s K_s \wedge D(X)), \mathbb{F}_2) \cong \hom^t_{\mathcal{A}}(H^*(\Sigma^s K_s), H^*(X)),$$

which induces after taking homology an isomorphism

$$\mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*(\Sigma^s K_s \wedge D(X)), \mathbb{F}_2)) \longrightarrow \mathrm{Ext}^{s,t}(H^*(\Sigma^s K_s), H^*(X)))$$

□

Definition 2.28. The cap product is defined as the composition

$$\begin{array}{c}
\mathrm{Ext}_{\mathcal{A}}^{s,-t}(H^*(E), H^*(X)) \otimes \mathrm{Ext}_{\mathcal{A}}^{u,v}(H^*(E) \otimes H^*(X), \mathbb{F}_2) \\
\downarrow \\
\mathrm{Ext}_{\mathcal{A}}^{s,-t}(H^*(E) \otimes H^*(D(X)) \otimes \mathrm{Ext}_{\mathcal{A}}^{u,v}(H^*(E) \otimes H^*(X), \mathbb{F}_2) \\
\downarrow \\
\mathrm{Ext}_{\mathcal{A}}^{u+s,v-t}(H^*(E) \otimes H^*(E) \otimes H^*(D(X)) \otimes H^*(X), \mathbb{F}_2) \\
\Big\downarrow {\scriptstyle (1\otimes\Delta_X)_*} \\
\mathrm{Ext}_{\mathcal{A}}^{u+s,v-t}(H^*(E) \otimes H^*(E) \otimes H^*(D(X)) \otimes H^*(X) \otimes H^*(X), \mathbb{F}_2) \\
\Big\downarrow {\scriptstyle \mathrm{Ext}_{\mathcal{A}}^{*,*}(\mu^*\otimes \mathrm{ev}^* \otimes 1)} \\
\mathrm{Ext}^{u+s,v-t}(H^*(E) \otimes H^*(X), \mathbb{F}_2),
\end{array}$$

where $\mu : E \wedge E \to E$ is the product in E, and $\mathrm{ev} : D(X) \wedge X \to S$ is the evaluation map.

Corollary 2.22 together with definition 2.28 has as consequence the following result, which is the cap structure that we will need in sections 4 and 5.

Theorem 2.29 (Cap Pairing for sub-hopf algebras)**.** *Let $\mathcal{A}$ be the Steenrod algebra $H^*(H)$, and let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$. Furthermore, let X be a pointed finite CW complex, and let E be a ring spectrum with $H^*(E) \cong \mathcal{A}//\mathcal{B}$. Then, the cap product pairing of Ext groups over the algebra $\mathcal{A}$ 2.14 is compatible via the change of rings isomorphism with the cap pairing*

$$\mathrm{Ext}_{\mathcal{B}}^{u,-v}(\mathbb{F}_2, H^*(X)) \otimes \mathrm{Ext}_{\mathcal{B}}^{s,t}(H^*(X), \mathbb{F}_2) \to \mathrm{Ext}_{\mathcal{B}}^{u+s,t-v}(H^*(X), \mathbb{F}_2).$$

Proof. Since the cap structure 2.14 is induced by the smash product, it is compatible with the change of rings isomorphism from Corollary 2.22. □

2.1. **The Atiyah-Hirzebruch Spectral Sequence.** Given a generalized cohomology theory $\mathcal{H}^*$ represented by a spectrum E, there exist spectral sequences converging to $\mathcal{H}^*$ (of cohomological type)and to $\mathcal{H}_*$ (of homological type). The differentials of the spectral sequence of cohomological type are natural transformations of cohomology theories, and their description will be needed to deduce differentials of the Adams spectral sequence in section 4 and 5.

Theorem 2.30. *Let E be a spectrum, and denote by $\mathcal{H}^*$ and $\mathcal{H}_*$ the cohomology, respectively homology theories associated to E. There exist spectral sequences*

$$E_2^{p,q} = H^p(X, \pi_q(E)),$$

$$E_{p,q}^2 = H_p(X, \pi_q(E)),$$

of cohomological, respectively homological grading, which converge to the cohomology theory $\mathcal{H}^*$, respectively $\mathcal{H}_*$

The construction of the Postnikov System for the spectrum KO, and hence for the connective version ko has as consequence the following result:

Lemma 2.31. *The primary differentials in the cohomological Atiyah-Hirzebruch Spectral sequence for ko are as follows:*

- $H^p(X, ko_{8q}) \xrightarrow{d_2} H^{p+2}(X, ko_{8q+2})$ *is* $Sq^2 \circ r : H^p(X, \mathbb{Z}) \to H^{p+2}(X, \mathbb{F}_2)$, *where* r *is the reduction.*
- $H^p(X, ko_{8q+1}) \xrightarrow{d_2} H^{p+2}(X, ko_{8q+2})$ *is* $Sq^2 : H^p(X, \mathbb{F}_2) \to H^{p+2}(X, \mathbb{F}_2)$.
- $H^p(X, ko_{8q+1}) \xrightarrow{d_3} H^{p+3}(X, ko_{8q+2})$ *is* $Sq^3 = Sq^1 \circ Sq^2 : H^p(X, \mathbb{F}_2) \to H^{p+3}(X, \mathbb{Z})$.
- $H^p(X, ko_{8q+4}) \xrightarrow{d_5} H^{p+3}(X, ko_{8q+8})$ *is* $Sq^5 = Sq^1 \circ Sq^4 \circ r : H^p(X, \mathbb{Z}) \to H^{p+5}(X, \mathbb{Z})$, *where* r *is the reduction.*

For the homological Atiyah-Hirzebruch spectral sequence, we have the following result:

Lemma 2.32. *The primary differentials in the homological Atiyah-Hirzebruch spectral sequence for* ko *are as follows:*

- $H_p(X, ko_{8q}) \xrightarrow{d^2} H_{s-2}(X, ko_{8q+1})$ *is* $Sq_2 \circ r : H_p(X, \mathbb{Z}) \to H_{p-2}(X, \mathbb{F}_2)$, *where* r *is the reduction.*
- $H_p(X, ko_{8q+1}) \xrightarrow{d^2} H_{p-2}(X, ko_{8q+2})$ *is* $Sq_2 : H_p(X, \mathbb{F}_2) \to H_{p-2}(X, \mathbb{F}_2)$.
- $H_p(X, ko_{8q+1}) \xrightarrow{d^3} H_{p-3}(X, ko_{8q+2})$ *is* $Sq_3 = Sq_1 \circ Sq_2 : H^p(X, \mathbb{F}_2) \to H^{p-3}(X, \mathbb{Z})$.
- $H_p(X, ko_{8q+4}) \xrightarrow{d_5} H_{p-5}(X, ko_{8q+8})$ *is* $Sq_5 = Sq_1 \circ Sq_4 \circ r : H_p(X, \mathbb{Z}) \to H_{p-5}(X, \mathbb{Z})$, *where* r *is the reduction.*

2.2. **The η-c-R sequence.** We will recall an exact sequence which relates connective complex and real k- theory.

Recall that the element $\eta : S^1 : ko$ defined above fits into a cofiber sequence of spectra maps

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} C(\eta) \longrightarrow \Sigma^2 ko,$$

where $C\eta \simeq S^2$ denotes the cone of multiplication by η. We will examinate in section 5 the behaviour of the η-c-R on both the Adams and the Atiyah-Hirzebruch spectral sequences.

3. Splittings, group cohomology and minimal resolutions.

Let X and Y be CW-complexes.

Lemma 3.1. *Let* X *and* Y *be suspension spectra.*

There exists a stable splitting for spaces X *and* Y

$$X \times Y \simeq X \vee Y \vee X \wedge Y,$$

inducing a weak homotopy equivalence

$$B\mathbb{Z}/4 \times B\mathbb{Z}/4 \simeq (B\mathbb{Z}/4) \vee (B\mathbb{Z}/4) \wedge (B\mathbb{Z}/4 \wedge B\mathbb{Z}/4).$$

Theorem 3.2. *For the group* $\mathbb{Z}/4 \times \mathbb{Z}/4$, *there exist isomorphisms*

$$ku_*(B\mathbb{Z}/4) \oplus ku_*(B\mathbb{Z}/4) \oplus ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4),$$

as well as

$$ko_*(B\mathbb{Z}/4) \oplus ko_*(B\mathbb{Z}/4) \oplus ko_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4).$$

The determination of the orders of the ko-homology groups of $B\mathbb{Z}/4$ was originally done by Bottvinik-Gilkey-Stolz using the Atiyah-Hirzebruch spectral sequence together with the proof of the Gromov-Lawson-Rosenberg conjecture in [6].

Here we will need the differentials of the Adams spectral sequence of the $\mathbb{Z}/4$ summand as input for the determination of the differentials in the Adams spectral sequence of the ku-homology groups of the smash product factor in section 4.

We begin by introducing two modules over the Steenrod algebra

FIGURE 3.1. The Module M_B

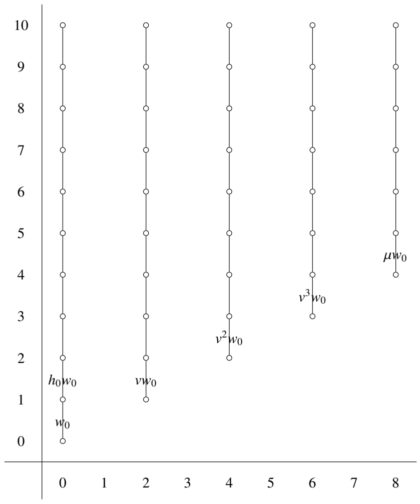


FIGURE 3.2. The graded Module $\mathrm{Ext}_{\mathcal{A}_1}(M_B, \mathbb{F}_2)$

Definition 3.3. The module M_B, called "bow", consists of the following data:

- Elements w_0, of degree 0, w_2 of degree 2.
- The action of the algebra $\mathcal{A}_1$ is determined by the fact that $Sq^1(w_0) = 0$, $Sq^1(w_1) = 0$, and $Sq^2(w_0) = w_2$.

The Ext term is depicted in 3.2

Lemma 3.4. *The* $\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(\mathbb{F}_2, \mathbb{F}_2)$*-module* $\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_B, \mathbb{F}_2)$ *is generated by classes* w_0, w_2, w_4, w_6 *of* $(t-s, s)$*-degree*

$$\mid w_0 \mid = (0,0),\ \mid w_2 \mid = (2,1),\ \mid w_4 \mid = (4,2),\ \mid w_6 \mid = (6,3).$$

with the relations

$$\eta w_i = 0$$

for $i = 0, 2, 4, 6$.

$$vw_{2i} = w_{2(i+1)},$$
$$\omega w_i = h_0 w_{i+4},$$
$$\mu \omega_i = \omega_{i+8}.$$

Definition 3.5. Let M_{SB} denote the $\mathcal{A}_1$- module

$$\mathbb{F}_2\langle d_0\rangle \oplus \mathbb{F}_2\langle d_{2,0}\rangle \oplus \mathbb{F}_2\langle d_{2,1}\rangle \oplus \mathbb{F}_2\langle d_4\rangle,$$

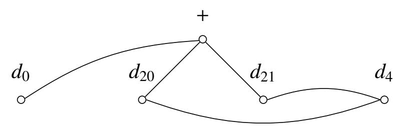


FIGURE 3.3. The module M_{SB}.

generated by elements d_0 of degree 0, d_{20} and d_{21} of degree 2, and d_4 of degree four, and the relations $Sq^2(d_0) = d_{20} + d_{21}$, $Sq^2(d_{20}) = Sq^2(d_{21}) = d_4$.

See picture 3.3 for the graphical depiction of the module M_{SB}.

Lemma 3.6. *The following holds for the $\mathcal{A}_1$-module M_{SB}.*

- *The graded group*
$$\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_{SB}, \mathbb{F}_2)$$
is freely generated as a module over $\mathbb{F}_2[h_0]$ by elements $a_{0,i}, a_{2,i}$ of bidegree $(2i, i)$, respectively $(2i + 2, i)$ for all $i = 0, 1, \ldots$.
- *They satisfy the relations*
$$h_2 a_{0,i} = h_0 a_{0,i+2}, \quad h_2 a_{2,i} = h_0 a_{2,i+2},$$
$$h_3 a_{0,i} = a_{0,i+4}, \quad h_3 a_{2,i} = a_{2,i+4}.$$
- *After restricting to $E(1)$, M_{SB} splits as a direct sum of trivial $E(1)$- modules $\mathbb{F}_2$.*
- *The graded group*
$$\mathrm{Ext}_{E(1)}(M_{SB}, \mathbb{F}_2)$$
is freely generated as a $\mathbb{F}_2[h_0]$- module by the elements $b_{0,i}$ of degree $(2i, i)$, $b_{20,i}$, $b_{21,i}$ of degree $(2i + 2, i)$, and $b_{4,i}$ of degree $(2i + 4, i)$.
- *Under the complexification map*
$$c : \mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_{SB}, \mathbb{F}_2) \to \mathrm{Ext}_{E(1)}(M_{SB}, \mathbb{F}_2),$$
$\nu^i a_{20}$ corresponds to $b_{20,i}$, and $\nu^i a_{21}$ corresponds to $b_{21,i}$. The image of $c : \mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_{SB}, \mathbb{F}_2) \to \mathrm{Ext}_{E(1)}(M_{SB}, \mathbb{F}_2)$ is generated as an $\mathbb{F}_2[h_0]$-module by
$$b_{0,2i}, b_{21,i} + h_0 b_{20,i},$$
and
$$b_{20,i} + b_{21,i}, \quad b_{20,2i+1} + b_{21,i} + h_0 b_{4,i}.$$

Lemma 3.7. *As $\mathbb{F}_2$-algebras, the mod 2 cohomology of the summands $B\mathbb{Z}/4$ are generated by*

- *Elements $x_0 \in H^1(B\mathbb{Z}/4)$ such that $x_0^2 = 0$ and $T_0 \in H^2(B\mathbb{Z}/4)$ corresponding to one factor.*
- *Elements $x_1 \in H^1(B\mathbb{Z}/4)$ such that $x_1^2 = 0$ and $T_1 \in H^2(B\mathbb{Z}/4)$ corresponding to the second factor.*

Lemma 3.8. *The minimal Adams resolution for $B\mathbb{Z}/4$ is given as follows:*

- $$\widetilde{H}^*(B\mathbb{Z}/4) \cong \Sigma^1\mathbb{F}_2 \oplus \bigoplus_{d\geq 0} \Sigma^{2d} M_B \oplus \Sigma^{2d+1} M_B.$$

  The $\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(\mathbb{F}_2,\mathbb{F}_2)$*-module* $\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_B,\mathbb{F}_2)$ *is generated by classes* w_0, w_2, w_4, w_6 *of* $(t-s,s)$*-degree*

  $$\mid w_0 \mid= (0,0),\ \mid w_2 \mid= (2,1),\ \mid w_4 \mid= (4,2),\ \mid w_6 \mid= (6,3).$$

  with the relations $\eta w_i = 0$ *for* $i = 0,2,4,6$.

  $$\omega w_i = h_0 w_{i+4}$$

- *The* $\mathrm{Ext}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)$*-module* $\mathrm{Ext}_{E(1)}(\tilde{H}^*(B\mathbb{Z}/4,\mathbb{Z}))$ *is freely generated by classes*

  $$z^k \text{ of degree } (2k,0),$$

Proof.

- A generating set as $\mathcal{A}_1$- module for $\widetilde{H}^*(X)$ is given by the set $\{x, z^{2d+1}, xz^{2d+1}\}$, where x is the generator in degree 1, and z s in degree 2. The element x generates an $\mathcal{A}_1$- module isomorphic to M_P, and the classes xz^{2d+1} generate modules isomorphic to M_P. The result of the lemma follows then from 3.4.
- The result is described in [8], Theorem 2.2.1 in page 34.

□

Lemma 3.9. *The* mod 2- *cohomology ring* $\widetilde{H}^*(B\mathbb{Z}/4)\times B\mathbb{Z}/4,\mathbb{F}_2)$ *has the following descomposition as* $\mathcal{A}_1$*-module,*

$$\tilde{H}^*(BG) \cong \Sigma^1\mathbb{F}_2^2 \oplus \Sigma^2(\mathbb{F}_2^2 \oplus M_B^2) \oplus \Sigma^3 M_B^4$$
$$\bigoplus_{k\geq 1} \Sigma^{4k}(M_B^2 \oplus M_{SB}^k) \oplus \Sigma^{4k+1} M_{SB}^{2k} \oplus \Sigma^{4k+2}(M_B^2 \oplus M_{SB}^k) \oplus \Sigma^{4k+3} M_B^4$$

has generators

- *In degree* $4k$, $x_0x_1T_1^{2k-1}$ *and* $x_0x_1T_0^{2k-1}$, *generating a copy of* M_B, *and* $T_0^{2l+1}T_1^{2(k-l)-1}$ *for* $l = 0,\dots k$, *which generate a copy of* M_{SB}.
- *In degree* $4k+1$, $x_1T_0^{2l+1}T_1^{2(k-l)-1}$, $x_0T^{2l+1}T_1^{2(k-l)-1}$ *for* $l = 0,\dots k-1$, *generating a copy of* M_{SB}.
- *In degree* $4k+2$, T_0^{2k+1} *and* T_1^{2k+1}, *generating a copy of* M_B, *and* $x_0x_1T_0^{2l+1}T_1^{2(k-l)-1}$, *generating a copy of* M_{SB}.
- *In degree* $4k+3$, $x_0T_0^{2k+1}$, $x_0T_1^{2k+1}$, $x_1T_0^{2k+1}$ *and* $x_1T_1^{2k+1}$ *generating a copy of* M_B.

Proof. With the relation of the classes x_0, x_1, T_0, T_1 we have that Sq^1 is zero for each word. On the other hand we have the following relations

$$Sq^2(T_0^iT_1^j) = \begin{cases} 0 & (i,j) \overset{2}{\equiv} (0,0) \\ T_0^{i+1}T_1^j & (i,j) \overset{2}{\equiv} (1,0) \\ T_0^iT_1^{j+1} & (i,j) \overset{2}{\equiv} (0,1) \\ T_0^{i+1}T_1^j + T_0^iT_1^{j+1} & (i,j) \overset{2}{\equiv} (1,1) \end{cases}$$

Hence the classes x_0, x_1 and x_0x_1 generate a copy of $\mathbb{F}_2$. The classes T_0^{2k+1}, T_1^{2k+1}, $x_0T_0^{2k+1}$, $x_1T_0^{2k+1}$, $x_0T_1^{2k+1}$, $x_1T_1^{2k+1}$ and $x_0x_1T_1^{2k+1}$ generate a copy of M_B. And the classes $T_0^{2l+1}T_1^{2(k-l)-1}$, $x_0T_0^{2l+1}T_1^{2(k-l)-1}$, $x_1T_0^{2l+1}T_1^{2(k-l)-1}$ and $x_0x_1T_0^{2l+1}T_1^{2(k-l)-1}$ generate a copy of M_{SB}.

□

Lemma 3.10. *As a graded module over* $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)$, $\mathrm{Ext}^{*,*}_{E}(1)H^*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4,\mathbb{Z})$ *is freely generated by classes*

- $x_0T_0^kT_1^l$, *of degree* $(1+2k+2l,0)$ *with* $k+l\geq 1$,
- $T_0^kx_1T_1^l$, *of degree* $(1+2k+2l,0)$, *with* $k+l\geq 1$,
- $T_0^kT_1^l$ *of degree* $(2k+2l,0)$ *with* $k+l\geq 2$, *and*
- $x_0T_0^kx_1T_1^l$ *of degree* $(2+2k+2l,0)$.

We will compute with the Adams spectral sequence the complex and real connective k- homology of the classifing space $B\mathbb{Z}/4$. This is needed for the computations for the smash factor performed later.

Recall that there exist a spherical fibration

$$S^1 \to L^\infty(4) \xrightarrow{p} \mathbb{C}P^\infty,$$

Where $L^\infty(4) = \mathrm{colim}_k\, S^{4k-1}/\mathbb{Z}/4$ is a model for $B\mathbb{Z}/4$.

Lemma 3.11. *Let y be the Euler class of the tautological complex line bundle over* $\mathbb{C}P^\infty$.

- *There exist unique classes* $\beta_i \in H_{2i}(\mathbb{C}P^\infty)$ *with the property that*
$$\langle v^j\beta_i, v^ly^k\rangle = \delta^i_k\delta^l_j.$$
- *For the pushforward map in complex K- homology, the following sequence is exact:*

$$0 \longrightarrow \widetilde{ku}_{2n}(\mathbb{C}P^\infty) \xrightarrow{4y\cap} \widetilde{ku}_{2n-2}(\mathbb{C}P^\infty) \xrightarrow{p_!} \widetilde{ku}_{2n-1}(L^\infty(4)) \longrightarrow 0.$$

Definition 3.12. The Hashimoto generators for $\widetilde{ku}_{2n-1}(L^\infty(4))$ are the $n-1$ elements

$$B_s = v^{n-s-1}(p_!(\beta_s)),$$

for $s=0,\dots,n-1$.

We will need another set of generator and relations for the solution of extension problems in the Adams spectral sequence.

Consider the abelian group of generators

$$M_G = \mathbb{Z}\langle B_0,\dots B_{n-1}\rangle,$$

The abelian group

$$M_G = \langle R_0,\dots,R_{n-1}\rangle,$$

and the group homomorphism

$$R_j \longmapsto \sum_{i=0}^{m}\binom{m}{i}B_{j-i},$$

adopting the convention that $B_i=0$ for negative i.

The homomorphism can be written as a matrix

$$A = \begin{pmatrix} \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots \\ 0 & \binom{m}{1} & \binom{m}{2} & \cdots \\ 0 & 0 & \binom{m}{1} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

By the Smith normal form, the cokernel of the matrix A is isomorphic to an abelian group

$$\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \dots.$$

The numbers d_i are called the elementary divisors of $\widetilde{ku}_{2n-1}(B\mathbb{Z}/4)$.

We have the following result of [19], Theorem 3.1, which gives the additive structure of $ku_*(B\mathbb{Z}/4)$ by determining the elementary divisors described above. The work of Hashimoto relies on previous work of Fujii, Kobayashi, Shimomura, and Sugawara [15]

Theorem 3.13. *Let $N := min\{n, 2^2 - 1\}$. The elementary divisor t_i, for $i = 1, 2, \dots, N$ of $\widetilde{ku}_{2n-1}(B\mathbb{Z}/4)$ are given as follows: Let $i = 2^s + d$ with $0 \leq d \leq 2^s$ and $n - 2^s + 1 = a_{s,n}2^s + b_{s,n}$ such that $0 \leq b_{s,n} < 2^s$, i.e., the leading digit of i in the $2-$adic representation is of order s. Moreover, define*

$$\overline{a} = \overline{a}(i, n) = \begin{cases} a_{s,n} + 1 & \text{if } d < b_{s,n} \\ a_{s,n} & \text{if } d \geq b_{s,n} \end{cases}$$

Then we have

$$t_i = 2^{1-s+\overline{a}}$$

and a basis for the elementary divisor is given by

$$B(i) = B(i, n) = \begin{cases} \sum_{k=1}^{2^s} \binom{2^s}{k} B_{n-k-d} & \text{if } d = b_{s,n} - 1 \\ \sum_{k=1}^{2^s} \left(\sum_{t=0}^{s} \sum_{j=1}^{2^t} (-1)2^t - j2^{(2^t-1)\overline{a}} \binom{2^t}{j} \binom{j2^{s-t}}{k} \right) B_{n-k-d}, & \text{otherwise.} \end{cases}$$

Hence we have

$$\widetilde{ku}_{2n-1}(B\mathbb{Z}/4) \cong \sum_{i=1}^{N} \mathbb{Z}/t_i \langle B(i) \rangle.$$

The advantage of the basis $\{B_i\}$ over the basis $B(i)$ is that for $B_i \in \widetilde{ku}_{2n-1}(B\mathbb{Z}/4)$, vB_i is $B_{i+1} \in \widetilde{ku}_{2n+1}(B\mathbb{Z}/4)$.

We have the following consequence for the differentials of the Adams spectral sequence which computes complex connective k-homology of $B\mathbb{Z}/4$.

Theorem 3.14. *The only non-zero differential in the Atiyah-Hirzebruch spectral sequence for $ku_*(B\mathbb{Z}/4)$ is d_2. It is given by the formula*

$$d_2(z^k) = h_0^2(xz^{k-1}) + h_0vxz^{k-2},$$
$$d_2(z) = h_0^2 x,$$
$$d_2(xz^k) = 0.$$

Proof. According to theorem 2.2.1 in page 34 of [8], the i-th Adams filtration quotient in terms of the Hashimoto generators 3.12 is

$$\langle 2^i B_n, 2^{i-1} B_{n-1}, \dots \rangle / \langle 2^{i+1} B_n, 2^i B_{n-1}, \dots \rangle.$$

This is $\mathbb{F}_2 \langle 2^i B_n, \dots B_{n-i} \rangle$ for $i < 2$, and

$$\mathbb{F}_2 \langle 2^2 B_n, \dots, B_{n-2} \rangle / \langle 2^2 B_n + 2B_{n-1} \rangle$$

for $i = 2$.

The reason is that the 2-adical valuation of $\binom{4}{i}$ is $3 - i$ for $i = 1, 2$, and the 2-adical valuation of $\binom{4}{i}$ is larger than $5 - i$ for $i = 3, 4$.

Hence, there are no relations between generators in

$$\langle 2^i B_n, 2^{i-1} B_{n-1}, \dots \rangle / \langle 2^{i-1} B_{n-1}, \dots \rangle$$

for $i < 2$, and there is a unique relation in

$$\langle 2^i B_n, 2^{i-1} B_{n-1} \dots \rangle / \langle 2^{i+1} B_n, \dots \rangle,$$

namely

$$2^2 B_n + 2^2 B_{n-1} = 0.$$

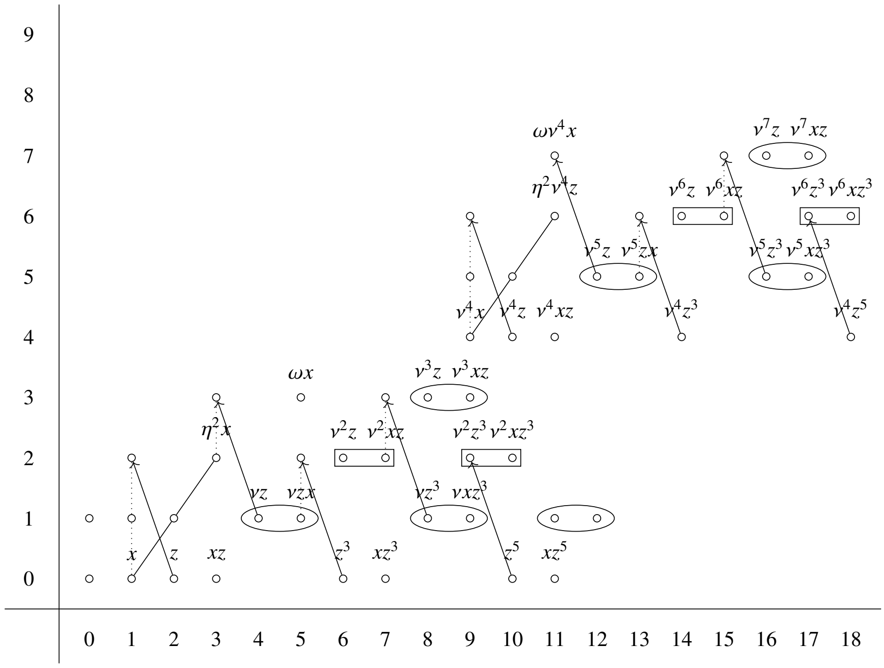


FIGURE 3.4. The E_2 term of the Adams spectral sequence for $ko_*(B\mathbb{Z}/4)$.

Since the differential is h_0 and v- linear, the formula follows. Moreover, there can not be any higher differentials, since there are no elements at the third page in even $(t-s)$ degree.

□

We obtain from the η-c-R exact sequence the following differentials for the Adams spectral sequence computing the real connective K-homology:

Theorem 3.15. *The differentials in the Adams spectral sequence for $ko_*(B\mathbb{Z}/4)$ are all zero except for d_2. It is given for $k > 0$ by*

- $z \mapsto h_0^2 x$
- $z^{2k+1} \mapsto h_0 v x z^{2k-1}$
- $v z^{2k+1} \mapsto h_0^3 x z^{2k+1} + h_0 v^2 x z^{2k-1}$
- $vz \mapsto h_0^3 xz,$
- $v x z^{2k+1} \mapsto 0$

The E_2 page is depicted with the differentials in figure 3.4. There are no further differentials, and the orders of the ko_*-groups are depicted in the following result.
The computation of the orders is stated in the following result.

Theorem 3.16. *For $n \geq 1$, the orders of the groups $\widetilde{ko}_n(B\mathbb{Z}/4)$ are follows:*

	$\log_2(\mid \widetilde{ko}_n(B\mathbb{Z}/4) \mid)$
$n = 8d$	0
$n = 8d + 1$	$(2d + 1) + 1$
$n = 8d + 2$	1
$n = 8d + 3$	$2d + 2$
$n = 8d + 4$	0
$n = 8d + 5$	$2d + 1$
$n = 8d + 6$	0
$n = 8d + 7$	$2d + 2$

More specifically, the groups are as follows:

Theorem 3.17. *The ko-homology groups of $B\mathbb{Z}/4$ for $d \geq 1$ are as follows:*

$$\widetilde{ko}_{8d+k}(B\mathbb{Z}/4) = \begin{cases} 0 & k = 0, 4, 6 \\ \mathbb{Z}/2^{2d+1} \oplus \mathbb{Z}/2 & k = 1 \\ \mathbb{Z}/2 & k = 2 \\ \mathbb{Z}/2^{4d+3} \oplus \mathbb{Z}/2^{2d+1} & k = 3 \\ \mathbb{Z}/2^{4d+5} \oplus \mathbb{Z}/2^{2d+1} & k = 7 \end{cases}$$

3.1. **Hidden Extensions in $\widetilde{ku}_{2n+1}(B\mathbb{Z}/4)$.** In the group $ku_{2n+1}(B\mathbb{Z}/4)$, the Adams filtration F_i is given by

$$\langle 2^i B_n, 2^{i-1}, B_{n-1}, \ldots \rangle \subset ku_{2n+1}(B\mathbb{Z}/4),$$

hence the filtration quotients are:

$$Q_i = \langle 2^i B_n, 2^{i-1} B_{n-1}, \ldots \rangle / \langle 2^{i+1} B_n, 2^i B_{n-1}, \ldots \rangle$$

Definition 3.18. Let $F_i = F_i ku_{2n+1}(B\mathbb{Z}/4)$ be the i-th filtration subgroup in the Adams Spectral sequence, and let $Q_i = F_i/F_{i+1}$ be the i-th filtration quotient. There exists a hidden extension between $x_i \in Q_i$ and $x_j \in Q_j$ if there is a natural number r with $2^r x = x_i$ in Q_i, and $2^{r+1} x = x_j$ in Q_j, with $j > i + 1$.

The following result shows that the extensions between consecutive filtration quotients are not hidden in the sense of definition 3.18.

Lemma 3.19. *Let $x_i \in F_i/F_{i+1}$ with $2x_i \in F_{i+1}/F_{i+2}$, then $h_0 x_i = x_{i+1}$.*

Proof. Let $x_i \in Q_i$, and let $x \in ku(B\mathbb{Z}/4)$ be a lift. Then, it is possible to write $x = x_i +$ higher filtration terms, we will write $x_i +$ h.o.t.. Then $2x_i +$ h.o.t is a lift of $h_0 x$, and hence $[2x_i] = h_0 x \in Q_i$

□

However, there exist hidden extensions in the complex K-homology of $B\mathbb{Z}/4$ which we will study in connection with the ones for $B\mathbb{Z}/4 \wedge B\mathbb{Z}/4$.

Example 3.20. Consider the group $\widetilde{ku}_7(B\mathbb{Z}/4)$, and the group element $x = 2B_3 + B_2$. It is 4-torsion, $2x = 4B_3 + 2B_2 \neq 0$. The element x is detected in the Adams spectral sequence, but the relation $x = 2x$ cannot be detected in the Adams spectral sequence, because $2x = 0$ in $Q_2 = F_2/F_3$. Since $2x = 0$ in Q_2, the element $2x$ is in F_3. This can be seen from the relation

$$2x = 4B_3 + 2B_3 = -(6B_2 + 4B_1 + B_0) + 2B_2) = -4B_2 - 4B_1 + B_0.$$

We will need in sections 4, and 5 the following description of the complex connective K-homology groups in terms of representation theory:

Theorem 3.21. *Let $R_{\mathbb{C}}(\mathbb{Z}/4)$ be the complex representation ring of $\mathbb{Z}/4$, and let $R^0_{\mathbb{C}}(\mathbb{Z}/4)$ be the augmentation ideal. Denote by α the standard complex one dimensional representation. Then, the group homomorphism*

$$\mathbb{Z}\langle B_0,\ldots,B_n\rangle/\langle\sum_{i=1}^{m}\binom{m}{i}B_{j-i}\rangle \longrightarrow RU^0_{\mathbb{C}}(\mathbb{Z}/4)/\big(RU^0_{\mathbb{C}}(\mathbb{Z}/4)\big)^{n+1}$$

given by

$$B_j \longmapsto (\alpha-1)^{n-j}$$

is an isomorphism.

4. Complex connective K- Theory computations on the smash summand

We turn now our attention to the smash factor.

Lemma 4.1. *The mod 2 cohomology of $B\mathbb{Z}/4\wedge B\mathbb{Z}/4$,*

$$\widetilde{H}^*(B\mathbb{Z}/4\wedge B\mathbb{Z}/4,\mathbb{F}_2).$$

Has as an $\mathbb{F}_2$-vector space basis the elements x_0x_1, $x_0T_1^n$, $x_1T_0^m$, $x_0x_1T_0^k$, $x_0x_1T_1^l$, $x_0T_0^rT_1^s$ and $x_1T_0^uT_1^v$. The modules over $\mathcal{A}_1$ which they generate are as follows

Degree	*Generators of suspended M_B*	*Generators of suspended M_{SB}*
$4k$	$x_0x_1T_1^{2k-1}$	$T_0^{2l+1}T_1^{2(k-l)-1}$
$4k+1$		$T_0^{2l+1}x_1T_1^{2(k-l-1)}$, $l=0,1\ldots,k-1$
$4k+2$		$x_0T_0^{2l+1}T_1^{2(k-l)-1}$
$4k+3$	$x_0T_1^{2k+1}, T_0^{2k+1}x_1$	

Let us recall the Universal Coefficient Theorem for integral coefficients in ordinary cohomology from [39], Theorem 13.10 in page 240.

Theorem 4.2. *Let X be a CW-complex of finite type. Then, there exists a natural short exact sequence of the form*

$$0\longrightarrow \mathrm{Ext}(\widetilde{H\mathbb{Z}}^{q+1}(X),\mathbb{Z})\longrightarrow \widetilde{H\mathbb{Z}}_q(X)\longrightarrow \mathrm{Hom}(\widetilde{H\mathbb{Z}}^{q}(X),\mathbb{Z}).\longrightarrow 0.$$

This theorem suggests the following notation

Definition 4.3. The generators in even degree $2n$ of $H\mathbb{Z}_*(B\mathbb{Z}/4\wedge B\mathbb{Z}/4)$ are tensor products. They are denoted by

$$\underline{x_0}\otimes\underline{T_1^{n-1}x_1},\underline{x_0T_0}^{n-2}\otimes\underline{T_1^{n-2}x_1},\ldots,\underline{x_0T_0^{n-1}}\otimes\underline{x_1}.$$

The class $\underline{x_0\otimes T_0}^l\otimes\underline{T_1^{n-1-l}\otimes x_1}$ is mapped to the class $\underline{x_0T_0^lT_1^{n-1}x_1}$ under the reduction modulo 2, where the homological class $\underline{x_0T_0^lT_1^{n-1-l}x_1}$ is the $\mathbb{F}_2$- vector space dual class to the mod 2-cohomology class $x_0T_0^lT_1^{n-1-l}x_1$.

Remark 4.4. To avoid clumsy notation, we will denote the classes

$$\underline{x_0T_0^lT_1^{n-1-l}x_1}$$

by

$$x_0T_0^lT_1^{n-1-l}x_1.$$

Recall the definintion of matrix Toda Brackets.

Definition 4.5. Let R be a commutative ring, and let M be an R-module. Assume given a matrix $A\in M_{n\times n}(R)$, and R-module morphisms $q:R^n\to M$, and $q^{'}:R^n\to M$ with the property that

$$A\circ q=0,\ A\circ q^{'}=0.$$

A Toda bracket is a triple, denoted by $\langle\tilde{v}|A|\tilde{w}\rangle$, consisting of vectors $v \in R^n$, $w \in R^n$ with the property that $q^{'}(\tilde{v}^t A) = 0$ and $q(\tilde{w}) = 0$.

Definition 4.6. The classes in $H_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ for odd degree $* = 2n+1$ are all torsion classes, denoted by

$$T_0 * T_1^n, T_0^2 * T_1^{n-1}, \ldots, T_0^n * T_1,$$

where $T_0^l * T_1^{n+1-l}$ is the Toda bracket $\langle T_0^l|1|T_1^{n+1-l}\rangle$.

The reduction mod 2 of the classes $T_0^l * T_1^{n+1-l}$ is $x_0 T_0^{l-1} T_1^{n+1-l} + T_0^l x_1 T_1^{n-l}$.

We will need later the following remarks concerning the induced homomorphisms from the subgroup $\mathbb{Z}/4$.

Lemma 4.7. *Let k and l be natural numbers. Given the group homomorphism $\mathrm{ind}_{k,l} : \mathbb{Z}/4 \to \mathbb{Z}/4 \times \mathbb{Z}/4$ defined by sending the generator 1 to the element (k,l), we will denote the induced homomorphism in odd homology degree by*

$$\mathrm{ind}_{\mathrm{k,l}_*} : H_*(\mathbb{Z}/4, \mathbb{Z}) \to H_*(\mathbb{Z}/4 \times \mathbb{Z}/4, \mathbb{Z}).$$

For the generators discussed above, the following correspondence determines the homomorphism :

$$y^i \mapsto \sum_{j+j^{'}=i} k^j l^{j^{'}} T_0^j * T_1^{j^{'}}.$$

Lemma 4.8. *Denote by $(a_0, \ldots, a_k)$ the coefficient vector for an element $x = \sum_{i=1}^{n} T_0^i * T_1^{i-1}$ The images of the induction homomorphisms are as follows:*

- *For the homomorphism $1 \mapsto (1,1)$, the coefficients of* $\mathrm{ind}_{1,1}(x)$ *are* $(1,1,\ldots,1)$.
- *For the homomorphism $1 \mapsto (1,-1)$, the coefficients of* $\mathrm{ind}_{(1,-1)}(x)$ *are* $(1,-1,\ldots,1,-1)$.
- *For the homomorphism $1 \mapsto (1,2)$, the coefficients of* $\mathrm{ind}_{1,2}(x)$ *are* $(2,0,\ldots,0)$.
- *For the homomorphism $1 \mapsto (2,1)$, the coefficients of* $\mathrm{ind}_{2,1}(x)$ *are* $(0,0,\ldots,2)$.

Corollary 4.9. *Every class in odd homological degree in $H\mathbb{Z}_*(B\mathbb{Z} \wedge B\mathbb{Z}/4)$ is induced by a group homomorphism. $\varphi : \mathbb{Z}/4 \to \mathbb{Z}/4 \times \mathbb{Z}/4$. In particular, the class in homological degree $2n+1$ defined by*

$$x_0 T_1 + T_0 x_1 T_1^{n-1} + x_0 T_0 T_1^{n-1} + T_0^n x_1$$

is in the image of the $\mathbb{F}_2$- vector space homomorphism induced by the group homomorphism $\varphi : \mathbb{Z}/4 \to \mathbb{Z}/4 \times \mathbb{Z}/4$ given by $1 \overset{\varphi}{\mapsto} (1,1)$.

Another ingredient is the result of the Künneth spectral sequence for ku-homology. in [32].

Theorem 4.10. *There exists a short exact sequence*

$$0 \longrightarrow \widetilde{ku}_*(B\mathbb{Z}/4) \otimes_{ku_*} \widetilde{ku}_*(B\mathbb{Z}/4) \longrightarrow \widetilde{ku}_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4) \longrightarrow \mathrm{Tor}^1_{ku_*}(\widetilde{ku}_*(B\mathbb{Z}/4), \widetilde{ku}_*(B\mathbb{Z}/4)) \longrightarrow 0$$

Proof. By the main theorem in page 173 of [32], there exists a natural Künneth spectral sequence such that the edge homorphism is the external product.

First notice that while the coefficient ring $ku_* = \mathbb{Z}[v]$ on the Bott generator has homological dimension two, the ku_* module $ku_*(B\mathbb{Z}/4)$ is of flat dimension 1 as a ku_*-module, and the complex connective K-theory groups are zero in even degree by lemma 1.4 in [19], p.767. Thus, there is no place for the possible non-zero differential producing potential Tor^2 terms, and the spectral sequence degenerates to the exact sequence above.

□

The ku-homology of the smash product $B\mathbb{Z}/4 \wedge B\mathbb{Z}/4$ is computed in table 1

degree	$\widetilde{ku}_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$	total degree
2	$(\mathbb{Z}/2^2)^1$	2
3	$(\mathbb{Z}/2^2)^1$	2
4	$(\mathbb{Z}/2^1)^1 \oplus (\mathbb{Z}/2^2)^2$	5
5	$(\mathbb{Z}/2^1)^2 \oplus (\mathbb{Z}/2^3)^1$	5
6	$(\mathbb{Z}/2^1)^3 \oplus (\mathbb{Z}/2^2)^3$	9
7	$(\mathbb{Z}/2^1)^5 \oplus (\mathbb{Z}/2^4)^1$	9
8	$(\mathbb{Z}/2^1)^5 \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^1$	14
9	$(\mathbb{Z}/2^1)^7 \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^2$	14
10	$(\mathbb{Z}/2^1)^7 \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^2$	19
11	$(\mathbb{Z}/2^1)^3 \oplus (\mathbb{Z}/2^2)^5 \oplus (\mathbb{Z}/2^6)^1$	19
12	$(\mathbb{Z}/2^1)^9 \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^2)^3$	24
13	$(\mathbb{Z}/2^1)^1 \oplus (\mathbb{Z}/2^2)^5 \oplus (\mathbb{Z}/2^3)^2 \oplus (\mathbb{Z}/2^7)^1$	24
14	$(\mathbb{Z}/2^1)^{11} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^4$	29
15	$(\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^5 \oplus (\mathbb{Z}/2^8)^1$	29
16	$(\mathbb{Z}/2^1)^{13} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^5$	34
17	$(\mathbb{Z}/2^2)^1 \oplus (\mathbb{Z}/2^3)^5 \oplus (\mathbb{Z}/2^4)^2 \oplus (\mathbb{Z}/2^9)^1$	34
18	$(\mathbb{Z}/2^1)^{15} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^6$	39
19	$(\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^4)^5 \oplus (\mathbb{Z}/2^{10})^1$	39
20	$(\mathbb{Z}/2^1)^{17} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^7$	44
21	$(\mathbb{Z}/2^3)^1 \oplus (\mathbb{Z}/2^4)^5 \oplus (\mathbb{Z}/2^5)^2 \oplus (\mathbb{Z}/2^{11})^1$	44
22	$(\mathbb{Z}/2^1)^{19} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^8$	49
23	$(\mathbb{Z}/2^4)^3 \oplus (\mathbb{Z}/2^5)^5 \oplus (\mathbb{Z}/2^{12})^1$	49
24	$(\mathbb{Z}/2^1)^{21} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^9$	54
8d	$(\mathbb{Z}/2^1)^{8d+3} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^{4d+3}$	20d-6
8d+1	$(\mathbb{Z}/2^{2d-2})^1 \oplus (\mathbb{Z}/2^{2d-1})^5 \oplus (\mathbb{Z}/2^{2d})^2 \oplus (\mathbb{Z}/2^{4d+1})^1$	20d-6
8d+2	$(\mathbb{Z}/2^1)^{8d+1} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^{4d-3}$	20d-1
8d+3	$(\mathbb{Z}/2^{2d-1})^3 \oplus (\mathbb{Z}/2^{2d})^5 \oplus (\mathbb{Z}/2^3)^{4d-1}$	20d-1
8d+4	$(\mathbb{Z}/2^1)^{8d-1} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^{4d-1}$	20d+4
8d+5	$(\mathbb{Z}/2^{2d-1})^1 \oplus (\mathbb{Z}/2^{2d})^5 \oplus (\mathbb{Z}/2^{2d+1})^2 \oplus (\mathbb{Z}/2^{4d+3})^1$	20d+4
8d+6	$(\mathbb{Z}/2^1)^{8d+3} \oplus (\mathbb{Z}/2^2)^3 \oplus (\mathbb{Z}/2^3)^{4d}$	20d+9
8d+7	$(\mathbb{Z}/2^{2d})^3 \oplus (\mathbb{Z}/2^{2d+1})^5 \oplus (\mathbb{Z}/2^{4d+4})^1$	20d+9

TABLE 1. Connective ku- homology according to UCT.

We will use this information to deduce information about the differentials in the Atiyah-Hirzebruch spectral sequence for $ku_*(B\mathbb{Z}/4)$. Let us recall the following definition, analogous to 3.12.

Definition 4.11. Given the Hashimoto generators B_i(see Definition 3.12) form the elements

$$B_{i,j} := B_i \otimes B_j \in ku_{2i+1}(B\mathbb{Z}/4) \otimes ku_{2j+1}(B\mathbb{Z}/4) \subset ku_{2(i+j)+2}(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4),$$

and extend the definition by v-periodicity by setting

$$B_{i,j}(d) := v^{d-i-j} B_{i,j}$$

Lemma 4.12. *In the Atiyah-Hirzebruch spectral sequence for computing $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$,*

- *For all i, j, d, e such that $i + j = e$, $1 \leq d \leq 4$, $2B_{i,j}(d+e) = 0$, but $B_{i,j}(d+e) \neq 0$.*
- *For all i, j, d, e such that $i + j = e$, $5 \leq d \leq 10$, $2^2 B_{i,j}(d+e) = 0$, but $2B_{i,j}(d+e) \neq 0$.*

Proof. Take i, j, d, e without the contitions on d. Since there are hidden extensions, the relation $2^s B_{i,j}(d+e)$ might hold in the $d-th$ filtration quotient. Consider for this the quotient

$$ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)/F_{d-1}(ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)) \tag{4.13}$$

In degree $2(d+e)+2$, this is generated by the elements $B_{i,j}(d+e)$ with $i+j \leq d+e$. Since the elements $B_{i,j}(d+e)$ with $i+j < d$ generate the subgroup

$$F_{d-1}(\widetilde{ku}_*(B\mathbb{Z}/4) \otimes_{ku_*} \widetilde{ku}_*(B\mathbb{Z}/4)))_{2(d+e)+2},$$

we get the relations $B_{i,j}(d+e) = 0$ with $i + j < d$, and the binomial relations producing 4.13. now, $2^s B_{i,j}(d+e) = 0$ in 4.13, and hence also in

$$F_d(ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4))/F_{d-1}(ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)) \tag{4.14}$$

in degree $2(d+e)+2$.

Now, we distinguish the two cases:

- For $1 \leq d \leq 4$, by capping with $T_0 T_1$, we obtain $2B_{0,0}(e) = 0$.
- For $5 \leq d \leq 10$, By capping the element $2^2 B_{i,j}(d+e)$ with $T_0^2 T_1^2$ to obtain $B_{0,0}(e) \neq 0$

□

Corollary 4.15. *The $E_\infty^{2e+2,2d}$ term of the Atiyah-Hirzebruch spectral sequence for computing $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ is*

- *$\mathbb{Z}/2^{e+1}$ for $1 \leq d \leq 4$.*
- *0 for $5 \leq d \leq 10$.*

The following result is an immediate consequence. Similar results have been studied in [9].

Lemma 4.16. *In the complex K-homology groups of the smash product $B\mathbb{Z}/4 \wedge B\mathbb{Z}/4$, the following holds:*

(i) Given a class of even degree $z \in ku_{2k+2l+2}(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ of the form $z = x_0 T_0^k T_1^l x_1$, the minimal integer N for which $v^N z = 0$ is $N = 4$.
(ii) The minimal N such that $2^N ku_{2n}(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)) = 0$ is $N = 3$.
(iii) In the Adams spectral sequence for $ku_{2n}(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$, there exist no elements in even degree $2n$ which have Adams filtration higher than 4.

Theorem 4.17. *In the Atiyah-Hirzebruch spectral sequence $E_{s,t}^n$ converging to $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$, the first non-trivial differentials are*

$$d_3(T_0^k * T_1^l) = 2\nu x_0 T_0^{k-1} T_1^{l-2} x_1 + 2\nu x_0 T_0^{k-2} T_1^{l-1} x_1,$$

where we adopt the convention $T_0^{-1} = 0 = T_1^{-1}$.

Proof. From lemma 4.16, we know that $2vx_0x_1 = 0$, and $vx_0x_1 \neq 0$. This implies that there must be a non-trivial differential

$$d_3 : (\mathbb{Z}/2^2\mathbb{Z})^2 \cong \langle T_0 * T_1^2, T_0^2 * T_1 \rangle = E_{5,0}^3 \to E_{2,2}^3 = \langle vx_0x_1 \rangle.$$

The possible values are $d_3(T_0 * T_1^2) = 2vx_0x_1$ and $d_3(T_0^2 * T_1) = 2vx_0x_1$. By part 3 of 2.32, both of them are non zero since they agree with Sq_3.

Now, using the cap structure for the Atiyah-Hirzebruch spectral sequence, since the classes T_0 and T_1 are both infinite cycles, by 2.12,

$$d^3(T_0 \cap T_0^k * T_1^l) = T_0 \cap d^3(T_0^k * T_1^l),$$

and analogously with the cap product with T_1. Notice that $T_0 \cap (T_0^k T_1^l) = T_0^{k-1} * T_1$. By induction over $n = k + l$, the result follows.

□

Definition 4.18. Define a filtration on the E_2 term of the Atiyah-Hirzebruch spectral sequence converging to $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ by setting

$$F^{q'} E^2_{2p-1,2m} := 2^{q'} E^2_{2p-1,2m}.$$

This induces a filtration on the E_9 term by setting

$$F^{q'} E^9_{2p-1,2m} := 2^{q'} E^9_{2p-1,2m}.$$

For an element $x \in E^9_{2p-1,2m}$, define x_0 as the image of x under the quotient

$$F^0 E^9_{2p-1,2m} / F^1 E^9_{2p-1,2m},$$

and inductively, let $\hat{x}_{q'-1}$ be the unique sum of distinct elements in

$$\{2^{q'-1} v^m T_0^l * T_1^k\}_{k+l=p}$$

such that the equivalence class of $\tilde{x}_{q-1}$, denoted by $x_{q'-1}$ in

$$F^q E^9_{2p-1,2m} / F^{q-1} E^9_{2p-1,2m}$$

equals $\hat{x}_{q'-1}$. Define $\tilde{x}_{q'} = \tilde{x}_{q'-1} - \hat{x}_{q'-1}$.

Remark 4.19. For an element $x \in E^2_{2p-1,2m}$ with the property that $x_0 = 0$, we obtain that $d_3(x) = 0$. Similarly, for an alement $x \in E^9_{2p-1,2m}$ such that $x_1 = 0$, $d_9(x) = 0$.

Theorem 4.20. *In the Atiyah-Hirzebruch spectral sequence converging to $ku_*(B\mathbb{Z}/4)$, there exist non-zero differentials (both from total odd degree to total even degree)*

- *d_3, with image $\langle 2v^{m+1} x_0 T_0^l T_1^k x_1 \rangle$*
- *d_9, with image $\langle v^{4+m} x_0 T_0^l T_1^k x_1 \rangle$.*

There is an isomorphism

$$E^2_{2p-1,2m} \cong \mathbb{Z}/4 \langle v^m T_0^l * T_1^k \rangle_{k+l=p}$$

Proof. The first part has been proved in 4.17. We proceed by induction on the degree n of the differential for the second part.

Assume that there is a non-trivial differential d_3 from even total degree to odd total degree. By using the cap structure and the fact that the differentials of the Atiyah-Hirzebruch spectral sequence preserve the cellular filtration, we can assume that the differential in question is

$$d^3 : E^3_{6,0} \to E^3_{3,2}.$$

Setting all v-multiples to zero in

$$ku_*(B\mathbb{Z}/4) \otimes_{ku_*} ku_*(B\mathbb{Z}/4)_6,$$

The only relations that are possible are

$$2^2 B_0 \otimes B_2 = 2^2 B_1 \otimes B_1 = 2^2 B_2 \otimes B_0 = 0.$$

Hence, we conclude that $d_3 = 0$. The same reasonings let us conclude that there is no differential from an even total degree to an odd total degree.

Consider now the differential d_n with $n \geq 4$. Assume that the differential from odd total degree to even total degree is non trivial. By v-linearity, we can restrict to

$$d_n : E_n^{n+k+1,0} \to E_n^{n,k}$$

for $k > 0$. If d_n is non-trivial, from 4.12, 4.15, we get that either $n = 3$, which is discarted or $n = 9$, since either $vx_0T_0^kT_1^lx_1 = 0$ in the 4-th filtration or $2^2v^4x_0T_0^kT_1^lx_1 = 0$ in the 8th filtration quotient, but there are no non-trivial relations between

$$\{2^qv^mx_0T_0^kT_1^lx_1\}$$

for $q = 1, 2$. We hence can assume that the differential is d_9.

Since we assumed that the premise of the theorem is valid for all differentials d^m with $3 < m < n$. From lemma 4.12, there is only a bounded number (independent of k) of elements in $E_{0,k}^n \subset \widetilde{H\mathbb{Z}k}(B\mathbb{Z}/4 \wedge \mathbb{Z}/4)$ for which their images $i(a)$ in the right hand side group are not divisible by 2, respectively. Moreover, denote by $i : E_2^{\mathrm{odd},0} \to H\mathbb{Z}_{\mathrm{odd}}(B\mathbb{Z}/4)$ the inclusion. Then,

$$\{i(a) \in H_{\mathrm{odd}}(B\mathbb{Z}/4), a \in E_2^{\mathrm{odd},0}\} \cong \mathbb{Z}/4\langle T_0 * T_1\rangle,$$

and

$$\{a \in E_n^{\mathrm{odd},0} \mid 2\,\mathrm{divides}\,i(a)\} \cong \mathbb{Z}/2^1\langle 2T_0^k * T_{1^l}\rangle,$$

It follows that

$$\langle 2^2T_0^k * T_1^l\rangle \subset \ker d_3,$$

or

$$\langle 2T_0^k * T_1^l\rangle \subset \ker d_9.$$

Hence, there must be an element $a^{'} \in E_{\mathrm{odd},0}^n$ which is a sum of elements

$$2T_0^k * T_1^l$$

with

$$d_n(a^{'}) = v^4x_0T_0^{k^{'}}T_1^{k^{'}}x_1$$

After capping with T_0 and T_1, we obtain an element $a \in E_{0,n+3}$, which is a sum of distinct elements in either $\{2^2T_0^k * T_1^l\}_{2(k+l)-1=n+2}$, for the d_3, or in $\{2T_0^k * T_1^l\}_{2(k+l)-1=n+2}$, for d_9 such that $d_n(a) = 2v^2x_0x_1$ or $d^n(a) = v^4x_0x_1$.

□

Remark 4.21. We will prove as a corollary of 4.27 that these differentials are the only non zero in the Atiyah-Hirzebruch spectral sequence by comparing with the Adams filtration.

Corollary 4.22. *In $\widetilde{ku}_*(B\mathbb{Z}/4)$, multiplication by v^4 annihilates $\widetilde{ku}_{even}(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$.*

We compare the Atiyah-Hirzebruch Spectral sequence with the Adams spectral sequence converging to $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$.

Recall that the Hurewicz homomorphism is a map of spectra to the integral Eilenberg-Maclane spectrum $h : ku \to H\mathbb{Z}$.

At the level of coefficients, the Hurewicz homomorphism satisfies

$$h_* : ku_* \to H\mathbb{Z}_*\,; 1 \mapsto 1,\, v \mapsto 0.$$

The following result summarizes the information on the Hurewicz homomorphism.

Lemma 4.23. *For the map induced by h, the following holds:*

(i) h maps all v-multiples to zero.

(ii) The restriction of h_* to the zeroth filtration of the Atiyah-Hirzebruch spectral sequence is inyective.
(iii) The restriction of h_* to the zeroth filtration of the Adams spectral sequence is injective.

Proof. The first claim follows directly from the behaviour on coefficients, and the fact that the differential is v-linear. The second claim is a direct consequence. For the statement on the Adams spectral sequence, recall that the mod 2-cohomology of the integral Eilenberg-Maclane spectrum is by Theorem 2.8,

$$H^*(H\mathbb{Z}) \cong \mathcal{A} \otimes_{\langle Sq^1 \rangle} \mathbb{F}_2 \cong \mathcal{A} \otimes_{E(0)} \mathbb{F}_2.$$

□

Due to the naturality of the Adams specral sequence with respect to maps of spectra, the E_∞ page for $\widetilde{H\mathbb{Z}}_*(B\mathbb{Z}/4 \wedge \mathbb{Z}/4)$ is generated over $\mathbb{F}_2[h_0]$ by

$$x_0 T_0^k T_1^l x_1, \; x_0 T_0^{k-1} T_1^l + T_0^k x_1 T_1^{l-1},$$

for $k + l \geq 1$. The only relations among those generators are

$$h_0^2 x_0 T_0^k T_1^l x_1 = 0$$

and

$$h_0^2 (x_0 T_0^{k-1} T_1^l + T_0^k x_1 T_1^{l-1}) = 0.$$

Definition 4.24. We will say that the element in the Adams spectral sequence $T_0^k * T_1^l$ is represented in the Atiyah-Hirzebruch spectral sequence by

$$h_0^i (x_0 T_0^{k-1} T_1^l + T_0^k x_1 T_1^{l-1}).$$

If the images in $\widetilde{H\mathbb{Z}_*}$ under the Hurewicz homomorphism of the i-th adams filtration quotient of $T_0^k * T_1^l$

$$2^i T_0^k * T_1^l,$$

and $h_0^i(x_0 T_0^{k-1} T_1^l + T_0^k x_1 T_1^{l-1})$ agree, and all higher Adams filtrations of $h_0^i(x_0 T_0^{k-1} T_1^l + T_0^k x_1 T_1^{l-1})$ are zero.

Using the correspondence of the elements and filtrations above we have the following result

Theorem 4.25. *The d_2 differentials of the Adams spectral sequence for $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ are given by*

- $T_0^k T_1^l \longmapsto (h_0^2 x_0 T_1^{k-1} + h_0 v x_0 T_1^{k-2}) T_1^l + T_0^k (h_0^2 x_1 T_1^{l-2})$.
- $x_0 T_0^k T_1^l \longmapsto x_0 T_0^k (h_0^2 x_1 T_1^{l-1} + h_0 v x_1 T_1^{l-2})$.
- $T_0^k x_1 T_1^l \longmapsto (h_0^2 x_0 T_1^{k-1} + h_0 v x_0 T_1^{k-1}) x_1 T_1^l$.

Theorem 4.26. *For n odd, the elements*

$$T_0 *_s T_1 := h_0^s (x_0 T_0^k T_1^{\frac{n-2k-1}{2}} + T_0^{k+1} x_1 T_1^{\frac{n-2k-3}{2}}) + v h_0^{s-1} (x_0 T_0^{k-1} T_1^{\frac{n-2k-1}{2}} + T_0^{k+1} x_1 T_1^{\frac{n-2k-5}{2}}),$$

for $k = 0, \ldots, \frac{n-1}{2}$, and $s > 0$, and their v-multiples generate the $E_3^{n,s}$ freely.

Theorem 4.27. *The third Adams differential in the spectral sequence for $ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ is*

$$d_3 : E_3^{11,1} \longrightarrow E_3^{10,6}.$$

The kernel of the differential is generated by

$$\{T_0^k * T_1^l \mid k + l = 6\} - \{T_0^2 * T_1^4, T_0^4 T_1^2\} \cup \{T_0^2 * T_1^4 + T_0^4 * T_1^2\}$$

The differential satisfies

(i)

$$T_0^2 * T_1^4 \longmapsto v^4 h_0 x_0 x_1.$$

(ii)

$$T_0^4 * T_1^2 \longmapsto v^4 h_0 x_0 x_1.$$

Using the cap structure , the third differentials are given in general by

$$d_3(T_0^k * T_1^l) = A_{k,l} + B_{k,l},$$

where

$$A_{k,l} = \begin{cases} v^4 x_0 T_0^{k-2} T_1^{l-4}, & k \geq 2,\, l \geq k \\ 0 & \text{else} \end{cases}$$

$$B_{k,l} = \begin{cases} v^4 x_0 T_0^{k-4} T_1^{l-2} & k \geq 4, l \geq 2 \\ 0 & \text{else} \end{cases}$$

We will need the following result for the final argument deducing the Adams differentials:

Lemma 4.28. *In the Adams spectral secuence for computing $ku_*(B\mathbb{Z}/4 \wedge \mathbb{Z}/4)$, the non- zero class in degree $(2n = 1, 0)$*

$$x_0 T_1^n + T_0 x_1 T_1^{n-1} + \ldots T_0^n x_0$$

is a permanent cycle.

Proof. The class is in the image for the induction map $ku_*(\mathbb{Z}/4) \to ku_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$, for the diagonal group homomorphism $1 \mapsto (1, 1)$. by 4.7, and the description of the Atiyah-Hirzebruch spectral sequence in 4.29, it is a permanent cycle. □

Corollary 4.29. *All differentials of degree 4 and higher in the Adams spectral sequence are zero.*

Proof. Consider the class in degree $(2n + 1, 0)$

$$x_0 T_1^n + T_0 x_1 T_1^{n-1} + \ldots T_0^n x_0.$$

Since the class is in the image of the induction map, it is a permanent cycle. It follows that the d_3 differentials such as $d_3 : E_3^{11,1} \to E_3^{10,4}$ are non trivial.

Since the classes are permanent cycles, there cannot be a differential from odd $t - s$ degree to even $t - s$ degree.

On the other hand, a non zero differential from $d_i : a \mapsto b$ even $(t - s)$ degree to odd $t - s$ degree, since otherwise, it is possible to multiply by a power of v, such that $v^N a = 0$, and $v^N b \neq 0$. Hence, the differentials d_4, d_5, … are all zero.

□

We finish this section by depicting the E_∞ term of the Adams spectral sequence converging to $\widetilde{ku}_*(B\mathbb{Z}/4)$ in figure 4.1.

5. Real connective K-Theory computations on the smash summand

The aim of this section is the detemination of the differentials of the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{A}_1}^{*,*}(H^*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4), \mathbb{F}_2) \Rightarrow ko_*(B\mathbb{Z}/4).$$

We will use the η-c-R exact sequence 5.1 and the compatibility of the adams spectral sequence to translate the information about the differentials for $ku_*(B\mathbb{Z}/4)$ into informations for these differentials. These groups fit in the exact sequence

$$.. \xrightarrow{h_1} \mathrm{Ext}_{\mathcal{A}_1}^{s,t}(H^*(X), \mathbb{F}_2) \xrightarrow{c} \mathrm{Ext}_{E(1)}^{s,t}(H^*(X), \mathbb{F}_2) \xrightarrow{R} \mathrm{Ext}_{\mathcal{A}_1}^{s,t-2}(H^*(X), \mathbb{F}_2) \xrightarrow{h_1} \ldots.$$

s																											
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4
11	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5
10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5	0	5
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5	0	5	0	5
7	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5	0	5	0	5	0	5
6	0	0	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5	0	5	0	5	0	5	0	5
5	0	0	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5	0	5	0	5	0	5	0	5	0	5
4	0	0	0	0	0	0	1	0	3	0	4	0	5	0	5	0	5	0	5	0	5	0	5	0	5	0	5
3	0	0	0	0	1	1	3	2	4	3	5	4	5	5	5	6	5	7	5	8	5	9	5	10	5	11	5
2	0	0	1	1	3	2	4	3	5	4	5	5	5	6	5	7	5	8	5	9	5	10	5	11	5	12	5
1	1	3	3	5	4	7	5	9	5	11	5	13	5	15	5	17	5	19	5	21	5	23	5	25	5	27	5
0	1	2	1	3	1	4	1	5	1	6	1	7	1	8	1	9	1	10	1	11	1	12	1	13	1	14	1
	t-s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26

FIGURE 4.1. Dimensions of the E_∞ term of the Adams spectral sequence for $ku_*(B\mathbb{Z}/4)$.

The procedure will determine the differentials in two steps, namely

(i) Using Sage, the differentials for the Adams spectral sequence converging to ko, $d_r : E_r \to E_r$ will be determined from the η-c-r exact sequence using the differentials for ku, and the behaviour of the maps in generators.

Given the diagram 5.1 with the dimensions of the $\mathbb{F}_2$-vector space corresponding to the E_2 term, and the information coming from the differential d_2 above, the program will compute the homology groups, thus giving the E_3 term 5.2, as well as a d_3 differential defined therein. We will compute the homology thus ending with the $E_4 = E_\infty$ term in figure 5.3.

This first step will concern the Adams degrees. $(s, t-s)$ for $0 \leq s \leq 17$, and $0 \leq t-s \leq 27$.

(ii) Using the cap structure for the Adams spectral sequence, the differentials for $t-s \geq 27$ are defined.

(iii) We state the orders of the ko-homology groups in Theorems 5.8, 5.9. This is the main input for the proof of the Gromov-Lawson-Rosenberg conjecture in section 6.

(iv) By analyzing the kernel and cokernel of multiplication with h_1 we will find hidden η extensions.

We recall that according to Lemma 4.1, the structure of $H^*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ as a module over the Steenrod algebra can be described as sums of the modules M_p, the point, M_B, bow and M_{SB}, as follows.

- In degree $4k$, $\Sigma^{4k} M_B \oplus \bigoplus_{i=1}^{k-1} \Sigma^{4k} M_{SB}$, generated by $x_0 x_1 T_1^{2k-1}$, in the case of M_B and the k elements $T_0^{2l+1} T_1^{2(k-l)-1}$ for $l = 0, 1, \dots k-1$.
- In degree $4k+1$, $\bigoplus_{i=1}^{n} \Sigma^{4k+1} M_B$, generated by the k elements $T_0^{2l+1} x_1 T_1^{2(k-l-1)}$ for $l = 0, \dots k-1$.
- In degree $4k+2$, $\bigoplus_{i=1}^{n} \Sigma^{4k+2} M_{SB}$
- In degree $4k+3$, $\bigoplus_{i=1}^{n} \Sigma^{4k+3} MB$.

We add to this information the result of the computations in 3.4 which give us the structure of the E_2 terms. Together with the complete information about differentials d_2 and d_3 of the Adams spectral sequence for $ku_*(B\mathbb{Z}/4)$ from section

4, by knowing the behaviour of the maps η, c, R, we are able to determine the differentials of the Adams spectral sequence for $ko_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$.

Lemma 5.1. *For the sequence maps of spectra*

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} C(\eta) \xrightarrow{R} \Sigma^2 ko,$$

the following holds:

- *The cone $C\eta$ is weakly equivalent to ku.*
- *The third map is equivalent to the realification map $R : ku \to \Sigma^2 ko$. Hence, the cofibre sequence is equivalent to*

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{R} \Sigma^2 ko,$$

- *The naturality of the Adams spectral sequence implies the existence of a long exact sequence*

$$.. \xrightarrow{h_1} \mathrm{Ext}^{s,t}_{\mathcal{A}_1}(H^*(X), \mathbb{F}_2) \xrightarrow{c} \mathrm{Ext}^{s,t}_{E(1)}(H^*(X), \mathbb{F}_2) \xrightarrow{R} \mathrm{Ext}^{s,t-2}_{\mathcal{A}_1}(H^*(X), \mathbb{F}_2) \xrightarrow{h_1} \dots .$$

Lemma 5.2. *In the $\eta - c - R$ exact sequence the maps are given in coefficients by*

- *For $\eta : \pi_*(ko) \to ko$, $\eta(\eta) = \eta^2$, $\eta(\eta^2) = 0$, $\eta(\omega) = 0$, $\eta(\beta) = 0$*
- *For $c : \pi_*(ko) \to \pi_*(ku)$, $c(\eta) = 0$, $c(\omega) = 2\nu^2$, $c(\mu) = \nu^4$.*
- *For R, $R(\nu) = 2$, $R(\nu^2) = \eta^2$, $R(\nu^2) = \omega$*

Consider the map $\varphi : M_B \oplus \Sigma^2 M_B \to M_B$ which sends the generators a and b in degrees 0 and 2 of $M_B \oplus \Sigma^2 M_B \to M_B$ to c and $Sq^2(c)$, respectively in M_B.

Recall that the Ext- groups $\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_B, \mathbb{F}_2) = \mathbb{F}_2[h_0]\langle x_0, x_1, \dots\rangle$ has generators x_i of degree $(2i, i)$, and
$\mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_B \oplus \Sigma^2 M_B, \mathbb{F}_2) = \mathbb{F}_2[h_0]\langle y_0, z_0, y_1, z_1, \rangle$ has generators y_i in degree $(2i, i)$ and $(2i+2, i)$, respectively.

The Yoneda Product with $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$ is given by $\nu y_i = y_{i+1}$ and $\nu z_i = z_{i+1}$.

The following result analyzes the behaviour of the Ext-groups for the modules M_B and M_{SB} under the $\eta - c - R$ exact sequence.

Lemma 5.3. *The homomorphism $\varphi^* : \mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_B, \mathbf{F}_2) \to \mathrm{Ext}^{*,*}_{\mathcal{A}_1}(M_B \oplus \Sigma^2 M_B, \mathbb{F}_2)$ sends the generators as follows:*

$$x_{2i} \mapsto y_{2i} \ \text{ and } \ x_{2i+1} \mapsto y_{2i+1} + h_0 z_{2i}.$$

As a consequence, the morphisms c, η, and R satisfy:

- *The complexification $c : \mathrm{Ext}^{s,t}_{\mathcal{A}_1}(\tilde{H}^*(B\mathbb{Z}/4), \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{E(1)}(\tilde{H}^*(B\mathbb{Z}/4), \mathbb{F}_2)$ is given by*

$$\begin{aligned} x &\mapsto x \\ \omega x &\mapsto h_0 \nu^2 x \\ \mu x &\mapsto \nu^4 x \\ \nu^{2l} z^{2k+1} &\mapsto \nu^{2l} z^{2k+1} \\ \nu^{2l} x z^{2k+1} &\mapsto \nu^{2l} x z^{2k+1} \\ \nu^{2l+1} x^{2k+1} &\mapsto \nu^{2l}(\nu x z^{2k+1} + h_0 x z^{2k+2}). \end{aligned}$$

- *The realification $R : \mathrm{Ext}^{s,t}_{E(1)}(\tilde{H}^*(B\mathbb{Z}/4, \Sigma^2\mathbb{F}_2)) \to \mathrm{Ext}^{s,t}(\tilde{H}^*(B\mathbb{Z}/4), \Sigma^2\mathbb{F}_2)$ is given for $k > 0$ by*

$$\begin{aligned} z^{2k+1} &\mapsto 0 \\ xz^{2k-1} &\mapsto 0 \\ z^{2k} &\mapsto z^{2k-1} \\ xz^{2k} &\mapsto xz^{2k-1} \\ \nu z^{2k-1} &\mapsto h_0 z^{2k-1} \end{aligned}$$

16																												
15																												1
14																										1	1	3
13																								1	1	3	2	4
12																						1	1	3	2	4	3	5
11																				1	1	3	2	4	3	5	4	6
10																		1	1	3	2	4	3	5	4	6	5	7
9																1	1	3	2	4	3	5	4	6	5	7	6	8
8														1	1	3	2	4	3	5	4	6	5	7	6	8	7	9
7												1	1	3	2	4	3	5	4	6	5	7	6	8	7	9	8	10
6										1	1	3	2	4	3	5	4	6	5	7	6	8	7	9	8	10	9	11
5								1	1	3	2	4	3	5	4	6	5	7	6	8	7	9	8	10	9	11	10	12
4						1	1	3	2	4	3	5	4	6	5	7	6	8	7	9	8	10	9	11	10	12	11	13
3			1	1	3	3	5	4	7	5	9	6	11	7	13	8	15	9	17	10	19	11	21	12	23	13	25	14
2			1	1	2	1	3	1	4	1	5	1	6	1	7	1	8	1	9	1	10	1	11	1	12	1	13	1
s	$t-s$		2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27

FIGURE 5.1. The dimensions of the E_2 term.

$$\nu x z^{2k-1} \mapsto h_0 x^{2k--1}$$
$$\nu z^{2k} \mapsto \nu z^{2k-1}$$
$$\nu x z^{2k} \mapsto \nu x z^{2k-1},$$

$$x \mapsto 0$$
$$\nu x \mapsto h_0 x$$
$$\nu^2 x \mapsto \eta^2 x$$
$$\nu^3 x \mapsto \omega x$$
$$\nu^5 x \mapsto h_0 \mu x.$$

We depict now in figure 5.1 the dimensions of the $\mathbb{F}_2$-vector space $\mathrm{Ext}^{t-s,s}\, H^*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ for $0 \leq t-s \leq 27$. We exclude h_1 multiples of the analysis at this stage.

We now add the d_2 and d_3 differentials and obtain the E_3 term. The result is shown in figure 5.2.

We have the followng result, needed for the determination of d_3-differentials.

Lemma 5.4. *The family of differentials from the $s = 1$ to the $s = 4$ line*

$$d_3 : E_3^{16,1} \to E_3^{15,4}, d_3 : E_3^{18,1} \to E_2^{17,4}$$

are surjective.

Proof. By direct computation, the rank of the differentials $d_3$starting in $t-s$-degree 12 and 14 are 2 and one, respectively. Let v_1 and v_2 be vectors in $E_3^{12,1}$ such that $d_3 \mid \langle v_1, v_2 \rangle$ has rank 2. Then $d_2 \mid_{\rangle, T_0^2 v_1, T_1^2 v_2, T_1^2 v_1, T_1^2 v_2 \langle}$ has rank three due to the compatibility of d_3 with the cap product, and the fact that the 3-dimensional vector space $E_3^{15,4}$ is generated by T_0^2 and T_1^2- multiples of elements in $E_3^{11,4}$. Inductively, we cap with higher powers of T_0^2 and T_1^2 to obtain that the bold and dashed differentials in figure 5.4 are all surjective. ☐

Corollary 5.5. *In the E_3 term of the Adams Spectral sequence for computing $ko_*(B\mathbb{Z}/4 \wedge B\mathbb{Z}74)$, the following holds:*

- *The d_3 differentials obtained by v- periodicity defined the $s = 2$ to the $s = 5$ line are surjective.*
- *The d_3 differentials from $t-s$ degree 2 to 5 hit all elements in degree $s > 3$ except multiples of the class $h_3^k x_0 x_1$.*

Proof.
- The argument is analogous to 5.4.

s \ t−s	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
13																									1	1
12																								1	1	2
11																					1	1		2	2	3
10																				1		2	2	2	2	3
9																	1	1		2	2	3		2	3	4
8																1	1	2	2	2	1	3	3	3	2	4
7													1	1		2	2	3	1	2	3	4		3	4	5
6												1		2	2	2	1	3	3	3	2	4	4	4	3	5
5									1	1		2	2	3	1	2	3	4	2	3	4	5	3	4	5	4
4								1	1	2	2	2	1	3	3	3	2	4	4	4	3	5	5	5	4	2
3					1	1		2	2	3	1	2	3	4	2	3	4	5	3	4	5	6	4	5	6	7
2				1		2	2	2	1	3	3	3	2	2	4	4	3	5	5	5	4	6	6	6	5	7
1	1	1	2	2	3	3	4	2	5	4	6	1	7	4	8	1	9	4	10	1	11	4	12	1	13	
0	1	1	2		1	1	3		2	1	4		3	1	5		4	1	6		5	1	7		6	1

FIGURE 5.2. The dimensions for the E_3 term of the Adams spectral sequence, omitting η-multiples.

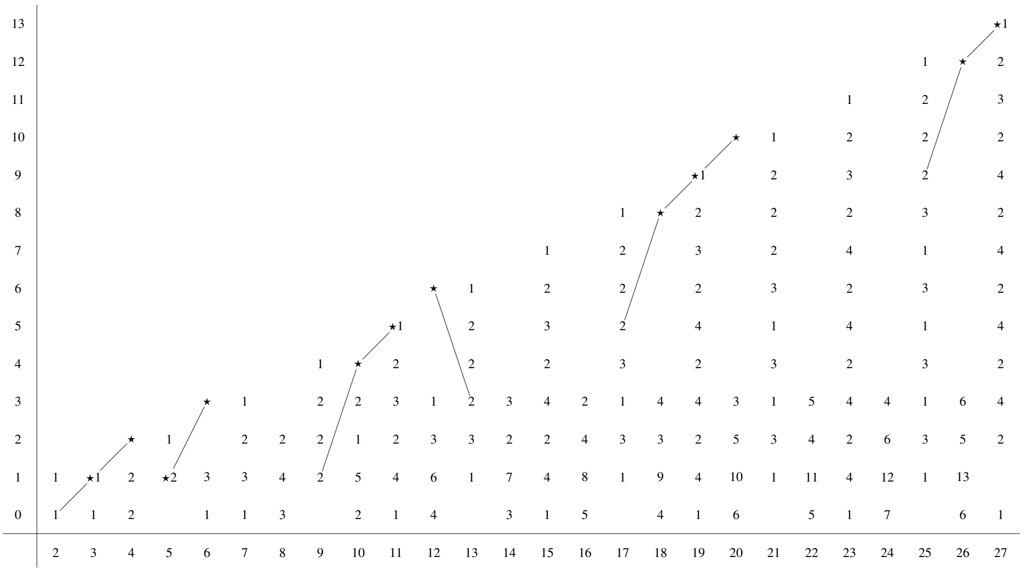


FIGURE 5.3. $E_4 = E_\infty$ term, now including η-multiples.

- By the compatibility of the complexification map with multiplication with v^2, and the description of 5.1, the differentials hit all elements in s degree > 3 and odd $t - s$ degree except the multiples of the class.

□

And finally the E_∞ term, now depicting the η-mutiples. We depict this information in figure 5.3.

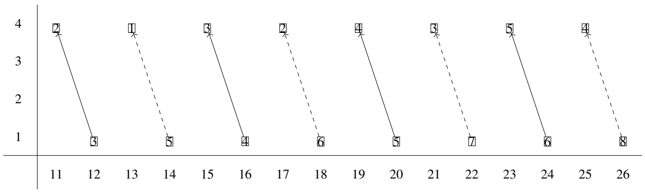


FIGURE 5.4. d_3 differentials for lemma 5.6.

Let us present here the conclusions

Lemma 5.6. *The dimensions of the E_3 term of the Adams spectral sequence (including η-multiples) converging to $ko_*(B\mathbb{Z}/4)$ is as in 5.4*

Theorem 5.7. *For $t-s \geq 7$ of $ko_{t-s}(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$, there is only one family of η-extensions: starting in $t-s$ degree $8k+1$ to $\beta^k x_0 x_1$ for $k = 1, 2, \ldots$.*

There exists a hidden η-extension from $(t-s, s)$ degree $(5,1)$ to $\alpha x_0 x_1$.

Theorem 5.8. *[Rank of lower ko-theory.] The group orders of $\tilde{ko}_n(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ are given by*

n	$\log_2(\mid \tilde{ko}_n(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4) \mid)$
2	2
3	3
4	5
5	3
6	4
7	7

Theorem 5.9. *For $n \geq 8$, the group orders of $\tilde{ko}_n(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$ are given by*

n	$\log_2(\mid \tilde{ko}_n(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4) \mid)$
$8d$	$10d-1$
$8d+1$	$8d-1$
$8d+2$	$10d+1$
$8d+3$	$12d+2$
$8d+4$	$10d+4$
$8d+5$	$8d+2$
$8d+6$	$10d+5$
$8d+7$	$12d+7$

5.1. **Hidden extensions.**

Theorem 5.10. *On the E_∞-term of the Adams spectral sequence for $ko_*(B\mathbb{Z}/4^{\wedge 2})$ there exist the following hidden extensions:*

- *There exists a hidden η-extension from $(t-s, s)$-degree $(5,1)$ to $\alpha x_0 x_1$.*
- *In odd degrees, there are no non-zero elements of Adams filtration ≥ 4 excepting β^k for $k \in \mathbb{N}$*
- *There exists a hidden η-extension from $E_\infty^{9,1}$ to $\beta x_0 x_1$.*

Proof. Consider the sum of dimensions of 5.2 and 5.1.

- Let us consider the η-c-R exact sequence in degrees 5 and 6. See tables 3 and 2 The ranks of the abelian groups are as follows: Because c does not decrease Adams filtration of the leading term, the η-multiple must be the

s	cokerh_1	ku_5	$(\ker h_1)_3$
2	1	1	
1	2	3	1
0		1	1

TABLE 2. Degree 5

s	cokerh_1	ku_6	$(\ker h_1)_3$
3	1		
2		1	1
1	3	5	2
0	1	3	2

TABLE 3. Degree 6

unique non-zero element in Adams filtration 3, which is $\alpha x_0 x_1$. It follows that there exists an element in $ku_5(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4)$, which is zero in the zeroth Adams filtration, non zero in the first Adams filtration and such that its η- multiple is represented by ax_0x_1, where a is as defined in 2.10. The hidden η extension cannot come from $(t-s,s)$ degree $(5,2)$, since there is no hidden extension from $(5,2)$ to $(6,1)$, In the same way, $\beta x_0 x_1$ is an η- multiple, and $\eta\beta x_0 x_1 \neq 0$, hence $\eta^2 \beta x_0 x_1$ must be the image of a differential supported in degree $t-s=13$. Due to the formula given above, it cannot be supported in s- degree 4. Moreover, the elements located in $(t-s,s)$-degrees $(17,2)$ and $(17,1)$ cannot hit the element. it follows that there exists a non-zero

$$d_3 : E_3^{13,3} \to E_3^{12,6}.$$

By considering the β- multiples of these differentials and th e hidden extensions, the comparison of number of generators fit, and hence there are no further differentials or hidden extensions.

□

6. Detection Theorems

The Atiyah-Patodi-Singer index Theorem [3] provides a formula for the index of the Dirac operator of a spin manifold W with boundary M, analogous to the usual index formula. Roughly speaking, after imposing the Atiyah-Patodi Singer boundary conditions, there exists a spectral function depending on the diffeomorphism type of M

$$\eta(s, M) = \sum_{\lambda} \text{sign}\,(\lambda)(\dim_{E_\lambda}) \mid \lambda \mid^{-s},$$

where the sum is over the point spectrum of the Dirac operator, E_λ denotes the finite dimensonal eigenvalue corresponding to λ, the sum converges for sufficiently large s, and has a meromorphic extension to the entire complex s-plane such that $\eta(0, M)$ is finite.

The equality

$$\eta(0, M) = \hat{A}(W) - \text{Index}(D)$$

holds, and it is the prototype of several index theorem for manifolds with boundary.

The η invariant can be defined for any self adjoint elliptic partial differential operator of order d acting on smooth sections of a spooth vector bundle V on a closed smooth manifold M. For a spectral resolution $\{\phi_n, \lambda_n\}$ of P, the associated η function is defined as

$$\eta(s, P) := \sum_{n\in\mathbb{N}} \text{sign}(\lambda_n) \mid \lambda_n \mid^{-s} + \text{dim ker}(P).$$

The η-invariant is defined as $\eta(0, P)$, and it will be denoted by $\eta(M)$ to strenghten the dependence on M (more precisely, its diffeomorphism type).

For a finite dimensional representation of a finite group π, denoted as ρ, and a map from a closed, smooth and spin manifold $f : M \to B\pi$, we can form a vector bundle over M, which will be denoted by $V_\rho : \tilde{M} \times_\pi \rho \to M$. Here $\tilde{M}$ is the cover of M classified by π. We define the Dirac operator acting on sections of V_ρ, $D(M, f, \rho)$, and we will consider its η-invariant

$$\eta(M, f)(\rho).$$

Recall that a Bott Manifold B^8 is a closed spin smooth manifold of dimension 8 and $\hat{A}$-genus equal to 1.

Up to spin bordism, a Bott Manifold L satisfies that the 4–times iterated connected sum $4L$ is bordant to the product of the Kummer surface $K \times K$, where K is the quadric

$$\{[x_0 : x_1 : x_2 : x_3] \in \mathbb{C}P^3 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0\}.$$

As a consequence of the determination of the alpha invariant [20], [36], there exists a natural transformation of homology theories consisting of natural isomorphisms

$$\Omega_n^{\text{Spin}}(X)/T_*(X)[B^{-1}] \longrightarrow KO_*(X).$$

In the previous expresion, $T_*(X)$ denotes the subgroup generated by the set of Spin manifolds which are Spin-bordant to $\mathbb{H}P^2$-bundles, and B denotes a Bott manifold. With this definition, the following theorem was proved as Theorem 4.1 in page 388 of [6].

Theorem 6.1. *Let ρ be a virtual representation of virtual dimension zero of the finite group π. Then the homomorphisms*

$$\Omega_n^{\text{spin}}(B\pi) \to \mathbb{R}/\mathbb{Z},\ KO_n(B\pi) \to \mathbb{R}/\mathbb{Z},$$

which send a spin bordism class $f : M \to B\pi$ to $\eta(M, f)(\rho)$ are well defined. Moreover, if ρ is of real type and n is congruent to 3 modulo 4, or if ρ is of quaternionic type and n is congruent to 7 modulo 8, the range can be replaced by $\mathbb{R}/2\mathbb{Z}$.

For a number of finite groups and several manifolds with them as fundamental group (notably the spherical space forms), see [16], the values for the η invariant are known.

We will need the following recorded values for lens spaces and bundles of lens spaces with fundamental group $\mathbb{Z}/4$, as well as some projective spaces.

Let $\mathbb{Z}/4$ be the subgroup of the multiplicative group of the non- zero complex numbers $\mathbb{C}^*$ defined as

$$\mathbb{Z}/4 = \{\lambda \in \mathbb{C} \mid \lambda^4 = 1\}.$$

For $i \in \{0, 1, 2, 3\}$, Let ρ_i be the irreducible, one dimensional complex representation. Notice that ρ_2 is of real type, and all other representations are of complex type.

Given a vector of integers $a = (a_1, \dots, a_{2k})$, and the representation

$$\tau_a : \mathbb{Z}/4 \to U(2k)$$

defined by

$$\mathbb{Z}/4 \ni \lambda \mapsto \begin{pmatrix} \lambda^{a_1} & & \\ & \ddots & \\ & & \lambda^{a_{2k}} \end{pmatrix} \in U(2k)$$

Denote by S^{4k-1} the set of norm one vectors in the standard hermitian norm on $\mathbb{C}^{2k}$. The $\mathbb{C}$- linear unitary representation τ_a restricts to a fixed point free action on S^{4k-1} if all a_i are odd.

Definition 6.2. Define the *lens space corresponding to* a by

$$L^{4k-1}(\tau_a) := S^{4k-1}/\tau_a(\mathbb{Z}/4).$$

And the *lens space bundle corresponding to* τ_a by

$$X^{4k+1}(\tau_a) := S(H^{\otimes 2} \oplus \mathbb{C}^{2k-1})/\tau_a(\mathbb{Z}/4),$$

where $H \to \mathbb{C}P^1$ is the Hopf line bundle.

We will be interested in the specific case of the representations τ_a given a $2k$-tuple $a = (1, 1, \dots, 2j+1)$ for j varying among positive integers. We will write

$$L_j^{4k-1} := L^{4k-1}(\tau_{1,1,\dots,2j+1}), \quad X_j^{4k+1} := X^{4k+1}(\tau_{1,1,\dots,2j+1}).$$

Lemma 6.3. *The following properties hold for the lens spaces, and are proved in* [13], [7] [4]*:*

- *Both* $L^{4k-1}(\tau_a)$ *and* $X^{4k+1}(\tau_a)$ *are spin manifolds of dimension* $4k-1$*, respectively* $4k+1$ *whenever* τ_a *is a fixed point free representation. This is the case if all* a_i *are odd.*
- *A spin structure can be chosen by picking a square root of the determinant representation* δ*. We will fix this choice as* $\frac{a_1+\dots+a_{4k}}{2}$ *. The* η*-invariant of* $L^{4k-1}(\tau_a)$ *for the standard (round) metric and spin structure satisfies*

$$\eta(L^{4k-1})(a)(\rho-\rho_0) = \frac{1}{4}\sum_{g=1}^{3} \frac{\mathrm{Tr}(\rho(g))(\delta(g)(\tau_a(g)))^{1/2}}{\det(\tau_a(g)-\mathrm{id})}.$$

This specializes to

$$\frac{1}{4}\sum_{g=1}^{3} \frac{\zeta_4^{g(d+j)}(\zeta_4^{gu}-1)}{(1-\zeta_4^g)^{2d-1}(1-\zeta_4^{g(1+2j)})}.$$

More specifically, for the irreducible representations ρ_u*:*

$$\eta(L_{1+2j}^{4d-1})(\rho_u-\rho_0) = \begin{cases} -1^{d+1}\cdot(\frac{1}{2^{d+1}}+\frac{1}{2^{2d+1}}) & j=0,\ u=1,3\,. \\ \frac{-1^{d+1}}{2^d} & u=2. \\ (-1^{d+1})\cdot(\frac{1}{2^{d+1}}-\frac{1}{2^{2d+1}}). & j=1,\ u=1,3. \end{cases}$$

Similarly, the η*-invariant of* X_j^{4k+1} *satisfies:*

$$\eta(X_j^{4k+1})(\rho_u) = \frac{1}{4}\sum_{g=1}^{3} \frac{\zeta_4^{g(k+j)}(1+\zeta_4^g)\zeta_4^{gu}}{(1-\zeta_4^g)^{2k}(1-\zeta_4^{g(1+2j)})}.$$

Which specializes to

$$\eta(X_{2j+1}^{4d+1})(\rho_u) = \begin{cases} \frac{-1^{d+1}}{2^{d+1}} & u=1. \\ 0 & u=2. \\ \frac{-1^d}{2^{d+1}} & u=3. \end{cases}$$

- *For the spin real projective spaces the η- invariants are as follows:*

$$\eta(\mathbb{R}P^{8d+3})(\rho_1 - \rho_0) = \frac{-1}{2^{4d+2}},$$

$$\eta(\mathbb{R}P^{8d+7})(\rho_1 - \rho_0) = \frac{-1}{2^{2d}} \in \begin{cases} \mathbb{R}/2\mathbb{Z} & 4d-1 \equiv 3 \ \ mod\ 8. \\ \mathbb{R}/\mathbb{Z} & 4d-1 \equiv 7 \ \ mod\ 8. \end{cases}$$

  Notice that this is defined for the only nontrivial virtual representation of $\mathbb{Z}/2$ of virtual dimension zero.
- *All of the three families of manifolds X^{4d+1}_{2j+1}, $\mathbb{R}P^d$, and L^{4d-1}_{2j+1} admit a metric of positive scalar curvature.*

Recall that the η invariant can also be defined for Spin(c), Pin, and even Pin$^+$, Pin$^-$, non-orientable manifolds. The following result can be found in [5] as Theorem 1.5 in page 224, and as Theorem 1.9.3 in page 106 of [16].

Lemma 6.4. *Let d be an integer greater or equal to zero.*

- *For the* Spin(c) *projective space $\mathbb{R}P^{4d-1}$, and the nontrivial irreducible representation of $\mathbb{Z}/2)$ ρ_1 the η-invariant satisfies*

$$\eta(\mathbb{R}P^{4d-1})(\rho_1) = \frac{1}{2^{3d-2}}.$$

- *For either one of the $\mathbb{Z}/4$- manifolds $\tilde{L}(1)$ (with free $\mathbb{Z}/4$- action, and S^1 (with trivial $\mathbb{Z}/4$-action, the manifolds $\mathbb{R}P^{4d-1} \times S^1$, and $\mathbb{R}P^{4d-1} \times \tilde{L(1)}$, the product manifolds are* Spin, *and their η-invariant with respect to the product metric and the spin structure satisfies*

$$\eta(\rho_u - \rho_0) = \begin{cases} \frac{1}{2^{3d-2}} & u = 0,\, 2. \\ 0 & else. \end{cases}$$

The following manifold will be relevant for the detection theorem 6.23.

Definition 6.5. Let i be a natural number,and let j be an even natural number.

Consider the manifold $N_{4i+1,j}$, defined as follows.

Take the tautological bundle L_i over $\mathbb{R}P^{4i+1}$, and form the vector bundle of even real rank $2_{L_i} \oplus \epsilon^{\oplus j} := 2L_i \bigoplus (\bigoplus_{i=1}^{j} \epsilon)$ over $\mathbb{R}P^{4i+1}$, where L_i is as before, and ϵ is the trivial line bundle.

Let the group $\mathbb{Z}/4$ act on the fiber of the vector bundle by the diagonal rotation of angle $\frac{\pi}{2}$

$$R_{\frac{\pi}{2}} \oplus \bigoplus_{i=1}^{\frac{j}{2}} R_{\frac{\pi}{2}},$$

Where every orthogonal transformation $R_{\frac{\pi}{2}}$ is a rotation in two dimensional euclidean space.

Consider the norm 1 sphere bundle with respect to a riemannian metric

$$S(2L_i \oplus \epsilon^{\oplus j}),$$

and form the fiber bundle over $\mathbb{R}P^{4i+1}$

$$S^{(2+\frac{j}{2})-1}/\mathbb{Z}/4 \to N_{i,j} \to \mathbb{R}P^{4i+1}.$$

Lemma 6.6. *Consider the lens space bundle over $\mathbb{R}P^{4i+1}$, $N_{4i+1,4(d-i-1)+2}$ with specific parameters $4i+1$ and $j = 4(d-i-1)-2$, where d is a natural number, and $0 \leq i \leq d-1$.*

(i) The mod 2 *cohomology of* $M_{4i+1,4(d-i-1)+2}$ *is the* $\mathbb{F}_2$*-truncated polynomial algebra with generators in degree 1*

$$H^*(M_{4i+1,4(d-1-i)+2}, \mathbb{F}_2) = \mathbb{F}_2[\hat{x}, \hat{y}]/\hat{x}^{4i+1}, \hat{y}^{4(d-i)} + \hat{x}^2\hat{y}^{4(d-1-i)+2}.$$

Moreover, the following relations hold:

$$\hat{x}\hat{y}^{4d-1} = \hat{x}^3\hat{y}^{4d-3} = \ldots = \hat{x}^{4i-1}\hat{y}^{4(d-i)+1} = \hat{x}^{4i+1}\hat{y}^{4(d-1-i)+3}.$$

(ii) For each $i \in \{0, 2\ldots, d-1\}$*, the* $4d$*-dimensional smooth manifold*

$$M_{4i+1,4(d-i-1)-2}$$

is spin.

Proof. • This follows from the Leray-Hirsch Theorem, as noticed in [22], section 6 in page 158.

Let $x \in H^1(S^{4(d-i)-1}/\mathbb{Z}/4)$ be the generator for the truncated polynomial algebra $\mathbb{F}_2[x]/x^{4(d-i)}$. Similarly, denote by $y \in H^1(\mathbb{R}P^{4i+1})$ the generator of the truncated polynomial algebra $\mathbb{F}_2[y]/y^{4i+2}$.

The statement about the relation follows for $\hat{x} = s(x)$, where s denotes a section of the map induced by the inclusion of a fiber, and $\hat{x} = p^*(x)$, for p the bundle projection.

• The following argument is due to M. Joachim and A. Malhotra[23].

Let Π be a vector bundle of rank n over $\mathbb{R}P^{4i+1}$ with a fibrewise unitary action of $\mathbb{Z}/4$. Denote by $L(\Pi)$ the bundle over $\mathbb{R}P^{4i+1}$ which has as fiber the quotient space of the unitary sphere under the $\mathbb{Z}/4$ action.

Recall [22], [12], that by the Leray-Hirsch Theorem, the bundle of lens spaces

Π has as mod2-cohomology ring

$$H^*(\mathbb{R}P\Pi) = H^*(\mathbb{R}P^{4i+1})[t]$$
$$/t^n + t^{n-1}w_1(\mathbb{R}P^{4i+1}) + \ldots t^1 w_{n-1}(\mathbb{R}P^{4i+1}) + w_n(\mathbb{R}P^{4i+1}).$$

In particular, the following equalities hold.

$$w_1(L(\Pi)) = w_1(\pi) + w_1(\mathbb{R}P^{4i+1}) + nt.$$

$$w_2(L(\Pi)) =$$
$$w_2(\Pi) + w_1(\mathbb{R}P^{4i+1})nt$$
$$+ w_1(\Pi)t + w_2(\Pi) + \frac{n(n-1)}{2}t^2 + (n-1)w_1(\Pi)t + w_2(\mathbb{R}P^{4i+1}).$$

'

For the vector bundle $2(L_{4i+1}) \oplus (4(d-1-i)-2)\epsilon$, and the mod 2 cohomology ring of $\mathbb{R}P^{4i+1}$, given as $\mathbb{Z}/2[\hat{x}]/\hat{x}^{4i+2}$, the following identities hold for the first and second Stiefel-Whitney classes :

$$w_1(2(L_{4i+1}) \oplus 4(d-1-i)+2)\epsilon) = \hat{x} + \hat{x} = 0,$$

$$w_2(2(L_{4i+1}) \oplus (4(d-1-i)-2)\epsilon) = \hat{x}^2,$$

as well as

$$w_k(2(L_{4i+1} \oplus 4(d-1-i)-2)\epsilon) = 0 \, \mathrm{k} > 2.$$

Using lemma 6.6, we conclude that

$$w_1(M_{4i+1,d-1-i-2}) = \\ w_1(\mathbb{R}P^{4i+1}) + w_1(2(L_{4i+1} \oplus 4(d-1-i)-2)\epsilon) + 4(d-1-i)\hat{y} = 0$$

as well as

$$w_2(M_{4i+1,d-1-i+2}) = \\ \hat{x}^2 + 4(d-i-1) + 2(4(d-i-1)+1)\hat{y}^2 + \hat{x}^2 = \\ 2\hat{x}^2 = 0.$$

□

Similarly to the lens space bundle consider now the action of $\mathbb{Z}/2$ on the fiber of the vector bundle

$$2L_i \bigoplus (\bigoplus_{i=1}^{j} \epsilon)$$

over $\mathbb{R}P^{4i+1}$, where L_i is as before, and ϵ is the trivial line bundle.

Definition 6.7. Let j be an even natural number, and let i be a natural number. Denote by $M_{i,j}$ the projectivized bundle of the vector bundle $2L_i \oplus \epsilon^{j\oplus}$ over $\mathbb{R}P^{4i+1}$. In symbols,

$$\mathbb{R}P^{2+\frac{j}{2}-1} \longrightarrow M_{i,j} \to \mathbb{R}P^{4i+1}.$$

Completely analogous to Lemma 6.6, the following property holds for the mod 2-homology of this manifold.

Lemma 6.8. *The smooth manifold $N_{i,j}$ is spin.*

Denote by $\hat{L}_{4k-1}(a)$ and $\hat{X}_{4k-1}(a)$ the manifolds with the trivial $\mathbb{Z}/4$-structure, meaning the homotopy class of the constant map $X \to B\mathbb{Z}/4$.

Definition 6.9. We fix the spin structure as before and consider the manifolds

- $\widetilde{L}_j^{4d-1} = L^{4d-1}(1,1,\dots,2j+1) - \hat{L}^{4d-1}(1,1,\dots,2j+1) \in \widetilde{\Omega}_{4d-1}^{\mathrm{Spin}}(B\mathbb{Z}/4)$.
- $\widetilde{X}_j^{4k+1} = X^{4k+1}(1,1,\dots,2j+1) - \hat{X}^{4k+1}(1,1,\dots,2j+1)$.

For integers j.

We denote the special case $L^1(1) = S^1/\mathbb{Z}/4$ together with the spin structure as before. And we denote by $\mathcal{M}_*(B\mathbb{Z}/4)$ the $\widetilde{\Omega}_*^{\mathrm{Spin}}(B\mathbb{Z}/4)$-submodule generated by the manifolds $\widetilde{L}^{4d-1}(1,1,\dots,1)$ and
$\widetilde{X}^{4k+1}(1,1,\dots,1)$.

Definition 6.10. We recall the generators for the kernel of the Gromov-Lawson-Rosenberg map for the cyclic group $\mathbb{Z}/4$ from [6], page 398.

- $Y^3 = \tilde{L}^3(1,1) - 3\tilde{L}^3(1,3)$.
- $Y^{8d+3} = Y^3 \times (B^8)^d$.
- $Z^3 = \tilde{L}^3(1,1)$.
- $Z^5 = \tilde{X}^5(1,1) - 3\tilde{X}^5(1,1)$
- $Z^7 = \tilde{L}^7(1,1,1,1) - 3\tilde{L}^7(1,1,1,3)$.
- $Z^9 = \tilde{L}^9(1,1,1,1) - 3\tilde{L}^9(1,1,1,3) - 3\tilde{L}^9(1,1,3,1) - 3\tilde{L}^9(1,1,3,3)$.
- $Z^n = Z^{n-8} \times B^8$ for $n > 9$.

For further reference we include here the key points of the estimate of the order of the image of D, which proves the Gromov-Lawson-Rosenberg conjecture for $\mathbb{Z}/4$.

The result was originally proved in [6] using the Atiyah-Hirzebruch spectral sequence to deduce the order of the relevant ko-groups. An additional proof using the Adams spectral sequence was given in [35].

Theorem 6.11. *For $n \geq 1$, the orders of the groups $\mathcal{M}_n(B\mathbb{Z}/4)$, generated by the manifolds of positive scalar curvature described in 6.10, and $\tilde{ko}_n(B\mathbb{Z}/4)$ relate as follows:*

	$\log_2(\mid \mathcal{M}_n(B\mathbb{Z}/4) \mid)$	$\log_2(\mid \tilde{ko}_n(B\mathbb{Z}/4) \mid)$
$n = 8d$	0	0
$n = 8d + 1$	$(2d + 1)$	$(2d + 1) + 1$
$n = 8d + 2$	0	1
$n = 8d + 3$	$2d + 2$	$2d + 2$
$n = 8d + 4$	0	0
$n = 8d + 5$	$2d + 1$	$2d + 1$
$n = 8d + 6$	0	0
$n = 8d + 7$	$2d + 2$	$2d + 2$

The following corollary was originally proved in [6], as consequence of Theorem 2.4 page 379. We give here the result as a consequence of the determination of the Adams differentials in 3.16. Previous alternative arguments have been given as part of Theorem 5.1 in [6].

Corollary 6.12. *The Gromov-Lawson-Rosenberg Conjecture holds for the group $\mathbb{Z}/4$*

Finally, we will need induction for the estimates of the orders of odd dimensional ko-homology groups.

The orthogonality relations for characters have as consequence the following result, proved as lemma 3.2.8 in page 297 of [16].

Lemma 6.13. *Let H be a subgroup of $\mathbb{Z}/4 \times \mathbb{Z}/4$.*

For a spin manifold M together with a map $M \to BH$ For the inclusion $i : H \to \mathbb{Z}/4 \times \mathbb{Z}/4$, the formula

$$\eta(\Omega_*^{\mathrm{Spin}}(i_*)(M))\rho_{u,\tilde{u}} = \eta(M)(\rho_{u,\tilde{u}} \mid_H^{\mathbb{Z}/4\times\mathbb{Z}/4})$$

holds.

We will consider the induction maps

$$\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/4) \xrightarrow{\varphi_{m,n}} \Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/4 \times \mathbb{Z}/4),$$

for group homomorphisms $\varphi : \mathbb{Z}/4 \to \mathbb{Z}/4 \times \mathbb{Z}/4$, which will be described below.

The group $\mathbb{Z}/4 \times \mathbb{Z}/4$ has the presentation

$$\mathbb{Z}/4 \times \mathbb{Z}/4 = \langle a, b \mid \quad a^4, \quad b^4, \quad aba^{-1}b^{-1} \rangle.$$

Definition 6.14. For a pair of integers m, n, we will denote by $H_{m,n}$ the cyclic subgroup generated by the element $a^m b^n$.

Lemma 6.15. *The following tables depict the isomorphism type of the restrictions of a representation $\rho_{n,m}$ to a subgroup $H_{i,j}$:*

	$\rho_{0,1}$	$\rho_{0,2}$	$\rho_{0,3}$	$\rho_{1,0}$	$\rho_{1,1}$	$\rho_{1,2}$	$\rho_{1,3}$
$H_{0,1}$	ρ_1	ρ_2	ρ_3	ρ_0	ρ_1	ρ_2	ρ_3
$H_{1,1}$	ρ_1	ρ_2	ρ_3	ρ_1	ρ_2	ρ_3	ρ_0
$H_{1,3}$	ρ_3	ρ_2	ρ_1	ρ_1	ρ_0	ρ_3	ρ_2
$H_{2,1}$	ρ_1	ρ_2	ρ_3	ρ_2	ρ_3	ρ_0	ρ_2
$H_{1,0}$	ρ_0	ρ_0	ρ_0	ρ_1	ρ_1	ρ_1	ρ_1
$H_{1,2}$	ρ_2	ρ_0	ρ_2	ρ_1	ρ_3	ρ_1	ρ_3

	$\rho_{2,0}$	$\rho_{2,1}$	$\rho_{2,2}$	$\rho_{2,3}$	$\rho_{3,0}$	$\rho_{3,1}$	$\rho_{3,2}$	$\rho_{3,3}$
$H_{0,1}$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_0	ρ_1	ρ_2	ρ_3
$H_{1,1}$	ρ_2	ρ_3	ρ_0	ρ_1	ρ_3	ρ_0	ρ_1	ρ_2
$H_{1,3}$	ρ_2	ρ_1	ρ_0	ρ_3	ρ_3	ρ_2	ρ_1	ρ_0
$H_{2,1}$	ρ_0	ρ_1	ρ_2	ρ_3	ρ_2	ρ_3	ρ_0	ρ_1
$H_{1,0}$	ρ_2	ρ_2	ρ_2	ρ_2	ρ_3	ρ_3	ρ_3	ρ_3
$H_{1,2}$	ρ_2	ρ_0	ρ_2	ρ_0	ρ_3	ρ_1	ρ_3	ρ_1

Given a group homomorphism $\varphi^{m,n} : \mathbb{Z}/4 \to \mathbb{Z}/4 \times \mathbb{Z}/4$, and a smooth spin manifold M with fundamental group $\mathbb{Z}/4$, we will denote the spin bordism class of the induced manifold by $\varphi_*^{m,n}(M)$. This represents the spin bordism class of a manifold with fundamental group $\mathbb{Z}/4 \times \mathbb{Z}/4$, and hence an element in the bordism group $\Omega_*^{\mathrm{Spin}}(B\mathbb{Z}/4 \times \mathbb{Z}/4)$.

General strategy for the proof of Theorem 1.2.

We will use Theorem 3.2 to split the arguments. At the level of connective ko-homology groups, the splitting appears as follows:

$$ko_*(B\mathbb{Z}/4 \times \mathbb{Z}/4) \cong \tilde{ko}_*(B\mathbb{Z}/4) \oplus \tilde{ko}_*(B\mathbb{Z}/4) \oplus \tilde{ko}_*(B\mathbb{Z}/4^{\wedge 2}).$$

We identify the kernels of the map $A \circ \mathrm{per}$ along this additive splitting.

Definition 6.16. Denote by $\ker_4$ the kernel of $A \circ \mathrm{per}$ on each summand $\widetilde{ko}_*(\mathbb{Z}/4)$.

Similarly, denote by $\ker_{4,4}$ the kernel of $A \circ \mathrm{per}$ on the smash summand $\widetilde{ko}_*(B\mathbb{Z}/4^{\wedge 2})$.

From the proof of the Gromov-Lawson-Rosenberg conjecture for $\mathbb{Z}/4$ we have that the order of $\ker_4$ agrees with the image of D, detected by η-invariants as described in the table inside the statement of 3.17.

Thus, we will concentrate in proving the following two statements to finish the proof of Theorem 1.2:

- For even degree, all classes in the mod 2 homology are realized by manifolds of positive scalar curvature, which are linearly independent. It follows from the Adams spectral sequence that the connective ko-homology is generated by fundamental classes of spin manifolds of positive scalar curvature.
- For odd degree, the order of $ker_{4,4}$ is equal to the rank of a matrix constructed with η-invariants of positive scalar curvature induced from the ones in 6.9, and 6.10.

6.1. **Odd Degree.** Consider the following ordered collection of cyclic subgroups in the notation of definition 6.14.

Definition 6.17.

$$SG_{4,4} = \{H_{0,1},\, H_{1,1},\, H_{1,3}, H_{2,1}, H_{1,0}, H_{1,2}\}.$$

We will show that the images under D of the induced manifolds with positive scalar curvature $\varphi_{m,n_*}(M)$ for

$$(m, n) \in \{(0, 1), (1, 1), (1, 3), (2, 1), (1, 0), (1, 2)\}$$

and M as in definitions 6.9 and 6.10 exhaust the kernel of the map $A \circ \mathrm{per}$ in odd degree. Our method will consist of estimating the order of the image of D by producing a matrix of η invariants as in [6], [25]. We then compare against the orders of the groups predicted by the computation of $\widetilde{ko}_*(B\mathbb{Z}/4 \times \mathbb{Z}/4)$.

Definition 6.18. Let C_n denote the abelian subgroup of $ko_n(B\mathbb{Z}/4 \times \mathbb{Z}/4)$ generated by the image under D of the induced manifolds

$$(\varphi^{k,l})_*(M),$$

Where M is n-dimensional belonging to the lists given in 6.9, 6.10.

Lemma 6.19. *For odd degree, the* $\log_2$ *orders of the subgroups* $\ker_4$, $\ker_{4,4}$, *and the order of the group defined in 6.18 relate as follows:*

$*$	$\ker_{4,4}$	$\ker_{4,4}$	C_*
$8d+1$	$2d+1$	$8d-2$	$12d$
$8d+3$	$6d+4$	$12d+2$	$24d+10$
$8d+5$	$2d+2$	$8d+2$	$12d+6$
$8d+7$	$6d+6$	$12d+7$	$24d+19$

In particular, according to the splitting from Theorem 3.2,

$$\mid \ker A \circ \mathrm{per} \mid = 2 \mid \ker_4 \mid + \mid \ker_{4,4} \mid .$$

Lemma 6.19 has as consequence theorem 1.2 for odd degrees. The rest of this subsection will deal with the verification of the assertions of the table in 6.19.

For the manifolds above, we will produce a 24×4 matrix containing η- invariants and their restrictions along the subgroups 6.17. According to lemma 6.13, this computes a lower bound for the order of the subgroup generated by the induction of the manifolds of 6.9 and 6.10.

Definition 6.20. Let $\rho_{(u,\tilde{u})}$ be an irreducible representation of $\mathbb{Z}/4 \times \mathbb{Z}/4$. Given a smooth, spin manifold M, form the 6×1 matrix with coefficients in $\mathbb{R}/\mathbb{Z}$,

$C_{u,\tilde{u}}$ with rows given by the restriction to the subgroups in the ordered list $SG_{4,4}$ as depicted in definition 6.17. In symbols:

$$C_{u,\tilde{u}} = \begin{pmatrix} \eta(M)(\rho_{u,\tilde{u}} \mid_{H_{0,1}}) \\ \eta(M)(\rho_{u,\tilde{u}} \mid_{H_{1,1}}) \\ \eta(M)(\rho_{u,\tilde{u}} \mid_{H_{1,3}}) \\ \eta(M)(\rho_{u,\tilde{u}} \mid_{H_{2,1}}) \\ \eta(M)(\rho_{u,\tilde{u}} \mid_{H_{1,0}}) \\ \eta(M)(\rho_{u,\tilde{u}} \mid_{H_{1,2}}) \end{pmatrix}$$

We now form the 24×4 matrix which is obtained by arranging the matrices $C_{u,\tilde{u}}$ acording to the lexicographic order. In condensed form

$$A(M) = \begin{pmatrix} C_{0,0} & C_{1,0} & C_{2,0} & C_{3,0} \\ C_{1,0} & C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,0} & C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,0} & C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}.$$

In expanded form the matrices $C_{i,j}$ are as follows:

$$C_{10} = \begin{pmatrix} \eta(M)(\rho_0) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \end{pmatrix}$$

$$C_{3,0} = \begin{pmatrix} \eta(M)(\rho_0) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \end{pmatrix}$$

$$C_{1,0} = \begin{pmatrix} \eta(M)(\rho_0) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \end{pmatrix}$$

$$C_{1,1} = \begin{pmatrix} \eta(M)(\rho_1) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_3) \end{pmatrix}$$

$$C_{1,2} = \begin{pmatrix} \eta(M)(\rho_2) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \end{pmatrix}$$

$$C_{1,3} = \begin{pmatrix} \eta(M)(\rho_3) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_3) \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \end{pmatrix} \qquad C_{3,0} \begin{pmatrix} \eta(M)(\rho_0) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \end{pmatrix}$$

$$C_{2,1} = \begin{pmatrix} \eta(M)(\rho_1) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \end{pmatrix} \qquad C_{3,1} = \begin{pmatrix} \eta(M)(\rho_1) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_1) \end{pmatrix}$$

$$C_{2,2} = \begin{pmatrix} \eta(M)(\rho_2) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_2) \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} \eta(M)(\rho_2) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \end{pmatrix}$$

$$C_{2,3} = \begin{pmatrix} \eta(M)(\rho_3) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_0) \end{pmatrix} \qquad C_{3,3} = \begin{pmatrix} \eta(M)(\rho_3) \\ \eta(M)(\rho_2) \\ \eta(M)(\rho_0) \\ \eta(M)(\rho_1) \\ \eta(M)(\rho_3) \\ \eta(M)(\rho_1) \end{pmatrix}$$

We will need the following modification in order to estimate the order of *ko*-groups of dimension 3 and 7 modulo 8, according to Theorem 6.1.

Definition 6.21. Let $\rho_{(u,\tilde{u})}$ be an irreducible representation of $\mathbb{Z}/4 \times \mathbb{Z}/4$. Given a smooth, spin manifold M, form the 6×1 matrix with coefficients in $\mathbb{R}/2\mathbb{Z}$,

$\tilde{C}_{u,\tilde{u}}$ with rows given by the restriction to the subgroups in the ordered list $SG_{4,4}$ as depicted in definition 6.17. In symbols:

$$\tilde{C}_{u,\tilde{u}} = \begin{pmatrix} \eta(M)(\rho_{u,\tilde{u}-u_{0,0}} \,|_{H_{0,1}}) \\ \eta(M)(\rho_{u,\tilde{u}-u_{0,0}} \,|_{H_{1,1}}) \\ \eta(M)(\rho_{u,\tilde{u}-u_{0,0}} \,|_{H_{1,3}}) \\ \eta(M)(\rho_{u,\tilde{u}-u_{0,0}} \,|_{H_{2,1}}) \\ \eta(M)(\rho_{u,\tilde{u}-u_{0,0}} \,|_{H_{1,0}}) \\ \eta(M)(\rho_{u,\tilde{u}-u_{0,0}} \,|_{H_{1,2}}) \end{pmatrix}$$

We now form the 24×4 matrix with coefficients in $\mathbb{R}/2\mathbb{Z}$, which is obtained by arranging the matrices $\tilde{C}_{u,\tilde{u}}$ according to the lexicographic order. In condensed form

$$A(M) = \begin{pmatrix} \tilde{C}_{0,0} & \tilde{C}_{1,0} & \tilde{C}_{2,0} & \tilde{C}_{3,0} \\ \tilde{C}_{1,0} & \tilde{C}_{1,1} & \tilde{C}_{1,2} & \tilde{C}_{1,3} \\ \tilde{C}_{2,0} & \tilde{C}_{2,1} & \tilde{C}_{2,2} & \tilde{C}_{2,3} \\ \tilde{C}_{3,0} & \tilde{C}_{3,1} & \tilde{C}_{3,2} & C_{3,3} \end{pmatrix}.$$

We now describe the matrices for each of the dimensions in odd degree.

Dimension $8d+1$

Put $d = 2d^{'}$ for $d^{'}$ a natural number. Consider the spin manifold X_j^{4d+1}. And form the 24×4 matrix over $\mathbb{R}/\mathbb{Z}$.

$$A^{8d+1} = \begin{pmatrix} C_{0,0} & C_{1,0} & C_{2,0} & C_{3,0} \\ C_{1,0} & C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,0} & C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,0} & C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}.$$

We introduce the notation

$$x_d = \frac{-1^{d+1}}{2^{d+1}},\ y_d = \frac{-1^{d+1}}{2^{2d+1}},\ z_d = \frac{-1^{d+1}}{2^d}.$$

The 6×1 matrices are given as the η-invariants of X_j^{8d+1}, as follows:

$$C_{10} = \begin{pmatrix} 0 \\ -x_d \\ -x_d \\ 0 \\ -x_d \\ -x_d \end{pmatrix} \qquad C_{1,1} = \begin{pmatrix} -x_d \\ 0 \\ 0 \\ x_d \\ -x_d \\ x_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{1,2} = \begin{pmatrix} 0 \\ x_d \\ x_d \\ 0 \\ -x_d \\ -x_d \end{pmatrix}$$

$$C_{3,0} = \begin{pmatrix} 0 \\ x_d \\ x_d \\ 0 \\ x_d \\ x_d \end{pmatrix}$$

$$C_{1,0} = \begin{pmatrix} 0 \\ -x_d \\ -x_d \\ 0 \\ -x_d \\ -x_d \end{pmatrix} \qquad C_{1,3} = \begin{pmatrix} x_d \\ 0 \\ 0 \\ -x_d \\ -x_d \\ x_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,0} = \begin{pmatrix} 0 \\ x_d \\ x_d \\ 0 \\ x_d \\ x_d \end{pmatrix}$$

$$C_{2,1} = \begin{pmatrix} -x_d \\ x_d \\ -x_d \\ x_d \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,1} = \begin{pmatrix} -x_d \\ 0 \\ 0 \\ x_d \\ x_d \\ -x_d \end{pmatrix}$$

$$C_{2,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} 0 \\ -x_d \\ -x_d \\ 0 \\ x_d \\ x_d \end{pmatrix}$$

$$C_{2,3} = \begin{pmatrix} x_d \\ -x_d \\ x_d \\ x_d \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,3} = \begin{pmatrix} x_3 \\ 0 \\ 0 \\ -x_d \\ x_d \\ -x_d \end{pmatrix}$$

Dimension $8d+3$

We recall the manifolds $Y^{8d+3} = \tilde{L}^3(1,1) - 3\tilde{L}^3(1,3) \times B^{8^d}$, and notice that the definition for the matrix here is 6.21.

The matrices $\tilde{C}_{i,j}$ take values over the ring $\mathbb{R}/2\mathbb{Z}$ are as follows

$$C_{1,0} = \begin{pmatrix} 0 \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2z_2 \\ -2x_d - 4y_d \end{pmatrix} \qquad C_{1,1} = \begin{pmatrix} -2x_d - 4y_d \\ -2z_d \\ 0 \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ -2z_d \\ -2z_d \\ 0 \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{1,2} = \begin{pmatrix} -2z_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ 0 \\ -2x_d - 4y_d \\ -2x_d - 4y_d \end{pmatrix}$$

$$C_{3,0} = \begin{pmatrix} 0 \\ -2x_d - 4y_d \\ -2xd - 4y_d \\ -2z_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \end{pmatrix}$$

$$C_{1,0} = \begin{pmatrix} 0 \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2z_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \end{pmatrix} \qquad C_{1,3} = \begin{pmatrix} -2x_d - 4y_d \\ 0 \\ -2z_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ -2z_d \\ -2z_d \\ 0 \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{3,0} = \begin{pmatrix} 0 \\ x_d - 4y_d \\ x_d - 4y_d \\ -2z_d \\ x_d - 4y_d \\ x_d - 4y_d \end{pmatrix}$$

$$C_{2,1} = \begin{pmatrix} -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{3,1} = \begin{pmatrix} x_d - 4y_d \\ 0 \\ -2z_d \\ x_d - 4y_d \\ x_d - 4y_d \\ x_d - 4y_d \end{pmatrix}$$

$$C_{2,2} = \begin{pmatrix} -2z_d \\ 0 \\ 0 \\ -2z_d \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} -2z_d \\ x_d - 4y_d \\ x_d - 4y_d \\ 0 \\ x_d - 4y_d \\ x_d - 4y_d \end{pmatrix}$$

$$C_{2,3} = \begin{pmatrix} -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2x_d - 4y_d \\ -2z_d \\ 0 \end{pmatrix} \qquad C_{3,3} = \begin{pmatrix} x_d - 4y_d \\ -2z_d \\ 0 \\ x_d - 4y_d \\ x_d - 4y_d \\ x_d - 4y_d \end{pmatrix}$$

Dimension $8d+5$ Recall that the proposed manifolds in this dimension are given by $\tilde{X}^5_j \times B^{8^d}$. The matrices $C_{i,j}$ from definition 6.20 are as follows:

$$C_{1,0} = \begin{pmatrix} 0 \\ -4x_d \\ -4x_d \\ 0 \\ -4x_d \\ -4x_d \end{pmatrix} \qquad C_{1,1} = \begin{pmatrix} -4x_d \\ 0 \\ 0 \\ -4x_d \\ -4x_d \\ -4x_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{1,2} = \begin{pmatrix} 0 \\ -4x_d \\ -4x_d \\ 0 \\ -4x_d \\ -4x_d \end{pmatrix}$$

$$C_{3,0} = \begin{pmatrix} 0 \\ -2x_d \\ -2x_d \\ 0 \\ -2x_d \\ -2x_d \end{pmatrix}$$

$$C_{1,0} = \begin{pmatrix} 0 \\ -4x_d \\ -4x_d \\ 0 \\ -4x_d \\ -4x_d \end{pmatrix} \qquad C_{1,3} = \begin{pmatrix} -4x_d \\ 0 \\ 0 \\ -4x_d \\ -4x_d \\ -4x_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,0} \begin{pmatrix} 0 \\ -4x_d \\ -4x_d \\ 0 \\ -4x_d \\ -4x_d \end{pmatrix}$$

$$C_{2,1} = \begin{pmatrix} -4x_d \\ -4x_d \\ -4x_d \\ -4x_d \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,1} = \begin{pmatrix} -4x_d \\ 0 \\ 0 \\ -4x_d \\ -4x_d \\ -4x_d \end{pmatrix}$$

$$C_{2,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} 0 \\ -4x_d \\ -4x_d \\ 0 \\ -4x_d \\ -4x_d \end{pmatrix}$$

$$C_{2,3} = \begin{pmatrix} -4x_d \\ -4x_d \\ -4x_d \\ -4x_d \\ 0 \\ 0 \end{pmatrix} \qquad C_{3,3} = \begin{pmatrix} -4x_d \\ 0 \\ 0 \\ -4x_d \\ -4x_d \\ -4x_d \end{pmatrix}$$

Dimension $8d+7$**.** Recall that the proposed manifolds in dimension 7 moulo 8 are $Z^7 \times B^{8^d}$, Where $Z^7 = \tilde{L}_1^7 - 3\tilde{L}_3^7$. Moreover, the matrix in this dimension is $\tilde{A}$, as in definition 6.21.

$$C_{1,0} = \begin{pmatrix} 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{1,0} = \begin{pmatrix} 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{1,1} = \begin{pmatrix} -2x_d - 2y_d \\ -2z_d \\ 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ 2z_d \\ 2z_d \\ 0 \\ 2z_d \\ 2z_d \end{pmatrix}$$

$$C_{1,2} = \begin{pmatrix} -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{3,0} = \begin{pmatrix} -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{1,3} = \begin{pmatrix} -2x_d - 2y_d \\ 0 \\ 2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{2,0} = \begin{pmatrix} 0 \\ -2z_d \\ -2z_d \\ 0 \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{3,0} \begin{pmatrix} 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{2,1} = \begin{pmatrix} -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{3,1} = \begin{pmatrix} -2x_d - 2y_d \\ 0 \\ -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{2,2} = \begin{pmatrix} -2z_d \\ 0 \\ 0 \\ -2z_d \\ -2z_d \\ -2z_d \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} -2z_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

$$C_{2,3} = \begin{pmatrix} -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2z_d \\ 0 \end{pmatrix} \qquad C_{3,3} = \begin{pmatrix} -2x_d - 2y_d \\ -2z_d \\ 0 \\ -2x_d - 2y_d \\ -2x_d - 2y_d \\ -2x_d - 2y_d \end{pmatrix}$$

The matrices are diagonalized with a Sage code introduced in section 7.

6.2. **Even Degree.** We will show below that for every positive even integer k, there exists a set

$$\mathfrak{M}_k$$

of spin smooth manifolds together with maps to $B\mathbb{Z}/4\times\mathbb{Z}/4$, with the property that any mod2-homology class in degree k is induced by a fundamental class of a smooth manifold with positive scalar curvature of dimension k. We collect this remark in the following result, whose proof will take the remaning part of the current section.

Definition 6.22. Let k be an even natural number. According to its mod 8 class, define the set of manifolds.

$$\mathfrak{M}_{8d} = \big\{L_1^{8d-1} \times \mathbb{R}P^1, L_1^{8d-1} \times L_1^1, N_{4i+1,4(2d-i)-2},\, i \in \{1, \dots, d-1\}\big\},$$

$$\mathfrak{M}_{8d+2} = \big\{L_1^{4i+3} \times \mathbb{R}P^{4(2d-i+1)+3},\, i \in \{0, \dots, d-1\}\big\},$$

$$\mathfrak{M}_{8d+4} = \big\{L_1^{8d-3} \times \mathbb{R}P^1, L_1^{8d-3} \times L_1^1, N_{4i+1,4(2d-i)-2},\, i \in \{1, \dots, d-1\}\big\},$$

$$\mathfrak{M}_{8d+6} = \big\{L_1^{4i+3} \times \mathbb{R}P^{4(d-i)+3},\, i \in \{0, \dots, d-1\}\big\}.$$

We state now the main theorem of this section

Theorem 6.23. *The previously introduced sets*

$$\mathfrak{M}_k$$

of smooth, spin manifolds with positive scalar curvature determine ko- homology classes in degree k for $\mathbb{Z}/4 \times \mathbb{Z}/4$.

The dimension of the $\mathbb{F}_2$*-vector space* $\mathrm{Ext}^{0,k}_{\mathcal{A}_1}(\mathbb{F}_2, H^*(B\mathbb{Z}/4 \wedge B\mathbb{Z}/4))$ *(the* E_2*-term of the Adams spectral sequence converging to* $ko_k(B\mathbb{Z}/4 \oplus B\mathbb{Z}/4)$*) is equal to the cardinality of the set* $\mathfrak{M}_k$.

We will prove theorem 6.23 by an indirect argument.

We will show first that the manifolds $N_{i,j}$ determine linearly independent elements of the mod 2 homology $H_*(B\mathbb{Z}/4 \times \mathbb{Z}/4)$ by analyzing their image under the $\mathbb{F}_2$ vector space homomorphism induced by the group quotient $\mathrm{pr}_{2,2} : H_*(B\mathbb{Z}/4 \times \mathbb{Z}/4) \to H_*(B\mathbb{Z}/2 \times \mathbb{Z}/2)$. The rest of the manifolds in the set $\mathfrak{M}_k$ are readily seen to generate elements in the homology of $\mathbb{Z}/4 \times \mathbb{Z}/4$.

Recall that the mod 2 cohomology of the group $\mathbb{Z}/4 \times \mathbb{Z}/4$ is given in Lemma 3.9.

Moreover, according to lemma 4.1, the mod2-cohomology of the smash product is

- In degree 2 by x_0x_1, generating a copy of $\mathbb{F}_2$.
- In degree $4d + 2$ for $d \geq 1$, $x_0x_1T_0^{2l+1}T_1^{2(d-l)-1}$, for $l = 0, \dots, d-1$, generating a copy of M_{SB}.
- In degree $4d$, $x_0x_1T_1^{2d-1}$ and $T_0^{2d-1}x_0x_1$, generating a copy of M_B, and $T_0^{2l+1}T_1^{2(d-l)-1}$ for $l = 0, \dots d$, which generate a copy of M_{SB}.

We introduce the follwowing notation for elements in the mod 2-homology of $\mathbb{Z}/4 \times \mathbb{Z}/4$ and $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Notation 6.24. Consider the $\mathbb{F}_2$- vector space

$$H^*(B\mathbb{Z}/4 \times \mathbb{Z}/4).$$

Given r, s, t, u natural numbers, we denote the following elements in the mod 2 dual vector space with respect to the monomial basis

$$\xi_{T_0^rT_1^s} = \left(T_0^rT_1^s\right)^* \in H_{2r+2s}(B\mathbb{Z}/4 \times \mathbb{Z}/4),$$

$$\xi_{x_0x_1T_0^t} \in H_{2t+2}(B\mathbb{Z}/4 \times \mathbb{Z}/4),$$

$$\xi_{x_0x_1T_0^u} \in H_{2u+2}(B\mathbb{Z}/4 \times \mathbb{Z}/4).$$

Similarly for the $\mathbb{F}_2$- vector space

$$H^*(B\mathbb{Z}/2 \times \mathbb{Z}/2,) = \mathbb{F}_2[x, y],$$

we introduce the notation

$$\xi_{i,j} = \left(x^iy^j\right)^* \in H_*(B\mathbb{Z}/2 \times \mathbb{Z}/2).$$

Lemma 6.25. *Under the group homomorphism*

$$\mathrm{pr}_{2,2_*} : H_*(B\mathbb{Z}/4 \times \mathbb{Z}/4) \longrightarrow H_*(B\mathbb{Z}/2 \times \mathbb{Z}/2),$$

the fundamental class of $N_{i,j}$ *is mapped to* $M_{i,j}$

Proof. Consider the fibrations

$$B\mathbb{Z}/4^{4(d-i)-1} \longrightarrow N_{4i+1,4(d-i-1)+2} \to \mathbb{R}P^{4i+1},$$

$$B\mathbb{Z}/2^{4(d-i)-1} \longrightarrow M_{4i+1,4(d-i-1)+2} \to \mathbb{R}P^{4i+1}.$$

The Serre spectral sequence converging to the homology of the total space of these fibrations have as E_2-terms

$$E^2_{p,q} = H_p(\mathbb{R}P^{4i+1}, H_q(B\mathbb{Z}/4^{4(d-i)-1})) \Rightarrow H_*(N_{4i+1,4(d-i-1)+2}),$$

and

$$F^2_{p,q} = H_p(\mathbb{R}P^{4i+1}, H_q(B\mathbb{Z}/2^{4(d-i)-1})) \Rightarrow H_*(M_{4i+1,4(d-i-1)+2}).$$

Where we have that $E^r_{p,q} = 0 = F^r_{p,q}$ for $p > 4i+1$ or $q > 4(d-i-1)+2$. Notice than no differential with source in $E_{4i+1,4(d-i-1)+2}$ can be non-zero. The same holds for $F_{4i+1,4(d-i-1)+2}$.

The map $\mathrm{pr}_{2,2_*}$ induces a map of Serre spectral sequences

$$E^r_{p,q} \longrightarrow E^r_{p,q}.$$

Which is surjective at the E_2 term. Since there are no further non zero differentials, the map sends the fundamental class of $N_{4i+1,4(d-i-1)+2}$ to $M_{4i+1,4(d-i-1)+2}$.

□

Consider the map $\iota_{4i+1,4(d-i-1+2)} : M_{4i+1,4(d-i-1)+2} \to B\mathbb{Z}/2 \times B\mathbb{Z}/2$, determined by $(j \circ p, c)$, where $p : M_{4i+1,4(d-i-1)+2} \to \mathbb{R}P^{4i+1}$ is the projection, $j : \mathbb{R}P^{4i+1} \to \mathbb{R}P^\infty = B\mathbb{Z}/2$ is the inclusion of the $4i+1$-skeleton, and c is the map classifying fundamental group $c : \mathbb{R}P^{4i+1} \to B\mathbb{Z}/2$. The following result states the behaviour in cohomology of this map, and it follows directly from the definitions. (See 6.8 for the notation.)

Lemma 6.26. *Under the map*

$$\iota_{4i+1,4(d-i-1+2)} : M_{4i+1,4(d-1-i)+2} \longrightarrow B\mathbb{Z}/2 \times \mathbb{Z}/2,$$

The following holds:

- *The classes $x^k y^{4d-k}$ are mapped to $\hat{x}^k \hat{y}^{4d-k}$ in cohomology. For k odd, this implies*

$$\iota^*_{4i+1,4(d-i-1+2)}(x^k y^{d-k}) = \hat{x}\hat{y}^{4d-1}.$$

- *In* mod 2 *homology, the fundamental class of $M_{4i+1,4(d-1-i)+2}$ is mapped to the linearly independent elements*

$$v_i = \sum_{k \geq 1 \,\mathrm{odd}}^{4+1} \xi_{k,4d-k}$$

This has a consequence the following corollary.

Corollary 6.27. *The classes $\xi_{1,4d-1}, \ldots v_1, \ldots v_{d-1}, \xi_{4d-1,1}$ are linearly independent .*

We are now in position to finish the proof of theorem 6.23.

Due to corollary 6.27, the manifolds defined there produce linearly independent classes in the even dimensional homology groups of $B\mathbb{Z}/4 \wedge B\mathbb{Z}/4$.

According to 4.1, the rank of the $\mathbb{F}_2$-vector subspace generated by the linearly independent equals the rank of the Mod 2 homology groups. Finally, by the Adams spectral sequence, we obtain that all classes in the even graded ko-homology groups of $B\mathbb{Z}^4 \wedge B\mathbb{Z}/4$ are obtained by this construction. This finishes the proof of theorem 6.23. Together with Theorem 6.1, this finishes the proof of Theorem 1.2.

7. Sage Code

7.1. Sage code for diagonalization of η- invariants. We present the Sage code needed to diagonalize the matrices of η-invariants. Computation of the echelon form for A^{8d+1}.

```
# Define the polynomial ring over the rationals with variable x_d
R.<x_d> = QQ[]

# Define the submatrices C_{i,j} as row block matrices
C_10 = matrix(R,[[0, -x_d, -x_d, 0, -x_d, -x_d]]).transpose()
C_20 = matrix(R,[[0, 0, 0, 0, 0, 0]]).transpose()
C_30 = matrix(R,[[0, x_d, x_d, 0, x_d, x_d]]).transpose()
C_11 = matrix(R,[[-x_d, 0, 0, x_d, -x_d, x_d]]).transpose()
C_12 = matrix(R,[[0, x_d, x_d, 0, -x_d, -x_d]]).transpose()
C_13 = matrix(R,[[x_d, 0, 0, -x_d, -x_d, x_d]]).transpose()
C_21 = matrix(R,[[-x_d, x_d, -x_d, x_d, 0, 0]]).transpose()
C_22 = matrix(R,[[0, 0, 0, 0, 0, 0]]).transpose()
C_23 = matrix(R,[[x_d, -x_d, x_d, x_d, 0, 0]]).transpose()
C_31 = matrix(R,[[-x_d, 0, 0, x_d, x_d, -x_d]]).transpose()
C_32 = matrix(R,[[0, -x_d, -x_d, 0, x_d, x_d]]).transpose()
C_33 = matrix(R,[[x_d, 0, 0, -x_d, x_d, -x_d]]).transpose()

# Construct the full matrix A^{8d+1} from the row block matrices
A_8d1 = block_matrix(R,[
    [block_matrix(R,1, 3, [C_10, C_20, C_30]), block_matrix(R,1, 3, [
                                   C_11, C_21, C_31])],
    [block_matrix(R,1, 3, [C_12, C_22, C_32]), block_matrix(R,1, 3, [
                                   C_13, C_23, C_33])]
])

# Transpose the matrix to get the desired form
A_8d1 = A_8d1.transpose()

# Compute the echelon form of the matrix
echelon_form_A_8d1 = A_8d1.echelon_form()

# Display the result
echelon_form_A_8d1
```

To simplify the computation we compute the echelon form of $\tilde{A}^{8d+3} := (-1)^{d+1}A^{8d+3}$ instead of A^{8d+3} using $\tilde{x}_d := (-1)^{d+1}x_d$, $\tilde{y}_d := (-1)^{d+1}y_d$, $\tilde{z}_d := (-1)^{d+1}z_d$. We do not add the prefix `tilde_` in the code for readability.

```
# Define the polynomial ring over the rationals with variable x_d
R.<x_d> = QQ[]

# Define the substitutions
y_d = 2*x_d^2
z_d = 4*x_d^2

# Define the submatrices C_{i,j} as row block matrices with the
                                   substitutions
C_10 = matrix(R,[[0, -2*x_d-4*y_d, -2*x_d-4*y_d, -2*x_d-4*y_d, -2*z_d,
                                   -2*x_d-4*y_d]]).transpose()
C_20 = matrix(R,[[0, -2*z_d, -2*z_d, 0, -2*z_d, -2*z_d]]).transpose()
C_30 = matrix(R,[[0, -2*x_d-4*y_d, -2*x_d-4*y_d, -2*z_d, -2*x_d-4*y_d,
                                   -2*x_d-4*y_d]]).transpose()
C_11 = matrix(R,[[-2*x_d-4*y_d, -2*z_d, 0, -2*x_d-4*y_d, -2*x_d-4*y_d,
                                   -2*x_d-4*y_d]]).transpose()
C_12 = matrix(R,[[-2*z_d, -2*x_d-4*y_d, -2*x_d-4*y_d, 0, -2*x_d-4*y_d,
                                   -2*x_d-4*y_d]]).transpose()
```

```
C_13 = matrix(R,[[-2*x_d-4*y_d, 0, -2*z_d, -2*x_d-4*y_d, -2*x_d-4*y_d,
                                     -2*x_d-4*y_d]]).transpose()
C_21 = matrix(R,[[-2*x_d-4*y_d, -2*x_d-4*y_d, -2*x_d-4*y_d, -2*x_d-4*
                                    y_d, -2*z_d, -2*z_d]]).transpose()
C_22 = matrix(R,[[-2*z_d, 0, 0, -2*z_d, -2*z_d, -2*z_d]]).transpose()
C_23 = matrix(R,[[-2*x_d-4*y_d, -2*x_d-4*y_d, -2*x_d-4*y_d, -2*x_d-4*
                                    y_d, -2*z_d, 0]]).transpose()
C_31 = matrix(R,[[x_d-4*y_d, 0, -2*z_d, x_d-4*y_d, x_d-4*y_d, x_d-4*
                                    y_d]]).transpose()
C_32 = matrix(R,[[-2*z_d, x_d-4*y_d, x_d-4*y_d, 0, x_d-4*y_d, x_d-4*
                                    y_d]]).transpose()
C_33 = matrix(R,[[x_d-4*y_d, -2*z_d, 0, x_d-4*y_d, x_d-4*y_d, x_d-4*
                                    y_d]]).transpose()

# Construct the full matrix A^{8k+3} from the row block matrices
A_8k3 = block_matrix(R, [
    [block_matrix(R, 1, 3, [C_10, C_20, C_30]), block_matrix(R, 1, 3,
                                        [C_11, C_21, C_31])],
    [block_matrix(R, 1, 3, [C_12, C_22, C_32]), block_matrix(R, 1, 3,
                                        [C_13, C_23, C_33])]
])

# Transpose the matrix to get the desired form
A_8k3 = A_8k3.transpose()

# Compute the echelon form of the matrix
echelon_form_A_8k3 = A_8k3.echelon_form()

# Display the result
echelon_form_A_8k3
```

Computation of the echelon form for A^{8d+5}.

```
# Define the polynomial ring over the rationals with variable x_d
R.<x_d> = QQ[]

# Define the submatrices C_{i,j} as row block matrices
C_10 = matrix(R,[[0, -4*x_d, -4*x_d, 0, -4*x_d, -4*x_d]]).transpose()
C_20 = matrix(R,[[0, 0, 0, 0, 0, 0]]).transpose()
C_30 = matrix(R,[[0, -2*x_d, -2*x_d, 0, -2*x_d, -2*x_d]]).transpose()
C_11 = matrix(R,[[-4*x_d, 0, 0, -4*x_d, -4*x_d, -4*x_d]]).transpose()
C_12 = matrix(R,[[0, -4*x_d, -4*x_d, 0, -4*x_d, -4*x_d]]).transpose()
C_13 = matrix(R,[[-4*x_d, 0, 0, -4*x_d, -4*x_d, -4*x_d]]).transpose()
C_21 = matrix(R,[[-4*x_d, -4*x_d, -4*x_d, -4*x_d, 0, 0]]).transpose()
C_22 = matrix(R,[[0, 0, 0, 0, 0, 0]]).transpose()
C_23 = matrix(R,[[-4*x_d, -4*x_d, -4*x_d, -4*x_d, 0, 0]]).transpose()
C_31 = matrix(R,[[-4*x_d, 0, 0, -4*x_d, -4*x_d, -4*x_d]]).transpose()
C_32 = matrix(R,[[0, -4*x_d, -4*x_d, 0, -4*x_d, -4*x_d]]).transpose()
C_33 = matrix(R,[[-4*x_d, 0, 0, -4*x_d, -4*x_d, -4*x_d]]).transpose()

# Construct the full matrix A^{8d+5} from the row block matrices
A_8d5 = block_matrix(R, [
    [block_matrix(R, 1, 3, [C_10, C_20, C_30]), block_matrix(R, 1, 3,
                                        [C_11, C_21, C_31])],
    [block_matrix(R, 1, 3, [C_12, C_22, C_32]), block_matrix(R, 1, 3,
                                        [C_13, C_23, C_33])]
])

# Transpose the matrix to get the desired form
A_8d5 = A_8d5.transpose()

# Compute the echelon form of the matrix
echelon_form_A_8d5 = A_8d5.echelon_form()
```

```
# Display the result
echelon_form_A_8d5
```

Computation of the echelon form for A^{8d+7}. We use the same conventions as for A^{8d+3}.

```
# Define the polynomial ring over the rationals with variable x_d
R.<x_d> = QQ[]

# Define the substitutions y_d = 2*x_d^2 and z_d = 4*x_d^2
y_d = 2 * x_d^2
z_d = 4 * x_d^2

# Define the submatrices C_{i,j} as row block matrices
C_10 = matrix(R, [[0, -2*x_d-2*y_d, -2*x_d-2*y_d, -2*z_d, -2*x_d-2*y_d
                                    , -2*x_d-2*y_d]]).transpose()
C_20 = matrix(R, [[0, 2*z_d, 2*z_d, 0, 2*z_d, 2*z_d]]).transpose()
C_30 = matrix(R, [[-2*z_d, -2*x_d-2*y_d, -2*x_d-2*y_d, -2*z_d, -2*x_d-
                                    2*y_d, -2*x_d-2*y_d]]).transpose()
C_11 = matrix(R, [[-2*x_d-2*y_d, -2*z_d, 0, -2*x_d-2*y_d, -2*x_d-2*y_d
                                    , -2*x_d-2*y_d]]).transpose()
C_12 = matrix(R, [[-2*z_d, -2*x_d-2*y_d, -2*x_d-2*y_d, 0, -2*x_d-2*y_d
                                    , -2*x_d-2*y_d]]).transpose()
C_13 = matrix(R, [[-2*x_d-2*y_d, 0, 2*z_d, -2*x_d-2*y_d, -2*x_d-2*y_d,
                                     -2*x_d-2*y_d]]).transpose()
C_21 = matrix(R, [[-2*x_d-2*y_d, -2*x_d-2*y_d, -2*x_d-2*y_d, -2*x_d-2*
                                    y_d, -2*z_d, -2*z_d]]).transpose()
C_22 = matrix(R, [[-2*z_d, 0, 0, -2*z_d, -2*z_d, -2*z_d]]).transpose()
C_23 = matrix(R, [[-2*x_d-2*y_d, -2*x_d-2*y_d, -2*x_d-2*y_d, -2*x_d-2*
                                    y_d, -2*z_d, 0]]).transpose()
C_31 = matrix(R, [[0, -2*x_d-2*y_d, -2*x_d-2*y_d, -2*z_d, -2*x_d-2*y_d
                                    , -2*x_d-2*y_d]]).transpose()
C_32 = matrix(R, [[-2*z_d, -2*x_d-2*y_d, -2*x_d-2*y_d, 0, -2*x_d-2*y_d
                                    , -2*x_d-2*y_d]]).transpose()
C_33 = matrix(R, [[-2*x_d-2*y_d, -2*z_d, 0, -2*x_d-2*y_d, -2*x_d-2*y_d
                                    , -2*x_d-2*y_d]]).transpose()

# Construct the full matrix A^{8k+7} from the row block matrices
A_8k7 = block_matrix(R, [
    [block_matrix(R, 1, 3, [C_10, C_20, C_30]), block_matrix(R, 1, 3,
                                        [C_11, C_21, C_31])],
    [block_matrix(R, 1, 3, [C_12, C_22, C_32]), block_matrix(R, 1, 3,
                                        [C_13, C_23, C_33])]
])

# Transpose the matrix to get the desired form
A_8k7 = A_8k7.transpose()

# Compute the echelon form of the matrix
echelon_form_A_8k7 = A_8k7.echelon_form()

# Display the result
echelon_form_A_8k7
```

References

[1] J. F. Adams. *Stable homotopy and generalised homology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[2] D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson. The structure of the Spin cobordism ring. *Ann. of Math. (2)*, 86:271–298, 1967.

[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[4] C. Bär. Dependence of the Dirac spectrum on the Spin structure. In *Global analysis and harmonic analysis (Marseille-Luminy, 1999)*, volume 4 of *Sémin. Congr.*, pages 17–33. Soc. Math. France, Paris, 2000.

[5] E. Barrera-Yanez. The eta invariant and the "twisted" connective K-theory of the classifying space for cyclic 2-groups. *Homology Homotopy Appl.*, 8(2):105–114, 2006.

[6] B. Botvinnik, P. Gilkey, and S. Stolz. The Gromov-Lawson-Rosenberg conjecture for groups with periodic cohomology. *J. Differential Geom.*, 46(3):374–405, 1997.

[7] B. Botvinnik and P. B. Gilkey. The eta invariant and metrics of positive scalar curvature. *Math. Ann.*, 302(3):507–517, 1995.

[8] R. R. Bruner and J. P. C. Greenlees. The connective K-theory of finite groups. *Mem. Amer. Math. Soc.*, 165(785):viii+127, 2003.

[9] D. Davis. Topological complexity of 2-torsion lens spaces and ku-(co)homology. *Morfismos*, 18(2), 2014.

[10] J. F. Davis and W. Lück. The topological K-theory of certain crystallographic groups. *J. Noncommut. Geom.*, 7(2):373–431, 2013.

[11] J. F. Davis and K. Pearson. The Gromov-Lawson-Rosenberg conjecture for cocompact Fuchsian groups. *Proc. Amer. Math. Soc.*, 131(11):3571–3578, 2003.

[12] A. Dold. Parametrized Borsuk-Ulam theorems. *Comment. Math. Helv.*, 63(2):275–285, 1988.

[13] H. Donnelly. Eta invariants for G-spaces. *Indiana Univ. Math. J.*, 27(6):889–918, 1978.

[14] W. Dwyer, T. Schick, and S. Stolz. Remarks on a conjecture of Gromov and Lawson. In *High-dimensional manifold topology*, pages 159–176. World Sci. Publ., River Edge, NJ, 2003.

[15] K. Fujii, T. Kobayashi, K. Shimomura, and M. Sugawara. KO-groups of lens spaces modulo powers of two. *Hiroshima Math. J.*, 8(3):469–489, 1978.

[16] P. B. Gilkey. *The geometry of spherical space form groups*, volume 28 of *Series in Pure Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, second edition, 2018.

[17] M. Gromov and H. B. Lawson, Jr. The classification of simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 111(3):423–434, 1980.

[18] M. Gromov and H. B. Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)*, 111(2):209–230, 1980.

[19] S. Hashimoto. On the connective K-homology groups of the classifying spaces $B\mathbf{Z}/p^r$. *Publ. Res. Inst. Math. Sci.*, 19(2):765–771, 1983.

[20] N. Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974.

[21] S. Hughes. On the equivariant K- and KO-homology of some special linear groups. *Algebr. Geom. Topol.*, 21(7):3483–3512, 2021.

[22] J. Jaworowski. Involutions in lens spaces. volume 94, pages 155–162. 1999. Special issue in memory of B. J. Ball.

[23] M. Joachim and A. Malhotra. On the gromov-lawson-rosenberg conjecture for elementary abelian 2-groups. In preparation.

[24] A. Lichnerowicz. Spineurs harmoniques. *C. R. Acad. Sci. Paris*, 257:7–9, 1963.

[25] A. Malhotra and K. Rodtes. The Gromov-Lawson-Rosenberg conjecture for the semi-dihedral group of order 16. *Glasg. Math. J.*, 57(2):365–386, 2015.

[26] J. McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[27] J. Milnor. The Steenrod algebra and its dual. *Ann. of Math. (2)*, 67:150–171, 1958.

[28] R. E. Mosher and M. C. Tangora. *Cohomology operations and applications in homotopy theory*. Harper & Row, Publishers, New York-London, 1968.

[29] D. C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.

[30] D. C. Ravenel. The stable homotopy theory of finite complexes. In *Handbook of algebraic topology*, pages 325–396. North-Holland, Amsterdam, 1995.

[31] R. Reinauer. *Real and complex connective K-homology of finite abelian* 2- *groups*. PhD thesis, University of Münster, 2020.

[32] C. A. Robinson. A Künneth theorem for connective K-theory. *J. London Math. Soc. (2)*, 17(1):173–181, 1978.

[33] J. Rosenberg. C^*-algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212, 1983.

[34] T. Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998.

[35] C. Siegemeyer. *On the Gromov-Lawson-Rosenberg Conjecture for finite abelian 2- groups of rank 2*. PhD thesis, University of Münster, 2013.

[36] S. Stolz. Simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 136(3):511–540, 1992.

[37] S. Stolz. Splitting certain M Spin-module spectra. *Topology*, 33(1):159–180, 1994.
[38] R. E. Stong. Determination of $H^*(\mathrm{BO}(k, \cdots, \infty), Z_2)$ and $H^*(\mathrm{BU}(k, \cdots, \infty), Z_2)$. *Trans. Amer. Math. Soc.*, 107:526–544, 1963.
[39] R. M. Switzer. *Algebraic topology—homotopy and homology*. Classics in Mathematics. Springer-Verlag, Berlin, 2002. Reprint of the 1975 original [Springer, New York; MR0385836 (52 #6695)].
[40] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
[41] K. Yano and S. Bochner. *Curvature and Betti numbers*, volume No. 32 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1953.

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