RIGIDITY OF ACTIONS ON METRIC SPACES CLOSE TO THREE DIMENSIONAL MANIFOLDS

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Abstract. In this article we propose a metric variation on the C^0-version of the Zimmer program for three manifolds.

After a reexamination of the isometry groups of geometric three-manifolds, we consider homomorphisms defined on higher rank lattices to them and establish a dichotomy betweeen finite image or infinite volume of the quotient.

Along the way, we enumerate classification results for actions of finite groups on three manifolds where available, and we give an answer to a metric variation on topological versions of the Zimmer program for aspherical three-manifolds, as asked by Weinberger and Ye, which are based on the dichotomy stablished in this work and known topological rigidity phenomena for three manifolds.

Using results by John Pardon and Galaz-García-Guijarro, the dichotomy for homomorphisms of higher rank lattices to isometry groups of three manifolds implies that a C_0-isometric version of the Zimmer program is also true for singular geodesic spaces closely related to three dimensional manifolds, namely three dimensional geometric orbifolds and Alexandrov spaces.

A topological version of the Zimmer Program is seen to hold in dimension 3 for Alexandrov spaces using Pardon's ideas.

1. Introduction

The question on the nature of group homomorphisms $\rho : \Gamma \to \mathrm{Diff}(M)$, between a finitely generated group and the group of diffeomorphisms of an n-dimensional smooth manifold, is interesting in many contexts. Particularly, in a series of questions and conjectures known as the Zimmer Program [Zimmer(1987)], [Fisher(2011)], [Fisher(2020)], concerning on the question whether the group homomorphism cannot have large image if the dimension of the manifold is small relative to the rank of the group. As an example of this, in the recent result [Brown et al.(2020)Brown, Fisher, and Hurtado], it is found that a homomorphism $\rho : SL_{k+1}(\mathbb{Z}) \to \mathrm{Diff(M)}$, factorizes through a finite group when $k \geq n-2$, or $k+1 \geq n$ if additionally, the action is known to preserve a finite volume form. This result is greatly generalized for other higher rank semisimple lattices on [Brown et al.(2021)Brown, Fisher, and Hurtado].

The C^0-version of the Zimmer Program, as suggested in [Weinberger(2011)], and [Ye(2020)], [Ye(2019)], asks roughly for changing the category of manifolds and morphisms in the Zimmer Program, from the smooth setting into a topological setting, that is, by considering a group homomorphism from a finitely generated group, and specifically a higher rank lattice, onto the group of homeomorphisms within a prescribed category (topological, smooth, piecewise linear). The following Conjecture is an example of a problem stated in this setting, found in [Ye(2020)]:

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Problem. Any group action of $Sl_n(\mathbb{Z})$ with $n \geq 3$ on a closed aspherical r- manifold by homeomorphisms factors trough a finite group if $r < n$.

We propose the following variation on the C^0-version of Zimmer's program. Let M be a three dimensional geometric manifold. Consider a group homomorphism from a discrete group Γ to the group of isometries of M,

$$\rho : \Gamma \to \mathrm{Isom}(M),$$

What can be said about ρ?

It turns out that there exists a similar pattern for the relationship of finiteness of the image of ρ to the growth of the rank when Γ is a lattice in a semisimple Lie group.

Previous examinations of a related question, focused on homologically infinite actions were performed in [Farb and Shalen(2000)]. The main conclusion, Theorem 1 in page 574 in loc.cit, is that the only homologically infinite actions of irreducible lattices in semisimple Lie groups of real rank greater or equal than two on three dimensional manifolds occur when the manifold is the three-dimensional torus, the lattice is up to conjugacy a finite index subgroup of $Sl_3(\mathbb{Z})$, and the action is isotypically standard.

Our main result below 1.3 gives a result which is valid for all Thurston geometries based on the study of the isometry groups of three dimensional manifolds. Moreover, the results stated for the isometry groups of three dimensional manifolds imply results for the isometry groups of singular spaces closely related to the given three manifold M, such as quotient orbifolds for the action of the lattice determined by the group homomorphism, and a class of singular geodesic length spaces called three dimensional Alexandrov spaces. In addition, the formulation for Alexandrov spaces allows for a topological version of the Zimmer Conjecture.

We now explain in a more detailed fashion the content of the main result. Let us begin by stating the content of the main result for three manifolds.

Among the three dimensional geometries, the most homogeneous ones are S^3, $\mathbb{H}^3$ and $\mathbb{R}^3$ and finite volume quotient of these models have either a finite group of isometries, in the hyperbolic case, or a 3-dimensional group of isometries: $Iso(S^3)$, respectively $\mathbb{R}^3/\mathbb{Z}^3$. The remaining five 3-dimensional homogeneous geometries present a more flexible description of their isometry groups, in such cases we have a fiber bundle structure

$$F \to X \to B,$$

where B is a two dimensional homogeneous geometry if X is one of $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL}_2(\mathbb{R})$ or Nil; and $B = \mathbb{R}$ if X is either Sol or $S^2 \times \mathbb{R}$. In this context, a discrete group acting on the homogeneous space X, acts on the base of the corrresponding fiber bundle, so that we can control the possible quotients of finite volume X by understanding the projected action on B and this is given by the following general dichotomy property:

Dichotomy 1.1. Let Γ be a discrete group of isometries of a any of the 3-dimensional geometric manifolds $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL}_2(\mathbb{R})$ or Nil; then either

- Γ projects to a discrete group of isometries of the base B of the fiber bundle, or
- the orbifold X/Γ has infinite volume.

Moreover, in the cases Sol and $S^2 \times \mathbb{R}$, the projection to the base space is always discrete. As a consequence of this dichotomy, it is possible to compute the isometry groups of finite volume quotients of homogeneous 3-dimensional manifolds

Theorem 1.2. *Let X be a simply connected, homogeneous 3-dimensional manifold and let Γ be a discrete group of isometries of X, such that X/Γ has finite volume. Then the isometry group $Iso(X/\Gamma)$ has finitely many connected components, such that its connected component of the identity is*

- *a closed subgroup of S^1, if X is either $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL}_2(\mathbb{R})$ or Nil;*
- *a closed subgroup of $SO(3) \times S^1$, if X is $S^2 \times \mathbb{R}$;*
- *a closed subgroup of $\mathbb{R}^3/\mathbb{Z}^3$, if $X = \mathbb{R}^3$;*
- *a closed subgroup of $SO(4)$, if $X = S^3$.*

Moreover, $Iso(X/G)$ is finite if X is either $\mathbb{H}^3$ or Sol.

Remark: 1. One should be careful, as the isometry group of a quotient X/G can differ a lot by changing the discrete subgroup G in the same geometry. As an example of this, $Iso(S^3) = O(4)$ realizes the biggest isometry group, but there are quotients S^3/G such that the isometry group $Iso(S^3/G)$ is trivial. Check each section to see examples and a more explicit description for the isometry groups of the corresponding geometry.

The point of view adopted in this note allows us to ask for the nature of group homomorphisms from finitely generated groups, in particular lattices, to the group of isometries of singular spaces which are closely related to three manifolds.

The following result is a consequence of Theorem 1.2

Phenomenon 1.3. If Λ is a non-uniform higher rank lattice in a semisimple Lie group without compact factors, such that Λ acts by isometries on a three dimensional orbifold X as in Theorem 1.2, then the action factors through a finite group.

This phenomenon is the particular form that the Zimmer Program takes for isometries of locally homogeneous three-dimensional orbifolds with finite volume, this is a natural extension to the isometric setting of the Zimmer conjecture obtained for example in [Brown et al.(2020)Brown, Fisher, and Hurtado].

A consequence of 1.3 and 1.2, is the positive answer to following metric variation of problem 1,

Corollary 1.4. *Any group action of $Sl_n(\mathbb{Z})$ with $n \geq 3$ on a closed aspherical r-manifold by isometries factors trough a finite group if $r < n$.*

We now explain the results for singular spaces obtained as a corollary of the results for three dimensional manifolds and their isometry groups.

A fruitful point of view in geometry of singular spaces such as orbifolds and geodesic metric spaces with synthetic notions of curvature has consisted in the analogy, oposition, reduction, and comparison with the case of smooth manifolds. There are many reasons for considering such generalizations, such as limit processes in the Gromov-Hausdorff notion of convergence to yield Alexandrov spaces [Burago et al.(2001)Burago, Burago, and Ivanov], and the conceptual generalizations of Cat(κ)- spaces and Gromov δ-hyperbolic spaces. An important motivation comes directly from classification problems in both the singular setting and the manifold case, where there are examples for progresses in one of the two parts which produce progress in the other one.

To mention some recent examples of reduction arguments let us recall in [Galaz-García et al.(2020)Galaz-Gar the proof of the geometrization of Alexandrov spaces characterizing sufficiently collapsed Alexandrov spaces with finite volume as the orbifold spaces we considered in Theorem 1.2, and in [Mecchia and Seppi(2019)] the proof of a version of the Smale conjecture for spherical orbifolds. For examples of the comparison to manifolds, rigidity results such as the ones related to the stable Cannon Conjecture [Ferry et al.(2019)Ferry, Lück, and Weinberger], and the proof of the Borel conjeture for sufficiently collapsed Alexandrov spaces [Bárcenas and Núñez Zimbrón(2021)], are a precedent to the argumentation of this work.

Finally, work by John Pardon originally oriented towards a proof of the Hilbert-Smith conjecture for three manifolds [Pardon(2013a)] can be extended to the singular case in the setting of Alexandrov spaces because the proof of the Hilberth-Smith conjecture [Repovs and Scepin(1997)] is reduced to a local behaviour(see theorem 36 below):

Theorem 2. If G is a locally compact, topological group, acting faithfully on a three dimensional Alexandrov space by homeomorphisms, then G is a Lie group.

Finally, under the assumption of co-compact quotients, we have the following result, which is proved in corollary 45

Corollary 3. Let X be a geometric 3-orbifold of finite volume, then X admits an isometric action of a higher rank lattice $\Gamma \subset G$ if and only if the group $Iso(X)$ contains the group $SO(3)$. Moreover, the semisimple Lie group G is isotypic of type $SO(3)$ and the lattice is uniform.

One of the final outcomes of the research is the following question:

To what exent a condition of prescribed curvature in metric spaces, such as the alexandrov condition, can be seen as a rigid structure, in the sense of Gromov [Fisher(2011)]**?**

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2. Structure of the paper

Contents

3. Generalities on discrete groups of isometries

If $\widetilde{X}$ is a complete, simply connected, Riemannian manifold and $\Gamma \subset Iso(\widetilde{X})$ a discrete subgroup of isometries, then X/Γ has the structure of a complete Riemannian orbifold. The covering map $\rho : \widetilde{X} \to X$ satisfies the property that $\rho(x) = \rho(y)$ if and only if $\gamma x = y$ for some $\gamma \in \Gamma$. An isometry $\phi : X \to X$ lifts to $\widetilde{\phi} : \widetilde{X} \to \widetilde{X}$ such that $\rho \circ \widetilde{\phi} = \phi \circ \rho$ and for every $\gamma \in \Gamma$ and $x \in \widetilde{X}$ we have

$$\rho \circ \widetilde{\phi}(\gamma x) = \phi \circ \rho(\gamma x) = \phi \circ \rho(x) = \rho \circ \widetilde{\phi}(x),$$

thus there exist $\gamma' \in \Gamma$ such that

$$\widetilde{\phi}(\gamma x) = \gamma' \widetilde{\phi}(x),$$

that is $\widetilde{\phi} \circ \gamma \circ \widetilde{\phi}^{-1} = \gamma'$ and $\widetilde{\phi} \in N = N_{Iso(\widetilde{X})}(\Gamma)$. This tells us that we have the isomorphism

$$Iso(X) \cong N_{Iso(\widetilde{X})}(\Gamma)/\Gamma.$$

Proposition 4. If G is a Lie group and $\Gamma \subset G$ is a discrete subgroup with associated normalizer and centralizer subgroups

$$N = N_G(\Gamma), \qquad Z = Z_G(\Gamma),$$

then the connected components of N and Z coincide. Moreover, if Z_0 denotes such connected component, the projection $\pi : N \to N/\Gamma$ is a covering Lie group homomorphism such $\pi(Z_0) \subset N/\Gamma$ is the connected component of the identity.

Proof. If $g_t \in N$ is a 1-parameter subgroup and $\gamma \in \Gamma$, then $g_t \gamma g_{-t} = \gamma_t$ is a 1-parameter group in Γ, but as Γ is discrete, $\gamma_t = \gamma$ and this tells us that $g_t \in Z$, so that $Z_0 = N_0$. Now, N is a Lie group having Γ as a normal, discrete subgroup so that the projection map

$$\pi : N \to N/\Gamma$$

is a homomorphism of Lie groups and a covering map. In particular, $\pi(N_0)$ is a connected, open Lie subgroup of the same dimension of N/Γ and thus it is the connected component of the identity. □

We are interested in the particular case where $\widetilde{X}$ is a homogeneous space, i.e. its group of isometries acts transitively and X has finite volume. A general setting where this is achieved is when we consider $G = \widetilde{X}$ a simply-connected Lie group with a right-invariant (or left-invariant) Riemannian metric and $X = G/\Gamma$, with $\Gamma \subset G$ a lattice subgroup (i.e. Γ is a discrete subgroup such that G/Γ has finite, left G-invariant volume). As there is an embedding $G \subset Iso(G)$, we have that $N_G(\Gamma) \subset N_{Iso(G)}(\Gamma)$, but it could happen that $N_G(\Gamma)/\Gamma$ is strictly smaller than $Iso(G/\Gamma)$. On the other hand, we can extend Γ to a discrete subgroup of $Iso(G)$ which is not completely contained in G, so that isometry group $Iso(G/\Gamma)$ is decreased.

In the following sections, we will examinate this phenomenon in the Thurston Geometries, and determine the possible isometry groups of the corresponding finite-volume orbifolds.

4. Euclidean Geometry

Recall that a three manifold is Euclidean if it is locally isometric to the Euclidean three dimensional space $\mathbb{R}^3$. The isometry group of the three dimensional space is the semidirect product $E(3) = \mathbb{R}^3 \rtimes O(3)$.

Let Γ be a discrete subgroup of isometries $E(3)$. It is a consequence of the Bieberbach Theorems, as interpreted by Nowacki [Nowacki(1934)], that there exists a free abelian group T of rank ≤ 3 and finite index in Γ.

End of proof of Theorem 1.2 for euclidean geometry. We will now verify the assertion of theorem 1.2 for the isometry group of R^3/Γ by examining the rank of the translation subgroup T.

- If the rank of T is one, then Γ is a finite extension of $\mathbb{Z}$, and $\mathbb{R}^3/T$ is either the interior of a solid torus of the interior of a solid Klein Bottle, depending on the orientability, where the generator of T acts as a screwdriver isometry (combination of a rotation around an axis and a translation along a parallel direction). It follows that $\mathbb{R}^3/T$ has infinite volume.
- If the rank of T is two, then $\mathbb{R}^3/T$ is the total space of a line bundle over either the torus or the Klein bottle, and $\mathbb{R}^3/T$ has infinite volume.
- If the rank of T is three, then the isometry group of $E(3)/\Gamma$ is a finite extension of a rank three torus by a finite subgroup.

□

The classification of (topological) finite group actions on the torus by isometries has been concluded by work of Lee, Shin and Yokura [Lee et al.(1993)Lee, Shin, and Yokura] and Ha, Jo, Kim and Lee [Ha et al.(2002)Ha, Jo, Kim, and Lee].

It follows from the Bieberbach theorems that any topological action on the three torus is topologically conjugated to an isometry; moreover, by the fact that the three dimensional torus is sufficiently large in the sense of Heil and Waldhausen, [Waldhausen(1968)], any homotopy equivalence is homotopic to a homeomorphism, and any two homotopic homeomorphisms are isotopic.

4.1. **Connected components.** The isometry group of co-compact euclidean orbifolds has been determined by Ratcliffe and Tschantz [Ratcliffe and Tschantz(2015)], in Theorem 1 and Corollaries 1 and 2 in pages 46 and 47, which we state now for later reference.

Theorem 4.1. *The isometry group of a cocompact euclidean orbifold* $\mathbb{R}^3/\Gamma$ *is a compact Lie group whose identity component is a Torus of dimension equal the first Betti number of the group* Γ*, which corresponds to the rank of the abelian group* $\Gamma/[\Gamma,\Gamma]$.

To understand why a compact quotients R^3/G could have as isometry group a torus of smaller dimension than 3, we can observe at two examples in dimension two:

Example 4.2. The group $\mathbb{Z}^2$ is a discrete subgroup of $Iso(\mathbb{R}^2)$, such that has the torus $N_{\mathbb{R}^2}(\mathbb{Z}^2)/\mathbb{Z}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ acting naturally by isometries, however the full isometry group $Iso(\mathbb{R}^2/\mathbb{Z}^2) \cong (\mathbb{R}^2/\mathbb{Z}^2) \rtimes D_4$ is bigger.

Example 4.3. We may extend the previous example to the group $\Lambda = \mathbb{Z}^2 \rtimes D_4$, which is a discrete subgroup of $Iso(\mathbb{R}^2)$, such that it is not completely contained in $\mathbb{R}^2$ and produces a compact quotient $\mathbb{R}^2/\Lambda$, homeomorphic to the 2-sphere S^2. To compute the isometry group, we observe the contentions

$$N_{\mathbb{R}^2}(\Lambda) = \{(n/2, n/2 + m) : n, m \in \mathbb{Z}\} \subset N_{\mathbb{R}^2}(\mathbb{Z}^2) = \mathbb{R}^2,$$

and $N_{Iso(\mathbb{R}^2)}(\Lambda) = Aut(\mathbb{Z}^2) \rtimes N_{\mathbb{R}^2}(\Lambda) \cong D_4 \rtimes N_{\mathbb{R}^2}(\Lambda)$, which gives us

$$N_{Iso(\mathbb{R}^2)}(\Lambda)/\Lambda \cong (D_4 \rtimes N_{\mathbb{R}^2}(\Lambda))/(D_4 \times \mathbb{Z}^2) \cong \mathbb{Z}_2.$$

This gives us a finite isometry group $Iso(\mathbb{R}^2/\Lambda) \cong \mathbb{Z}_2$. Observe that if $\sigma \in D_4$ and $v \in \mathbb{Z}^2$, then the commutator of these elements is $[\sigma, v] = \sigma(v) - v \in \mathbb{Z}^2$ and we can produce two linearly independent elements. Thus, $[\Gamma, \Gamma]$ contains a lattice in $\mathbb{R}^2$ and $\Gamma/[\Gamma, \Gamma]$ is finite, verifying Theorem 4.1.

5. Nil-geometry

5.1. **Riemannian geometry of the Heisenberg group.** If $\mathbb{F}$ is a commutative ring, denote by $H_{\mathbb{F}}$ the group of 3×3 upper triangular matrices over $\mathbb{F}$ with 1 in the diagonal, that is

$$H_{\mathbb{F}} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F} \right\}.$$

The group $H_{\mathbb{R}}$ is a Lie group called the three dimensional Heisenberg group that fits into the exact sequence

$$1 \to \mathbb{R} \to H_{\mathbb{R}} \to \mathbb{R}^2 \to 1,$$

where $\mathbb{R} \subset H_{\mathbb{R}}$ is its center. The three matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

determine a canonical basis of the tangent space at the identity $T_I(H_{\mathbb{R}})$, so that its translations by left-multiplications gives us a basis of left invariant vector fields denoted by X_j with $X_j(I) = e_j$. For a fixed element

$$g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in H_{\mathbb{R}},$$

the vector fields at $T_g(H_{\mathbb{R}})$ have expresions

$$X_1(g) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2(g) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3(g) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we consider the global coordinates

$$\mathbb{R}^3 \to H_{\mathbb{R}}, \qquad (x, y, z) \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

then a vector $v \in T_g(H_{\mathbb{R}})$ decomposes as

$$v = v_1 e_1 + v_2 e_2 + v_3 e_3 = v_1 X_1(g) + v_2 X_2(g) + (v_3 - x v_2) X_3(g),$$

so that the left-invariant metric in $H_{\mathbb{R}}$ having $X_j(g)$ as an orthonormal basis is given in this coordinates as $ds^2 = dx^2 + dy^2 + (dz - xdy)^2$. Being left-invariant, this metric has $H_{\mathbb{R}}$ as a subgroup of isometries given by left multiplication

$$L_g : H_{\mathbb{R}} \to H_{\mathbb{R}}, \qquad L_g(h) = gh,$$

for every $g \in H_{\mathbb{R}}$, but there are other isometries that don't come from left multiplication of $H_{\mathbb{R}}$, such isometries form a group isometric to the orthogonal group $O(2)$ generated by twisted rotations

$$m : S^1 \times H_{\mathbb{R}} \to H_{\mathbb{R}}, \qquad m_\theta(x, y, z) = (\rho_\theta(x, y), z + \eta_\theta(x, y)),$$

where ρ_θ is a rotation in the (x, y)-plane with angle θ, η_θ is a polynomial function in x and y and trigonometric in θ, and the reflection $R(x, y, z) = (x, -y, -z)$. The whole isometry group $Iso(H_{\mathbb{R}})$ can be described either as a semi-direct product $H_{\mathbb{R}} \rtimes O(2)$, because

$$m_\theta \circ L_g \circ m_\theta^{-1} = L_{m_\theta(g)}, \qquad R \circ L_g \circ R = L_{R(g)},$$

or via the exact exact sequence

$$1 \to \mathbb{R} \to Iso(H_{\mathbb{R}}) \to Iso(\mathbb{R}^2) \to 1,$$

induced by the action on the quotient by the center $H_{\mathbb{R}}/\mathbb{R} \cong \mathbb{R}^2$, see [Scott(1983)] for more details.

5.2. **Examples.** In this section we describe a series of ilustrative examples that capture the behaviour of every discrete subgroup of $Iso(H_{\mathbb{R}})$.

Example 1. The group $H_{\mathbb{Z}} \subset H_{\mathbb{R}}$ is a discrete subgroup so that the exact sequence determining $H_{\mathbb{R}}$ induces the fiber-bundle structure

$$\mathbb{R}/\mathbb{Z} \to H_{\mathbb{R}}/H_{\mathbb{Z}} \to \mathbb{R}^2/\mathbb{Z}^2$$

and thus $H_{\mathbb{Z}}$ is a lattice subgroup of $H_{\mathbb{R}}$ such that $H_{\mathbb{R}}/H_{\mathbb{Z}}$ is a compact Riemannian manifold. As the conjugation can be computed as

$$g = (x, y, z), \qquad g\ (n, m, p)\ g^{-1} = (n, m, p + xm - yn),$$

the normalizer in $H_{\mathbb{R}}$ is $N_{H_{\mathbb{R}}}(H_{\mathbb{Z}}) = \{(n, m, p) : n, m \in \mathbb{Z},\ p \in \mathbb{R}\}$ gives us the isometries in the quotient

$$S^1 \cong N_{H_{\mathbb{R}}}(H_{\mathbb{Z}})/H_{\mathbb{Z}} \hookrightarrow Iso(H_{\mathbb{R}}/H_{\mathbb{Z}}),$$

but the bigger normalizer in $Iso(H_{\mathbb{R}})$ is computed as

$$N_{Iso(H_{\mathbb{R}})}(H_{\mathbb{Z}}) = \{(n, m, p) : n, m \in \mathbb{Z},\ p \in \mathbb{R}\} \rtimes D_4,$$

where the Dihedral group D_4 is generated by the isometries

$$m_{\pi/2}(n, m, p) = (-m, n, p - nm), \qquad R(n, m, p) = (n, -m, -p),$$

so that what we get is $Iso(H_{\mathbb{R}}/H_{\mathbb{Z}}) \cong S^1 \rtimes D_4$. We can modify this example by adding the Dihedral group to the lattice, so that we have the fiber bundle structure

$$\mathbb{R}/\mathbb{Z} \to H_{\mathbb{R}}/(H_{\mathbb{Z}} \rtimes D_4) \to \mathbb{R}^2/(\mathbb{Z}^2 \rtimes D_4) \cong S^2,$$

and we decreased the normalizer

$$N_{Iso(H_{\mathbb{R}})}(H_{\mathbb{Z}} \rtimes D_4) = \{(n, m, l) : n, m, 2l \in \mathbb{Z}\} \rtimes D_4,$$

and thus, $Iso(H_{\mathbb{R}}/(H_{\mathbb{Z}} \rtimes D_4)) \cong \mathbb{Z}_2$.

Example 2. Fix a positive integer $p \in \mathbb{N}$ and consider the lattice

$$G_p = \left\{ \left(n, m, \frac{l}{p} \right) : n, m, l \in \mathbb{Z} \right\} \subset H_{\mathbb{R}},$$

which has as normalizer group in $H_{\mathbb{R}}$ the group

$$N_{H_{\mathbb{R}}}(G_p) = \left\{ \left(\frac{n}{p}, \frac{m}{p}, r \right) : n, m \in \mathbb{Z},\ r \in \mathbb{R} \right\},$$

and normalizer group in $Iso(H_{\mathbb{R}})$, the group $N_{H_{\mathbb{R}}}(G_p) \rtimes D_4$, with Dihedral group $D_4 = \langle m_{\pi/2}, R\rangle$ as before. The isometry group is characterized by the exact sequence

$$1 \to S^1 \to Iso(H_{\mathbb{R}}/G_p) \to D_4 \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p) \to 1$$

and we recover the previous example by taking $p = 1$.

Example 3. Fix a positive integer $p \in \mathbb{N}$ and consider the lattice

$$L_p = \left\{ \left(\frac{n}{2} + m, \frac{\sqrt{3}n}{2}, \frac{\sqrt{3}l}{2p} \right) : n, m, l \in \mathbb{Z} \right\} \subset H_{\mathbb{R}},$$

so that it has normalizer group in $H_{\mathbb{R}}$

$$N_{H_{\mathbb{R}}}(L_p) = \left\{ \left(\frac{n}{2p} + \frac{m}{p}, \frac{\sqrt{3}n}{2p}, r \right) : n, m \in \mathbb{Z},\ r \in \mathbb{R} \right\}.$$

As the group L_p projects to a hexagonal lattice in $\mathbb{R}^2$, we should expect to have a dihedral group D_6 normalizing L_p, however, the rotation $m_{\pi/3} : H_{\mathbb{R}} \to H_{\mathbb{R}}$ given by

$$m_{\pi/3}(x, y, z) = \left(\frac{1}{2}\left(x - \sqrt{3}y\right), \frac{1}{2}\left(y + \sqrt{3}x\right), z + \frac{\sqrt{3}}{8}\left(y^2 - x^2 - 2\sqrt{3}xy\right) \right),$$

doesn't preserve L_p. To fix this, we must add a translation mixed with the rotation, more precisely, if $g = \left(\frac{1}{8}, -\frac{\sqrt{3}}{8}, 0\right) \in H_{\mathbb{R}}$, then $\varphi = m_{\pi/3} \circ L_g \in N_{Iso(H_{\mathbb{R}})}(L_p)$, which can be verified using the relation $\varphi \circ L_h \circ \varphi^{-1} = L_{m_{\pi/3}(ghg^{-1})}$. We can describe the normalizer group of L_p in $Iso(H_{\mathbb{R}})$ via its generators as

$$N_{Iso(H_{\mathbb{R}})}(L_p) = \langle L_g, \varphi, R : g \in N_{H_{\mathbb{R}}}(L_p)\rangle,$$

where $R(x, y, z) = (x, -y, -z)$, and so, we have the isometry group

$$1 \to S^1 \to Iso(H_{\mathbb{R}}/L_p) \to (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes D_6 \to 1,$$

where the dihedral group D_6 is generated by $\langle \varphi, R\rangle$ (mod $\mathbb{R}$).

Example 4. The previous examples can be generalized as follows: Fix $p \in \mathbb{N}$ and $u, v \in \mathbb{R}^2$ linearly independent, so that $\Gamma = \{nu + mv : n, m \in \mathbb{Z}\} \subset \mathbb{R}^2$ is a lattice. If $(u, 0) \times (v, 0) = (0, 0, \lambda) \in \mathbb{R}^3$, then the group

$$M_p = \left\{ \left(nu + mv, \frac{\lambda}{p} l \right) : n, m, l \in \mathbb{Z} \right\} \subset H_{\mathbb{R}}$$

is a lattice having normalizer group in $H_{\mathbb{R}}$

$$N_{H_{\mathbb{R}}}(M_p) = \left\{ \left(\frac{n}{p}u + \frac{m}{p}v, r \right) : n, m \in \mathbb{Z},\ r \in \mathbb{R} \right\}.$$

The lattice Γ has an automorphism group $Aut(\Gamma) \in \{0, \mathbb{Z}_2, D_4, D_6\}$, which is, if non-trivial, generated by a rotation with angle θ and a reflection. So we can compute the whole normalizer group via its generators as

$$N_{Iso(H_{\mathbb{R}})}(M_p) = \langle L_g, \varphi, R : g \in N_{H_{\mathbb{R}}}(M_p)\rangle,$$

where $\varphi = m_\theta \circ L_w$. Here, $w = (w_0, 0) \in H_{\mathbb{R}}$ must be chosen so that if

$$w \times (u, 0) = (0, r_1), \qquad w \times (v, 0) = (0, r_2),$$

then $m_\theta(u, r_1), m_\theta(v, r_2) \in M_p$. Thus, we have an isometry group of the quotient given by the exact sequence

$$1 \to S^1 \to Iso(H_{\mathbb{R}}/M_p) \to (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes Aut(\Gamma) \to 1.$$

Remark: 5. The previous examples give us the general strategy to compute the isometry group of a quotient $H_{\mathbb{R}}/G$, for $G \subset Iso(H_{\mathbb{R}})$ a discrete group of isometries. This strategy is as follows: G projects to a discrete subgroup $\Gamma \subset Iso(\mathbb{R}^2)$ which has a finite index subgroup $\Gamma_0 \subset \mathbb{R}^2$, corresponding to a finite index subgroup $G_0 = G \cap H_{\mathbb{R}} \subset G$ and a lattice in $H_{\mathbb{R}}$. The normalizer of G_0 projects again a lattice in $\mathbb{R}^2$ and thus $Iso(H_{\mathbb{R}}/G_0)$ is an extension of a finite group $\mathbb{Z}_p \times \mathbb{Z}_p$ by S^1. The isometry group $Iso(H_{\mathbb{R}}/G)$ is just the previous group with an extra finite group of isometries, comming from the automorphisms of the lattice $Aut(\Gamma)$. This strategy fails when the projection to $\mathbb{R}^2$ is non-discrete, as shown in the following two examples, but in the next section we will see that in the case where $H_{\mathbb{R}}/G$ has finite volume, then this doesn't happen.

Here we add two examples of discrete groups which project onto the action on $\mathbb{R}^2$ to non-discrete groups, these examples capture the general behaviour of discrete groups having this property as we'll see in the next section.

Example 5. The isometry group $Iso(H_{\mathbb{R}})$ preserves the fiber structure

$$\mathbb{R} \to H_{\mathbb{R}} \to \mathbb{R}^2$$

and in particular, induces an action on the Euclidean plane $H_{\mathbb{R}}/\mathbb{R} \cong \mathbb{R}^2$. In the previous examples, the subgroups act on $\mathbb{R}^2$ as the lattice $\mathbb{Z}^2$ and in particular the action is discrete, but this is not always the case for discrete subgroups of $Iso(\mathbb{R}^2)$. As an example of this, consider $\varphi : \mathbb{N} \to S^1$, a homomorphism with dense image and $g = (0, 0, 1) \in H_{\mathbb{R}}$ a generator of the center, so that the group

$$\{(g^n, \varphi(n)) : n \in \mathbb{N}\} \subset H_{\mathbb{R}} \rtimes S^1 \cong Iso(H_{\mathbb{R}})$$

is a discrete subgroup of isometries of $H_{\mathbb{R}}$ with dense projection onto $S^1 \cong SO(2)$ and in particular, with a non-discrete action $\mathbb{R}^2$. In this example, the projected group leaves fixed the point $p = \frac{1}{1-\lambda} \in \mathbb{C} \cong \mathbb{R}^2$, where $\lambda = \varphi(1)$ and in particular, it is a group of rotations around such point.

Example 6. Another example of this is the following: Given a scaling $0 < \varepsilon < 1$, consider the group generated by $(1, 0, 1), (\varepsilon, 0, 1) \in H_{\mathbb{R}} \subset Iso(H_{\mathbb{R}})$ and $-1 \in \mathbb{S}^1 \subset Iso(H_{\mathbb{R}})$. This is a discrete subgroup of $Iso(H_{\mathbb{R}})$, which projects to a non-discrete subgroup of $Iso(\mathbb{R}^2)$ leaving fixed the line $\{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$.

5.3. **Classification of discrete subgroups of isometries.** In this section $H_{\mathbb{R}}$ denotes the Heisenberg Lie group considered as a Riemannian manifold with respect to the left-invariant metric constructed in the previous section. In this section, we describe the conditions on which a discrete group on $Iso(H_{\mathbb{R}})$ induces a discrete action on the Euclidean plane $\mathbb{R}^2$.

Proposition 6. If G is a discrete subgroup of isometries of $H_{\mathbb{R}}$, then the exact sequence

$$1 \to \mathbb{R} \to Iso(H_{\mathbb{R}}) \to Iso(\mathbb{R}^2) \to 1$$

induces an exact sequence

$$1 \to K \to G \to \Gamma \to 1,$$

where $\Gamma \subset Iso(\mathbb{R}^2)$ is either discrete or it is an abelian group leaving fixed either a point or a line. Moreover, if $\Gamma \subset Iso(\mathbb{R}^2)$ has a finite index lattice, then $K \subset \mathbb{R}$ is a non-trivial discrete subgroup and if Γ is non-discrete and leaves fixed a line, then there is a finite index subgroup of G which is contained in $H_{\mathbb{R}}$.

Proof. Observe first that $K = G \cap \mathbb{R}$ is a discrete subgroup of isometries of $\mathbb{R}$ and so, if non-trivial, there is an isomorphism $\mathbb{R}/K \cong S^1$. The exact sequence

$$1 \to S^1 \to H_{\mathbb{R}}/K \to \mathbb{R}^2 \to 1$$

gives us

$$1 \to S^1 \to Iso(H_{\mathbb{R}})/K \to Iso(\mathbb{R}^2) \to 1,$$

which has compact Kernel and thus, any discrete group in $Iso(H_{\mathbb{R}})/K$ projects to a discrete group in $Iso(\mathbb{R}^2)$. This argument tells us that if K is non-trivial then Γ is discrete in $Iso(\mathbb{R}^2)$, because it is the projection of G/K with compact kernel, and G/K is always discrete in $Iso(H_{\mathbb{R}})/K$. Suppose from now on that K is trivial. If we identify $\mathbb{R}^2 \cong \mathbb{C}$ as a Euclidean space, then we can realise the group of orientation preserving isometries of the plane $\mathbb{R}^2$ as the matrix group

$$Iso^+(\mathbb{R}^2) \cong \left\{ \begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix} : \lambda, z \in \mathbb{C},\ |\lambda| = 1 \right\}$$

with action

$$\begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda w + z \\ 1 \end{pmatrix}.$$

Observe that the restriction $Iso^+(\mathbb{R}^2) \subset Iso(\mathbb{R}^2)$ reduces the discusion to a subgroup of index 2, which doesn't alter the property of discreteness. We recall two important properties on commutators. First, commutators of two isometries give elements of pure translation part

$$\left[\begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \beta & w \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & (z-w) + (\lambda w - \beta z) \\ 0 & 1 \end{pmatrix},$$

which tells us that $[G, G]$ projects to a subgroup of $Iso(\mathbb{R}^2)$ with only translation part, and so $[G, G] \subset H_{\mathbb{R}}$. Second, the commutator in $H_{\mathbb{R}}$ satisfies the relation

$$\left[\begin{pmatrix} 1 & x & r \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u & s \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 & xv - uy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which has the geometric interpretation: if two elements of $H_{\mathbb{R}}$ project to the vectors (x, y) and (u, v), then its commutator is an element of the center $\mathbb{R} = Z(H_{\mathbb{R}})$ whose magnitud is the area of the projected vectors. As we are under the supposition that $G \cap \mathbb{R}$ is trivial, the two previous relations on commutators tells us that $[G, G]$ is a commutative group and the corresponding projected group satisfies

$$[\Gamma, \Gamma] \subset \left\{ \begin{pmatrix} 1 & rz_0 \\ 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\}$$

for some $z_0 \in \mathbb{C}$. Suppose first that Γ is non-commutative. The commutation relation

$$\left[\begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & (\lambda - 1)z_0 \\ 0 & 1 \end{pmatrix},$$

and the hypothesis that all the translation elements of $[\Gamma, \Gamma]$ are linearly dependent give us the condition $r = (1 - \lambda)$ for some $r \in \mathbb{R}$ and as $|\lambda| = 1$, the only options are $\lambda = \pm 1$. As Γ is non-commutative, there is at least one element that is not a translation, that is

$$\begin{pmatrix} -1 & z \\ 0 & 1 \end{pmatrix} \in \Gamma,$$

and without loss of generality, we can change Γ by $h\Gamma h^{-1}$ (where h is the translation by $1/2z$) so that in fact

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \Gamma,$$

this conjugation leaves $[\Gamma,\Gamma]$ invariant. Observe also that

$$\left[\begin{pmatrix} \beta & w \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 2w \\ 0 & 1 \end{pmatrix} \subset [\Gamma,\Gamma],$$

implies by the same argument that $\beta = \pm 1$ and $w = sz_0$ for some $s \in \mathbb{R}$, and thus Γ preserves the line generated by z_0. If on the other hand Γ is commutative and contains an element of the form

$$\begin{pmatrix} \lambda & z \\ 0 & 1 \end{pmatrix}, \qquad \lambda \neq 1,$$

this element has as a unique fixed point $\frac{-z}{\lambda-1}$. As Γ is a commutative group, every element of Γ must fix $\frac{-z}{\lambda-1}$, and thus, it consists of rotations around this point. If no such element exists, Γ consists of elements with purely translation part, which tells us that $G \subset H_{\mathbb{R}}$. We observe that in this last case, two elements $a, b \in G$ which project to two linearly independent vectors in Γ must satisfy that $e \neq [a,b] \in G \cap K$, which can't happen by hypothesis, so Γ is a subgroup of the group $\{r\omega : r \in \mathbb{R}\}$ for some $\omega \in \mathbb{C}$ and thus Γ preserves the line generated by ω. □

Lemma 7. Let G be a discrete subgroup of isometries of $H_{\mathbb{R}}$ together with the projection to the isometry group of $\mathbb{R}^2$

$$G \to \Gamma \subset Iso(\mathbb{R}^2).$$

If Γ preserves either a line or a point in $\mathbb{R}^2$, then the orbifold $H_{\mathbb{R}}/G$ has infinite volume.

Proof. Suppose first that Γ preserves the line $\mathbb{R}v$, then as a consequence of either Bieberbach's Theorem if Γ is discrete, or as a consequence of the proof of Proposition 6 if Γ is non-discrete, G has a finite index subgroup that is contained in $H_{\mathbb{R}}$. Passing to a finite index subgroup doesn't change the property of having finite co-volume so without loss of generality we may suppose that $G \subset H_{\mathbb{R}}$. There is a fundamental domain that has non-empty interior, given for example by the Dirichlet's fundamental domain $\{q \in H_{\mathbb{R}} : d(q_0, q) < d(q_0, \gamma(q)),\ \gamma \in G \setminus \{e\}\}$, with respect to the Riemannian distance d, see [Ratcliffe(2019)]. In particular there is a subset of the form

$$D = \{(tv + sv^{\perp}, \lambda r_0) \subset \mathbb{C} \times \mathbb{R} : (s,t,\lambda) \in (-\varepsilon,\varepsilon)^3 + (s_0, t_0, \lambda_0)\} \subset H_{\mathbb{R}}$$

such that no two elements of D can be identified with an element of G. As Γ preserves the line $\mathbb{R}v$, then we can see that no two elements of $\widetilde{D}$ can be identified with an element of G, where

$$\widetilde{D} = \{(tv + sv^{\perp}, \lambda r_0) \subset \mathbb{C} \times \mathbb{R} : (s,t,\lambda) \in \mathbb{R} \times (-\varepsilon,\varepsilon)^2 + (0, t_0, \lambda_0)\}$$

but $\widetilde{D} = \bigcup_j D_j$, where

$$D_j = \{(tv + sv^{\perp}, \lambda r_0) \subset \mathbb{C} \times \mathbb{R} : (s,t,\lambda) \in (-\varepsilon,\varepsilon)^3 + (s_j, t_0, \lambda_0)\} \subset H_{\mathbb{R}}$$

and every D_j can be obtained by translating D with an element of $H_{\mathbb{R}}$, thus

$$Vol(N/G) \geq Vol(\widetilde{D}) = \sum_j Vol(D_j) = \sum_j Vol(D) = \infty.$$

The second possibility is when Γ is a commutative group preserving a point, that is, Γ is conjugated to a subgroup of $SO(2)$. Again there is a fundamental domain of G with non-empty interior and in particular, there is a subset

$$\Omega = \{(re^{i\theta}, sr_0) \subset \mathbb{C} \times \mathbb{R} : (r,\theta,s) \in (-\varepsilon,\varepsilon)^3 + (a,b,c)\} \subset H_{\mathbb{R}},$$

such that no two elements of Ω can be identified with an element of G. As Γ acts only as rotations in the $\mathbb{C}$ plane, we can enlarge as before Ω to the subset

$$\widetilde{\Omega} = \{(re^{i\theta}, sr_0) \subset \mathbb{C} \times \mathbb{R} : (r, \theta, s) \in \mathbb{R}_{>0} \times (-\varepsilon, \varepsilon)^2 + (0, b, c)\},$$

so that no two elements of $\widetilde{\Omega}$ can be identified with an element of G. As before, we have a countable union of disjoints sets contained in $\widetilde{\Omega}$ that are translated copies of Ω, that is

$$\bigcup_i \Omega + (\omega_j, 0) \subset \widetilde{\Omega}$$

and $Vol(N/G) \geq Vol(\widetilde{\Omega}) \geq \sum_j Vol(\Omega + (\omega_j, 0)) = \sum_j Vol(\Omega) = \infty$. □

Lemma 8. If $u, v \in \mathbb{R}^2$ are two linearly independent vectors, with $(u, 0) \times (v, 0) = (0, 0, \lambda) \in \mathbb{R}^3$ and $n \in \mathbb{N}$, $r, s \in \mathbb{R}$, then the group

$$G = \left\langle (u, r), (v, s), \left(0, 0, \frac{\lambda}{n}\right) \right\rangle \subset H_{\mathbb{R}}$$

is a lattice in $H_{\mathbb{R}}$. Conversely, every lattice in $H_{\mathbb{R}}$ can be obtained like this.

Proof. Observe that the center of G is the subgroup $K = \left\{ \frac{\lambda p}{n} : p \in \mathbb{Z} \right\}$ and if $(x, y, z), (x, y, z') \in G$, then

$$(x, y, z)^{-1} \cdot (x, y, z') = (0, 0, z' - z) \in K,$$

so that for $k, l \in \mathbb{N}$ fixed, and

$$(u, r)^k \cdot (v, s)^l = (ku + lv, r_{k,l})$$

the level set

$$\{(w, z) \in G : w = ku + lv\} = \left\{ \left(ku + lv, r_{n,m} + \frac{\lambda p}{n} \right) : p \in \mathbb{Z} \right\}$$

is discrete and thus, G is a discrete subgroup of $H_{\mathbb{R}}$. If $\Gamma = \{ku + lv : k, l \in \mathbb{Z}\}$ denotes the projection of G onto $\mathbb{R}^2$, then there is an exact sequence

$$1 \to K \to G \to \Gamma \to 1$$

which induces the fiber bundle structure

$$S^1 \cong \mathbb{R}/K \to H_{\mathbb{R}}/G \to \mathbb{R}^2/\Gamma \cong S^1 \times S^1,$$

which tells us that $H_{\mathbb{R}}/G$ is compact and thus, G is a lattice in $H_{\mathbb{R}}$. Suppose now that $L \subset H_{\mathbb{R}}$ is a lattice, then by Lemma 7, L projects to a lattice subgroup of $\mathbb{R}^2$, generated by two linearly independent vectors $u', v' \in \mathbb{R}^2$ such that $(u', 0) \times (v', 0) = (0, 0, \lambda')$, with $0 \neq \lambda' \in \mathbb{R}$ and observe that if $g = (u', r'), h = (v', s') \in L$, then their commutator is $[g, h] = (0, 0, \lambda')$. As the intersection $K' = G \cap Z(H_{\mathbb{R}})$ is discrete and contains the non-trivial element $(0, 0, \lambda') \in K'$, then there is an integer $n' \in \mathbb{N}$ such that $K' = \left\{ \frac{\lambda' p}{n'} : p \in \mathbb{Z} \right\}$ and thus, the lattice L is generated by the set $\left\{ (u', r'), (v', s'), \left(0, 0, \frac{\lambda'}{n'}\right) \right\}$. □

Theorem 9. If $G \subset Iso(H_{\mathbb{R}})$ is a discrete subgroup such that $H_{\mathbb{R}}/G$ has finite volume, then there is an exact sequence

$$1 \to C \to Iso(H_{\mathbb{R}}/G) \to F \to 1,$$

where F is a finite group, and $C \subset S^1$ is a closed subgroup. In particular, either $Iso(H_{\mathbb{R}}/G)$ is finite, or it is a finite extension of S^1.

Proof. By proposition 6 and Lemma 7, the projection of G to $Iso(\mathbb{R}^2)$ has a lattice $\Gamma \subset \mathbb{R}^2$ as a finite index subgroup. This is equivalent to the fact that $L = G \cap H_{\mathbb{R}}$ is a lattice in $H_{\mathbb{R}}$ and a finite index subgroup in G. By Lemma 8, there are $u, v \in \mathbb{R}^2$, $\lambda, r, s \in \mathbb{R}$, with $\lambda \neq 0$, and $n \in \mathbb{N}$, such that $\Gamma = \{ku + lv : k, l \in \mathbb{Z}\}$ and

$$L = \left\langle (u, r), (v, s), \left(0, 0, \frac{\lambda}{n}\right) \right\rangle \subset H_{\mathbb{R}}.$$

As seen in Example 4, the group $N_{H_{\mathbb{R}}}(L) = \left\{ \left(\frac{n}{p}u + \frac{m}{p}v, r\right) : n, m \in \mathbb{Z},\ r \in \mathbb{R} \right\}$ is the normalizer of L in $H_{\mathbb{R}}$. Denote by $Aut(\Gamma) \subset O(2)$ the subgroup that preserves the lattice Γ and observe that an element $\varphi = \sigma \circ L_g \in Iso(H_{\mathbb{R}})$ satisfies that $\varphi \circ L_h \circ \varphi^{-1} = L_{\sigma(ghg^{-1})}$. As $\sigma(ghg^{-1})$ and $\sigma(h)$ have the same projection onto Γ, then if φ normalizes L, $\sigma \in Aut(\Gamma)$ and we have that

$$1 \to K \to G \to F' \ltimes \Gamma \to 1,$$

for some subgroup $F' \subset Aut(\Gamma)$ and $K = L \cap \mathbb{R}$. As $H_{\mathbb{R}}$ is normal in $Iso(H_{\mathbb{R}})$, we see that $N_{Iso(H_{\mathbb{R}})}(G) \subset N_{H_{\mathbb{R}}}(G)$, and thus, by applying a trick as in Example 4.3, we may describe the greater normalizer as

$$1 \to H \to N_{Iso(H_{\mathbb{R}})}(G) \to F'' \ltimes \Lambda \to 1,$$

with $F'' \subset Aut(\Gamma)$ a finite group and $\Lambda \subset \mathbb{R}^2$ a lattice containing Γ. Thus, the isometry group is calculated as

$$1 \to C = H/K \to N_{Iso(H_{\mathbb{R}})}(G)/G \to F = (F'' \ltimes \Lambda) / (F' \ltimes \Gamma) \to 1,$$

so that C is either finite, cyclic or S^1 and F is finite. □

Remark: 10. The most symmetric lattices in $\mathbb{R}^2$ are the square and hexagonal lattices, having linear symmetry groups D_4 and D_6. Theorem 9 tells us that the generalizations of these lattices to $H_{\mathbb{R}}$, described in Example 2 and Example 3 are the most symmetric finite volume quotients $H_{\mathbb{R}}/G$, with isometry groups

$$1 \to S^1 \to Iso(H_{\mathbb{R}}/G) \to (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes D \to 1,$$

with D equal to D_4 and D_6 respectively, and $n \in \mathbb{N}$.

6. Spherical Geometry

This section is largely expository due to the fact that the verification of 1.2 consists of the comparison of the statement with the (fundamentally algebraic) classification of groups acting by isometries on three dimensional spherical manifolds and orbifolds. These groups are restricted by results related to the positive answer of the Smale conjeture, and the fact that a three dimensional Alexandrov space of positive curvature is either a spherical three-manifold or a suspension of a real projective space. This concerns specifically the quotient orbifold of an action of a discrete group on a spherical three-manifold, that is, a quotient of the form

$$M = S^3/\Gamma,$$

for Γ a finite subgroup of isometries of the three dimensional sphere $O(4)$.

The following is a consequence of the classification of isometry groups of spherical 3- manifolds in [McCullough(2002)], tables 2 and 3 in pages 173 and 176, relying on work of Mccullough and collaborators and ultimately going back to Seifert, Threlfall, Hopf and Hattori. See [Hong et al.(2012)Hong, Kalliongis, McCullough, and Rubinstein], chapter 1 for an account of these facts.

Lemma 6.1. *Up to finite subgroups, the isometry groups of spherical three manifolds are:*

- $SO(3)$.
- $O(2)$.
- $O(4)$.
- $SO(4)$.
- $SO(3)$.
- $O(2) \times O(2)$
- $S^1 \times_{\mathbb{Z}/2} S^1$.

In particular, they are all closed subgroups of $O(4)$.

For a complete list of isometry groups of spherical orbifolds, see Chapter 3 of [Mecchia and Seppi(2019)].

6.1. **The Smale Conjecture.** An important result by Hatcher [Hatcher(1983)], originally conjectured by Smale states that the inclusion of the isometry group of S^2 into the group of diffeomorphisms is a homotopy equivalence.

The following result with contributions of many persons including (at least) Asano, Boileau, Bonahon, Birman, Cappell, Ivanov, Rubinstein, and Shaneson, is a consequence of research in mapping class groups and three- dimensional spherical manifolds. It is discussed with comments about attribution in [McCullough(2002)], Theorem 1.1 in page 3.

Theorem 6.2. *Let M be a spherical manifold, then the inclusion of the group of isometries of M on the group of Diffeomorphisms induces a bijection on path components.*

As of 2022, the following result in page 2 of [Bamler and Kleiner(2019)] is a consequence of the study via Ricci flow methods of the homotopy type of the spaces of positive scalar curvature and the subspace of metrics which are locally isometric to either the round sphere S^3 or the round cylinder $S^2 \times \mathbb{R}$.

Theorem 6.3. *Let (M, g) be a riemannian manifold which is an isometric quotient of the three dimensional round sphere. Then, the inclusion of the isometry group into the diffeomorphism group is a homotopy equivalence.*

6.2. **Connected components of isometries in spherical orbifolds.** The following theorem was proved in [Mecchia and Seppi(2019)], using previous analysis of the authors of Seifert fibrations for spherical orbifolds. It is a consequence of tables 2 in page 1302, table 3 in page 1304 and table 4 in page 1308.

Theorem 6.4. *Let X be a spherical three manifold, and let Γ be a discrete group of X.*

- *The isometry groups of the orbifold X/Γ are either closed subgroups of SO_4 or PSO_4, if the action is orientation preserving.*
- *The identity component of the isometry groups are S^1, $S^1 \times S^1$ or trivial for the orientation preserving case.*

End of the proof of Theorem 1.2 and Phenomenon 1.3 for the sherical geometry. Recall that an orbifold quotient of an action of a discrete group on a spherical three manifold is an Alexandrov space of positive curvature It follows from corollary 2.2 in [Galaz-Garcia and Guijarro(2015)] (see Theorem 12.2), that any Alexandrov space of positive curvature is homeomorphic to a spherical three manifold or suspension of $\mathbb{R}P^2$. The result thus follows from Lemma 6.1. □

7. $S^2 \times \mathbb{R}$.

A three dimensional manifold is said to have $S^2 \times \mathbb{R}$- geometry if its universal covering is homeomorphic to $S^2 \times \mathbb{R}$.

Recall [Kobayashi and Nomizu(1996)], Chapter VI, Theorem 3.5 that given a product of riemannian manifolds $M \times N$ with M of constant sectional curvature and N flat, the isometry group of $M \times N$ decomposes as a direct product $\mathrm{Isom}(M) \times \mathrm{Isom}(N)$. It follows that for a discrete subgroup $\Gamma \leq \mathrm{Isom}(S^2 \times \mathbb{R}) \cong O(3) \times (\mathbb{R} \times \mathbb{Z}_2)$, the projection onto the second factor $\pi_{\mathbb{R}}(\Gamma) \leq \mathbb{R} \rtimes \mathbb{Z}/2$ is a discrete subgroup.

Remark: 11. Due to the classification theorem of manifolds covered by $S^2 \times \mathbb{R}$ by Tollefson in page 61 of [Tollefson(1974)], there exist only four such three-manifolds, namely: $S^2 \times S^1$, the non orientable S^2- bundle over S^1, $\mathbb{R}P^1 \times S^1$, and $\mathbb{R}P^3 \# \mathbb{R}P^2$. Moreover, the finite groups which act freely on $S^2 \times S^1$ are classified in [Tollefson(1974)], Corollary 2. They are:

- $\mathbb{Z}/p$, producing quotient spaces homeomorphic to $S^2 \times S^1$ in the odd case, and $\mathbb{R}P^2$ in the even case as quotient space.
- $\mathbb{Z}/p \times \mathbb{Z}/2$, for p even, producing a quotient space homeomorphic to $\mathbb{R}P^2$, and
- D_n, the dihedral group of order $2n$, producing $\mathbb{R}P^3 \# \mathbb{R}P^3$ as quotient space.

We may observe that the projection onto the S^2 factor of a discrete group of isometries need not be discrete as the following example shows:

Example 7.1. If $\sigma \in SO(3)$ is a rotation with irrational angle along a fixed axis, so that the orbit $\{\sigma^n(p) : n \in \mathbb{N}\}$ is dense in a circle, orthogonal to the rotation axis, for almost every $p \in S^2$, then the group given by twisted translations

$$\{(\sigma^n, n) \in O(3) \times \mathbb{R} : n \in \mathbb{N}\}$$

is a discrete subgroup of $Iso(S^2 \times \mathbb{R})$ with non-discrete projection on $Iso(S^2)$.

The previous example gives us the general behaviour for discrete groups of isometries on $S^2 \times \mathbb{R}$ as seen by the following Lemma

Lemma 12. If $\Gamma \subset Iso(S^2 \times \mathbb{R})$ is a discrete subgroup, then there is a finite group $F \subset O(3)$ and $\lambda \in \mathbb{R}$ such that the exact sequence

$$1 \to O(3) \to Iso(S^2 \times \mathbb{R}) \to \mathbb{R} \rtimes \mathbb{Z}_2$$

induces an exact sequence $1 \to F \to \Gamma \to L$, where L is either $\lambda\mathbb{Z}$ or $\lambda\mathbb{Z} \rtimes \mathbb{Z}_2$.

Proof. As the group $O(3)$ is compact, the projection of the discrete group Γ onto $Iso(\mathbb{R})$ is discrete, so it is of the form $\lambda\mathbb{Z}$ or $\lambda\mathbb{Z} \rtimes \mathbb{Z}_2$, for some $\lambda \in \mathbb{R}$. As $O(3)$ can be seen as a closed subgroup of $Iso(S^2 \times \mathbb{R})$, then the intersection of Γ with $O(3)$ is a finite group, which we denote by F. The result thus follows from the product structure of $Iso(S^2 \times \mathbb{R})$. In fact, Γ is generated by F, $\mathbb{Z}_2 \subset Iso(\mathbb{R})$ and the twisted translation subgroup $\{(\sigma^n, n\lambda) \in O(3) \times \mathbb{R} : n \in \mathbb{N}\}$, for some $\sigma \in O(3)$. □

Theorem 7.2. *If $\Gamma \subset Iso(S^2 \times \mathbb{R})$ is a discrete subgroup, such that $(S^2 \times \mathbb{R})/\Gamma$ is compact, then $Iso((S^2 \times \mathbb{R})/\Gamma)$ is up to finite index, a closed subgroup of $SO(3) \times S^1$. In particular, the connected component of the identity of the isometry group of the quotient can only be one of the three possibilities:*

$$SO(3) \times S^1, \quad S^1 \times S^1, \quad or \quad S^1.$$

Proof. By Lemma 12, the discrete group Γ is generated by a finite group of $O(3)$ and a twisted translation as in Example 7.1. The isometry group is compact, so it has a finite number of connected components and by Proposition 4, the connected component of the identity can be computed using the centralizer, which always contains the $\mathbb{R}$-factor, so the result follows by examining the possible connected, closed subgroups of $SO(3)$. □

8. Solv-geometry

8.1. **The geometry.** Solv-geometry is given by the solvable Lie group of upper-triangular matrices

$$S = \left\{ \begin{pmatrix} e^t & 0 & x \\ 0 & e^{-t} & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\},$$

which decomposes as a semidirect product $S = [S, S] \rtimes (S/[S, S]) \cong \mathbb{R}^2 \rtimes \mathbb{R}$. In global coordinates, the vector fields

$$X_1(x, y, t) = (e^t, 0, 0), \quad X_2(x, y, t) = (0, e^{-t}, 0), \quad X_3(x, y, t) = (0, 0, 1),$$

define a basis of left-invariant vector fields. We choose the left invariant Riemannian metric in S having this basis as orthonormal basis, so that in our global coordinates, the metric has the expresion

$$ds^2 = e^{-2t}dx^2 + e^{2t}dy^2 + dt^2.$$

The isometry group of this metric is generated by left translations

$$L_g : S \to S, \qquad L_g(h) = gh,$$

and the group of reflections $(x, y, z) \to (\pm x, \pm y, \pm z)$, isomorphic to the Dihedral group D_4. In particular, $Iso(S)$ has eight connected components, with the connected component of the identity isomorphic to S [Scott(1983)].

8.2. **Existence of lattices.** We start by observing that a Lie group admits a lattice subgroup if and only if it is unimodular [Raghunathan(2007)], and so, not every solvable group admit lattice subgroups.

Example 7. A solvable group which is close to S considered here, is the group of orientation preserving affine transformations on $\mathbb{R}$, given by

$$\mathrm{Aff}^+(\mathbb{R}) \cong \left\{ \begin{pmatrix} e^t & x \\ 0 & 1 \end{pmatrix} : x, t \in \mathbb{R} \right\}.$$

We could try for example, to exponentiate the set a lattice in $\mathbb{R}^2$ as

$$\Lambda = exp\left(\left\{ \begin{pmatrix} n & m \\ 0 & 0 \end{pmatrix} : n, m \in \mathbb{N} \right\} \right) = \left\{ \begin{pmatrix} e^n & (m/n)(e^n - 1) \\ 0 & 1 \end{pmatrix} : n, m \in \mathbb{N} \right\},$$

however, such discrete set is not a subgroup and the group which generates is not discrete. The problem is that the group $\mathrm{Aff}^+(\mathbb{R})$ is not unimodular, and in fact its modular function has the expression

$$\Delta : \mathrm{Aff}^+(\mathbb{R}) \to \mathbb{R}_+, \qquad \Delta \begin{pmatrix} e^t & x \\ 0 & 1 \end{pmatrix} = e^t,$$

which is non-trivial.

The solvable group S is unimodular, so that it admits a lattice subgroup and an explicit way to construct a lattice is as follows: Consider a matrix $A \in SL_2(\mathbb{Z})$, such that $tr(A) > 2$ and the group

$$\Gamma_A = \left\{ \begin{pmatrix} A^n & Z \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z},\ Z \in M_{2\times 1}(\mathbb{Z}) \right\} \cong \mathbb{Z}^2 \rtimes_A \mathbb{Z}.$$

Lemma 13. For every $A \in SL_2(\mathbb{Z})$, with $\mathrm{tr}(A) > 2$, Γ_A is conjugated in $SL_3(\mathbb{R})$ to a lattice subgroup in S, moreover, every lattice subgroup of S is conjugated to one of such groups.

Proof. Suppose first that $A \in SL_2(\mathbb{Z})$, with $\mathrm{tr}(A) > 2$, then there is a matrix $B \in SL_3(\mathbb{R})$ such that

$$BAB^{-1} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix},$$

for some $\lambda \neq 0$. We may define $A^t = B^{-1} \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{-t\lambda} \end{pmatrix} B$, so that Γ_A is a discrete subgroup of the group

$$S_A = \left\{ \begin{pmatrix} A^t & Z \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R},\ Z \in M_{2\times 1}(\mathbb{R}) \right\} \cong \mathbb{R} \ltimes \mathbb{R}^2,$$

such that

$$1 \to \mathbb{R}^2/\mathbb{Z}^2 \to S_A/\Gamma_A \to \mathbb{R}/\mathbb{Z} \to 1,$$

thus, Γ_A is a lattice in S_A. Observe that we have an isomorphisms of Lie groups via the conjugation

$$S_A \to S, \qquad X \mapsto \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} B^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and thus, a lattice in S.

Before proceeding with the proof of this Lemma, we need to prove the following discrete projection Lemma:

Lemma 14. If $\Gamma \subset S$ is a discrete subgroup, then its projection $\overline{\Gamma} \subset S/[S,S] \cong \mathbb{R}$ is also discrete.

Proof. An element $\gamma = (x, y, t)$, with $t \neq 0$, acts discretely by translations on the line $\left\{ \left(\frac{x}{1-e^t}, \frac{y}{1-e^{-t}}, s \right) : s \in \mathbb{R} \right\} \subset S$, as $\gamma^n \left(\frac{x}{1-e^t}, \frac{y}{1-e^{-t}}, s \right) = \left(\frac{x}{1-e^t}, \frac{y}{1-e^{-t}}, s + nt \right)$. Thus, if Γ is commutative, either $\Gamma \subset \mathbb{R}^2$ and its projection is trivial, or Γ preserves a unique line on which it acts as translations and the action in this line is precisely the action on $\mathbb{R}$ of its projection, which must be discrete. If Γ is non-commutative, then at least it has two elements $u = (a, b, 0)$ and $\gamma = (x, y, t)$ with $t \neq 0$. Observe that if $b = 0$, then $\gamma u \gamma^{-1} = (e^t a - a, 0, 0)$ and by iterating conjugation we get a non-discrete subgroup of $\mathbb{R}^2 \cap \Gamma$ which is impossible and the same goes for the case $a = 0$. Thus $a, b \neq 0$ and $\Gamma \cap \mathbb{R}^2$ contains two linearly independent vectors, say u and $v = \gamma u \gamma^{-1} = (e^t a, e^{-t} b, 0)$, which implies that $\Gamma \cap \mathbb{R}^2$ is cocompact in

$\mathbb{R}^2$. $\Gamma/(\Gamma\cap\mathbb{R}^2)$ is discrete in $S/(\Gamma\cap\mathbb{R}^2)$ and the projection $S/(\Gamma\cap\mathbb{R}^2)\to S/\mathbb{R}^2$ has compact Kernel $\mathbb{R}^2/(\Gamma\cap\mathbb{R}^2)\cong S^1\times S^1$, thus the corresponding projection of $\Gamma/(\Gamma\cap\mathbb{R}^2)$ into $S/[S,S]$ is discrete. □

Suppose now that $\Gamma\subset S$ is a lattice subgroup, then by Lemma 14, the Γ projects to a non-trivial discrete group $\mathbb{R}$, generated by an element $e^{n\beta}$, with $\beta\neq 0$. The intersection $\Gamma\cap\mathbb{R}^2$ is a lattice, so that there are $u,v\subset\mathbb{R}^2$ linearly independent, such that $\Gamma\cap\mathbb{R}^2=\{nu+mv : n,m\in\mathbb{Z}\}$. Take $C\in GL_2(\mathbb{R})$ the matrix sending $\Gamma\cap\mathbb{R}^2$ onto the canonical lattice $\mathbb{Z}^2$ and define the matrix $A'=B\begin{pmatrix} e^{\beta} & 0\\ 0 & e^{-\beta}\end{pmatrix}B^{-1}$. An element $g=\begin{pmatrix} B^{-1}A'B & W\\ 0 & 1\end{pmatrix}\in\Gamma$ must preserve $\Gamma\cap\mathbb{R}^2$, so that the element $h=\begin{pmatrix} A' & BW\\ 0 & 1\end{pmatrix}\in\Gamma$ must preserve $\mathbb{Z}^2$. Observe that the action of h in an element $v=(n,m)\in\mathbb{Z}^2$ is $A'v+BW\in\mathbb{Z}^2$, this implies that $BW\in\mathbb{Z}^2$ and $A'\in SL_2(\mathbb{Z})$. In particular, the group Γ is isomorphic to the group $\Gamma_{A'}$ and the isomorphism is obtained by conjugation. □

Remark: 15. The existence of lattices in the Lie group S is related to the existence of a $\mathbb{Q}$-structure on S. More precisely, if $A\in SL_2(\mathbb{Z})$, with $\mathrm{tr}(A)>2$, and $c=\sqrt{\mathrm{tr}(A)^2-4}$, then A is diagonalizable over the field $\mathbb{Q}(c)$, that is, there is a matrix $B\in SL_2(\mathbb{Q}(c))$ such that BAB^{-1} is diagonal. If $Q_{ij}(X)=X_{ij}$ is the linear map that gives the (i,j)-entry, then the group

$$\mathbb{G}(k)=\left\{\begin{pmatrix} X & Z\\ 0 & 1\end{pmatrix} : Z\in M_{2\times 1}(k),\ Q_{ij}(BXB^{-1})=0,\ i\neq j, i,j\in\{1,2\}\right\},$$

is algebraic subgroup of $SL_3(k)$, defined by polynomial equations with coefficients over $\mathbb{Q}(c)$, such that $\mathbb{G}(\mathbb{R})\cong S$ and $\mathbb{G}(\mathbb{Z})=\Gamma_A$. Moreover, the Galois automorphism $\sigma:\mathbb{Q}(c)\to\mathbb{Q}(c)$, defined by $\sigma(c)=-c$, has a natural extension to automorphisms of matrices and polynomials, so that we have the embedding

$$SL_3(\mathbb{Q}(c))\to SL_3(\mathbb{R})\times SL_3(\mathbb{R}),\qquad Y\mapsto (Y,\sigma(Y)),$$

and a polynomial condition $Q(Y)=0$ on $Y\in SL_3(\mathbb{Q}(c))$ is equivalent to the pair of polynomial conditions $Q(Y)+\sigma(Q)(Y')=0$ and $Q(Y)\sigma(Q)(Y')=0$ on $(Y,Y')\in SL_3(\mathbb{R})\times SL_3(\mathbb{R})$, but the latter are polynomials with coefficients over $\mathbb{Q}$ (this trick is called “restriction of scalars” [Morris(2015)]).

Lemma 16. *S has trivial center and the centralizer of a lattice group $\Gamma\subset S$ is also trivial.*

Proof. Take $\gamma=(x,y,t)$ in the centralizer of Γ in S, then as in the previous proposition $\Gamma\cap[S,S]$ has a rank two subgroup, thus it contains at least a vector $u=(a,b,0)$ such that $a,b\neq 0$ and we have

$$\gamma u\gamma^{-1}=(e^t a,e^{-t}b,0)=(a,b,0),$$

which implies that $t=0$. As Γ projects to a lattice group in $S/[S,S]\cong\mathbb{R}$, then there is a $\beta\in\Gamma$ such that $\beta=(c,d,s)$ with $s\neq 0$ and thus

$$\beta\gamma\beta^{-1}=(e^s x,e^{-s}y,0)=\gamma=(x,y,0)$$

which implies that $x=y=0$ and γ is the identity. A completely analogous computation shows that S has trivial center. □

Corollary 17. *If Γ is a discrete group of isometries of S such that S/Γ has finite volume, then S/Γ is compact and has finite isometry group.*

Proof. As the connected component of the isometry group of S is S itself acting by left multiplications, Γ is modulo a finite index subgroup a lattice in S and it lies in an exact sequence

$$1 \to \Gamma_0 \to \Gamma \to \Gamma_1 \to 1$$

where $\Gamma_0 = \Gamma \cap [S,S]$ and $\Gamma/\Gamma_0 \cong \Gamma_1 \subset \mathbb{R}$. By Proposition 14 Γ_1 is a discrete subgroup, so this exact sequence induces a the fiber bundle

$$\mathbb{R}^2/\Gamma_0 \to S/\Gamma \to R/\Gamma_0,$$

so that S/Γ has finite volume if and only if $\mathbb{R}^2/\Gamma_0$ and $\mathbb{R}/\Gamma_1$ are torus of the corresponding dimension and S/Γ is compact. The isometry group of S/Γ is a compact Lie group with connected component of the identity determined by the centralizer of Γ in S (Proposition 4) which is the trivial group by Lemma 16, thus the isometry group is a compact, zero-dimensional Lie group, i.e. finite. □

Example 8. For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $n \in \mathbb{N}$, consider the lattice $\Gamma_{A^n} = \mathbb{Z}^2 \rtimes_{A^n} \mathbb{Z}$. A matrix $Y = \begin{pmatrix} M & W \\ 0 & 1 \end{pmatrix} \in GL_3(\mathbb{R})$ normalizes Γ_{A^n} if and only if $M = A^k$ for some $k \in \mathbb{Z}$ and $(I - A^n)W \in \mathbb{Z}^2$, so that if $\Lambda_n = (I - A^n)^{-1}\mathbb{Z}^2$, then the normalizer is $N_{Iso(S)}(\Gamma_{A^n}) = \mathbb{Z} \ltimes_A \Lambda_n$ and the isometry group is computed as

$$Iso(S/\Gamma_{A^n}) = (\Lambda_n/\mathbb{Z}^2) \rtimes_A \mathbb{Z}_n.$$

Three ilustrative cases are

(i) $\Lambda_1 = \mathbb{Z}^2$, so that $Iso(S/\Gamma_A)$ is trivial;
(ii) $\det(I - A^2) = -5$, so that $\mathbb{Z}^2 \leq \Lambda_2 \leq \frac{1}{5}\mathbb{Z}^2$ and each contention is of index 5, in particular we have that $Iso(S/\Gamma_2) = \mathbb{Z}_5 \rtimes \mathbb{Z}_2$;
(iii) $\Lambda_5 = \frac{1}{11}\mathbb{Z}^2$, so that $Iso(S/\Gamma_5) = (\mathbb{Z}_{11} \times \mathbb{Z}_{11}) \rtimes_A \mathbb{Z}_5$.

Remark: 18. The previous example exhibits an isometric action of each finite cyclic group on a three dimensional manifold.

Notice that the action is not necesarilly free, since there exists a very rigid classification of free actions of finiite groups on three dimensional manifolds with Nil and Sol structure, based on p-rank estimates and P.A. Smith Theory, [Jo and Lee(2010)], [Koo et al.(2017)Koo, Oh, and Shin].

9. Hyperbolic geometry

9.1. **Normalizers of Fuchsian groups.** Denote by $\mathbb{H}^n$ the n-dimensional hyperbolic space and recall that the isometry group $Iso(\mathbb{H}^n)$ is a non-compact semisimple Lie group that can be identified with the group $PO(n,1)$.

Lemma 19. If $\Gamma \subset Iso(\mathbb{H}^n)$ is a discrete subgroup such that $\mathbb{H}^n/\Gamma$ has finite volume, then the normalizer group

$$\Lambda = \{g \in Iso(\mathbb{H}^n) : ghg^{-1} = h, \ \forall\ h \in \Gamma\} \subset Iso(\mathbb{H}^n)$$

is discrete and $Iso(\mathbb{H}^n/\Gamma)$ is a finite group.

Proof. Passing to a finite cover doesn't alter the outcome, so we may suppose that $\Gamma, \Lambda \subset O(n,1)$. By Proposition 4, the connected component of Λ lies inside of the centralizer of Γ in $O(n,1)$. Let $g \in O(n,1)$ centralizing Γ, then the polynomial

$$P_t : M_{n+1}(\mathbb{R}) \to M_{n+1}(\mathbb{R}), \qquad P_t(X) = gXg^{-1} - X$$

vanishes at Γ but by Borel's density Theorem (see [Furstenberg(1976)]), Γ is Zariski dense in $O(n,1)$ and thus $P_t(O(n,1)) = 0$ which tells us that g lies in the center of $O(n,1)$, which is finite. This tells us that Λ is a discrete group that contains the lattice Γ, so Λ is also a lattice in $O(n,1)$. If $F_\Lambda, F_\Gamma \subset \mathbb{H}^n$ are fundamental domains of the groups Λ and Γ correspondingly, so we have that

$$|Iso(\mathbb{H}^n/\Gamma)| = |\Lambda/\Gamma| = Vol(F_\Gamma)/Vol(F_\Lambda) < \infty.$$

□

Remark: 20. The previous result is stated for hyperbolic manifolds in Corollary 3, Section 12.7 of [Ratcliffe(2019)] and for hyperbolic orbifolds in [Ratcliffe(1999)], where the hypotheses are that the discrete group is nonelementary, geometrically finite and without fixed m-planes, for $m < n-1$. In Lemma 19 we presented an argument using Zariski-density of the lattice group in $Iso(\mathbb{H}^n)$, which implies for example the non-existence of fixed m-planes. As seen in [Greenberg(1974)], every finite group can be realized as the isometry group of a compact hyperbolic surface as in Lemma 19.

Lemma 21. If Σ is a compact, orientable surface of genus $g \geq 2$, then there are no faithful actions of the compact group S^1 on Σ.

Proof. Suppose there is a faithful action $S^1 \times \Sigma \to \Sigma$, then perhaps after an averaging process, we may suppose that the action is isometric with respect to a Riemannian metric h. The existe of isothermal coordinates [Umehara and Yamada(2017)] tells us that there exists a complex structure in Σ such that in holomorphic coordinates $z = x + iy$, the vector fields ∂_x and ∂_y are h-orthogonal. As the S^1-action is h-isometric, it preserves angles and orientation in the isothermal coordinates and thus it is an action by holomorphic transformations. By the uniformization Theorem, the universal cover of Σ is the hyperbolic semiplane $\mathbb{H}^2 \subset \mathbb{C}$ and the holomorphic automorphisms of Σ lift to holomorphic automorphisms of $\mathbb{H}^2$ which also are isometric automorphisms with respect to the hyperbolic metric. As a consequence of this, we have that the S^1-action preserves a hyperbolic metric in Σ which has finite volume, because Σ is compact, but this contradicts Lemma 19. □

Corollary 22. If Σ is a compact, orientable surface of genus $g \geq 2$ and h is a Riemannian metric in Σ, then the isometry group $Iso(\Sigma, h)$ is finite.

Proof. As Σ is compact, then the isometry group $G = Iso(\Sigma, h)$ is a compact Lie group. If $\mathfrak{g}$ denotes the Lie algebra of G, then for every $X \in \mathfrak{g}$, the one parameter group $\{exp(tX)\}$ is a commutative group whose closure is a compact, commutative Lie group with connected component of the identity isomorphic to a product $S^1 \times \cdots \times S^1$. As a consequence of this and the fact that G has only has finitely many connected components, if G is infinite, then it has a closed subgroup isomorphic to S^1, but this is impossible as is shown in Lemma 21. □

9.2. **Finner classification of hyperbolic isometries.** Recall that in dimension two, we have the action of $SL_2(\mathbb{R})$ on $\mathbb{H}^2$ by isometries via Möbius transformations, so that we have a realization of the orientation preserving isometries as $Iso(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$. An element $A \in SL_2(\mathbb{R})$ has as a characteristic polynomial $p_A(x) = x^2 - tr(A)x + 1$, and discriminant $tr(A)^2 - 4$. Thus, there are three dynamically different possibilities for the isometry of $\mathbb{H}^2$ generated by A, characterized by the sign of $tr(A) - 2$:

- $tr(A) - 2 > 0$, where the matrix is conjugated to a diagonal matrix over $\mathbb{R}$, and thus, the conjugated isometry is contained in the one parameter group of isometries generated by
$$\left\{ \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}.$$
One isometry of this type is called hyperbolic, and the one-parameter group generated by this matrix is characterized by the property of having two fixed points in the boundary $S^1 = \partial \mathbb{H}^2$ and preserves a foliation determined by the two points and guided by the geodesic that joints the two points (in the case of diagonal matrices, this is $\{0, \infty\}$).
- $tr(A) - 2 = 0$, where the matrix is conjugated over $\mathbb{R}$ to an upper triangular matrix, and thus, the conjugated isometry is contained in the one parameter group of isometries generated by
$$\left\{ \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$
One isometry of this type is called parabolic, and the one-parameter group generated by this matrix is characterized by the property of having one fixed point in the boundary $\partial \mathbb{H}^2$ and preserving the foliation of horocycles tangent to the fixed point (in the upper triangular case, the horocycles that are tangent to ∞ are just horizontal lines).
- $tr(A) - 2 < 0$, where the matrix is conjugated over $\mathbb{R}$ to a rotation matrix, so that the conjugated isometry is contained in the one parameter group of isometries generated by
$$\left\{ \exp \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} : t \in \mathbb{R} \right\}.$$
One isometry of this type is called elliptic, and the one-parameter group generated by this matrix is characterized by the property of having one fixed point in the interior of $\mathbb{H}^2$ and preserving a foliation of circles.

Lemma 23. If $\alpha, \beta \in PSL_2(\mathbb{R})$ are two non-trivial elements, then

(i) α and β commute if and only if $Fix(\alpha) = Fix(\beta)$,
(ii) $C(\alpha) = \{\gamma \in PSL_2(\mathbb{R}) : \alpha\beta = \beta\alpha\} = \{exp(tX) : t \in \mathbb{R}\}$, for some $X \in \mathfrak{sl}_2(\mathbb{R})$. In particular $C(\alpha)$ is isomorphic to either $\mathbb{R}$ or S^1.

Proof. Suppose $\alpha\beta = \beta\alpha$, then $\beta(Fix(\alpha)) = Fix(\alpha)$ and $\alpha(Fix(\beta)) = Fix(\beta)$. If α is parabolic or elliptic, then it has only one fixed point and thus $Fix(\alpha) = Fix(\beta)$ and the same applies for β either parabolic or elliptic. In the case where both α and β are hyperbolic, we observe that β cannot interchange two distinct elements of the boundary S^1, thus the property $\beta(Fix(\alpha)) = Fix(\alpha)$ implies $Fix(\alpha) = Fix(\beta)$. On the other hand, if α and β have the same set of fixed points, then they are elements of the same one-parameter group, this is obvious when the fixed points are in standard configuration, that is $\{0, \infty\}$, $\{\infty\}$ or $\{i\}$ according if the element is hyperbolic, parabolic or elliptic; and in general it can be seen via a conjugation of matrices by sending the fixed points to the standard configuration. In particular $\alpha\beta = \beta\alpha$, because a one-parameter group is commutative and the result follows. □

Corollary 24. If $\Gamma \subset PSL_2(\mathbb{R})$ is a subgroup such that it has the identity element as an accumulation point (equivalently Γ is not a discrete subgroup) and $\Lambda \subset \Gamma$ is a non-trivial, normal and discrete subgroup, then there exist $\Gamma_1 \subset \Gamma$ commutative subgroup of finite index.

Proof. **Λ is cyclic.** As Λ is normal, for every $\gamma \in \Gamma$, the conjugation induces an automorphism

$$\Lambda \to \Lambda, \qquad g \mapsto \gamma g \gamma^{-1},$$

and as Λ is discrete and Γ has the identity element as an accumulation point, for every $F \subset \Lambda$ finite set, there exist $\gamma \in \Gamma$ close enough to the identity such that $\gamma \neq e$ and $\gamma g = g\gamma$, for every $g \in F$. By the Lemma 23, the group generated by F is a discrete subgroup of the one-parameter group $C(\gamma)$ and thus it is a cyclic group. For $F_1 \subset F_2 \subset \Lambda$ any two distinct finite subsets, there are elements $g_j \in \Lambda$ such that $\langle g_j \rangle = \langle F_j \rangle$ and $\langle F_1 \rangle \subset \langle F_2 \rangle$ which implies that $g_1 = g_2^k$ for some k and in particular $0 < |g_2| < |g_1|$. Now Λ must be cyclic because otherwise we would have a sequence $\{g_j\} \subset \Lambda$ obtained as the generators of subgroups generated by an increasing tower of finite subsets of Λ that converge to the identity.

Existence of Γ_1. Take $\alpha \subset \Lambda$ a generator of the group and as $\gamma\alpha\gamma^{-1}$ is again a generator of Λ, for every $\gamma \in \Gamma$, then the subgroup

$$\Gamma_1 = \{\gamma \in \Gamma : \gamma\alpha\gamma^{-1} = \alpha\}$$

is a finite index subgroup of Γ ($[\Gamma : \Gamma_1] \leq 2$ if $\Lambda \cong \mathbb{Z}$, and $[\Gamma : \Gamma_1] \leq |\Lambda|$ if $\Lambda \cong \mathbb{Z}/m\mathbb{Z}$). Finally, by the Lemma 23, Γ_1 is commutative and the result follows. □

10. $\mathbb{H}^2 \times \mathbb{R}$

Recall [Kobayashi and Nomizu(1996)], Chapter VI, Theorem 3.5 that given a product of riemannian manifolds $M \times N$ with M of constant sectional curvature and N flat, the isometry group of $M \times N$ decomposes as a direct product, $\mathrm{Isom}(M) \times \mathrm{Isom}(N)$. The following give us the isometry groups of finite volume quotients of $\mathbb{H}^2 \times \mathbb{R}$ (see Theorem 31 for another proof):

Theorem 25. If $G \subset Iso(\mathbb{H}^2 \times \mathbb{R})$ is a discrete subgroup such that $(\mathbb{H}^2 \times \mathbb{R})/G$ has finite volume, then the group $Iso((\mathbb{H}^2 \times \mathbb{R})/G)$ is a finite extension of S^1

Proof. Consider the exact sequence

$$1 \to K \to G \to \Gamma \to 1,$$

where $K = G \cap Iso(\mathbb{R})$ is a discrete subgroup of G and $\Gamma \cong G/K$ is a subgroup of isometries of $\mathbb{H}^2$. If Γ is discrete as a subgroup of $Iso(\mathbb{H}^2)$, then $\mathbb{H}^2/\Gamma$ is an hyperbolic orbifold such that

$$\mathbb{R}/K \to (\mathbb{H}^2 \times \mathbb{R})/G \to \mathbb{H}^2/\Gamma$$

is a locally trivial fiber bundle and as $(\mathbb{H}^2 \times \mathbb{R})/G$ has finite volume, then $\mathbb{R}/K \cong S^1$ and Γ is a Lattice subgroup of $Iso(\mathbb{H}^2)$. In this case, we have an exact sequence of isometry groups

$$1 \to Iso(S^1) \to Iso(\mathbb{H}^2 \times \mathbb{R}/G) \to Iso(\mathbb{H}^2/\Gamma) \to 1,$$

where $Iso(\mathbb{H}^2/\Gamma)$ is a finite group by Lemma 19 and thus $Iso(\mathbb{H}^2 \times \mathbb{R}/G)$ is a finite extension of S^1.

If Γ is not discrete as a subgroup of $Iso(\mathbb{H}^2)$, we can see that the quotient $(\mathbb{H}^2 \times \mathbb{R})/G$ cannot have finite volume. To see this, first observe that we have another exact sequence

$$1 \to \Lambda \to G \to L \to 1,$$

where $\Lambda = G \cap Iso(\mathbb{H}^2) \subset \Gamma$ is a discrete, normal subgroup and $G/\Lambda \cong L \subset Iso(\mathbb{R})$. If $\Lambda = 0$, then $G \cong L$ is commutative and thus Γ is commutative. If instead Λ is non-trivial, then Corollary 24 tells us again that Γ is commutative (perhaps after passing to a finite index subgroup). In any case, G leaves a closed surface $\zeta \times \mathbb{R} \subset \mathbb{H}^2 \times \mathbb{R}$ fixed, where ζ is a geodesic, an horocycle or a circle (corresponding to the type of the isometries of Γ). If Γ consists of parabolic or hyperbolic elements, then Γ acts discretely by Euclidean automorphisms in $\zeta \times \mathbb{R} \cong \mathbb{R}^2$ so that by Bieberbach Theorem [Ratcliffe(2019)], Γ contains a finite index subgroup isomorphic to a subgroup of $\mathbb{Z}^2$ and in particular the fundamental domain of the G-action in $\mathbb{H}^2 \times \mathbb{R}$ contains a subset isometric to

$$\{x + iy : a < x < b\} \times [c, d] \subset \mathbb{H}^2 \times \mathbb{R},$$

this implies that $(\mathbb{H}^2 \times \mathbb{R})/G$ doesn't have finite volume. If Γ consists of elliptic elements, then G acts discretely by Euclidean automorphisms in $\zeta \times \mathbb{R} \cong S^1 \times \mathbb{R}$, and thus as in the previous case, the G-action has a fundamental domain containing an open subset isomorphic to

$$\{(se^{i\theta}, r) \in \mathbb{D} \times \mathbb{R} : a < \theta < b,\ c < r < d\},$$

where $\mathbb{D} \cong \mathbb{H}^2$ is the poincaré disc model of the hyperbolic plane, and again $(\mathbb{H}^2 \times \mathbb{R})/G$ doesn't have finite volume. □

11. SL_2-GEOMETRY

11.1. **Riemannian structure of $PSL_2(\mathbb{R})$.** Recall that given a Riemannian manifolds (M, g), there is a natural construction of a Riemannian tensor on the tangent bundle TM constructed as follows: if $(p, x) \in TM$, and $(c(t), v(t)) \in TM$ is a smooth curve such that $c(0) = p$ and $v(0) = x$, then

$$\|(c'(0), v'(0))\|^2_{(p,x)} = \|d\pi_{(p,x)}((c'(0), v'(0)))\|^2_p + \left\| \frac{D}{dt}_{|t=0} v(t) \right\|^2_p,$$

where $\pi : TM \to M$ is the projection, $\frac{D}{dt}v(t)$ is the covariant derivative along the curve $c(t)$ and $g(u, u)_p = \|u\|^2_p$. If $X = c'(0)$ and $Z = v'(0)$, in local coordinates we have the formula

$$\|(X, Z)\|^2_{(p,x)} = \|X\|^2_p + \left\|Z + X^j v^i \Gamma^k_{ij} \partial_k\right\|^2_p.$$

The vector (X, Z) is called horizontal if $c(t)$ is constant, and thus $X = 0$, it is called vertical if it is orthogonal to every horizontal vector in which case $Z = -X^j v^i \Gamma^k_{ij} \partial_k$. So, we have a decomposition in horizontal and vertical components as

$$(X, Z) = (0, Z + X^j v^i \Gamma^k_{ij} \partial_k) + (X, -X^j v^i \Gamma^k_{ij} \partial_k).$$

If we take the global coordinates $(x, y) \mapsto x + iy$ of the hyperbolic plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\},$$

with corresponding metric tensor $ds^2 = \frac{dx^2+dy^2}{y^2}$, then the Christoffel symbols at a point $x + iy$ are given by

$$-\Gamma^2_{11} = \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -1/y.$$

There is a natural identification of the tangent bundle

$$\mathbb{H}^2 \times \mathbb{C} \cong \mathrm{T}\mathbb{H}^2, \qquad (z, w) \mapsto \frac{d}{dt}_{|t=0}(z + tw)$$

and so the projection $\pi : T\mathbb{H}^2 \to \mathbb{H}^2$ is just given by the projection in the first factor and we have global coordinates in each tangent plane $\partial_1 = 1$ and $\partial_2 = i$. If as before, (X, Z) is a tangent vector to $T\mathbb{H}^2$ at the point $(p, v) = (i, 1)$, then the orthogonal decomposition in horizontal and vertical components is given by

$$(X, Z) = (0, Z - X^2 + X^1 i) + (X, X^2 - X^1 i).$$

The isometric action by Möbius transformations of $SL_2(\mathbb{R})$ in $\mathbb{H}^2$, induces the action in the tangent bundle

$$SL_2(\mathbb{R}) \times T\mathbb{H}^2 \to T\mathbb{H}^2, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, w) = \left(\frac{az+b}{cz+d}, \frac{w}{(cz+d)^2} \right).$$

This action is transitive in the unitary tangent bundle $T^1\mathbb{H}^2 = \{(z, w) \in \mathbb{H}^2 : \|w\|_z = 1\}$, so the orbit of the point $(i, 1) \in T^1\mathbb{H}^2$ induces the diffeomorphism $\phi : PSL_2(\mathbb{R}) \to T^1\mathbb{H}^2$ given explicitly by the formula

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{ai+b}{ci+d}, \frac{1}{(ci+d)^2} \right).$$

As this action is also isometric with respect to the previously defined metric, it will define a left invariant metric in $PSL_2(\mathbb{R})$ that corresponds to an inner product in its tangent vector to the identity, naturally identified with the Lie algebra

$$\mathfrak{sl}_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) : \mathrm{tr}(A) = 0\}.$$

More precisely, if we consider the derivative $d\phi$, we get the identification

$$\Psi : \mathfrak{sl}_2(\mathbb{R}) \to T_{(i,1)}(T\mathbb{H}^2), \qquad \Psi(X) = \frac{d}{dt}_{|t=0} \phi(exp(tX)).$$

A basis of $\mathfrak{sl}_2(\mathbb{R})$ is given by

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $g_{t,j} = exp(tX_j)$, then $\phi(g_{t,1}) = (e^{2t}i, e^{2t})$, $\phi(g_{t,2}) = (i, e^{2it})$ and

$$\phi(g_{t,3}) = \left(\frac{ch(t)i + sh(t)}{ch(t) + ish(t)}, \frac{1}{(ch(t) + ish(t))^2} \right),$$

where $ch(t)$ and $sh(t)$ denote the hyperbolic cosine and the hyperbolic sine correspondingly. If $\widehat{X}_j = \Psi(X_j)$, we have

$$\widehat{X}_1 = (2i, 2), \quad \widehat{X}_2 = (0, 2i), \quad \widehat{X}_3 = (2, -2i),$$

where we immediatly see that $\widehat{X}_2$ is vertical and a direct computation tells us that $\widehat{X}_1$ and $\widehat{X}_3$ are horizontal and orthogonal. Thus $\{\frac{1}{2}X_1, \frac{1}{2}X_2, \frac{1}{2}X_3\}$ is an orthonormal basis in the corresonding inner product in $\mathfrak{sl}_2(\mathbb{R})$.

As the $PSL_2(\mathbb{R})$-action is given by holomorphic maps, it commutes with the action of given by rotations in each tangent plane

$$S^1 \times T^1\mathbb{H}^2 \to T^1\mathbb{H}^2, \qquad \eta \cdot (z, w) = (z, \eta w),$$

as well as with the map $(z, w) \mapsto (\overline{z}, \overline{w})$. It is immediate the previous maps act by isometries and in fact generate the whole isometry group. Thus, the isometry group $Iso(PSL_2(\mathbb{R}))$ is isomorphic to $PSL_2(\mathbb{R}) \times (S^1 \rtimes \mathbb{Z}_2)$, see [Scott(1983)].

Theorem 26. If $\Gamma \subset Iso(PSL_2(\mathbb{R}))$ is a discrete group such that $PSL_2(\mathbb{R})/\Gamma$ has finite volume, then

$$Iso(PSL_2(\mathbb{R})/\Gamma) \cong S^1 \rtimes F,$$

where F is a finite group.

Proof. Consider the projection into the simple factor

$$P : Iso(PSL_2(\mathbb{R})) \to PSL_2(\mathbb{R}),$$

as the Kernel of P is compact and Γ is a discrete subgroup, then $\Gamma_0 = P(\Gamma)$ is a discrete subgroup of $PSL_2(\mathbb{R})$ and $\Gamma_0 \cong \Gamma/F_0$, with $F_0 = Ker(P) \cap \Gamma$ a finite subgroup of $S^1 \rtimes \mathbb{Z}_2$. Observe that $\pi : PSL_2(\mathbb{R}) \to \mathbb{H}^2$ is a fiber bundle with fiber S^1 such that $\pi(\gamma x) = P(\gamma)\pi(x)$, so we have an induced projection

$$\pi : PSL_2(\mathbb{R})/\Gamma \to \mathbb{H}^2/\Gamma_0,$$

which implies that $\mathbb{H}^2/\Gamma_0$ has finite hyperbolic area. As we also have the identification $\pi : PSL_2(\mathbb{R})/\Gamma_0 \to \mathbb{H}^2/\Gamma_0$, we have that Γ_0 is a Lattice in $PSL_2(\mathbb{R})$. By Lemma 19, we have that Γ_0 has finite index in $\Lambda = N_{PSL_2(\mathbb{R})}(\Gamma_0)$. Observe that if $F_1 = \Lambda/\Gamma_0$, then we have that $\Gamma \subset \Lambda \times S^1 \rtimes \mathbb{Z}_2$ and a bijection of sets

$$(\Lambda \times S^1 \rtimes \mathbb{Z}_2)/\Gamma \cong \frac{(\Lambda \times S^1 \rtimes \mathbb{Z}_2)/F_0}{\Gamma/F_0} \hookrightarrow (\Lambda \times S^1 \rtimes \mathbb{Z}_2)/\Gamma_0 \cong F \ltimes S^1,$$

where F is either F_1, or $F_1 \times \mathbb{Z}_2$, deppending on whether Γ contains the map $(z, w) \mapsto (\overline{z}, \overline{w})$ or not. Thus, we have that

$$S^1 \subset N_{Iso(PSL_2(\mathbb{R}))}(\Gamma)/\Gamma \subset (\Lambda \times S^1 \rtimes \mathbb{Z}_2)/\Gamma \cong F \ltimes S^1,$$

and the result follows. □

Remark: 27. As seen in Remark 20, we can obtain every finite group as an isometry group of an hyperbolic surface, so that, the finite factor of the isometry group in Theorem 26, can be any finite group.

11.2. **The universal cover $\widetilde{SL}_2(\mathbb{R})$.** The Lie group $PSL_2(R)$ is topologically the product $S^1 \times \mathbb{R}$, so that there is a simply connected Lie group denoted by $\widetilde{SL}_2(\mathbb{R})$ which is the topological universal cover of $PSL_2(\mathbb{R})$ and algebraically it is a central extension by a cyclic group $\mathbb{Z}$, more precisely, there is an exact sequence

$$1 \to \mathbb{Z} \to \widetilde{SL}_2(\mathbb{R}) \to PSL_2(\mathbb{R}) \to 1,$$

where $\mathbb{Z} \subset \widetilde{SL}_2(\mathbb{R})$ is the center. We can pull-back the metric tensor of $PSL_2(\mathbb{R})$ we constructed in the previous section to $\widetilde{SL}_2(\mathbb{R})$ to obtain the model of the homogeneous 3-dimensional geometry denoted by SL_2.

Remark: 28. The isometry group of $\widetilde{SL}_2(\mathbb{R})$ can be characterized into three different ways. First, we have the homomorphism

$$\widetilde{SL}_2(\mathbb{R}) \times \mathbb{R} \to Iso(\widetilde{SL}_2(\mathbb{R}))$$

given by left and right multiplications, here $\mathbb{R} \cong \widetilde{SO}(2)$ is the universal cover of the rotation group $SO(2) \subset SL_2(\mathbb{R})$, with Kernel $\mathbb{Z} = \mathbb{R} \cap \widetilde{SL}_2(\mathbb{R})$ being precisely the center of $\widetilde{SL}_2(\mathbb{R})$, so that $Iso(\widetilde{SL}_2(\mathbb{R}))$ has two connected components and

$$Iso(\widetilde{SL}_2(\mathbb{R}))_0 \cong \big(\widetilde{SL}_2(\mathbb{R}) \times \mathbb{R}\big)/\mathbb{Z}$$

is the component of the identity. In fact, we have an epimorphism

$$Iso(\widetilde{SL}_2(\mathbb{R})) \to Iso(PSL_2(\mathbb{R})) \cong PSL_2(\mathbb{R}) \times (S^1 \rtimes \mathbb{Z}_2),$$

with Kernel isomorphic to $\mathbb{Z}$, however, the group $Iso(\widetilde{SL}_2(\mathbb{R}))$ is no longer a product group. The left projection of the previous product gives us the second description in terms of a short exact sequence

$$1 \to \mathbb{R} \to Iso(\widetilde{SL}_2(\mathbb{R}))_0 \to PSL_2(\mathbb{R}) \to 1,$$

and if we consider the groups $\widetilde{SL}_2(\mathbb{R})$ and $\mathbb{R}$ as closed subgroups of $Iso(\widetilde{SL}_2(\mathbb{R})$, then we have the third description

$$Iso(\widetilde{SL}_2(\mathbb{R}))_0 = L(\widetilde{SL}_2(\mathbb{R}))R(\mathbb{R}),$$

where $L(\cdot)$ and $R(\cdot)$ represent left and right multiplications in the group $\widetilde{SL}_2(\mathbb{R})$.

A discrete subgroup $\Gamma \subset Iso(PSL_2(\mathbb{R}))$ can be lifted to a discrete subgroup $\widetilde{\Gamma} \subset Iso(\widetilde{SL}_2(\mathbb{R}))$, so that $\widetilde{SL}_2(\mathbb{R})/\widetilde{\Gamma} \cong PSL_2(\mathbb{R})/\Gamma$ and thus, we can compute $Iso(\widetilde{SL}_2(\mathbb{R})/\widetilde{\Gamma})$ with Theorem 26, however, not every discrete group of $Iso(\widetilde{SL}_2(\mathbb{R}))$ can be obtained this way. In the next section we prove discuss the general setting for discrete groups of isometries in $\widetilde{SL}_2(\mathbb{R})$.

11.3. **General orbifolds modelled in $\widetilde{SL}_2(\mathbb{R})$.** The following Lemma is well known and holds for every Lie group, but we include a proof of the case we need for the sake of completeness.

Lemma 29. If G is a Lie group locally isomorphic to $\mathbb{R} \times SL_2(\mathbb{R})$, for example G can be the isometry group of $\widetilde{SL}_2(\mathbb{R})$ or $\mathbb{H}^2 \times \mathbb{R}$, then there exists a neighborhood of the identity $U \subset G$ such that $[U, U] \subset U$.

Proof. Observe first that this is a local property, so we only need to prove this for linear groups. As the $\mathbb{R}$ factor lies in the center, we have that

$$[gg_0, hh_0] = [g, h], \qquad \forall\ g_0, h_0 \in \mathbb{R}$$

and thus we only need to prove this for $SL_2(\mathbb{R})$. The commutator

$$\left[\begin{pmatrix} a & x \\ y & b \end{pmatrix}, \begin{pmatrix} c & z \\ w & d \end{pmatrix}\right] = \begin{pmatrix} t_1 & t_3 \\ t_4 & t_2 \end{pmatrix},$$

is defined by the relations

- $t_1 = 1+xy+zw+xyzw+wxac+w^2x^2-adxw-a^2zw+bxwd-yd^2x-azyd,$
- $t_2 = 1+xy+zw+xyzw-xwbc-c^2xy+ayzc-zb^2w-zbcy+zybd+z^2y^2,$
- $t_3 = xac(d-c) - cx^2w + acz(a-b) - xwbz + zydx + z^2ya,$
- $t_4 = w^2xb + wxcy + bdw(b-a) - awyz + bdy(c-d) - dy^2z.$

So that if $0 \leq |x|, |y|, |z|, |w| < \varepsilon$ and $1-\varepsilon < a, b, c, d < 1+\varepsilon$, then there is a constant $C > 0$ independent of ε such that $|t_3|, |t_4| < C\varepsilon^2$ and $|t_1 - 1|, |t_2 - 1| < C\varepsilon^2$. Thus, by choosing $\varepsilon > 0$ such that $C\varepsilon^2 < \varepsilon$, the neighborhood

$$U_\varepsilon = \left\{\begin{pmatrix} a & x \\ y & b \end{pmatrix} : |x|, |y| < \varepsilon,\ |a-1|, |b-1| < \varepsilon\right\}$$

is stable under taking commutators. □

Proposition 30. Let H be a Lie group which is a central extension of $PSL_2(\mathbb{R})$ of the form

$$1 \to \mathbb{R} \to H \to PSL_2(\mathbb{R}) \to 1.$$

If $G \subset H$ is a discrete subgroup with induced exact sequence

$$1 \to K \to G \to \Gamma \to 1,$$

with $K \subset \mathbb{R}$, then either $\Gamma \subset PSL_2(\mathbb{R})$ is discrete or is an abelian subgroup leaving fixed a point, a geodesic or a horocycle in $\mathbb{H}^2$.

Proof. Denote by $p : H \to PSL_2(\mathbb{R})$ and consider $U \subset H$ a neighborhood of the identity such that $[U, U] \subset U$ and $U \cap G = \{e\}$. We have that the group $L = \langle p(U) \cap \Gamma \rangle$ is a commutative subgroup of $PSL_2(\mathbb{R})$, to see why this is true take two elements $\alpha, \beta \in G$ such that $p(\alpha), p(\beta) \in p(U)$, then we may write those elements as $\alpha = \alpha_0\alpha_1$ and $\beta = \beta_0\beta_1$, where

$$\alpha_1, \beta_1 \in \mathbb{R}, \qquad \alpha_0, \beta_0 \in U.$$

As $\mathbb{R}$ lies in the center of H we have that $[\alpha_0, \beta_0] = [\alpha, \beta] \in G \cap U$ and thus α and β commute. Now, for every $\alpha \in G$, choose a neighborhood of the identity $U_\alpha \subset G$ such that $[\alpha, U_\alpha] \subset U$, so that the elements of $\Gamma \cap p(U_\alpha)$ commute with $p(\alpha)$ (same argument as with the commutativity of L). Suppose that Γ is non-discrete, then L is a non-trivial commutative subgroup and for every $\gamma = p(\alpha) \in \Gamma$, we have that $\Gamma \cap p(U_\alpha)$ is a non-trivial subset that generates the group L and commutes with γ. So, Γ commutes with L and thus, there exists an element $X \in \mathfrak{sl}_2(\mathbb{R})$ such that

$$\Gamma \subset \overline{L} = \{exp(tX) : t \in \mathbb{R}\}$$

and Γ leaves fixed a point, a geodesic or a horocycle, depending on the type of X. □

Theorem 31. If G is a discrete subgroup of isometries of X either $\widetilde{SL}_2(\mathbb{R})$ or $\mathbb{H}^2 \times \mathbb{R}$ such that X/G has finite volume, then the isometry group $Iso(X/G)$ is a finite extension of S^1.

Proof. The exact sequence

$$1 \to \mathbb{R} \to Iso(X) \to PSL_2(\mathbb{R}) \to 1$$

induces the sequence

$$1 \to K \to G \to \Gamma \to 1,$$

if Γ is non-discrete, then it preserves a geodesic, a point or a horocycle by Proposition 30 and we can see that X/G doesn't have finite volume (as we did in Theorem 25). So, Γ is a discrete subgroup of isometries of the hyperbolic plane and we have a fiber bundle structure

$$\mathbb{R} \to X \to \mathbb{H}^2$$

so that the volume form decomposes as

$$\int_{\mathbb{R}} \int_{\mathbb{H}^2} f d\mu dt = \int_X f dvol_X$$

where $d\mu$ is the hyperbolic area form. If $D \subset \mathbb{H}^2$ is a fundamental domain of Γ, then $\pi^{-1}(D) = \widehat{D}$ is such that $g\widehat{D} \cap \widehat{D} \neq \emptyset$ only for $g \in K = G \cap \mathbb{R}$. Thus for $\Omega \subset \mathbb{R}$ fundamental domain of K in $\mathbb{R}$ we have that $\Omega \times D$ is a fundamental domain for G which implies that

$$Vol(X/G) \geq \int_\Omega \int_D \xi = |\Omega| \times \mu(D),$$

and we have that $\mu(A) < \infty$ and $|\Omega| < \infty$ which implies that $K = \mathbb{Z}$. Take $\widetilde{N} = N_{Iso(X)}(G)$ and $N = N_{PSL_2(\mathbb{R})}(\Gamma)$, so that we have the exact sequence

$$1 \to N_0 \to \widetilde{N} \to N \to 1$$

(because $gGg^{-1} = G$ projects to $\overline{g}\Gamma\overline{g}^{-1} = \Gamma$ and $N_0 = \mathbb{R}$ because $\mathbb{R}$ normalizes $\widetilde{SL}_2(\mathbb{R})$), this sequence induces the exact sequence

$$1 \to N_0/\mathbb{Z} \to \widetilde{N}/G \to N/\Gamma \to 1$$

(to see that this sequence is exact observe that $\pi(gG) = \overline{g}\Gamma$ is well defined and surjective, the condition $\pi(gG) = \Gamma$ holds if and only if $\overline{g} \in \Gamma$ and thus $g = [A, r]$ with $A \in \Gamma$, this is because there is an element $h = [A, s] \in G$, thus $gG = gh^{-1}G$ and

$gh^{-1} \in i(N_0/\mathbb{Z})$. This implies that the kernel of π is $i(N_0/\mathbb{Z})$. Finally $i(r\mathbb{Z}) = G$ if and only if $r \in G$, but $G \cap \mathbb{R} = \mathbb{Z}$, so that $r\mathbb{Z} = \mathbb{Z}$ and thus i is injective). This exact sequence can be written as

$$1 \to S^1 \to Iso(X/G) \to Iso(\mathbb{H}^2/\Gamma) \to 1$$

which implies the result because of the Lemma 19. □

12. Three dimensional Alexandrov Spaces

We will need some preliminaries on three dimensional Alexandrov spaces. For a general reference see [Burago et al.(2001)Burago, Burago, and Ivanov].

Alexandrov spaces are a synthetic generalization of complete riemannian manifolds with a lower bound on sectional curvature. The generalization uses comparison triangles with respect to the model spaces S^2_k, which are simply connected, two dimensional complete riemannian manifolds of constant curvature k. More precisely, for $k > 0$, S^2_k is the sphere of radius $\frac{1}{\sqrt{k}}$, for $k < 0$, S^2_k is the hyperbolic plane $\mathbb{H}^2(\frac{1}{\sqrt{-k}})$ of constant curvature k, and for $k = 0$, S^2_k is the euclidean space $\mathbb{R}^2$.

Given a geodesic triangle in a geodesic length space (X, d), with vertices $p, q, r \in X$, a comparison triangle in S^2_k is a geodesic triangle $\bar{p}\bar{q}\bar{r}$ having the the same side lengths. The geodesic length space (X, d) is said to satisfy the Topogonov property for $k \in \mathbb{R}$, if for each triple $p, q, r \in X$ of vertices of a geodesic triangle, and each point s on the geodesic from q to r, the inequality $d(p, s) \geq d(\bar{p}, \bar{s})$ holds, where $\bar{s}$ is the point on the geodesic side $\bar{q}\bar{r}$ of the comparison triangle with $d(\bar{p}\bar{s}) = d(p, s)$.

Definition 12.1. A n-dimensional k-Alexandrov space is a complete, locally compact, length space of finite Hausdorff dimension n, such that the Topogonov Property is satisfied locally for k.

Topogonov's globalization theorem tells us that the local and global Toponogov property are equivalent in k-Alexandrov spaces. By Gromov's precompactness theorem, Alexandrov n-dimensional spaces arise as Gromov-Hausdorf limits of compact riemannnian manifolds of dimension n for which the sectional curvature is bounded below by k, and the diameter above by some fixed positive number D.

The class of k-Alexandrov spaces includes riemannian manifolds of sectional curvature bounded below by k, and several constructions including more general geodesic length spaces such as euclidean cones, suspensions, joins, quotients by isometric actions of compact Lie groups, and glueings along a submetry, see [Galaz-García(2016)] section 2.2. From now on, we will omit the k from the notation.

There exists a notion of angle between geodesics of an Alexandrov space, and a space of tangent directions at a given point p, denoted by Σ_p can be defined as the completion of the metric space of equivalence classes of geodesics making a zero angle.

The space of tangent directions at a point p in an Alexandrov space X , denoted by Σ_p, has the structure of a 1-Alexandrov space of Hausdorf dimension $\dim(X)-1$. There is a set $R_X \subset X$, called the *set of metrically regular points*, where a point p belongs to R_X if its direction space Σ_p is isometric to the radius one sphere. The complement is called the set of metric singular points and denoted by $S_X = X \setminus R_X$. There are examples of Alexandrov spaces whose space of metrically singular points is dense, as seen in an example constructed in [Otsu and Shioya(1994)] as a limit of Alexandrov spaces, using baricentric subdivisions of a tetrahedron. However, for every Alexandrov space X, there is a dense subset of topologically regular points, whose space of directions are *homeomorphic* to a sphere (the set of topologically singular points is the complement of the set of topologically regular points). By Perelman's conical neighborhood theorem, every point p in an Alexandrov space has a neighborhood pointed homeomorphic to the euclidean cone over Σ_p, so that a locally compact, finite dimensional Alexandrov space has a dense subset which is a topological manifold.

In the specific case of dimension three, there are only two possibilities for the homeomorphic type of the space of directions, which is the two sphere S^2, for the topologically regular points and the real projective space $\mathbb{RP}^2$ for the topologically singular points. Let us summarize the basic structure of three dimensional Alexandrov spaces due to Galaz-García and Guijarro, compare Theorem 1.1 in page 5561 of [Galaz-Garcia and Guijarro(2015)], and Theorem 3.1 and 3.2 in page 1196 of [Galaz-Garcia and Guijarro(2013)].

Theorem 12.2. *Let X be three dimensional Alexandrov space.*

- *The set of metrically regular points is a riemannian three manifold.*
- *The set of topologically singular points is a discrete subset of X.*
- *If X is closed, and positively curved Alexandrov space, that contains a topologically singular point, then X is homeomorphic to the suspension of $\mathbb{R}P^2$.*

A closed Alexandrov space is *geometric* if it can be written as a quotient of one of the eight geometries of Thurston under a cocompact lattice. The following theorem was proved as Theorem 1.6 in [Galaz-Garcia and Guijarro(2015)] in page 5563.

Theorem 12.3. *A three dimensional Alexandrov space admits a geometric decomposition into geometric pieces, along spheres, projective planes, tori and Klein bottles.*

We now direct our attention to the isometry group of three dimensional Alexandrov spaces.

Theorem 12.4. *Let X be an n-dimensional Alexandrov space of Hausdorff dimension n.*

- *The Isometry group of X is a Lie Group. It is compact if X is.*
- *The dimension of the group of Isometries of X is at most*

$$\frac{n(n+1)}{2},$$

and the bound is attained if and only if X is a riemannian manifold.

Proof.
- The first part is proved as the main Theorem, 1.1 in [Fukaya and Yamaguchi(1994)]. The second part follows from the Van Dantzig-Van der Waerden Theorem [Dantzig and Van der Waerden(1928)].
- This is proved as Theorem 3.1 in page 570.

□

Remark: 32. It is proved in [Bagaev and Zhukova(2007)] that the same lower bound for the dimension of the isometry group holds in general for Riemannian orbifolds.

13. Hilbert-Smith conjecture

The following conjecture was formulated as an extension of Hilbert's 5th Problem:

Conjecture 13.1 (Hilbert-Smith conjecture)**.** *If G is a locally compact, topological group, acting faithfully on a topological manifold, then G is a Lie group.*

As a consequence of structural theorems of locally compact groups, such as Gleason-Yamabe theorem and some theorems of Newman, a counter-example to such conjecture must contain a copy of a p-adic group $\widehat{\mathbb{Z}}_p$, for some p, see [Lee(1997)], thus giving the equivalent conjecture

Conjecture 13.2 (Hilbert-Smith conjecture p-adic version)**.** *For every prime p, there are no faithful actions of the p-adic group $\widehat{\mathbb{Z}}_p$ on a topological manifold.*

Conjecture 13.2 has been proved in different contexts. For example, if there is a notion of dimension which must be preserved, such as bi-Lipschitz actions of $\widehat{\mathbb{Z}}_p$ on Riemannian manifolds, where three notions of dimension coincide: Hausdorff dimension, cohomological dimension with integer coefficients and topological dimension. In such setting, the bi-Lipschitz condition tells us that the Hausdorff dimension on the quotient cannot decrease, but on the other hand a theorem by Yang [Yang(1960)], tells us that the cohomological dimension of the quotient increases by two, leading to the following result:

Theorem 33 (Repovš-Ščepin [Repovs and Scepin(1997)])**.** There are no faithful actions by bi-Lipschitz maps of the p-adic group $\widehat{\mathbb{Z}}_p$ on a Riemannian manifold.

The stronger setting of the topological actions is much harder and has been proved only for small dimensions

Theorem 34 ([Pardon(2019)], [Pardon(2013b)])**.** For every prime p, there are no faithful actions by homeomorphisms of the p-adic group $\widehat{\mathbb{Z}}_p$ on a topological manifold of dimension $n \leq 3$.

Remark: 35. The p-adic group can be described as

$$\widehat{\mathbb{Z}}_p = \left\{ \sum_{n=0}^{\infty} a_n p^n : a_n \in \{0, 1, \cdots, p-1\}, \right\}$$

so that $p^k\widehat{\mathbb{Z}}_p \subset \widehat{\mathbb{Z}}_p$ is an open, normal subgroup, with $\widehat{\mathbb{Z}}_p / p^k\widehat{\mathbb{Z}}_p \cong \mathbb{Z}_{p^k\mathbb{Z}}$, giving the inverse limit description $\widehat{\mathbb{Z}}_p = \lim_{\leftarrow} \mathbb{Z}_{p^k}$, moreover, the group $\widehat{\mathbb{Z}}_p$ is homeomorphic to the Cantor space $\{0, \cdots, p-1\}^{\mathbb{N}}$. Observe that there is a topological 2-manifold with the cantor space $2^{\mathbb{N}}$ as its ends space, which is $\Sigma = S^2 \setminus C$, where $C \subset S^2$ is a closed subset homeomorphic to $2^{\mathbb{N}}$. Thus, there is a faithful action of $\widehat{\mathbb{Z}}_2$ on $End(\Sigma) \cong 2^{\mathbb{N}}$ and every homeomorphism of $End(\Sigma)$ extends to a homeomorphism of the surface Σ, however, by Theorem 34, such extensions cannot be promoted to an action of $\widehat{\mathbb{Z}}_2$ on the Freudenthal compactification.

As seen in the previous section, an Alexandrov space X has a closed subset S_X, corresponding to topologically singular points and such that the set of regular points $R_X = X \setminus S_X$ is an open-dense subset, having the structure of a topological manifold. An action by homeomorphisms on X must preserve the decomposition $X = S_X \cup R_X$ and a continuous action of $\widehat{\mathbb{Z}}_p$ which is trivial on the regular points, is trivial on the whole space X.

Hence, the weaker version of the p-adic Hilbert-Smith conjecture for Alexandrov spaces holds, and we can consider the following conjecture:

Conjecture 13.3. *If G is a locally compact, topological group, acting faithfully on a finite dimensional Alexandrov space by homeomorphisms, then G is a Lie group.*

A consequence of Theorem 34, gives us

Theorem 36. If G is a locally compact, topological group, acting faithfully on a three dimensional Alexandrov space by homeomorphisms, then G is a Lie group.

Remark: 37. As observed in previous section, there is a subset of metrically regular points which admits a compatible Riemannian metric, constructed in [Otsu and Shioya(1994)]. Thus, we have as a consequence of Theorem 33, that the p-adic group $\widehat{\mathbb{Z}}_p$ cannot act faithfully by bi-Lipschitz homeomorphisms. However, we should be careful, as the set of metrically singular points can be dense, as seen in an example constructed in [Otsu and Shioya(1994)] as a limit of Alexandrov spaces, using baricentric subdivisions of a tetrahedron.

14. Lattices on semisimple Lie groups of higher rank

Recall that an algebraic $\mathbb{R}$-group is a subgroup $\mathbb{G}_{\mathbb{C}} \subset GL_m(\mathbb{C})$ obtained as solutions of polynomial equations with coefficients over $\mathbb{R}$ and $\mathbb{G}_{\mathbb{R}} = \mathbb{G}_{\mathbb{C}} \cap GL_m(\mathbb{R})$ is a real Lie group. In this context we say that $\mathbb{G}_{\mathbb{R}}$ is a real form of $\mathbb{G}_{\mathbb{C}}$ or that $\mathbb{G}_{\mathbb{C}}$ is a complexification of $\mathbb{G}_{\mathbb{R}}$. The local structure of a Lie group is captured by its Lie algebra, so that two groups are locally isomorphic if and only if they have isomorphic Lie algebras, and thus, they can be obtained one from the other by taking connected components and topological covers.

The class of semisimple Lie groups can be defined as the class of Lie groups which are constructed up to covers and connected components from algebraic $\mathbb{R}$-groups which split as products $G_1 \times \cdots \times G_k$, where each factor G_j is simple. This definition is equivalent to other definitions of semisimple Lie groups aviailable in the literature, see [Zimmer(1984)].

Remark: 38. Not every semisimple Lie group is an algebraic group as the group $SL_2(\mathbb{R})$ has a universal cover, denoted by $\widetilde{SL}_2(\mathbb{R})$, which is homeomorphic to $\mathbb{R}^3$ and it cannot be embedded in any linear group $GL_m(\mathbb{C})$ as a Lie subgroup. In the same way, not every semisimple Lie group splits as a product of simple Lie groups, as the example $SO(4)$ shows, but its universal cover is isomorphic to the product $SU(2) \times SU(2)$. In general, given a connected semisimple Lie group G, with center $Z(G)$, then the quotient $G/Z(G)$ is a connected, linear algebraic group which splits as a product of simple groups and it is locally isomorphic to G. Thus it is common for some results to ask for the group to be centerless.

In the context of algebraic groups defined over a field k, the concept of k-rank is the maximal abelian subgroup which can be diagonalized over k. Thus, for a complex algebraic group, the $\mathbb{C}$-rank is the dimension of a maximal subgroup isomorphic to a complex torus $(\mathbb{C}^*)^l$ and we are particularly interested in the real rank of a real form. We can observe that the real rank of a product $G_1 \times \cdots \times G_k$ is the sum of the real rank of its factors G_j and we can give some explicit examples.

Example 14.1. The following is a complete list, up to local isomorphism, of complex, simple Lie groups and some examples of their real forms:

(i) The group $SL_n(\mathbb{C})$, has $\mathbb{C}$-rank $n-1$ and has the groups $SU(p,q)$ and $SL_n(\mathbb{R})$ as real forms, with real rank equal to $\min\{p,q\}$ and $n-1$ respectively.
(ii) The group $SO(n,\mathbb{C})$ has $\mathbb{C}$-rank $\lfloor \frac{n}{2} \rfloor$ and has the groups $SO(p,q)$ as real forms, having real rank equal to $\min\{p,q\}$.
(iii) The group $Sp(2n,\mathbb{C})$ has $\mathbb{C}$-rank n and has the groups $Sp(p,q)$ and $Sp(2n,\mathbb{R})$ as real forms, with real rank equal to n and $\min\{p,q\}$ respectively.
(iv) The exceptional complex groups $G_2(\mathbb{C})$, $F_4(\mathbb{C})$, $E_6(\mathbb{C})$, $E_7(\mathbb{C})$, $E_8(\mathbb{C})$ have $\mathbb{C}$-rank determined by the corresponding subindex.

Remark: 39. Between the possible real forms of a complex semisimple Lie group, there is one and only one compact real form up to conjugacy and such compact form has a compact universal cover, so the compactness property survives in the process of passing to a cover. We can thus, speak of the compact factors of a real semisimple Lie group. Moreover, the rank of a compact Lie group, defined as the dimension of a maximal torus $(S^1)^l$ contained in the group, equals the rank of its complexification and has real rank equal to 0. Finally, given a compact, connected, Lie group C, there is a finite cover of C that splits as $G \times T$, where G is an algebraic semisimple Lie group, and T is a torus.

Definition 14.2. A semisimple Lie group is said to have higher rank if its real rank is greater or equal to 2. Moreover, if a semisimple Lie group has a complexification whose simple factors are all locally isomorphic, the group is called isotypic.

Isotypic Lie groups are important, because we can construct irreducible lattices in them, which don't split as a product of lattices in the simple factors.

Example 9. If $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ is the non-trivial Galois automorphism and $Q(x,y,z,t) = x^2 + y^2 - \sqrt{2}z^2 - \sqrt{2}t^2$, $\sigma(Q) = x^2 + y^2 + \sqrt{2}z^2 + \sqrt{2}t^2$. The groups $G = SO(Q,\mathbb{R}) \cong SO(2,2)$ and $K = SO(\sigma(Q),\mathbb{R}) \cong SO(4)$ are semisimple Lie groups, with K compact and G of real rank equal to 2. If we consider the integral points in G, that is, the group $\Gamma = SO(Q,\mathbb{Z}(\sqrt{2})) \subset G$, then the group,

$$\widehat{\Gamma} = \{(g,\sigma(g)) \in G \times K : g \in \Gamma\}$$

is discrete. In fact there is an $\mathbb{R}$-group $\mathbb{H}_{\mathbb{C}} \subset GL_m(\mathbb{C})$ such that $\mathbb{H}_{\mathbb{R}} = G \times K$ and $\mathbb{H}_{\mathbb{Z}} = \widehat{\Gamma}$, in particular, it is a lattice which is co-compact. As the projection $G \times K \to G$ has compact Kernel and maps $\widehat{\Gamma}$ onto Γ, thus $\Gamma \subset G$ is discrete and thus, a co-compact lattice in G.

Remark: 40. The previous example captures the general behaviour of irreducible lattices in isotypic semisimple Lie groups. In fact, isotypic, semisimple Lie groups are the only cases of semisimple Lie groups admiting irreducible lattices and such lattices are constructed with the method of the previous example, but with higher degree extension fields $k/\mathbb{Q}$. See [Morris(2015)], Section 5.6 for the details of the previous example and the construction in general.

Remark: 41. As a consequence of the previous discussion, for every semisimple Lie group G such that $G \times SO(4)$ is isotypic[1], there is an irreducible lattice $\Gamma \subset G$ and an homomorphism $\Gamma \to SO(4)$ with dense image. In particular, such lattice acts by isometries on the round sphere S^3 with dense orbits. This tells us that there are no restriction on the dimension of the higher rank lattices which can act on the round sphere, but the type of such lattice is restricted. The same applies to the 3-orbifolds of the type $S^2 \times S^1$, as the first factor has isometry group $SO(3)$ which is simple.

We have in fact a converse of Remark 41, given by the following Theorem:

Theorem 42. If $\Gamma \subset G$ is a lattice in a higher rank, semisimple Lie group, K is a compact Lie group and $\varphi : \Gamma \to K$ is a homomorphism with dense image, then the group $G \times K$ is isotypic and $\Gamma \subset G$ is cocompact.

Theorem 42 is "well known to the experts", but a sketch of the first part of its proof is made in [Brown et al.()Brown, Fisher, and Hurtado], section 2.3. The fact that the lattice Γ is cocompact is a consequence of Godement's compactness criterion.

Corollary 43. If $\Gamma \subset G$ is a lattice in a higher rank, simple Lie group, K is a compact Lie group and $\varphi : \Gamma \to K$ is a homomorphism with infinite image, then $G \times L$ is isotypic, with $L = \overline{\varphi(\Gamma)}$. In particular, $\dim(G) \leq \dim(K)$ and Γ is cocompact in G.

As an immediate consequence, non-cocompact lattices don't appear in this setting and we have

Corollary 44. Let X be a geometric 3-orbifold of finite volume, and Γ a non-cocompact higher rank lattice in a semisimple Lie group G, then any action of Γ in X factors through a finite group.

As a particular example of the previous, any action of $SL_n(\mathbb{Z})$ in a geometric 3-orbifold of finite volume, factors through a finite group.

Corollary 45. Let X be a geometric 3-orbifold of finite volume, then X admits an isometric action of a higher rank lattice $\Gamma \subset G$ if and only if the group $Iso(X)$ contains the group $SO(3)$. Moreover, the semisimple Lie group G is isotypic of type $SO(3)$ and the lattice is uniform.

Observe that the group $SO(4)$ factors locally as the product $SO(3) \times SO(3)$ and in fact, there is a copy of $SO(3)$ inside $SO(4)$, so that the previous Corollary includes at the same time examples like $X = S^3/\Lambda$ and $X = (S^2 \times \mathbb{R})/\Lambda$.

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[1]For example, any product $G = G_1 \times \cdots \times G_k$, where each G_j is one of $SO(3,1)$, $SO(2,2)$ or $SO(4,\mathbb{C})$.

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