THE COMPLETION THEOREM IN TWISTED EQUIVARIANT K-THEORY FOR PROPER ACTIONS.

NOÉ BÁRCENAS AND MARIO VELÁSQUEZ

Abstract. We compare different algebraic structures in twisted equivariant K-Theory for proper actions of discrete groups. After the construction of a module structure over untwisted equivariant K-Theory, we prove a completion Theorem of Atiyah-Segal type for twisted equivariant K-Theory. Using a Universal coefficient Theorem, we prove a cocompletion Theorem for Twisted Borel K-Homology for discrete Groups.

The Completion Theorem in equivariant K-theory by Atiyah and Segal [6] had a remarkable influence on the development of topological K-theory and computational methods related to it.

Twisted equivariant K-theory for proper actions of discrete groups was defined in [9] and further computational tools, notably a version of Segal's spectral sequence have been developed by the authors and collaborators in [10], and [11].

In this work, we examinate Twisted equivariant K-theory with the above mentioned methods as a module over its untwisted version and prove a generalization of the completion theorem by Atiyah and Segal.

It turns out that in the case of groups which admit a finite model for the classifying space for proper actions $\underline{E}G$, the ring defined as the zeroth (Untwisted) G-equivariant K-theory ring $K^0_G(\underline{E}G)$ is Noetherian. Hence, usual commutative algebraic methods can be applied to deal with completion problems on noetherian modules over it, as it has been done in other contexts in the literature, [6], [21], [14], [18].

Using a universal coefficient theorem developed in the analytical setting [23], we prove a version of the co-completion theorem in twisted Borel Equivariant K-homology, thus extending results in [17] to the twisted case.

This work is organized as follows:

In section 1, we collect results on the multiplicative (twist-mixing) structures on twisted equivariant K-theory following its definition in [9]. We also recall in this section the spectral sequence of [10] and the needed notions of Bredon-type cohomology and G-CW complexes.

In section 2, we examine the ring Structure over the ring $K^0_G(\underline{E}G)$, and establish the noetherian condition for certain relevant modules over it given by twisted equivariant K-theory groups.

The main theorem, 3.6 is proved in section 3.

Theorem. *Let G be a group which admits a finite model for $\underline{E}G$, the classifying space for proper actions. Let X be a finite, proper G-CW complex. Then, the pro-homomorphism*

$$\varphi_{\lambda,p} : \left\{K^*_G(X,P)/\mathbf{I}_{G,\underline{E}G}{}^n K^*_G(X,P)\right\} \longrightarrow \left\{K^*_G(X\times EG^{n-1}, p^*(P))\right\}$$

is a pro-isomorphism. In particular, the system $\left\{K^*_G(X\times EG^{n-1}, p^*(P))\right\}$ satisfies the Mittag-Leffler condition and the $\lim^1$ term is zero.

Finally, section 4 deals with the proof of the cocompletion theorem 4.6 involving Twisted Borel K-homology.

Theorem. *Let G be a discrete group. Assume that G admits a finite model for $\underline{E}G$. Let X be a finite G-CW complex and $P \in H^3(X \times_G EG, \mathbb{Z})$. Let $\mathbf{I}_{G,\underline{E}G}$ be the augmentation ideal. Then, there exists a short exact sequence*

$$\operatorname{colim}_{n\geq 1} \operatorname{Ext}^1_{\mathbb{Z}}(K^*_G(X,P)/\mathbf{I}^n_{G,\underline{E}G}, \mathbb{Z}) \to K_*(X \times_G EG, p^*(P)) \to \operatorname{colim}_{n\geq 1} K^*_G(X,P)/\mathbf{I}^n_{G,\underline{E}G}$$

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1. Preliminaries on (twisted) Equivariant K-theory for Proper and Discrete actions

Definition 1.1. Recall that a G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup_{-1 \leq n} X_n$ with $X_1 = \emptyset$, $X_0 = A$ where every space is inductively obtained from the previous one by attaching cells in pushout diagrams

$$\begin{array}{ccc} \coprod_i S^{n-1} \times G/H_i & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_i D^n \times G/H_i & \longrightarrow & X_n \end{array}$$

We say that a proper G-CW complex is finite if it is constructed out of a finite number of cells $G/H \times D^n$.

We recall the notion of the classifying space for proper actions:

Definition 1.2. Let G be a discrete group. A model for the classifying space for proper actions is a G-CW complex $\underline{E}G$ with the following properties:

- All isotropy groups are finite.
- For any proper G-CW complex X there exists up to G-homotopy a unique G-map $X \to \underline{E}G$.

The classifying space for proper actions always exists, it is unique up to G-homotopy and admits several models. The following list contains some examples. We remit to [19] for further discussion.

- If G is a compact group, then the singleton space is a model for $\underline{E}G$.
- Let G be a group acting properly and cocompactly on a Cat(0) space X. Then X is a model for $\underline{E}G$.
- Let G be a Coxeter group. The Davis complex is a model for $\underline{E}G$.
- Let G be a mapping class group of a surface. The Teichmüller space is a model for $\underline{E}G$.

Let G be a discrete group. a model for the classifying space for free actions EG is a free contractible G-CW complex. Given a model EG for the classifying space for free actions, the space BG is the CW-complex EG/G.

The following result is proved in [17], lemma 26 in page 6.

Lemma 1.3. *Let X be a finite proper G-CW complex. Then $X \times_G EG$ is homotopy equivalent to a CW complex of finite type.*

Twisted equivariant K-Theory. Twisted Equivariant K-Theory for proper actions of discrete groups was introduced in [9]. In what follows we will recall its definition using Fredholm bundles and its properties following the above mentioned article. The crucial diference to [9] is the use of graded Fredholm bundles, which are needed for the definition of the multiplicative structure.

Let $\mathcal{H}$ be a separable Hilbert space and

$$\mathcal{U}(\mathcal{H}) := \{U : \mathcal{H} \to \mathcal{H} \mid U \circ U^* = U^* \circ U = \mathrm{Id}\}$$

the group of unitary operators acting on $\mathcal{H}$. Let $\mathrm{End}(\mathcal{H})$ denote the space of endomorphisms of the Hilbert space and endow $\mathrm{End}(\mathcal{H})_{c.o.}$ with the compact open topology. Consider the inclusion

$$\begin{aligned} \mathcal{U}(\mathcal{H}) &\to \mathrm{End}(\mathcal{H})_{c.o.} \times \mathrm{End}(\mathcal{H})_{c.o.} \\ U &\mapsto (U, U^{-1}) \end{aligned}$$

and induce on $\mathcal{U}(\mathcal{H})$ the subspace topology. Denote the space of unitary operators with this induced topology by $\mathcal{U}(\mathcal{H})_{c.o.}$ and note that this is different from the usual compact open topology on $\mathcal{U}(\mathcal{H})$. Let $\mathcal{U}(\mathcal{H})_{c.g}$ be the compactly generated topology associated to the compact open topology, and topologize the group $P\mathcal{U}(\mathcal{H})$ from the exact sequence

$$1 \to S^1 \to \mathcal{U}(\mathcal{H})_{c.g.} \to P\mathcal{U}(\mathcal{H}) \to 1.$$

Let $\mathcal{H}$ be a Hilbert space. A continuous homomorphism a defined on a Lie group G, $a : G \to P\mathcal{U}(\mathcal{H})$ is called stable if the unitary representation $\mathcal{H}$ induced by the homomorphism $\widetilde{a} : \widetilde{G} = a^*\mathcal{U}(\mathcal{H}) \to \mathcal{U}(\mathcal{H})$ contains each of the irreducible representations of $\widetilde{G}$

Definition 1.4. Let X be a proper G-CW complex. A projective unitary G-equivariant stable bundle over X is a principal $P\mathcal{U}(\mathcal{H})$-bundle

$$P\mathcal{U}(\mathcal{H}) \to P \to X$$

where $P\mathcal{U}(\mathcal{H})$ acts on the right, endowed with a left G action lifting the action on X such that:

- the left G-action commutes with the right $P\mathcal{U}(\mathcal{H})$ action, and
- for all $x \in X$ there exists a G-neighborhood V of x and a G_x-contractible slice U of x with V equivariantly homeomorphic to $U \times_{G_x} G$ with the action

  $$G_x \times (U \times G) \to U \times G, \quad k \cdot (u, g) = (ku, gk^{-1}),$$

  together with a local trivialization

  $$P|_V \cong (P\mathcal{U}(\mathcal{H}) \times U) \times_{G_x} G$$

  where the action of the isotropy group is:

  $$\begin{array}{rcl} G_x \times ((PU(\mathcal{H}) \times U) \times G) & \to & (P\mathcal{U}(\mathcal{H}) \times U) \times G \\ (k, ((F, y), g)) & \mapsto & ((f_x(k)F, ky), gk^{-1}) \end{array}$$

  with $f_x : G_x \to PU(\mathcal{H})$ a fixed stable homomorphism.

Definition 1.5. Let X be a proper G-CW complex. A G-Hilbert bundle is a locally trivial bundle $E \to X$ with fiber on a Hilbert space $\mathcal{H}$ and structural group the group of unitary operators $\mathcal{U}(\mathcal{H})$ with the strong* operator topology. Note that in $\mathcal{U}(\mathcal{H})$ the strong* operator topology and the compact open topology are the same [25]. The Bundle of Hilbert-Schmidt operators with the strong topology between Hilbert Bundles E and F will be denoted by $L_{HS}(E, F)$.

The following result resumes some facts concerning projective unitary stable G-equivariant bundles.

Lemma 1.6. *(i) Given a projective unitary, stable G-equivariant Bundle P, there exists a G-Hilbert bundle $E \to X$ such that the bundle $End_{HS}(E, E)$ has an associated $P\mathcal{U}(\mathcal{H})$ principal, stable G-equivariant bundle isomorphic to P, where $P\mathcal{U}(\mathcal{H})$ carries the *-strong topology.*

(ii) Given projective unitary stable G-equivariant bundles P_1 and P_2, the isomorphism class of the $P\mathcal{U}(\mathcal{H})$ bundle associated to $L_{\mathsf{HS}}(E_1^*, E_2)$ does not depend on the choice of the Hilbert bundles E_i.

Proof. (i) Given a central extension $1 \to S^1 \to \widetilde{G} \to G \to 1$ of G, consider the Hilbert space $L^2_{S^1}(\widetilde{G}) \subset L^2(\widetilde{G})$ defined as the closure of the direct sum of all V-isotypical subspaces, where V is a $\widetilde{G}$-representation where S^1 acts by multiplication. Form the completed sum $\mathcal{H}$ indexed by isomorphism classes of S^1-central extensions $\widetilde{G}$ of G. In symbols:

$$\mathcal{H} = \bigoplus_{\widetilde{G} \in Ext(G, S^1)} L^2_{S^1}(\widetilde{G}) \otimes l^2(\mathbb{N}),$$

and consider the trivial bundle $E = X \times \mathcal{H} \to X$. Form the Bundle of Hilbert endomorphisms $End_{HS}(E, E)$ in the $*$-strong topology [25].

The stability of the projective unitary bundle P gives a group homeomorphism between $P(\mathcal{U}(\mathcal{H})_{c.g}$ and the structural group of the bundle $End_{HS}(E, E^*)$, which is $P(\mathcal{U}(\mathcal{H}))$.

(ii) Follows from the reduction of the structural group $\mathcal{U}(\mathcal{H})$ in the *-strong topology to $P\mathcal{U}(\mathcal{H})$(in the *-strong topology, since the central S^1 acts trivially on $L_{\mathsf{HS}}(E_1^*, E_2)$.) The equivalence of principal bundles and associated bundles, as well as the classification of projective unitary, stable G-equivariant bundles from [9] finish the argument.

□

Definition 1.7. Define $P_1 \otimes P_2$ as the principal $P\mathcal{U}(\mathcal{H})$-bundle associated to $L_{\mathsf{HS}}(E_1^*, E_2)$.

In [9], Theorem 3.8, the set of isomorphism classes of projective unitary stable G-equivariant bundles, denoted by $Bun_{st}^G(X, P\mathcal{U}(\mathcal{H}))$ was seen to be in bijection with the third Borel cohomology groups with integer coefficients $H^3(X \times_G EG, \mathbb{Z})$.

Proposition 1.8. *The map*

$$Bun_{st}^G(X, P\mathcal{U}(\mathcal{H})) \to H^3(X \times_G EG, \mathbb{Z})$$

is an abelian group isomorphism if the left hand side is furnished with the tensor product as additive structure.

Proof. In [9], a classifying G-space $\mathcal{B}$, a universal projective unitary stable G-equivariant bundle $\mathcal{E} \to B$, as well as a homotopy equivalence

$$f : Maps(X, \mathcal{B})^G \to Maps(X \times_G EG, BP\mathcal{U}(\mathcal{H}))$$

were constructed in Theorem 3.8. (This was only stated for π_0 there, but the argument goes over to higher homotopy groups). On the other hand, Theorem 3.8 in [9] gives an isomorphism of sets to the equivalence classes of projective unitary stable G-equivariant bundles $Bun_{st}^G(X, P\mathcal{U}(\mathcal{H}))$. On the isomorphic sets $\pi_0\big(Maps(X, \mathcal{B})^G\big) \cong \pi_0\big(Maps(X \times_G EG, BP\mathcal{U}(\mathcal{H}))\big)$ define the operations

- The operation $*$, given by the unique H-space structure in $BP\mathcal{U}(\mathcal{H}) = K(\mathbb{Z}, 3)$, and
- The operation $\star$, defined in $\pi_0(Maps(X, \mathcal{B})^G)$ as follows. Given maps f_0 and f_1 consider the projective unitary stable G-equivariant bundles $f_i^*(\mathcal{E})$, where $\mathcal{E}$ is the universal bundle and form the classifying map ψ of the projective unitary stable, G-equivariant bundle $f_1^*(\mathcal{E}) \otimes f_2^*(\mathcal{E})$. Define $f_1 \star f_2 = \psi$.

The classification of bundles yields that these operations are mutually distributive and associative, and have a common neutral element given by the constant map. The two operations agree then because of the standard Lemma, see for example Lemma 2.10.10, page 56 in [1].

□

Definition 1.9. Let X be a proper G-CW complex and let $\mathcal{H}$ be a separable Hilbert space. The space $\mathrm{Fred}'(\mathcal{H})$ consists of pairs (A, B) of bounded operators on $\mathcal{H}$ such that $AB - 1$ and $BA - 1$ are compact operators. Endow $\mathrm{Fred}'(\mathcal{H})$ with the topology induced by the embedding

$$\begin{array}{rcl} \mathrm{Fred}'(\mathcal{H}) & \to & \mathsf{B}(\mathcal{H}) \times \mathsf{B}(\mathcal{H}) \times \mathsf{K}(\mathcal{H}) \times \mathsf{K}(\mathcal{H}) \\ (A, B) & \mapsto & (A, B, AB - 1, BA - 1) \end{array}$$

where $\mathsf{B}(\mathcal{H})$ denotes the bounded operators on $\mathcal{H}$ with the compact open topology and $\mathsf{K}(\mathcal{H})$ denotes the compact operators with the norm topology.

We denote by $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ a $\mathbb{Z}_2$-graded, infinite dimensional Hilbert space.

Definition 1.10. Let $U(\widehat{\mathcal{H}})_{c.g.}$ be the group of even, unitary operators on the Hilbert space $\widehat{\mathcal{H}}$ which are of the form

$$\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix},$$

where u_i denotes a unitary operator in the compactly generated topology defined as before.

We denote by $P\mathcal{U}(\widehat{\mathcal{H}})$ the group $U(\widehat{\mathcal{H}})_{c.g.}/S^1$ and recall the central extension

$$1 \to S^1 \to \mathcal{U}(\widehat{\mathcal{H}}) \to P\mathcal{U}(\widehat{\mathcal{H}}) \to 1$$

Definition 1.11. Let X be a proper G-CW complex. The space $\mathrm{Fred}''(\widehat{\mathcal{H}})$ is the space of pairs $(\widehat{A}, \widehat{B})$ of self-adjoint, bounded operators of degree 1 defined on $\widehat{\mathcal{H}}$ such that $\widehat{A}\widehat{B} - I$ and $\widehat{B}\widehat{A} - I$ are compact.

Given a $\mathbb{Z}/2$-graded, stable Hilbert space $\widehat{\mathcal{H}}$, the space $\mathrm{Fred}''(\widehat{\mathcal{H}})$ is homeomorphic to $\mathrm{Fred}'(H)$.

Definition 1.12. We denote by $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ the space of self-adjoint degree 1 Fredholm operators A in $\widehat{\mathcal{H}}$ such that A^2 differs from the identity by a compact operator, with the topology coming from the embedding $A \mapsto (A, A^2 - I)$ in $\mathcal{B}(\mathcal{H}) \times \mathcal{K}(\mathcal{H})$.

The following result was proved in [3], Proposition 3.1 :

Proposition 1.13. *The space* $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ *is a deformation retract of* $\mathrm{Fred}''(\widehat{\mathcal{H}})$.

The above discussion can be concluded telling that $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$ is a representing space for K-theory. The group $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.}$ of degree 0 unitary operators on $\widehat{\mathcal{H}}$ with the compactly generated topology acts continuously by conjugation on $Fred^{(0)}(\widehat{\mathcal{H}})$, therefore the group $P\mathcal{U}(\widehat{\mathcal{H}})$ acts continuously on $Fred^{(0)}(\widehat{\mathcal{H}})$ by conjugation. In [9] twisted K-theory for proper actions of discrete groups was defined using the representing space $\mathrm{Fred}'(\mathcal{H})$, but in order to have multiplicative structure we proceed using $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Let us choose the operator

$$\widehat{I} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

as the base point in $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Choosing the identity as a base point on the space $\mathrm{Fred}^{'}(\mathcal{H})$, gives a diagram of pointed maps

$$\begin{array}{ccccc} \mathrm{Fred}^0(\widehat{\mathcal{H}}) & \xrightarrow{i} & \mathrm{Fred}^{''}(\widehat{\mathcal{H}}) & \xrightarrow{f} & \mathrm{Fred}^{'}(\mathcal{H}) \ , \\ & & \downarrow{\scriptstyle r} & & \\ & & \mathrm{Fred}^0(\widehat{\mathcal{H}}) & & \end{array}$$

where i denotes the inclusion, r is a strong deformation retract and f is a homeomorphism. Moreover, the maps are compatible with the conjugation actions of the groups $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.}, \mathcal{U}(\mathcal{H})_{c.g.}$ and the map $\mathcal{U}(\widehat{\mathcal{H}})_{c.g.} \to \mathcal{U}(\mathcal{H})_{c.g.}$.

Let X be a proper compact G-ANR and let $P \to X$ be a projective unitary stable G-equivariant bundle over X. Denote by $\widehat{P}$ the projective unitary stable bundle obtained by performing the tensor product with the trivial bundle $\mathbb{P}(\widehat{\mathcal{H}})$, $\widehat{P} = P \otimes \mathbb{P}(\widehat{\mathcal{H}})$.

The space of Fredholm operators is endowed with a continuous right action of the group $P\mathcal{U}(\widehat{\mathcal{H}})$ by conjugation, therefore we can take the associated bundle over X

$$\mathrm{Fred}^{(0)}(\widehat{P}) := \widehat{P} \times_{P\mathcal{U}(\widehat{\mathcal{H}})} \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}),$$

and with the induced G action given by

$$g \cdot [(\lambda, A))] := [(g\lambda, A)]$$

for g in G, λ in $\widehat{P}$ and A in $\mathrm{Fred}^{(0)}(\widehat{\mathcal{H}})$.

Denote by

$$\Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))$$

the space of sections of the bundle $\mathrm{Fred}^{(0)}(\widehat{P}) \to X$ and choose as base point in this space the section which chooses the base point $\widehat{I}$ on the fibers. This section exists because the $P\mathcal{U}(\widehat{\mathcal{H}})$ action on $\widehat{I}$ is trivial, and therefore

$$X \cong \widehat{P}/P\mathcal{U}(\widehat{\mathcal{H}}) \cong \widehat{P} \times_{P\mathcal{U}(\widehat{\mathcal{H}})} \{\widehat{I}\} \subset \mathrm{Fred}^{(0)}(\widehat{P});$$

let us denote this section by s.

Definition 1.14. Let X be a connected G-space and P a projective unitary stable G-equivariant bundle over X. The *Twisted G-equivariant K-theory* groups of X twisted by P are defined as the homotopy groups of the G-equivariant sections

$$K_G^{-p}(X; P) := \pi_p \left(\Gamma(X; \mathrm{Fred}^{(0)}(\widehat{P}))^G, s \right)$$

where the base point $s = \widehat{I}$ is the section previously constructed.

1.1. **Topologies on the space of Fredholm Operators.** In [24] a Fredholm picture of twisted K-theory is introduced, using the strong-* operator topology on the space of Fredholm Operators. For the sake of completness, we establish here the isomorphism of these twisted equivariant K-theory groups with the ones described here.

Denote by $\mathrm{Fred}'(\mathcal{H})_{s*}$ the space whose elements are the same as $\mathrm{Fred}'(\mathcal{H})$ but with the strong *-topology on $B(\mathcal{H})$.

Definition 1.15. [24, Thm. 3.15] Let X be a connected G-space and P a projective unitary stable G-equivariant bundle over X. The *Twisted G-equivariant K-theory* groups of X (in the sense of Tu-Xu-Laurent) twisted by P are defined as the homotopy groups of the G-equivariant strong*-continuous sections

$$\mathbb{K}_G^{-p}(X; P) := \pi_p \left(\Gamma(X; \mathrm{Fred}'(P)_{s^*})^G, s \right).$$

The bundle $\mathrm{Fred}'(P)_{s^*}$ is defined in a similar way as $\mathrm{Fred}'(P)$.

We will prove that the functors $K_G^*(-, P)$ and $\mathbb{K}_G^*(-, P)$ are naturally equivalent.

Lemma 1.16. *The spaces* $\mathrm{Fred}'(\mathcal{H})$ *and* $\mathrm{Fred}'(\mathcal{H})_{s^*}$ *are $PU(\mathcal{H})$-weakly homotopy equivalent.*

Proof. The strategy is to prove that $\mathrm{Fred}'(\mathcal{H})_{s^*}$ is a representing of equivariant K-theory. The same proof for $\mathrm{Fred}'(\mathcal{H})$ in [3, Prop. A.22] applies. In particular $GL(\mathcal{H})_{s^*}$ is G-contractible because the homotopy h_t constructed in [3, Prop. A.21] is continuous in the strong*-topology and then the proof applies. □

Using the above lemma one can prove that the identity map defines an equivalence between (twisted) cohomology theories $K_G^*(-, P)$ and $\mathbb{K}_G^*(-, P)$. Then we have that the both definitions of twisted K-theory are equivalents. Summarizing

Theorem 1.17. *For every proper G-CW-complex X and every projective unitary stable G-equivariant bundle over X. We have an isomorphism*

$$K_G^{-p}(X;P) \cong \mathbb{K}_G^{-p}(X;P).$$

Remark 1.18. In order to simplify the notation from now on we denote by $\mathcal{H}$ a $\mathbb{Z}_2$-graded separable Hilbert space and we denote by $\mathrm{Fred}^{(0)}(P)$ the bundle $\mathrm{Fred}^{(0)}(\widehat{P})$.

1.2. **Additive structure.** There exists a natural map

$$\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^G \times \Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^G \to \Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^G,$$

inducing an abelian group structure on the twisted equivariant K- theory groups, which we will define below. Consider for this the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) \times \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) & \xrightarrow{f\circ i} & \mathrm{Fred}^{'}(\widehat{\mathcal{H}}) \times \mathrm{Fred}^{'}(\widehat{\mathcal{H}}) \\ & & \Big\downarrow \circ \\ \mathrm{Fred}^{(0)}(\widehat{\mathcal{H}}) & \xleftarrow{f^{-1}\circ r} & \mathrm{Fred}^{'}(\widehat{\mathcal{H}}) \end{array}$$

where the vertical map denotes composition. As the maps involved in the diagram are compatible with the conjugation actions of the groups $\mathcal{U}(\widehat{\mathcal{H}})_{c.g}$, respectively $\mathcal{U}(\mathcal{H})_{c.g}$ and G, for any projective unitary, stable G-equivariant bundle P, this induces a pointed map

$$\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^G, s) \times (\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^G, s) \to (\Gamma(X;\mathrm{Fred}^{(0)}(\widehat{P}))^G, s).$$

Which defines an additive structure in $K_G^{-p}(X;P)$.

1.3. **Multiplicative structure.** We define an associative product on twisted K-theory.

$$K_G^{-p}(X;P) \times K_G^{-q}(X;P') \to K_G^{-(p+q)}(X;P\otimes P')$$

Induced by the map

$$(A, A') \mapsto A\widehat{\otimes} I + I\widehat{\otimes} A'$$

defined in $\mathrm{Fred}^0(\widehat{\mathcal{H}})$, and $\widehat{\otimes}$ denotes the graded tensor product, see [7] in pages 24-25 for more details. We denote this product by $\bullet$.

Let 0 be the projective unitary, stable G-equivariant bundle associated to the neutral element in $H^3(X\times_G EG, \mathbb{Z})$. The groups $\pi_*\big(\Gamma^G(Fred(0)\big)$ define *untwisted*, equivariant, representable K-Theory in negative degree for proper actions. The extended version via Bott periodicity agrees with the usual definitions of *untwisted*, equivariant K-theory groups for compact G-CW complexes [22], [21] as a consequence of Theorem 3.8, pages 8-9 in [16].

Bredon Cohomology and its Čech Version. (Untwisted) Bredon cohomology has been an useful tool to approximate equivariant cohomology theories with the use of spectral sequences of Atiyah-Hirzebruch type [15], [10].

We will recall a version of Bredon cohomology with local coefficients which was introduced in [10] and compared there to other approaches. These approaches fit all into the general approach of spaces over a category [15], [8].

Let $\mathcal{U} = \{U_\sigma \mid \sigma \in I\}$ be an open cover of the proper G-CW complex X which is closed under intersections and has the property that each open set U_σ is G-equivariantly homotopic to an orbit $G/H_\sigma \subset U_\sigma$ for a finite subgroup H_σ. The existence of such a cover, sometimes known as *contractible slice cover*, is guaranteed for proper G-ANR's by an appropriate version of the slice Theorem (see [2]).

Definition 1.19. Denote by $\mathcal{N}_G\mathcal{U}$ the category with objects $\mathcal{U}$ and where a morphism is given by an inclusion $U_\sigma \to U_\tau$. A twisted coefficient system with values on R-Modules is a contravariant functor $\mathcal{N}_G\mathcal{U} \to R - \mathrm{Mod}$.

Definition 1.20. Let X be a proper G-space with a contractible slice cover $\mathcal{U}$, and let M be a twisted coefficient system. Define the Bredon equivariant homology groups with respect to $\mathcal{U}$ as the homology groups of the category $\mathcal{N}_G\mathcal{U}$ with coefficients in M,

$$H^n_G(X,\mathcal{U};M) := H^n(\mathcal{N}_G\mathcal{U}, M).$$

These are the homology groups of the chain complex defined as the R-module

$$C^{\mathbb{Z}}_*(\mathcal{N}_G\mathcal{U}) \otimes_{\mathcal{N}_G\mathcal{U}} M,$$

given as the balanced tensor product of the contravariant, free $\mathbb{Z}\mathcal{N}_G\mathcal{U}$-chain complex $C^{\mathbb{Z}}_*(\mathcal{N}_G\mathcal{U})$ and M. This is the R-module

$$\bigoplus_{U_\sigma \in \mathcal{N}_G\mathcal{U}} R \otimes_R M(U_\sigma)/K$$

where K is the R-module generated by elements

$$r \otimes x - r \otimes i^*(x),$$

for an inclusion $i : U_\sigma \to U_\tau$.

Remark 1.21 (Coefficients of twisted equivariant K-Theory on contractible Covers)**.** Let $i_\sigma : G/H_\sigma \to U_\sigma \to X$ be the inclusion of a G-orbit into X and consider the Borel cohomology group $H^3(EG \times_G G/H_\sigma, \mathbb{Z})$. Given a class $P \in H^3(EG \times_G X, \mathbb{Z})$, we will denote by $\widetilde{H_{P_\sigma}}$ the central extension $1 \to S^1 \to \widetilde{H_{P_\sigma}} \to H_\sigma \to 1$ associated to the class given by the image of P under the maps

$$\omega_\sigma : H^3(EG \times X, \mathbb{Z}) \xrightarrow{i^*_\sigma} H^3(EG \times_G G/H_\sigma, \mathbb{Z}) \xrightarrow{\cong} H^3(BH_\sigma, \mathbb{Z}) \xrightarrow{\cong} H^2(BH_\sigma, S^1).$$

Restricting the functors $K^0_G(X,P)$ and $K^1_G(X,P)$ to the subsets U_σ gives contravariant functors defined on the category $\mathcal{N}_G\mathcal{U}$.

As abelian groups, the functors $K^*_G(X,P)$ satisfy:

$$K^*_G(U_\sigma, P) = \begin{cases} R_{S^1}(\widetilde{H_{P_\sigma}}) \text{ If } j = 0 \\ 0 \text{ If } j = 1 \end{cases}$$

The Symbol $R_{S^1}(\widetilde{H_{P_\sigma}})$ denotes the subgroup of the abelian group of isomorphisms classes of complex $\widetilde{H_{P_\sigma}}$-representations, where S^1 acts by complex multiplication.

We recall the key result from [10], proposition 4.2

Proposition 1.22. *spectral sequence associated to the locally finite and equivariantly contractible cover $\mathcal{U}$ and converging to $K^*_G(X,P)$, has for second page $E^{p,q}_2$ the cohomology of $\mathcal{N}_G\mathcal{U}$ with coefficients in the functor $\mathcal{K}^0_G(?, P|_?)$ whenever q is even, i.e.*

$$E^{p,q}_2 := H^p_G(X,\mathcal{U};\mathcal{K}^0_G(?, P|_?)) \tag{1.23}$$

and is trivial if q is odd. Its higher differentials

$$d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

vanish for r even.

2. Module Structure for twisted Equivariant K-theory

Let X be a proper G-CW complex, and let P be a stable projective unitary G-equivariant bundle over X. Recall that up to G-equivariant homotopy, there exists a unique map $\lambda : X \to \underline{E}G$. The map λ together with the multiplicative structure give an abelian group homomorphism

$$K_G^0(\underline{E}G) \to K_G^0(X, P),$$

which gives $K_G^0(X, P)$ the structure of a module over the ring $K_G^0(\underline{E}G)$.

We will analyze the structure of $K_G^0(\underline{E}G)$ as a ring. The results in the following lemma are proved inside the proofs of Theorem 4.3, page 610 in [21], and Theorem 6.5, page 21 in [20].

Proposition 2.1. *Let G be a group which admits a finite model for the classifying space for proper actions $\underline{E}G$. Then,*

- *$K_G^0(\underline{E}G)$ is isomorphic to the Grothendieck Group of G-equivariant, finite dimensional complex vector bundles.*
- *The ring $K_G^0(\underline{E}G)$ is noetherian*
- *Let $\mathsf{Or}_{\mathcal{FIN}}(G)$ be the orbit category consisting of homogeneous spaces G/H with H finite and G-equivariant maps. Denote by $R(?)$ the contravariant $\mathsf{Or}_{\mathcal{FIN}}(G)$-module given by assigning to an object G/H the complex representation ring $R(H)$ and to a morphism $G/H \to G/K$ the restriction $R(K) \to R(H)$. Then, there exists a ring homomorphism*
  $$K_G^0(\underline{E}G) \to \lim_{\mathsf{Or}_{\mathcal{FIN}}(G)} R(?)$$
  which has nilpotent kernel and cokernel.
- *Given a prime number p, there exists a vector bundle E of dimension prime to p, such that for every point $x \in \underline{E}G$, the character of the G_x representation $E\mid_x$ evaluated on an element of order not a power of p is 0.*

Proof.

- This is proved in [21], [22], [16], 3.8 in pages 8-9.
- Given a finite proper G-CW complex X, there exists an equivariant Atiyah-Hirzebruch spectral sequence abutting to $K_G^*(X)$ with E_2 term given by $E_2^{p,q} = H^p_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(G)}(X, K^q(G/?))$, where the right hand side denotes *untwisted* Bredon cohomology, defined over the Orbit Category $\mathsf{Or}_{\mathcal{FIN}}(G)$ rather than over the category $\mathcal{N}_G\mathcal{U}$.

  The group $E_2^{p,q}$ can be identified with Bredon cohomology with coefficients on the representation ring if q is even and is zero otherwise.

  Since the Bredon cohomology groups of the spectral sequence are finitely generated if $\underline{E}G$ is a finite G-CW complex, this proves the first assumption
- The edge homomorphism of the Atiyah-Hirzebruch spectral sequence of [15] gives a ring homomorphism $K_G^0(X) \to H^0_{\mathbb{Z}\mathsf{Or}_{\mathcal{FIN}}(G)}(X, R^?)$. The right hand side can be identified with the ring $\lim_{\mathsf{Or}_{\mathcal{FIN}}(G)} R(?)$. The rational collapse of the equivariant Atiyah-Hirzebruch spectral sequence gives the second part.
- Let m be the least common multiple of the orders of isotropy groups H in $\underline{E}G$. For any finite subgroup H, pick up a homomorphism $\alpha_H : H \to \Sigma_m$ corresponding to a free action of H on $\{1, \dots m\}$. Let n be the order of the group $\Sigma_m / Syl_p(\Sigma_m)$ and let $\rho : \Sigma_m \to U(n)$ be the permutation representation. Consider the element $\{V_H\} = \{\mathbb{C}^n[\rho \circ \alpha_H]\}$ in the inverse limit $\lim_{\mathsf{Or}_{\mathcal{FIN}}(G)} R(?)$. According to the second part, there exists a vector bundle E which is mapped to some power $\{{V_H}^{\otimes^k}\}$. The Vector bundle satisfies the required properties.

□

Lemma 2.2. *Let G be a discrete group admiting a finite model for $\underline{E}G$ and P be a stable projective unitary G-bundle over a finite G-CW complex X. Then, the $K^0_G(\underline{E}G)$-modules $K^i_G(X,P)$ are noetherian for $i=0,1$.*

Proof. There exists [10](Theorem 4.9 in page 14), a spectral sequence abutting to $K^*_G(X,P)$. Its E_2 term consists of groups $E_2^{p,q}$, which can be identified with a version of Bredon cohomology associated to an open, G-invariant cover $\mathcal{U}$ consisting of open sets wich are G-homotopy equivalent to proper orbits.

These groups are denoted by $H^p_{\mathbb{Z}\mathcal{N}_G\mathcal{U}}(X, K^q_G(\mathcal{U}))$ and are zero if q is odd. Since X is a proper, compact G-CW complex, the cover can be assumed to be finite. Given an element of the cover U, The group $K^0_G(U)$ is a finitely generated, free abelian group, as it is seen from A.3.4, page 40 in [9], where the groups $K^0_G(U)$ are identified with groups of projective complex representations. Compare also remark 1.21.

In particular the groups $H^p_{\mathbb{Z}\mathcal{N}_G\mathcal{U}}(X, K^q_G(\mathcal{U})$ in the spectral sequence abutting to $K^*_G(X,P)$ are finitely generated. By induction, the groups $E_r^{p,q}$ are finitely generated for all r and hence the term E_∞. Hence $K^i_G(X,P)$ is it for $i=0,1$. Since $K^0_G(\underline{E}G)$ is a noetherian ring, the result follows. □

3. The completion Theorem

Definition 3.1 (Augmentation ideal)**.** Let G be a discrete group. Given a proper G-CW complex, the augmentation ideal $\mathbf{I}_{G,X} \subset K^0_G(X)$ is defined to be the kernel of the homomorphism

$$K_0^G(X) \to K^0_G(X_0) \to K^0_{\{e\}}(X_0)$$

defined by restricting to the zeroth skeleton and restricting the acting group to the trivial group.

Proposition 3.2. *Let X be an n-dimensional proper G-CW complex. Then, any product of n+1 elements in $\mathbf{I}_{G,X}$ is zero.*

Proof. This is proved in [21], lemma 4.2 in page 609. □

We fix now our notations concerning pro-modules and pro-homomorphisms.

Let R be a ring. A pro-module indexed by the integers is an inverse system of R-modules.

$$M_0 \overset{\alpha_1}{\leftarrow} M_1 \overset{\alpha_2}{\leftarrow} M_2 \overset{\alpha_3}{\leftarrow} M_3, \ldots$$

We write $\alpha_n^m = \alpha_{m+1} \circ \ldots \circ \alpha_n : M_n \to M_m$ for $n > m$ and put $\alpha_n^n = \mathrm{id}_{M_n}$.

A *strict* pro-homomorphism $\{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ consists of a collection of homomorphisms $\{f_n : M_n \to N_n\}$ such that $\beta_n \circ f_n = f_{n-1} \circ \alpha_n$ holds for each $n \geq 2$. A pro R-module $\{M_n, \alpha_n\}$ is called pro-trivial if for each $m \geq 1$ there is some $n \geq m$ such that $\alpha_n^m = 0$. A strict homomorphism f as above is called a pro isomorphism if $\ker(f)$ and coker(f) are both pro-trivial. A sequence of strict homomorphisms

$$\{M_n, \alpha_n\} \overset{\{f_n\}}{\to} \{M_n^{'}, \alpha_n^{'}\} \overset{\{g_n\}}{\to} \{M_n^{''}, \alpha_n^{''}\}$$

is called pro-exact if $g_n \circ f_n = 0$ holds for $n \geq 1$ and the pro-R-module $\{\ker(g_n)/\mathrm{im}(f_n)\}$ is pro-trivial. The following lemmas are proved in [5], Chapter 10, section 2, see also [21]:

Lemma 3.3. *Let $0 \to \{M^{'}, \alpha_n^{'}\} \to \{M_n, \alpha_n\} \to \{M_n^{''}, \alpha_n^{''}\} \to 0$ be a pro-exact sequence of pro-R-modules. Then there is a natural exact sequence*

$$0 \to \mathrm{invlim} M_n^{'} \overset{\mathrm{invlim} f_n}{\longrightarrow} \mathrm{invlim} M_n \overset{\mathrm{invlim} g_n}{\longrightarrow} \mathrm{invlim} M_n^{''} \overset{\delta}{\to}$$
$$\mathrm{invlim}^1 M_n^{'} \overset{\mathrm{invlim}^1 f_n}{\longrightarrow} \mathrm{invlim}^1 M_n \overset{\mathrm{invlim}^1 g_n}{\longrightarrow} \mathrm{invlim}^1 M_n^{''}$$

In particular, a pro-isomorphism $\{f_n\} : \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ *induces isomorphisms*

$$\mathrm{invlim}_{n\geq 1} f_n : \mathrm{invlim}_{n\geq 1} \overset{\cong}{\to} \mathrm{invlim}_{n\geq 1} N_n$$
$$\mathrm{invlim}^1_{n\geq 1} f_n : \mathrm{invlim}^1_{n\geq 1} \overset{\cong}{\to} \mathrm{invlim}^1_{n\geq 1} N_n$$

Lemma 3.4. *Fix any commutative noetherian ring R and any ideal $I \subset R$. Then, for any exact sequence $M^{'} \to M \to M^{''}$ of finitely generated R-modules, the sequence*
$$\{M^{'}/I^n M^{'}\} \to \{M/I^n M\} \to \{M^{''}/I^n M^{''}\}$$
of pro-R-modules is pro-exact.

Definition 3.5 (Completion Map)**.** Let X be a proper G-CW complex. Let $p : X \times EG \to X$ be the projection to the first coordinate. The up to G-homotopy unique map $\lambda : X \to \underline{E}G$, combined with Proposition 3.2 defines a pro-homomorphism

$$\varphi_{\lambda,p} : \left\{ K_G^*(X,P)/\mathbf{I}_{G,\underline{E}G}{}^n K_G^*(X,P) \right\} \longrightarrow \left\{ K_G^*(X \times EG^{n-1}, p^*(P)) \right\}$$

Theorem 3.6. *Let G be a group which admits a finite model for $\underline{E}G$, the classifying space for proper actions. Let X be a finite, proper G-CW complex. Then, the pro-homomorphism*
$$\varphi_{\lambda,p} : \left\{ K_G^*(X,P)/\mathbf{I}_{G,\underline{E}G}{}^n K_G^*(X,P) \right\} \longrightarrow \left\{ K_G^*(X \times EG^{n-1}, p^*(P)) \right\}$$
is a pro-isomorphism. In particular, the system $\left\{ K_G^*(X \times EG^{n-1}, p^*(P)) \right\}$ satisfies the Mittag-Leffler condition and the $\lim^1$ *term is zero.*

Proof. Due to propositions 2.1 and 2.2, we are dealing with a noetherian ring $K_G^0(\underline{E}G)$ and the noetherian modules $K_G^*(X,P)$ over it. Hence, we can use lemmas 3.4 and 3.3, and the 5-lemma for pro-modules and pro-homomorphisms to prove the result by induction on the dimension of X and the number of cells in each dimension.

Assume that $X = G/H$ for a finite group H. Then, the completion map fits in the following diagram

$$\begin{array}{ccc}
\left\{ K_G^*(G/H,P)/\mathbf{I}^n_{G,\underline{E}G} \right\} & \longrightarrow & \left\{ K_G^0(G/H \times EG^{n-1}, p^*(P)) \right\} . \\
\mathrm{ind}_{H\to G} \downarrow \cong & & \cong \downarrow \mathrm{ind}_{H\to G} \\
\left\{ K_H^*(\{\bullet\}, P\mid_{eH})/J^n \right\} & & \left\{ K_H^*(EH^{n-1}, p^*(P)) \right\} \\
\downarrow & & \downarrow = \\
\left\{ K_H^*(\{\bullet\}, P\mid_{eH})/\mathbf{I}^n_{H,\{\bullet\}} \right\} & \longrightarrow & \left\{ K_H^*(EH^{n-1}, p^*(P)) \right\}
\end{array}$$

The higher vertical maps are induction isomorphisms, and the ideal J is generated by the image of $\mathbf{I}_{G,\underline{E}G}$ under the map $\mathrm{ind}_{H\to G} \circ \lambda$. The lower horizontal map is a pro-isomorphism as a consequence of the Atiyah-Segal Completion Theorem for Twisted Equivariant K-theory of finite groups, Theorem 1, page 1925 in [18], where it is proved even for compact Lie groups. We will analyze now the lower vertical map and verify that it is a pro-isomorphism of pro-modules. This amounts to prove that $\mathbf{I}_{H,\{\bullet\}}/J$ is nilpotent. Since the representation ring of H, $R(H)$ is noetherian, this holds if every prime ideal which contains J also contains $\mathbf{I}_{H,\{\bullet\}}$. For an element $v \in H$, denote by χ_v the characteristic funcion of the conjugacy class of v. Let H be a finite group. Let ζ be the primitive $|H|$-root of unity given by $e^{\frac{2\pi i}{|H|}}$. Put $A = Z[\zeta]$.

Recall [4], lemma 6.4 in page 63, that given a finite group H, and a prime ideal of the representation ring $\mathcal{P}$, there exists a prime ideal $\mathbf{p} \subset A$ an an element in H, v such that $\mathcal{P} = \chi_v^{-1}(\mathbf{p})$.

Let $\mathcal{P}$ be a prime ideal containing J. We can assume that there exist $s, t \in H$ with $\chi_s^{-1}(t) \in \mathbf{p}$ and such that if p is the characteristic of the field $A/\mathbf{p}$, then the order of s is prime to p.

According to part 3 of proposition 2.1, there exists a complex vector bundle E over $\underline{E}G$ such that p is prime to $\dim_{\mathbb{C}} E$, and the character $\chi_{E|_x}$ is zero after evaluation at the conjugacy class of s. Let $k = \dim E$. Then, $\mathbb{C}^k - E\mid_{\lambda(G/H)}$ is in $\mathbf{I}_{H,\{\bullet\}}$. It follows that $\mathcal{P}$ contains $\mathbf{I}_{H,\{\bullet\}}$.

This proves that the lower horizontal arrow is a pro-isomorphism, the $\lim^1$ term is zero, and the theorem holds for 0-dimensional G-CW complexes X. Assume that the theorem holds for all $n-1$-dimensional, finite proper G-CW complexes. Given a k-dimensional, finite, proper G-CW complex, X there exists a pushout

$$\begin{array}{ccc} \coprod_\alpha S^{k-1} \times G/H & \longrightarrow & \coprod_\alpha D^k \times G/H \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where Y is a k-dimensional, finite proper G-CW complex. The Mayer-Vietoris sequence for twisted equivariant K-theory gives pro-homomorphisms

$$\begin{aligned} \dots \left\{K_G^*(X,P)/\mathbf{I}_{G,\underline{E}G}{}^n\right\} &\longrightarrow \\ \left\{K_G^*(Y,P)/\mathbf{I}_{G,\underline{E}G}{}^n\right\} \bigoplus \bigoplus_\alpha \left\{K_G^*(D^k \times G/H, P)/\mathbf{I}_{G,\underline{E}G}{}^n\right\} &\longrightarrow \\ \bigoplus_\alpha \left\{K_G^*(S^{k-1} \times G/H, P)/\mathbf{I}_{G,\underline{E}G}{}^n\right\} \longrightarrow \{K_G^{*+1}(X,P)/\mathbf{I}_{G,\underline{E}G}{}^n\right\} \dots \end{aligned}$$

By induction, the completion maps for the $n-1$-dimensional G-CW complexes are isomorphisms. By the 5-lemma for pro-groups, the completion map for X is an isomorphism.

□

Corollary 3.7. *Let G be a discrete group with a finite model for $\underline{E}G$. Let $P \in H^3(BG,\mathbb{Z}) \cong H^3(\underline{E}G \times_G EG,\mathbb{Z})$ be a discrete torsion twisting. Consider $I = I_G(\underline{E}G)$ Then ,*

$$K^*(BG, p^*(P)) \cong K_G^*(\underline{E}G, P)_{\hat{\mathbf{I}}}$$

4. The cocompletion Theorem

Given a CW complex X, and a class $P \in H^3(X,\mathbb{Z})$, the twisted K-homology groups are defined in terms of Kasparov bivariant groups involving continuous trace algebras. We remit the reader for preliminaries on Kasparov KK-Theory and its relation to K-homology and Brown-Douglas-Fillmore Theory of extensions to [12], Chapter VII.

Let $\mathcal{H}$ be a separable Hilbert space. Let $\mathcal{K}$ be the C^*-algebra of compact operators in $\mathcal{H}$. Recall that the automorphism group of the C^*-algebra $\mathcal{K}$ consists of the unitary operators with the norm topology $\mathcal{U}(\mathcal{H})$ and the inner automorphisms can be identified with the central S^1. Hence, there is an action of the group $PU(\mathcal{H}) = \mathcal{U}(\mathcal{H})$ on the algebra $\mathcal{K}$.

Remark 4.1. The norm topology and the compactly generated topology agree on compact operators, hence, there is also a conjugation action of the group $\mathcal{U}(\mathcal{H})_{c.g}$ of unitary operators in the compactly generated topology, as well as a group homomorphism $P\mathcal{U}(\mathcal{H}) \to \mathrm{out}(\mathcal{K})$ to the outer automorphism group of the C^*-algebra algebra of compact operators.

Definition 4.2 (Continuous trace Algebras)**.** Let X be a CW complex. Given a cohomology class in the third cohomology group, $H^3(X, \mathbb{Z})$, represented by a principal projective unitary bundle $P : E \to X$, the continuous trace algebra associated to P is the algebra A_P of continuous sections of the bundle $\mathcal{K} \times_{P\mathcal{U}(\mathcal{H})} E \to X$.

Definition 4.3 (KK-picture of twisted K-homology)**.** Let X be a locally compact space and P be a $P(\mathcal{U}(\mathcal{H}))$-principal bundle. The twisted equivariant K-homology groups associated to the projective unitary principal bundle P are defined as the KK-groups

$$K_*(X, P) = KK_*(A_P, \mathbb{C})$$

Continuous trace algebras, used in the operator theoretical definition of twisted K-theory and K-homology belong to the Bootstrap class [13] Proposition IV.1.4.16, in page 334. Hence, the following form of the Universal Coefficient Theorem for KK-Groups holds. It was proved in [23], page 439, Theorem 1.17:

Theorem 4.4 (Universal coefficient Theorem for Kasparok KK-Theory)**.** *Let A be a C^*-algebra belonging to the smallest full subcategory of separable nuclear C^* algebras and which is closed under strong Morita equivalence, inductive limits, extensions, ideals, and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. Then, there is an exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(K^*(A), K^*(B)) \to KK_*(A, B) \to \mathrm{Hom}_{\mathbb{Z}}(K^*(A), K^*(B)) \to 0$$

Where K^* denotes the topological K-theory groups for C^*-algebras.

Specializing to the algebras A_P one has:

Theorem 4.5. *Let X be a locally compact space and P be a $P(\mathcal{U}(\mathcal{H}))$-principal bundle. Then, there is an exact sequence*

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(K^{*-1}(X, P), \mathbb{Z}) \to K_*(X, P) \to \mathrm{Hom}_{\mathbb{Z}}(K^*(X, P), \mathbb{Z}) \to 0$$

We will prove the following cocompletion Theorem, inspired by the methods and results of [17].

Theorem 4.6. *Let G be a discrete group. Assume that G admits a finite model for $\underline{E}G$. Let X be a finite G-CW complex and $P \in H^3(X \times_G EG, \mathbb{Z})$. Let $\mathbf{I}_{G,\underline{E}G}$ be the augmentation ideal. Then, there exists a short exact sequence*

$$\mathrm{colim}_{n\geq 1} \mathrm{Ext}^1_{\mathbb{Z}}(K^*_G(X, P)/\mathbf{I}^n_{G,\underline{E}G}, \mathbb{Z}) \to K_*(X \times_G EG, p^*(P)) \to \mathrm{colim}_{n\geq 1} K^*_G(X, P)/\mathbf{I}^n_{G,\underline{E}G}$$

Proof. Choose a CW complex Y of finite type and a cellular homotopy equivalence $f : Y \to X \times_G EG$. Let $f^n : Y^n \to X \times_G EG^n$ be the map restricted to the skeletons. The pro-homomorphisms

$$\left\{K^*(X \times_G EG^n, p^*(P))\right\} \longrightarrow \left\{K^*(Y^n, p^*(P) \mid Y_n)\right\}$$

are a pro-isomorphism of abelian pro-groups. On the other hand, due to the completion theorem, 3.6, there is a pro-isomorphism

$$\varphi_{\lambda,p} : \left\{K^*_G(X,P)/\mathbf{I}_{G,\underline{E}G}{}^n K^*_G(X,P)\right\} \longrightarrow \left\{K^*_G(X \times_G EG^{n-1}, p^*(P))\right\}$$

Using 4.5, one gets the exact sequence
$0 \to \mathrm{Ext}_{\mathbb{Z}}(K_{*-1}(Y, p^*(P)), \mathbb{Z}) \to K^*(Y, p^*(P)) \to \mathrm{Hom}_{\mathbb{Z}}(K_*(Y, p^*(P)), \mathbb{Z}) \to 0.$

Combining this exact sequence with the pro-isomorphisms given previously, one has the exact sequence

$$\begin{aligned}\mathrm{colim}_{n\geq 1}\, \mathrm{Ext}^1_{\mathbb{Z}}(K^*_G(X,P)/\mathbf{I}^n_{G,\underline{E}G}, \mathbb{Z}) \to \\ K_*(X \times_G EG, p^*(P)) \to \mathrm{colim}_{n\geq 1} K^*_G(X,P)/\mathbf{I}^n_{G,\underline{E}G}\end{aligned}$$

□

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Centro de Ciencias Matemáticas. UNAM, Ap.Postal 61-3 Xangari. Morelia, Michoacán MÉXICO 58089

E-mail address: barcenas@matmor.unam.mx

URL: http://www.matmor.unam.mx/~barcenas

Centro de Ciencias Matemáticas. UNAM, Ap.Postal 61-3 Xangari. Morelia, Michoacán MÉXICO 58089

E-mail address: mavelasquezm@gmail.com

URL: https://sites.google.com/site/mavelasquezm/