A survey on Computations of Bredon Cohomology

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ABSTRACT. We present an overview of computational methods for Bredon cohomology with a special focus on infinite groups.

1. Different meanings for Bredon Cohomology

Bredon cohomology is one of the most prominent cohomology theories for spaces with an action of a group.

We reserve the notion of equivariant ordinary cohomology - as understood traditionally and in this volume- for the cohomology of the Borel construction of a space with an action of a compact Lie or finite group. We will speak, however, of Bredon cohomology as an equivariant cohomology theory in a sense to be defined below.

Historically, the construction of Bredon cohomology goes back to the announcement [**13**] and the extended version [**14**], and it is strongly based on the notion of a G-CW complex, which we will review in this note.

It is this relation which explains the use of Bredon cohomology in the study of finiteness properties in group cohomology [**45**], [**6**]. We will not extend in the discussion of this subject and rather refer to the excellent survey [**48**], and to [**53**] specifically to the Eilenberg-Ganea problem for families which is phrased in terms of Bredon cohomology of classifying spaces for families.

Simplicial versions of Bredon cohomology were provided by Bröcker and Illman [**15**], [**32**], with the outcome of the posibility of considering actions of (Hausdorff, locally compact) topological groups, based on an equivariant version of simplicial complexes, which mimics the definition of G-CW complexes.

Homotopical versions of Bredon cohomology, which even allow a description in the parametrized equivariant setting are described in [**52**] for *Naïve equivariant* cohomology theories, in the sense of equivariant homotopy theory. For the specific case of Bredon cohomology with local (twisted) coefficients in the complex twisted representation group, a construction has been written in detail in [**8**], where also the

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relation to the Čech versions has been established in order to provide a description of the Segal/Atiyah-Hirzebruch spectral sequence.

Genuine equivariant versions of Bredon cohomology have been considered in equivariant homotopy theory [**42**]. Classical computations of these theories include seminal work of Gaunce Lewis [**43**] (atributed to unpublished work of Stong), based on explicit cellular structures of the relevant examples, the $RO(G)$- graded Künneth and universal coefficient spectral sequences [**44**], the implicit use of classifying spaces for families, such as the Tits building in [**4**], the explicit use of classifying spaces for families of proper subgroups [**37**], and a more recent development of a variety of tools whose interest goes back to the role of such computations in the proof of the Kervaire invariant one problem in [**27**].

The methods include (without the intention to be exhaustive in their enumeration) those based on the slice filtration [**28**], the homotopy fixed point spectral sequences [**30**], often in combination with ad-hoc cellular constructions [**36**], as well as parametrized homotopy theory considerations [**16**], and the notion of freeness of [**29**]. They are quoted here with the idea of giving a representative example of an application of each kind of method.

Finally, a genuine proper equivariant version of Bredon cohomology has been defined in terms of equivariant homotopy theory in example 3.2.16 in [**18**], where also the relation of the extensions of the gradings from $\mathbb{Z}$ to the equivariant KO^0-theory of the classifying space for proper actions (more general than $RO(G)$ is adressed.

We will focus on computational methods for the determination of *naive* versions of Bredon cohomology, with an emphasis on infinite discrete groups, extending the content of the lecture delivered at the AMS sectional meeting with a number of references and additional examples expanding the exposition.

Contents

2. Bredon Cohomology

Let G be a (possibly infinite) discrete group. A G-CW-complex is a CW-complex with a G-action permuting the cells and such that if a cell is sent to itself, it is done by the identity map. We call the G-action proper if all cell stabilizers are finite subgroups of G.

Definition 2.1. Recall that a G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup_{-1\leq n} X_n$, $X_{-1} = \emptyset, X_0 = A$ and for which every space X_n is inductively obtained from the previous one by attaching cells in pushout diagrams of the form

$$\begin{array}{ccc} \coprod_i S^{n-1} \times G/H_i & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i D^n \times G/H_i & \longrightarrow & X_n \end{array}$$

We say that a proper G-CW complex is finite if it consists of a finite number of (orbits of) cells $G/H \times D^n$.

Example 2.2. [Examples of G-CW decompositions]

- It is a consequence of the equivariant triangulation theorem that there exists equivariant triangulations of smooth manifolds with a proper smooth action of a lie Group. [**33**], [**34**]. Such a triangulation produces a G-simplicial complex.

  This notion is described in [**34**], section 5 as a triangulation of the orbit space X/G with an extra compatibility condition with respect to the quotient map $\pi : X \to X/G$, namely: the inverse image of an n-dimensional simplex Δ_n on X/G is a *standard equivariant simplex*, denoted $(\Delta_n, H_0, \ldots, H_n)$, which is a quotient of the product of a free G-orbit of the standard n-dimensional simplex $\Delta_n \times G$, where pairs (x, g) and $(x, g^{'})$ are identified if for each $k = 0, \ldots n$ x belongs to the boundary of one of the k-dimensional simplices in Δ_n, the cosets for the compact subgroup H_k gH_k and $g^{'}H_k$ agree, and the sequence of compact subgroups satisfy $H_{i+1} \subset H_i$.

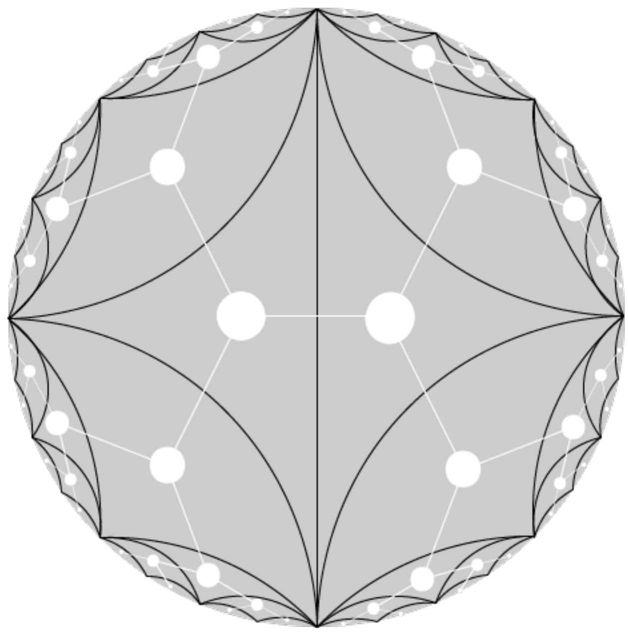
  Notice that the data provided by this identification is equivalent to G-equivariant maps from orbits of i-dimensional simplices $G/H_i \times \Delta_i \to X$, on which the inclusions of standard simplices $\Delta_i \subset \Delta_{i+1}$ are made compatible with inclusions up to G-conjugacy $H_{i+1} \subset H_i$.

  Such a map $\Delta_i \times G/H_i \to X$ can be identified with a cell. The full details are worked out in [**34**], propositon 11.5, page 170.
- A G-Absolute Neighborhood Retract with a proper action of a Lie group, or more generally, a locally compact group has the homotopy type of a G-CW complex. This is is a consequence of the slice theorem [**55**].
- Equivariant cell decomposition for the action of $Sl_2(\mathbb{Z})$ on the hyperbolic plane and the 1- dimensional deformation retract.

  The group $Sl_2(\mathbb{Z})$ acts by isometries on the hyperbolic plane. The dual of the Farey tesselation provides an example of a one- dimensional G-CW complex (the Bass-Serre tree) for $Sl_2(\mathbb{Z})$), consisting of two orbits of zero dimensional cells of type C_4, C_6, and one dimensional cell of type C_2. See figure 2.2

  More generally, Li, Lück and Kasprovski have constructed in [**35**] a G-CW structure for the flag complex associated to a group G given as a graph product (Examples include right angled Artin groups and right angled Coxeter groups).
- Equivariant cell decomposition for the action of $Sl_3(\mathbb{Z})$ on the homogeneous space $Sl_3(\mathbb{R})/SO_3$. There exists a triangulation of the quotient of an equivariant deformation retract of this homogeneous space described

Figure 1. Dual of the Farey triangulation.



in [**64**], but also in [**58**], which is the main input for the computations of twisted equivariant K-Theory and K-homology in [**9**], [**10**].

Let Q be the space of real, positive definite 3×3-square matrices. Multiplication by positive scalars gives an action whose quotient space $Q/\mathbb{R}^+$ is homotopy equivalent to the quotient $Sl_3(\mathbb{R})/SO_3/SL_3(\mathbb{Z})$.

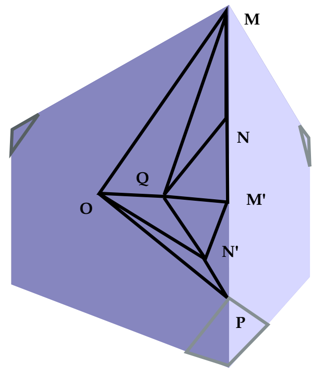
We describe its orbit space. Let C be the truncated cube of $\mathbb{R}^3$ with centre $(0,0,0)$ and side length 2, truncated at the vertices $(1,1,-1),(1,-1,1),(-1,1,1)$ and $(-1,-1,-1)$, trough the mid-points of the corresponding sides. As stated in [**64**], every matrix A admits a representative of the form

$$\begin{pmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{pmatrix}$$

which may be identified with the corresponding point (x,y,z) inside the truncated cube. We introduce the following notation for the vertices of the cube:

$$\begin{array}{rr} O=(0,0,0) & Q=(1,0,0) \\ M=(1,1,1) & N=(1,1,1/2) \\ M^{'}=(1,1,0) & N^{'}=(1,1/2,-1/2) \\ P=(2/3,2/3,-2/3) & \end{array}$$

FIGURE 2. Triangulation for the fundamental region.



Note that the elements of $Sl_3\mathbb{Z}$

$$q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} q_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

send the triangle (M, N, Q) to the triangle $(M^{'}, N^{'}Q)$ and the quadrilateral

$(N, N^{'}, M^{'}, Q)$ to $(N^{'}, N, M^{'}, Q)$. Thus, the following identification must be performed in the quotient: $M \cong M^{'}$, $N \cong N^{'}$, $QM \cong QM^{'}$, $QN \cong QN^{'}$, $MN \cong M^{'}N^{'} \cong M^{'}N$ and $QMN \cong QM^{'}N \cong QM^{'}N^{'}$.

Following [**58**] we now describe the orbits of cells and corresponding stabilizers. This can be found also in Theorem 2 of Soulé's article [**64**] (although we use a cellular structure instead of a simplicial one). We have changed the chosen generators. We summarize the information on Table 1. We use the following notations: $\{1\}$ denotes the trivial group, C_n the cyclic group of n elements, D_n the dihedral group with $2n$ elements and S_n the Symmetric group of permutations on n objects.

vertices				2-cells			
v_1	O	$g_2,\ g_3$	S_4	t_1	OQM	g_2	C_2
v_2	Q	$g_4,\ g_5$	D_6	t_2	$QM'N$	g_1	$\{1\}$
v_3	M	$g_6,\ g_7$	S_4	t_3	$MN'P$	g_{12}, g_{14}	$C_2 \times C_2$
v_4	N	$g_6,\ g_8$	D_4	t_4	$OQN'P$	g_5	C_2
v_5	P	$g_5\ ,\ g_9$	S_4	t_5	$OMM'P$	g_6	C_2
edges				3-cells			
e_1	OQ	$g_2,\ g_5$	$C_2 \times C_2$	T_1	g_1	$\{1\}$	
e_2	OM	g_6, g_{10}	D_3				
e_3	OP	g_6, g_5	D_3				
e_4	QM	g_2	C_2				
e_5	QN'	g_5	C_2				
e_6	MN	g_6, g_{11}	$C_2 \times C_2$				
e_7	$M'P$	g_6, g_{12}	D_4				
e_8	$N'P$	g_5, g_{13}	D_4				

The first column is an enumeration of equivalence classes of cells; the second lists a representative of each class; the third column gives generating elements for the stabilizer of the given representative; and the last one is the isomorphism type of the stabilizer. The generating elements referred to above are

$$g_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \qquad g_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad g_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad g_6 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$g_7 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad g_8 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \qquad g_9 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$g_{10} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad g_{11} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad g_{12} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$g_{13} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad g_{14} = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

We fix an orientation; namely, the ordering of the vertices $O < Q < M < M' < N < N' < P$ induces an orientation in E and also in the quotient $\underline{B}Sl_3\mathbb{Z} = E/\equiv$. The cells coboundaries are detemined in section 5 of [**9**] and include restriction of representations and signs coming from the prescribed orientation chosen above.

Bredon cohomology admits a description in terms of functors depending on the orbit category of a group.

Similar to the discussion of part 1 in 2.2, the subconjugacy relations of subgroups, say from H to K by an element of G, denoted by g satisfying $gHg^{-1} = H^{'} \subset K$, has the consequence that there exists a G-equivariant map $G/H \to G/K$ given by asigning to the coset $g^{'}H$ the coset $gg^{'}g^{-1}K$.

Such equivariant maps determine geometric information when considered as part of the intrinsically given data in a G-CW structure. In particular, such maps origin inclusions of orbits of cells in such a way that the lowest dimension of cells is associated to the biggest groups in terms of inclusion up to G-conjugacy.

There exist two important categories associated to a group, which are relevant to the codification of such relations among the subgroups of G, the Orbit category, and the Conjugation homomorphism category. We will assume for the rest of this section that the group G is discrete.

Definition 2.3 (Orbit Category). Denote by $\mathcal{O}_G$ the orbit category of G; a category with one object G/H for each subgroup $H \subseteq G$ and where morphisms are given by G-equivariant maps. There exists a morphism $\phi : G/H \to G/K$ if and only if H is conjugate in G to a subgroup of K. More generally, given a family of subgroups $\mathcal{F}$, which is closed under intersection and conjugation, we can form the full subcategory $\mathcal{O}_G^{\mathcal{F}}$ where the objects are of the form G/H with H in $\mathcal{F}$.

Definition 2.4 (Conjugation Homomorphism Category). Given a Group G, the conjugation-homomorphism category $\mathcal{S}_G$ is the category where the objects are subgroups of G, and the set of morphisms between two objects H and K is the quotien, denoted by $\mathrm{Conhom}(H, K)/\mathrm{Inn(K)}$ of the set $\mathrm{conhom}(H, K)$ of group homomorphisms $\varphi : H \to K$ for which there exists an element $g \in G$ such that φ is given as conjugation by g, and the group $\mathrm{Inn}(K)$ of inner automorphisms of K acts by composition.

There exists a projection functor $\mathrm{pr} : \mathcal{O}_G \to \mathcal{S}_G$, which asigns to each orbit G/H the subgroup H and to a G- map $f : G/H \to G/K$ the homomorphism $H \to K$ defined as conjugation by an element g satisfying $gHg^{-1} \cong H^{'} \leq K$.

Remark 2.5. The set $\mathrm{Mor}_{\mathcal{S}_G}(H, K)$ is isomorphic to the quotient of the action of the centralizer of the subgroup H in G, $C_G(H) = \{g \in G \mid gh = hg \text{ for all } h \in H \}$ on the set of G-equivariant maps $\mathrm{Mor}_{\mathcal{O}_G}(\mathrm{G/H}, \mathrm{G/K})$, where an element of the centralizer $g \in C_G(H)$ acts by composition with the right multiplication $R_{g^{-1}} : G/H \to G/H$, $g^{'}H \mapsto g^{'}g^{-1}H$.

The main advantage in considering this category is that the automorphism group of a finite group H is finite.

In the orbit category, the automorphism group of an object G/H is the quotient of the nomalizer susbgroup in G by the subgroup $H \cdot C_H(G)$ consisting of elements of the form hc, where $h \in H$, and c is an elment of the centralizer in G of H. We will denote this group by $W_G(H) = N_G(H)/H \cdot C_H(G)$.

Both the orbit category and the conjugation-homomorphism category are EI-categories, in the sense that every endomorphism of an object is invertible.

Example 2.6. [Orbit Categories for infinite groups]

- The orbit category for the family of finite subgroups for the group $Sl_2(\mathbb{Z})$ has three objects: $Sl_2(\mathbb{Z})/C_6$, $Sl_2(\mathbb{Z})/C_4$, and $Sl_2(\mathbb{Z})/C_2$. There exist two G- equivariant maps $Sl_2(\mathbb{Z})/C_2 \to Sl_2(\mathbb{Z})/C_6$ and $Sl_2(\mathbb{Z})/C_2) \to Sl_2(\mathbb{Z})/C_4$.

- The triangulation of the quotient of the action of $Sl_3(\mathbb{Z})$ on the space constructed by Soulé and discused in 2.2, part 4, has the consequence of a complete description of the orbit category for finite stabilizer subgroups in the group: There exist eight maximal finite subgroups, which are the stabilizers of the zero dimensional cells, there are as many morphisms between them as the one dimensional cells of the triangulation having as edges the vertices, the composition of pairs of such morphisms are related by the obvious rule given by the two dimensional cells, and finally, there exists a unique composition of length three in the orbit category for finite isotropy subgroups of $Sl_3(\mathbb{Z})$. See the table at the end of the previous section.

Definition 2.7. (Bredon homology) Let X be a G-CW complex. The contravariant functor $\underline{C}_*(X) : \mathcal{O}_G \to \mathbb{Z}-\mathsf{CHCOM}$ assigns to every object G/H the cellular $\mathbb{Z}$- chain complex of the H-fixed point complex $\underline{C}_*(X^H) \cong C_*(\mathrm{Map}_G(G/H, X))$ with respect to the cellular boundary maps $\underline{\partial}_*$.

We will use homological algebra to define Bredon homology and cohomology functors.

A contravariant Bredon Module is a contravariant functor $N : \mathcal{O}_G^{\mathcal{F}} \to \mathbb{Z} - \mathsf{MODULES}$, where $\mathcal{F}_G$ is the full subcategory of the orbit category of G, $\mathcal{O}_G$ generated by the objects G/H for a family of subgroups $H \in \mathcal{F}_G$.

Given a contravariant Bredon module M, the Bredon cochain complex $C_G^*(X; M)$ is defined as the abelian group of natural transformations of functors defined on the orbit category $\underline{C}_*(X) \to M$. In symbols,

$$C_G^n(X; M) = Hom_{\mathcal{F}_G}(\underline{C}_n(X), M)$$

Where $\mathcal{F}_G$ is a subcategory containing the isotropy groups of X.

Given a set $\{e_\lambda\}$ of orbit representatives of the n-cells of the G-CW complex X, and isotropy subgroups S_λ of the cells e_λ, the abelian groups $C_G^n(X, M)$ satisfy:

$$C_G^n(X, M) = \bigoplus_\lambda Hom_{\mathbb{Z}}(\mathbb{Z}[e_\lambda], M(G/S_\lambda))$$

with one summand for each orbit representative e_λ. They afford a differential $\delta^n : C_G^n(X, M) \to C_G^{n-1}(X, M)$ determined by $\underline{\partial}_*$ and maps $M(\phi) : M(G/S_\mu) \to M(G/S_\lambda)$ for morphisms $\phi : G/S_\lambda \to G/S_\mu$.

For a covariant functor

$$N : \mathcal{F}_G \to \mathbb{Z} - \mathsf{MODULES},$$

the chain complex

$$C_*^{\mathcal{O}_G} = \underline{C}^n(X) \underset{\mathcal{F}_G}{\otimes} N = \bigoplus_\lambda \mathbb{Z}[e_\lambda] \otimes N(G/S_\lambda)$$

admits differentials $\delta_* = \partial_* \otimes N(\phi)$ for morphisms $\phi : G/S_\lambda \to G/S_\mu$ in $\mathcal{F}_G$.

Definition 2.8 (Bredon homology). The Bredon homology groups with coefficients in N , denoted by $H_*^{\mathcal{O}_G}(X, N)$. are defined as the homology groups of the chain complex $\left(C_*^{\mathcal{O}_G}(X, N), \delta_*\right)$

Definition 2.9 (Bredon cohomology). The Bredon cohomology groups with coefficients in M, denoted by $H^*_{\mathcal{O}_G}(X, M)$ are the cohomology groups of the cochain complex $\big(C^*_G(X, M), \delta^*\big)$.

Remark 2.10 (Bredon cohomology in terms of the conjugation homomorphism category). Let X be a G-CW complex with finite cell stabilizers. Consider the contravariant functor defined on the conjugation homomorphism category $\mathcal{S}_G$ and taking values in the category of chain complexes, where we asign to a subgroup H the cellular chain complex of the quotient space $X^H/C_G(H)$. We will denote by $C^{\mathcal{S}_G}_*$ the obtained functor, and define for every contravariant functor defined on $\mathcal{S}_G$ and values on the category of R-modules the Cochain complex of natural transformations from $C^{\mathcal{S}_G}_*$ to M as $Hom_{\mathcal{S}_G}(C^{\mathcal{S}_G}_*, M)$. We can apply the cohomology funtor with respect to the cellullar cochain maps and we will denote the obtained modules by $H^n_{\mathcal{S}_G}(C^{\mathcal{S}_G}_*, M)$.

As a consequence of remark 2.5, the projection functor $\mathrm{pr} : \mathcal{O}_G \to \mathcal{S}_G$ induces for every contravariant functor M defined on the conjugation isomorphism category and taking values in R-modules a bijective pair of natural transformations

$$pr^* : \mathrm{pr}^* M \to M,$$

between the composition of functors, and

$$pr_* : C^{\mathcal{O}_G}_* \to C^{\mathcal{S}_G}_*$$

between the chain complexes, which produce an isomorphism on the level of cohomology groups

$$H^n_{\mathcal{O}_G}(X, \mathrm{pr}^* M) \cong H^n_{\mathcal{S}_G}(C^{\mathcal{S}_G}_*, M)$$

Example 2.11 (Examples of Bredon Modules). We will now enumerate some important examples of Bredon modules.

(1) Constant coefficients. Given a fixed R-Module M, we consider the constant functor with value M for each orbit.
(2) Free Bredon functors. A contravariant functor defined on the orbit category is said to be free if it is a direct sum of representable contravariant functors; that is, there exist a number of objects G/H_α such that the functor is given as

$$\bigoplus_{G/H_\alpha} R[\mathrm{Mor}(\quad, G/H_\alpha)].$$

Chain complexes associated to G-CW complexes provide examples of free contravariant Bredon modules.
(3) The complex representation ring as a Bredon module.

We restrict to the family of finite subgroups of the possibly infinite group G, and associate to the object in the orbit category G/H with a finite subgroup H, the complex representation ring $R_{\mathbb{C}}(H)$. Recall that the subjacent abelian group is free in the set of conjugacy classes of H, but it is not free as a functor over the orbit category as defined in the example above.

On the complex representation ring we can consider a covariant structure, assigning to a G-map $G/H \to G/K$ the induction homomorphism

$R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(K)$ constructed as follows. Let $H^{'} \leq K$ be the subgroup of K which is conjugated to H.

Let now V be a representation of H, and consider it as an $H^{'}$- representation. Consider now the K-representation $V \otimes_{\mathbb{C}} \mathbb{C}(K)$.

There exists a contravariant structure on the representation ring, defined by assigning to a morphism $G/H \to G/K$ the homomorphism $R_{\mathbb{C}}(K) \to R_{\mathbb{C}(H')} \cong R_{\mathbb{C}}(H)$ given by restriction.

(4) Twisted complex representation rings

Let H be a finite group and V be a complex vector space. Given a cocycle $\alpha : H \times H \to S^1$ representing a class in $H^2(H, S^1) \cong H^3(H, \mathbb{Z})$, an α-twisted representation is a function $P : H \to Gl(V)$ satisfying:

$$P(e) = 1$$

$$P(x)P(y) = \alpha(x, y)P(xy)$$

The isomorphism type of an α-twisted representation only depends on the cohomology class in $H^2(H, S^1)$.

Definition 2.12. Let H be a finite group and $\alpha : H \times H \to S^1$ be a cocycle representing a class in $H^2(H, S^1) \cong H^3(H, \mathbb{Z})$. The α-twisted representation group of H, denoted by ${}^{\alpha}\mathcal{R}(H)$ is the Grothendieck group of isomorphism classes of complex, α-twisted representations with direct sum as binary operation.

Let H be a finite group. Given a cocycle $\alpha \in H^2(H, S^1)$ representing a torsion class of order n, the normalization procedure gives a cocycle β cohomologous to α such that $\beta : H \times H \to S^1$ takes values in the subgroup $\mathbb{Z}/n \subset S^1$ generated by a primitive n-th rooth of unity η. Associated to a normalized cocycle, there exists a central extension

$$1 \to \mathbb{Z}/n \to H^* \to H \to 1$$

with the property that any twisted representation of H is a linear representation of H^*, with the additional property that $\mathbb{Z}/n$ acts by multiplication with η. Such a group is called a Schur covering group for H.

(5) Burnside Ring. We restrict again to the family of finite groups for the isotropy of objects in the orbit category. Given a finite group H, the Burnside ring $A(H)$ is defined to be the Grothendieck ring of the set of isomorphism classes of finite sets with an action of H.

Similar to the representation ring, there exist two structures on the Burnside Ring: one covariant, given by induction of actions, and a contravariant one, which is defined by restriction.

Example 2.13 (Computation of Bredon homology of $Sl_2(\mathbb{Z})$ from the definition.). Let us consider the complex representation ring as Bredon module. We will restrict to the family of finite subgroups here. Directly from the 1- dimensional cellular structure for the (Bass Serre) graph given as de dual of the Farey triangulation, we obtain a Chain complex computing the Bredon homology where the depicted map is d_1.

$$0 \to \mathcal{R}(C_2) \overset{\left(\mathrm{ind}_{C_2}^{C_6}), -\mathrm{ind}_{C_2}^{C_4}\right)}{\to} \mathcal{R}(C_6) \oplus \mathcal{R}(C_4) \to 0.$$

Notice that the rank of the abelian group on the left is 2, the rank of the group on the right is 6+4=10, and the map is injective.

Thus, if we denote by T the 1-dimensional $Sl_2(\mathbb{Z})$- complex obtained.

$$H_0\mathcal{O}_{Sl_2(\mathbb{Z})}(T, \mathcal{R}) = \mathbb{Z}^8,$$

and all other Bredon homology groups are zero.

REMARK 2.14 (Further accesible examples of low dimension). We mention for the sake of utility the following illustrative examples of computations of Bredon cohomology in low dimension: [**23**], [**22**].

REMARK 2.15 (Computations for $SL_3(\mathbb{Z})$ based on the CW decomposition.). The triangulation proposed by Soulé has been extensively exploited for computations of Bredon cohomology.

The first computation of the Bredon chain and cochain complex was done in [**58**]. In detail, he determination of stabilizers, and conjugacy relations, as well as the differentials have been used as input for computations of Bredon cohomology with coefficients in complex representations in [**58**], for coefficients in twisted representations in [**9**], [**10**], and more recently, with the coefficients of equivariant real K- Theory in [**31**].

The output is that the spectral sequence of Atiyah-Hirzebruch type collapses, and the computation amounts to a computation of the left hand side of the Baum-Connes conjecture.

REMARK 2.16 (Equivariant Obstruction Theory on G-CW complexes and Bredon cohomology). One of the first succeses of Bredon cohomology was the development of equivariant Obstrution Theory. Consider the equivariant version of the Obstruction problem: Given G-Cw complexes X and Y, and a G-map $f : X_{n-1} \to Y$ defined on the $n-1$- skeleton, the main theorem states that f can be redefined to be G- homotopic over the $n-2$ skeleton to a map which extends over an $n+1$-dimensional additional orbit of a cell $G/H \times D^{n+1}$ if an obstruction class vanishes

$$\mathfrak{o}(f) \in H^n_{\mathcal{O}_G}(X, \pi_{n+1}(Y^H)).$$

See [**14**] Chapter II and [**12**] for the use in combinatorial geometry in the specific case of free actions.

REMARK 2.17 (Bredon cohomology in Topological complexity). Lower bounds for topological complexity of Eilenberg-Maclane spaces have been obtained with computations of Bredon cohomology in [**21**].

REMARK 2.18 (Hecke Operators and Bredon cohomology). There exists a framework for the study of Hecke operators actng on the Bredon cohomology to obtain a Hecke action on the reduced C^* algebra of the group in [**51**].

REMARK 2.19 (Motivic Version of Bredon cohomology). There exists a computation of a motivic version of Bredon Cohomology in [**26**].

REMARK 2.20 (Algebraic properties of the Abelian Category of Bredon Modules and its objects). We briefly mentioned in the previous example that (by definition of free object) the chain complex associated to a G-CW complex provides an example of a free functor over the orbit category. We will examinate some algebraic properties of the categories of modules and chain categories.

The category where the objects are functors on the orbit category in a category of Modules is an abelian category, where a morphism is a natural transformation between them. A pais of consecutive morphisms is said to be exact if it is exact on every object.

The notions of projective and injective module can be given as usual in terms of Hom functors, and free and projective resolutions exist as a consequence of the Yoneda Lemma.

See [**53**] for an introduction to Bredon homological algebra in connection with group cohomology, and specifically finiteness properties.

Remark 2.21 (Computation of Bredon cohomology in practice). The coefficient systems considered in this note yield chain complexes, respectively cochain complexes of free abelian groups with preferred bases to compute both Bredon homology and cohomology.

Notice that if we have a complex of free abelian groups

$$\cdots \to \mathbb{Z}^{\oplus n} \xrightarrow{f} \mathbb{Z}^{\oplus m} \xrightarrow{g} \mathbb{Z}^{\oplus k} \to \cdots$$

with f and g represented by matrices A and B for some fixed basis, then the homology at $\mathbb{Z}^{\oplus m}$ is

$$ker(g)/im(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z} \oplus \mathbb{Z}^{\oplus(m-s-r)},$$

where $r = rank(B)$ and $d_1, \ldots, d_s$ are the elementary divisors of A.

This has a consecuence the Torsion Freeness chriterion for Bredon homology, see Theorem 5.2 in page 1496 of [**41**].

Effective approaches for implementing Bredon cohomology in computations (mainly in GAP) include [**60**], [**56**].

2.1. Comparison to other constructions. While the simplicial and cellular versions of Bredon cohomology are compared by Illman in [**32**], the Čech versions is seen to agree for a proper G-ANR with the *naive*-homotopical versions of Bredon cohomology in the more general parametrized setting in Theorem 5.3 of [**8**]. The theorem is phrased for the specific example of (twisted) complex representation ring, but the argument holds for any Bredon module.

Remark 2.22 (Contrast to genuine equivariant homotopical versions of Bredon cohomology). There exist versions of Bredon cohomology obtained by considering a *genuine equivariant* version of the Eilenberg Maclane spectrum in a category of equivariant spectra.

Some examples of these computations include [**44**], [**4**] [**37**] [**27**], [**28**], [**30**] [**36**], [**16**].

We refer to example 3.2.16 in [**18**] for the relation of naive graded equivariant cohomology theories to extensions of them to genuine ones, which apply in particular to Bredon cohomology.

2.2. Computations based on the algebraic properties of the Category of Mackey functors. We will present now a decomposition of Bredon cohomology in terms of information concerning automorphism groups of each object for the Bredon module. The relation is crucial to decompositions which refine equivariant cohomological Chern characters.

References for this section are [**63**], [**47**], [**46**], from where the totality of arguments and definitions are extracted with no claim of originality.

It turns out that there exists a close relation between the algebraic properties of being injective in the category of functors over the orbit category, and the possibility of the decomposition in terms of the stabiizer groups associated to the isomorphism classes of objects.

Consider for this, in order to fix notation, the inclusion of an object in the orbit category $i_{G/H} : G/H \to \mathcal{O}_G$, and recall that the automorphism group is $W_H(G)$.

There is the *restriction* functor i^* which restricts a contravariant module M to the R- Module $i^*(M) = M(G/H)$, and which gives an $R[W_G(H)]$-module structure by considering theautomorphisms of G/H.

Given an $R[W_H(G)]$- module N, the *induced module* i_*N, is the $\mathcal{O}_G$- R module defined as quotient of the tensor with the contravariant free object

$$G/K \mapsto N(G/H) \otimes R[\mathrm{Mor}(G/K, G/H)],$$

where we declare equivalent to zero the submodule of $M(G/H) \otimes R[\mathrm{Mor}(G/K, G/H)]$ generated by elements of the form $m \otimes f_*(x) - f^*m \otimes x$, for all $f \in W_G(H)$.

Given a $R[W_G(H)]$- module N, the *coinduction* functor assigns (contravariantly!) to an object in the orbit categrory G/K the R-module

$$i_{G/H_!} = \mathrm{Hom}_{R[W_H(G)]} R[\mathrm{mor}_{\mathcal{O}_G}(G/H, G/K)], N)]$$

Notice the adjunctions for every pair consisting of an $\mathcal{O}_G$- functor M and N a $R[W_G(H)]$-module.

$$\mathrm{hom}_{R[W_G(H)]}({i_{G/H}}^* M, N) \cong \mathrm{hom}_{\mathcal{O}_G}(M, i_{G/H_!} N)$$

$$\mathrm{hom}_{\mathcal{O}_G}({i_{G/H}}_* N, M) \cong \mathrm{hom}_{R[W_H(G)]}(N, i^*_{G/H}(M)).$$

Definition 2.23 (Projective and Injective Splitting functors). Let G/H be an object in the Orbit category.

(1) The projective splitting functor $S_{G/H}$ associates to a contravariant functor defined over the orbit category M, the $R[W_G(H)]$ module defined as the cokernel of the map

$$\prod_{\substack{f: G/H \to G/K \\ f \text{ not an isomorphism}}} M(G/K) \to M(G/H).$$

(2) The injective splitting functor $T_{G/H}$ associates to a covariant functor defined over the orbit category N, the $R[W_G(H)]$- module defined as the kernel of the map

$$\bigoplus_{\substack{f: G/H \to G/K \\ f \text{ not an isomorphism}}} N(G/H) \to N(G/K).$$

The projective splitting functor comes with a canonical projection $M(G/H) \to S_{G/H}(M)$. Given any $R[W_G(H)]$- section, the inclusion of the object G/H into the orbit category produces a natural transformation ${i_{(G/H)}}_* S_{G/H}(M) \to {i_{G/H}}_*(M)$. Dually, the projective splitting functor comes with a canonical injection $T_{G/H} M \to M(G/H)$, an any $W_G(H)$- retraction $M(G/H) \to T_{G/H} M$ produces a natural transformation

$$i_{G/H_!} M(G/H) \to {i_{G/H}}_* T_{G/H} M.$$

For an object G/H in either one of the categories $W_G(H)$ or $\mathcal{O}_G$, the length of an object G/H is the supremum of all l for which there exists a sequence of morphisms $G/H_0 \to G/H_1 \ldots G/H_l$ with $G/H_l = G/H$ and none of the morphism is an isomorphism, dually, the colength is the supremum over all l for which there exists a sequence $G/H_0 \to G/H_1 \to \ldots G/H_l$ with $G/H_0 = G/H$, and none of the morphisms is an isomorphism. A category has finite length, respectively finite colength, if each object has finite length or colengt.

The Structure Theorem [**47**], 2.2 in page 1035 reads as follows:

THEOREM 2.24. *(1) Suppose that $\mathcal{O}_G$ has finite colength, and that M is a covariant Bredon functor with the property that $S_{G/H}M$ is a projective $RW_H(G)]$- module for each object G/H. Let $\sigma_{G/H} : S_{G/H}M \to M$ be an $R[W_H(G)]$- section for the canonical projection, and consider the map of Bredon functors*

$$\mu(M) : \bigoplus_{G/H \in \mathrm{Iso}(\mathcal{O}_G)} i_{G/H_*} S_{G/H} M \overset{\bigoplus_{G/H \in \mathrm{Iso}(\mathcal{O}_G)} i_{G/H_*}\sigma_{G/H} M}{\longrightarrow} i_{G/H_*} M(G/H) \overset{\bigoplus_{G/H \in \mathrm{Iso}(\mathcal{O}_G)} \alpha_{G/H}}{\longrightarrow} M.$$

Where $\alpha : i_{G/H_*} M(G/H) = i_{G/H_*} i^*_{G/H} M \to M$ is the adjoint of the identity.

The map is always surjective. It is bijective if and only if M is a projective Bredon module.

(2) Suppose that $\mathcal{O}_G$ has finite length. Let M be a contravariant $R_{\mathbb{O}_G}$-module such that the $R[W_G(H)]$- module $M(G/H)$ is injective for every G/H. Let $\rho_{G/K} : M(G/K) \to T_{G/K}M$ be an $R[W_G(H)]$- retraction of the canonical injection $T_{G/H}M \to M$ and consider the natural transformation

$$\nu(M) : M \overset{\prod_{G/K \in \mathrm{Iso}(\mathcal{O}_G)} \beta_{G/K}}{\longrightarrow} \prod_{G/K \in \mathrm{Iso}(\mathcal{O}_G)} i_{G/H_!} M(G/K) \overset{\prod_{G/K \in \mathrm{Iso}(\mathcal{O}_G)} i_{G/K_!}\rho_{G/K}}{\longrightarrow} i_{G/K_*} T_{G/K} M.$$

The map is always injective. It is bijective if M is an injective Bredon module.

In the important example of complex representation rings, we noticed the fact that there exists a covariant and contravariant structure on fuctors which agree on objects.

On the other hand side, the second part of Theorem 2.24, a condition appears as for the map $\nu_{G/H}$ to be surjective, which is equivalent to the fact that the Bredon module is injective.. This has the consequence for a contravariant functor M that the composition with the projection functor to the conjugation homomomorphism category $pr^*(M) : \mathcal{S}_{\mathcal{G}}$ gives an injective R- module after evaluation on each object.

The following notion is an equivalent characterizetion of this property, which and will be the most relevant algebraic tool to the rational computation of Bredon cohomology.

DEFINITION 2.25. Let FGINJ be the category of finitely generated groups and injective group homomorphisms. Let M^*, M_* be a bifunctor to the category of R-modules; that is, a pair consisting of a contravariant functor M^* and a covariant functor M_* agreeeing on objects. We will denote by ind$f$ the covariantly induced homomorphism, and by res$f$ the contravariantly induced homomorphism. For inclusions of a subgroup $H \to G$, we will write res_G^H and ind_H^G.

M is said to be a Mackey functor if

- For an inner automorphism $c(g) : G \to G$, we have $M_*(c_g) : M(G) \to M(G)$ is the identity.
- For an isomorphism of groups $f : G \xrightarrow{\cong} H$, the composites $\mathrm{res}f \circ \mathrm{ind}f$ and $\mathrm{ind}f \circ \mathrm{res}f$ are the identity.
- Double coset formula. For two subgroups $H, K \subset G$,

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G = \sum_{KgH \in G/H/K} \mathrm{ind}_{c_g : H \cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H \cap g^{-1}Kg},$$

where c_g denotes conjugation with g.

The following result was proved in [**47**], Theorem 5.2 in page 1046.

THEOREM 2.26 (Injectivity and Mackey functors). *Let G be a Group and let R be a commutative ring such that the order of every finite subgroup is invertible in R. Assume that M is a mackey functor.*

Suppose that the $R[W_G(H)]$-module $T_{G/H}M$ is injective as a $R[W_G(H)]$- module for each object. Then, M is injective as a $\mathcal{S}_G$- module, and the map ν is bijective.

A finer structure occurs for cohomology with coefficients in modules over the Green functor of the rational representation ring. The following corollary is even true for Bredon cohomology with coefficients in such modules. See [**46**], sections 6 and 7.

The following theorem was proved as 6.3 in [**46**], page 221 for Bredon homology

THEOREM 2.27. *Let (X, A) be a proper G-CW pair. Let M be a Mackey functor with module structure over the Green ring of rational representations. Then, there exists a decomposition*

$$H_p^{\mathcal{O}_G}(X, A) \cong \bigoplus_{H \in I} H_p(X^H, A^H / C_G(H)) \otimes_{R[W_G(H)]} S_{G/H} M,$$

where I denotes the set of conjugacy classes of finite subgroups.

We quote now the most complete result which uses the Module structure over the Green Ring of the rational representation ring. This appeared as Theorem 0.2 in [**46**].

THEOREM 2.28. *Let M be a Mackey functor which admits a module structure over the Green functor of rational representations. For any group and any G-CW pair (X, A) there exists a direct sum decomposition*

$$H_p^{\mathcal{O}_G}(X, A) \cong \bigoplus_{H \in I} H_p((X^C, A^C)/C_G(H)) \otimes R[W_G(C)] \theta_C^C M).$$

Here $\theta_C^C M$ denotes mutiplication with an idempotent in the rational representation ring, and this image equals:

$$\mathrm{coker} \bigoplus_{D \subsetneq C} M(D) \overset{\mathrm{ind}_D^C}{\to} M(C).$$

The explicit use of such idempotents plays a role in delocalized Chern characters.

2.3. Computations based on elementary homological algebra over the orbit category. The fact that Bredon cohomology can be defined as a Hom construction (limit) obtaining a cochain complex from which Bredon cohomology is obtained as usual homology allows that usual constructions (based on the existence of resolutions and the concept of derived functor) in homological algebra often have a generalization to Bredon versions.

We present two instances of these constructions: the Universal Coefficient Theorem for Bredon cohomology of [**9**] and the Künneth theorem of [**59**].

The following result appeared in [**59**], Theorem 3.1 in page 776. The main hypothesis asks for the property that the evaluations of the Bredon Module are free modules over a commutative ring. Notice that this does not mean that the funtor is free in the sense defined above.

Theorem 2.29 (Künneth Theorem for Bredon cohomology). *Let X be a G-CW complex and let Y be an H-CW complex. Let $\mathfrak{F}$ and $\mathfrak{F}^{'}$ be families of subgroups of G and Y containing the isotropy groups of cells in X, respectively Y. Assume that M and N are covariant Bredon functors defined on the orbit categories $\mathcal{O}_G$, respectively $\mathcal{O}_H$, with the property that $M(G/G^{'})$, respectively $N(H/H^{'})$ are free modules for each pair of objects $G/G^{'}$, $H/H^{'}$. Denote by $\mathfrak{F}\times\mathfrak{F}^{'}$ the family of subgroups of the product which is given as products of subgroups of G and H, and let $M\otimes N$ be the Bredon module defined in this category.*

Then the product $X\times Y$ with the diagonal action is a $G\times H$-CW complex, and there exists a short exact sequence

$$0\to\bigoplus_{i+j=n}H_i^{\mathcal{O}_{\mathfrak{F}}}(X)\otimes H_j^{\mathcal{O}_{\mathfrak{F}^{'}}}(Y)$$
$$\to H_n^{\mathfrak{F}\times\mathfrak{F}^{'}}(X\times Y, M\otimes N)\to\bigoplus_{i+j=n}\mathrm{Tor}(H_i^{\mathcal{O}_{\mathfrak{F}}}(X), H_j^{\mathcal{O}_{\mathfrak{F}^{'}}}(Y))\to 0.$$

For the Universal Coefficient Theorem for Bredon cohomology, in addition to the freenes of the evaluation of the functor on each object, there is a requirement of a basis compatibility in a dual basis which is a direct consequence of Frobenius reciprocity for the complex representation ring with the characters as basis. We give the definition below.

Condition 2.30. Let G be a discrete group, Let $M_?$ and $M^?$ be covariant, respectively contravariant functors defined on a subcategory $\mathcal{F}_G$ of the orbit category $\mathcal{O}$ agreeing on objects. Suppose that

- There exists for every object G/H a choice of a finite basis $\{\beta_{i^H}\}$ expressing $M_?(G/H)=M^?(G/H)$ as the finitely generated, free abelian group on $\{\beta_{i^H}\}$ and isomorphisms $a_H : M^?(G/H)\overset{\cong}{\to}\mathbb{Z}[\{\beta_{i^H}\}]\overset{\cong}{\leftarrow}M_?(G/H) : b_H$.
- For the covariant functor $\widehat{M}:=Hom_{\mathbb{Z}}(M^?(\ \ ),\mathbb{Z})$, the dual basis $\{\widehat{\beta}_{i^H}\}$ of $Hom_{\mathbb{Z}}(\mathbb{Z}[\{\beta_{i^H}\}],\mathbb{Z})$ and the isomorphisms a_H and b_H, the following

diagram is commutative:

$$\begin{array}{ccccccc}
M_?(G/H) & \xrightarrow{b_H} & \mathbb{Z}[\{\beta_{i^H}\}] & \xrightarrow{D_H} & \mathbb{Z}[\{\widehat{\beta}_{i^H}\}] & \xrightarrow{\widehat{a_H}} & \widehat{M}(G/H) \\
\Big\downarrow{\scriptstyle M_?(\phi)} & & & & & & \Big\downarrow{\scriptstyle \widehat{M}(\phi)} \\
M_?(G/K) & \xrightarrow[b_K]{} & \mathbb{Z}[\{\beta_{j^K}\}] & \xrightarrow[D_K]{} & \mathbb{Z}[\{\widehat{\beta}_{j^K}\}] & \xrightarrow[\widehat{a_K}]{} & \widehat{M}(G/K)
\end{array}$$

Where D_H, D_K are the duality isomorphisms associated to the bases and $\phi : G/H \to G/K$ is a morphism in the orbit category.

Conditions 2.30 are satisfied in some cases:

- Constant coefficients $\mathbb{Z}$.
- The complex representation ring functors defined on the family $\mathcal{FIN}$ of finite subgroups, $\mathcal{R}^?$, $\mathcal{R}_?$. A computation using characters as bases and Frobenius reciprocity yields conditions 2.30.
- Consider a discrete group G and a normalized torsion cocycle
$$\alpha \in Z^2(G, S^1),$$
take the α- and α^{-1} twisted representation ring functors $\mathcal{R}^{\alpha}_?$ ${\mathcal{R}_\alpha}^?$ defined on the objects G/H, where H belongs to the family $\mathcal{FIN}$ of finite subgroups. Consider for every object G/H the cocycles $i_H^*(\alpha)$, where $i_H : H \to G$ is the inclusion, and assume without loss of generality that they are normalized and correspond to a family of Schur covering groups in central extensions $1 \to \mathbb{Z}/n_H \to H^* \to H \to 1$.

  We select the set $\{\beta_H\}$ given as the set of characters of irreducible representations of H^* where $\mathbb{Z}/n_H$ acts by multiplication with a primitive n_H-th root of unity. Given a choice of sections for the quotient maps $H^* \to H$, one can construct isomorphisms ${}^{i^*(\alpha)}\mathcal{R}(G/H) \xrightarrow{\cong} \mathbb{Z}[\{\beta_H\}]$. The orthogonality relations and Frobenius reciprocity for their twisted characters guarantee that conditions 2.30 yield.

Theorem 2.31 (Universal Coefficient Theorem for Bredon Cohomology). *Let X be a proper, finite G-CW complex. Let $M^?$ and $M_?$ be a pair of functors satisfying conditions 2.30. Then, there exists a short exact sequence of abelian groups*

$$0 \to \operatorname{Ext}_{\mathbb{Z}}(H^{\mathcal{O}_G}_{n-1}(X, M_?), \mathbb{Z}) \to H^n_{\mathcal{O}_G}(X, M^?) \to \operatorname{Hom}_{\mathbb{Z}}(H^{\mathcal{O}_G}_n(X, M_?), \mathbb{Z}) \to 0$$

2.4. Computations based on structural properties of orbit categories of groups. There exist examples of conditions on a group which have consequences on the particular shape that an orbit category might take.

We give for this the example of the family of finite groups of a group which is a central extension by a finite cyclic group of a discrete group which is classified by a second degree cohomology class with coefficients on the finite cyclic group.

Under this condition, there exists a bijective correspondence between finite subgroups of the central extensions and inverse images of finite subgroups in the original group.

Having the aim of computing Bredon cohomology with coefficients in twisted complex representations, there exist an *untwisting procedure* to change twisted coefficients in favor of the extension group, and a certain class of representations. Let us recall the needed definitions.

DEFINITION 2.32. Let $1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{H} \to H \to 1$ be a central extension. Let k be a natural number with $0 \leq k \leq n$. Let V be a complex vector space. A k-central representation of $\tilde{H}$ is a homomorphism $\tilde{H} \to \mathrm{GL}(V)$, where the generator $t \in \mathbb{Z}/n\mathbb{Z}$ acts by multiplication by $e^{2\pi i k/n}$.

DEFINITION 2.33. The k-central representation group of $\tilde{H}$, denoted by $R_k(\tilde{H})$, is the Grothendieck group of isomorphism classes of k-central representations of $\tilde{H}$.

The k-central representation group is a contravariant coefficient system. Given a central extension of discrete groups, $1 \to \mathbb{Z}/n\mathbb{Z} \to \tilde{G} \to H \to 1$, we denote by $\mathcal{R}^?$ the functor

$$\mathcal{R}^? : \mathsf{Or}_{\mathcal{FIN}}(\tilde{G}) \to \mathbb{Z}\text{ - }\mathsf{MODULES}$$
$$\tilde{G}/\tilde{H} \mapsto R_k(\tilde{H}).$$

The following theorem appeared as 4.4 in page 57 of [**10**]. It was originally stated for the classifying space for proper actions, but it holds for any proper G-CW complex.

It is the main input for the untwisting argument for twisted equivariant K-Theory of discrete torsion twists described below.

THEOREM 2.34. *Let G be a discrete group and let $\alpha \in Z^2(G; S^1)$ be a cocycle taking values in $\mathbb{Z}/n\mathbb{Z} \subseteq S^1$. Consider the extension associated to α*

$$1 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow G_\alpha \xrightarrow{\rho} G \longrightarrow 1.$$

Denote by X a G-CW complex with finite groups as cell stabilizers.

Then, the map ρ gives an isomorphism of abelian groups between the Bredon cohomology groups of X with coefficients in the α-twisted representation group and the G_α- equivariant Bredon cohomology groups of X with coefficients in the so-called 1-central group representation Bredon module (defined in 2.33). In symbols,

$$H^*_{\mathcal{O}_G}(X; \mathcal{R}^G_\alpha) \xrightarrow{\rho^*} H^*_{\mathcal{O}_{G_\alpha}}(X; \mathcal{R}^{G_\alpha}_1)$$

is an isomorphism.

Further instances of computations based on knowledge about the orbit category of specific examples are often stated in terms of the existence of particular model for classifying spaces.

These computations hold more generally, for any equivariant cohomology or homology theory, including Bredon cohomology, and we mention the following instances:

- Conditions M and NM of page 294 in [**45**], which hold together for Fuchsian groups, One relator groups, and extensions $1 \to \mathbb{Z}^n \to G \to F$ for finite F acting freely outside 0.
- The computation of equivariant homology theories for classifying spaces of families of Graph product groups of [**35**].
- Condition C of [**3**].

3. Bredon Co-homology as recipient for equivariant Chern Characters

An equivariant Chern Character is a natural transformation between equivariant cohomology theories. Let us briefly recall the notion of an equivariant cohomology theory. We stress that the equivariant cohomology theories considered here are *naive*, and remit to [**18**],

DEFINITION 3.1. Let G be a group and fix an associative ring with unit R. A G-Cohomology Theory with values in R-modules is a collection of contravariant functors $\mathcal{H}_G^n$ indexed by the integer numbers $\mathbb{Z}$ from the category of G-CW pairs together with natural transformations $\partial_G^n : \mathcal{H}_G^n(A) := \mathcal{H}_G^n(A, \emptyset) \to \mathcal{H}_G^{n+1}(X, A)$, such that the following axioms are satisfied:

(1) If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW pairs, then $\mathcal{H}_G^n(f_0) = \mathcal{H}_G^n(f_1)$ for all n.

(2) Given a pair (X, A) of G-CW complexes, there is a long exact sequence

$$\dots \stackrel{\mathcal{H}_G^{n-1}(i)}{\to} \mathcal{H}_G^{n-1}(A) \stackrel{\partial_G^{n-1}}{\to} \mathcal{H}_G^n(X, A) \stackrel{\mathcal{H}_G^n(j)}{\to} \mathcal{H}_G^n(X)$$

$$\stackrel{\mathcal{H}_G^n(i)}{\to} \mathcal{H}_G^n(A) \stackrel{\partial_G^n}{\to} \mathcal{H}_G^{n+1}(X, A) \stackrel{\mathcal{H}_{n+1}(j)}{\to} \dots$$

where $i : A \to X$ and $j : X \to (X, A)$ are the inclusions.

(3) Let (X, A) be a G-CW pair and $f : A \to B$ be a cellular map. The canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism

$$\mathcal{H}_G^n(X \cup_f B, B) \stackrel{\cong}{\to} \mathcal{H}_G^n(X, A)$$

(4) Let $\{X_i \mid i \in \mathcal{I}\}$ be a family of G-CW-complexes and denote by $j_i : X_i \to \coprod_{i\in\mathcal{I}} X_i$ the inclusion map. Then the map

$$\Pi_{i\in\mathcal{I}}\mathcal{H}_G^n(j_i) : \mathcal{H}_G^n(\coprod_i X_i) \stackrel{\cong}{\to} \Pi_{i\in\mathcal{I}}\mathcal{H}_G^n(X_i)$$

is bijective for each $n \in \mathbb{Z}$.

A G-Cohomology Theory is said to have a multiplicative structure if there exist natural, graded commutative $\cup$- products

$$\mathcal{H}_G^n(X, A) \otimes \mathcal{H}_G^m(X, A) \to \mathcal{H}_G^{n+m}(X, A)$$

Let $\alpha : H \to G$ be a group homomorphism and X be a H-CW complex. The induced space $\mathrm{ind}_\alpha X$, is defined to be the G-CW complex defined as the quotient space $G \times X$ by the right H-action given by $(g, x) \cdot h = (g\alpha(h), h^{-1}x)$.

An Equivariant Cohomology Theory consists of a family of G-Cohomology Theories $\mathcal{H}_G^*$ together with an induction structure determined by graded ring homomorphisms

$$\mathcal{H}_G^n(\mathrm{ind}_\alpha(X, A)) \to \mathcal{H}_H^n(X, A)$$

which are isomorphisms for group homomorphisms $\alpha : H \to G$ whose kernel acts freely on X satisfying the following conditions:

(1) For any n, $\partial_H^n \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_G^n$.

(2) For any group homomorphism $\beta : G \to K$ such that $\ker \beta \circ \alpha$ acts freely on X, one has

$$\mathrm{ind}_{\alpha\circ\beta} = \mathcal{H}^n_K(f_1 \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha) : \mathcal{H}^n_K(\mathrm{ind}_{\beta\circ\alpha}(X, A)) \to \mathcal{H}^n_H(X, A)$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha \to \mathrm{ind}_{\beta\circ\alpha}$ is the canonical G-homeomorphism.

(3) For any $n \in \mathbb{Z}$, any $g \in G$, the homomorphism

$$\mathrm{ind}_{c(g):G\to G} : \mathcal{H}^n_G(\mathrm{ind})_{\mathrm{c(g):G\to G}}(\mathrm{X, A})) \to \mathcal{H}^{\mathrm{n}}_{\mathrm{G}}(\mathrm{X, A})$$

agrees with the map $\mathcal{H}^n_G(f_2)$, where $f_2 : (X, A) \to \mathrm{ind}_{c(g):G\to G}$ sends x to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in G.

EXAMPLE 3.2 (Examples of equivariant cohomology theories). We now describe the cohomology theories which will be relevant for the computations below.

(1) Complex Equivariant K-theory was defined via vector bundles for finite proper G-CW complexes in [**49**]. For any proper orbit G/H one has

$$KU^*(G/H) = \begin{cases} \mathcal{R}_{\mathbb{C}}(H) \; * = 2k \\ 0 * = 2k + 1 \end{cases} .$$

(2) Complex, twisted equivariant K-Theory with a twist given by a torsion element in $H^3(BG, \mathbb{Z})$ was defined in [**20**]. This is an equivariant cohomology theory which is a submodule over untwisted, complex K- theory in the sense of Oliver and Lück described above.

Twisted equivariant K- Theory for any twist in $H^3(EG \times_G X, \mathbb{Z})$ was defined in the Fredholm picture in [**7**]. The aproppriate axiomatic for twisted equivariant K-theory for any third cohomology twist is that of parametrized cohomology theories [**8**] and are more general than the viewpoint adopted here. The equivariant Chern Character, however, is not a rational isomorphism in the twited case.

3.1. The Atiyah-Hirzebruch Spectral Sequence. The Atiyah-Hirzebruch spectral sequence for equivariant cohomology theories was developed by Davis and Lück in [**17**]. A detailed deduction and a presentation of the relevant details is available in [**18**], page 108 construction 3.214.

THEOREM 3.3. *Let $\mathcal{H}^*$ be an equivariant cohomology theory. Then, there exists a spectral sequence which has E_2-term Bredon cohomology with coefficients in the functor*

$$\mathcal{H}^q : G/H \mapsto \mathcal{H}^q_G(G/H)$$

$$E_2^{p,q} = H^q_{\mathcal{O}_G}(X, \mathcal{H}^q),$$

which converges contidionally to the equivariant cohomology theory modules

$$\mathcal{H}^*_G(X).$$

It is a consequence of the existence of the equivariant Chern character, that the Atiyah-Hirzebruch spectral sequence rationally collapses.

REMARK 3.4 (The $RO(G)$-graded Atiyah-Hirzebruch spectral sequence). There is a discussion of a spectral sequence to compute $RO(G)$- graded cohomology theories out of $RO(G)$-graded Bredon cohomology in [**38**].

REMARK 3.5 (Third differential of the equivariant Atiyah Hirzebruch spectral sequence for Equivariant complex K-Theory). Let us restrict to complex equivariant K- theory. While the equivariant Atiyah-Hirzebruch spectral sequence rationally collapses, and for non-equivariant complex K-theory there exist closed formulas for the first non-vanishing differential, d_3 (and even for higher degree ones, in terms of secondary cohomology operations), as of 2022 there exists no closed formula for the third differential

$$d_3 : H^p_{\mathcal{O}_G}(X, \mathcal{R}_{\mathbb{C}}) \to H^{p+3}_{\mathcal{O}_G}(X, \mathcal{R}_{\mathbb{C}}).$$

See [**19**] for a discussion of the failure of the integral differential to be an isomorphism.

Work by Uribe and Gómez [**25**] introduced a decomposition of (non twisted) equivariant complex K- theory of finite groups G which have a normal abelian subgroup A which acts trivially on a finite G-CW complex X in summands of twisted equivariant K- Theory corresponding to a number of twists corresponding to irreducible representations of A.

The outcome is that they are able to identify the third differential of the (untwisted) equivairiant Atiyah-Hirzebruch spectral sequence with a special instance of the third differential of the twisted Segal spectral sequence constructed in [**8**].

While the differentials of the equivariant Atiyah -Hirzebruch spectral sequence are a natural transformation, and even in the parametrized setting they are identified by Theorem 5.7 in [**5**] in homotopy theoretical terms as maps between classifying spectra for Bredon cohomology, and there exist constructions of the cohomology operations in [**24**], it has not been possible to give a complete list of candidates for the relevant cohomological operations between Bredon cohomology groups.

In the $RO(G)$-graded setting, the definition of the adequate version of the Steenrod Algebra goes back to Oruç [**54**], and efforts to address the analogous problem of determining the possible operations are [**57**], and [**61**].

3.2. Equivariant cohomological Chern Characters.

The equivariant Chern character for was addressed first by Slominska for equivariant K-Theory of finite groups in [**62**].

In the context of the Baum-Connes conjecture, the need for decomposition of equivariant K-homology into informations of fixed point sets of finite cyclic grubgroups led to a specific construction, named the delocalized Chern character in [**11**], and the formalization in the terms refered here was done by Lück mainly in the articles [**46**], [**47**].

The following result was proved as Theorem 4.2 in page 1041 of [**47**].

THEOREM 3.6 (The equivariant Chern Character). *Let R be a ring containing the Rational numbers. Let $\mathcal{H}^*$ be a proper equivariant cohomology theory with valuer in R-modules. Suppose that the $\mathcal{S}_G$- module $\mathcal{H}^q \circ \mathrm{pr}$ is injective as $\mathcal{S}_G$-module for every group G and every $q \in \mathbb{Z}$. Then, we obtain a transformation of proper equivariant cohomology theories*

$$\mathrm{ch}^n : \mathcal{H}^n_? : \prod_{p+q=n} H^p_{\mathcal{S}_G}(X, \mathcal{H}^q).$$

The R- map is bijective for all proper relatively finite G-CW pairs (X, A). if $\mathcal{H}^*$ satisfies the disjoint union axiom, then the R- map is bijective for all proper G-CW pairs (X, A).

The natural transformation is constructed using a composition of eight different construction in page 1040 of Lück, to where we refer for further details.

Remark 3.7 (Orbifold versions). Adem and Ruan introduced an orbifold bersion of both twisted complex K-theory and Bredon cohomology for a discrete torsion twist.

The latter one turns out to be isomorphic to Chen-Ruan cohomology of orbifolds in [**2**]. See [**1**], 3.3 and 3.10 in pages 60 and 77, and for a more detailed exposition.

Remark 3.8 (Delocalized versions). In connection with the rationalized Baum-Connes assembly map, delocalization refers to modified versions of the eqivariant Chern character, which within a geometric setting can be thought of being defined before inverting the Thom class of normal bundles of inclusions of fixed point sets, [**11**], paragraph 9. A good discussion of these versions of the Chern Character,including the relation to the equivariant Chern character presented here is given in [**50**].

Remark 3.9 (Homotopy theoretical refinements). As of 2022, the most refined homotopy theoretical versions of the Chern Character are presented in [**39**]. The results initiated in the Author's Ph. D Thesis and include a study of external duality, and a homology representation theorem. See also [**40**].

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