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Mathematical Logic

Mathias–Prikry and Laver type forcing; summable ideals, coideals, and +-selective filters

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Abstract We study the Mathias–Prikry and the Laver type forcings associated with filters and coideals. We isolate a crucial combinatorial property of Mathias reals, and prove that Mathias–Prikry forcings with summable ideals are all mutually bi-embeddable. We show that Mathias forcing associated with the complement of an analytic ideal always adds a dominating real. We also characterize filters for which the associated Mathias–Prikry forcing does not add eventually different reals, and show that they are countably generated provided they are Borel. We give a characterization of ω-hitting and ω-splitting families which retain their property in the extension by a Laver type forcing associated with a coideal.

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Introduction

The Mathias–Prikry and the Laver type forcings were introduced in [17] and [11] respectively. Recently, properties of these forcings were characterized in terms of properties of associated filters, see [1,5,10,12]. We continue this line of research, and investigate forcings associated with coideals.

1 Preliminaries

Our notation and terminology is fairly standard. We give here an overview of basic notions used in this paper. We sometimes neglect the formal difference between integer singletons and integers, if no confusion is likely to occur. We are mostly concerned with filters and ideals on ω and on the set of finite sets of integers $\mathrm{fin} = [\omega]^{<\omega}$. If a domain of a filter or ideal is not specified or obvious, it is assumed that the domain is ω. All filters and ideals are assumed to be proper and to extend the Fréchet filter.

For $a, b \subseteq \omega$ we write $a \subset^* b$ if $a \setminus b \in \mathrm{fin}$, $a =^* b$ if $a \subset^* b$ and $b \subset^* a$, $a < b$ if $n < m$ for each $n \in a$ and $m \in b$, and $a \sqsubseteq b$ if there is $n \in \omega$ such that $a = b \cap n$.

A *tree* T will usually be an initial subtree of the tree of finite sequences of integers $(\omega^{<\omega}, \subseteq)$ with no leaves. The space of maximal branches of T is denoted $[T]$. For $t \in T$ we denote by $T[t]$ the subtree consisting of all nodes of T compatible with t. An element $r \in T$ is called the *stem* of T if r is the maximal node of T such that $T = T[r]$. For $a \subseteq \omega$ we denote by $T^{[a]}$ the set of all nodes $t \in T$ such that $|t| \in a$ (i.e. the nodes from levels in a). A node $t \in T$ is a *branching node* of T if t has at least two immediate successors in T. For $\mathcal{X} \subset \mathcal{P}(\omega)$ we call t an $\mathcal{X}$-*branching* node if $\left\{ i \in \omega \mid t^\frown i \in T \right\} \in \mathcal{X}$. A tree is an $\mathcal{X}$-*tree* if every node of T is $\mathcal{X}$-branching.

For $\mathcal{X} \subset \mathcal{P}(\omega)$ and $A \subseteq \omega$ we write $\mathcal{X} \restriction A$ for the set $\{ X \cap A \mid X \in \mathcal{X} \}$. For a filter $\mathcal{F}$ we denote by $\mathcal{F}^*$ the dual ideal, and by $\mathcal{F}^+$ the complement of $\mathcal{F}^*$ (i.e. the $\mathcal{F}$ positive sets). For an ideal $\mathcal{I}$ we denote $\mathcal{I}^*$ the dual filter, $\mathcal{I}^+ = (\mathcal{I}^*)^+$. A complement of an ideal is called a *coideal*. We will generally not distinguish between terminology for properties of a filter and of the dual ideal, i.e. statements "$\mathcal{F}$ is φ" and "$\mathcal{F}^*$ is φ" are often regarded as synonymous. We will sometimes speak of filters on general countable sets as if they were filters on ω. In these cases statements about these filters are understood as statements about filters on ω isomorphic with them.

We call an ideal $\mathcal{I}$ *summable* if there is a function $\mu\colon \omega \to \mathbb{R}$ such that $\mathcal{I} = \left\{ I \subseteq \omega \mid \sum \{ \mu(i) \mid i \in I \} < \infty \right\}$. We say that $\mathcal{I}$ is tall if $\mathcal{I} \cap [A]^\omega \neq \emptyset$ for each $A \in [\omega]^\omega$. An ideal $\mathcal{I}$ is below an ideal $\mathcal{J}$ in the *Rudin–Keisler order*, $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $I \in \mathcal{I}$ iff $f^{-1}[I] \in \mathcal{J}$ for each $I \subseteq \omega$. We say that $\mathcal{I}$ is *Rudin–Blass* below $\mathcal{J}$, $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ if the witnessing function f is finite-to-1. The Rudin–Keisler and Rudin–Blass ordering on filters is defined in the same way as on ideals. Note that for ideals is $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ iff $\mathcal{I}^* \leq_{\mathrm{RK}} \mathcal{J}^*$, and similarly for $\leq_{\mathrm{RB}}$.

For a filter $\mathcal{F}$ we will consider the filter $\mathcal{F}^{<\omega}$ generated by sets $[F]^{<\omega}$ for $F \in \mathcal{F}$. If $\mathcal{F}$ is a filter on ω, then $\mathcal{F}^{<\omega}$ is a filter on fin. Notice that for $X \subset \text{fin}$ is $X \in \mathcal{F}^{<\omega+}$ iff for each $F \in \mathcal{F}$ there is $a \in X$ such that $a \subset F$, and iff for each $F \in \mathcal{F}$ there are infinitely many $a \in X$ such that $a \subset F$. The elements of $\mathcal{F}^{<\omega+}$ are sometimes called the *$\mathcal{F}$-universal sets*.

A filter $\mathcal{F}$ is a *P^+-filter* if for every sequence $\{X_n \mid n \in \omega\} \subseteq \mathcal{F}^+$ there is a sequence $Y = \{y_n \in [X_n]^{<\omega} \mid n \in \omega\}$ such that $\bigcup Y \in \mathcal{F}^+$.

The ideal of all meager sets of reals is denoted by $\mathscr{M}$. For $f, g \in \omega^\omega$ write $f =^\infty g$ if $\{n \in \omega \mid f(n) = g(n)\}$ is infinite. Recall that $\text{cov}(\mathscr{M}) = \mathfrak{b}(\omega^\omega, =^\infty)$ and $\text{non}(\mathscr{M}) = \mathfrak{d}(\omega^\omega, =^\infty)$. Let V be a model of set theory. We say that $e \in \omega^\omega$ is an *eventually different* real (over V) if $e \neq^\infty f$ for each $f \in \omega^\omega \cap V$. We say that $d \in \omega^\omega$ is a *dominating* real (over V) if $f <^* d$ for each $f \in \omega^\omega \cap V$. Every dominating real is an eventually different real. For every $f \in \omega^\omega$ the set $\{g \in \omega^\omega \mid g =^\infty f\}$ is a dense G_δ subset of ω^ω. The following proposition is well known, the proof is analogous to the proof of [2, Lemma 2.4.8].

Proposition 1 *Let V be a model of set theory. The set $V \cap \omega^\omega$ is meager in ω^ω if and only if there exists an eventually different real over V.*

The Cohen forcing for adding a subset of a set $X \subseteq \omega$ will be denoted $\mathbb{C}_X$, and $\mathbb{C}$ denotes $\mathbb{C}_\omega$. The conditions of $\mathbb{C}_X$ are finite subsets of X ordered by $\sqsupseteq$. A Cohen generic real is the union of a generic filter on $\mathbb{C}_X$.

Let $\mathcal{X}$ be a family of subsets of ω, typically a filter or a coideal. The *Mathias–Prikry forcing* $\mathbb{M}(\mathcal{X})$ associated with $\mathcal{X}$ consists of conditions of the form (s, A) where $s \in \text{fin}$ and $A \in \mathcal{X}$. Although we usually assume $s < A$, we do not require it. The ordering is given by $(s, A) \leq (t, B)$ if $t \sqsubseteq s$, $A \subseteq B$, and $s \setminus t \subset B$. Given a generic filter on $\mathbb{M}(\mathcal{X})$, we call the union of the first coordinates of conditions in the generic filter the $\mathbb{M}(\mathcal{X})$ *generic real*. Given $\mathcal{X}$ and $r \subset \omega$, we denote $G_r(\mathcal{X}) = \{(s, A) \in \mathbb{M}(\mathcal{X}) \mid s \sqsubseteq r, r \subseteq A\}$. It is easy to see that r is an $\mathbb{M}(\mathcal{X})$ generic real iff $G_r(\mathcal{X})$ is a generic filter on $\mathbb{M}(\mathcal{X})$. Properties of $\mathbb{M}(\mathcal{F})$ when $\mathcal{F}$ is an ultrafilter were studied in [4] and for $\mathcal{F}$ a general filter in [5,12]. Since $\mathbb{M}(\mathcal{F})$ is σ-centered, it always adds an unbounded real. On the other hand, it was shown that $\mathbb{M}(\mathcal{F})$ can be weakly ω^ω-bounding and even almost ω^ω-bounding. Filters for which $\mathbb{M}(\mathcal{F})$ is weakly ω^ω-bounding are called Canjar, and these are exactly those filters for which $\mathcal{F}^{<\omega}$ is a P^+-filter.

The *Laver type forcing* associated with $\mathcal{X}$ is denoted by $\mathbb{L}(\mathcal{X})$. Conditions in this forcing is trees $T \subseteq \omega^{<\omega}$ with stem t such that every node $s \in T$, $t \leq s$, is $\mathcal{X}$-branching. The ordering of $\mathbb{L}(\mathcal{X})$ is inclusion. Given a generic filter on $\mathbb{L}(\mathcal{X})$, the generic real is the union of stems of conditions in the generic filter. The generic real is a function dominating $\omega^\omega \cap V$, unless $\mathcal{X} \cap \text{fin} \neq \emptyset$. Properties of $\mathbb{L}(\mathcal{F})$ for $\mathcal{F}$ filter were studied in [1,12].

For an ideal $\mathcal{I}$ on ω, the forcing $(\mathcal{P}(\omega)/\mathcal{I}, \subset)$ adds a generic V-ultrafilter on ω containing $\mathcal{I}^*$, which will be denoted $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$. The superscript will be omitted when $\mathcal{I}$ is apparent from the context.

A family $\mathcal{X}$ is *ω-hitting* (also called *ω-tall*) if for each countable sequence $\{A_n \in [\omega]^\omega \mid n \in \omega\}$ exists $X \in \mathcal{X}$ such that $A_n \cap X$ is infinite for each $n \in \omega$.

A family $\mathcal{X}$ is *ω-splitting* if for each countable sequence $\{ A_n \in [\omega]^\omega \mid n \in \omega \}$ exists $X \in \mathcal{X}$ such that both $A_n \cap X$ and $A_n \setminus X$ are infinite for each $n \in \omega$.

To conclude the preliminaries let us recall a useful characterization of F_σ ideals. A *lower semicontinuous submeasure* is a function $\varphi\colon \mathcal{P}(\omega) \to [0, \infty]$ such that $\varphi(\emptyset) = 0$; if $A \subseteq B$, then $\varphi(A) \leq \varphi(B)$ (monotonicity); $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ (subadditivity); and $\varphi(A) = \sup \{ \varphi(A \cap n) \mid n \in \omega \}$ for every $A \subseteq \omega$ (lower semicontinuity).

Proposition 2 (Mazur) *Let $\mathcal{I}$ be an F_σ ideal on ω. There is a lower semicontinuous submeasure φ such that $\varphi(\{ n \}) = 1$ for every $n \in \omega$, and $\mathcal{I} = \mathrm{fin}(\varphi)$.*

2 Mathias like reals and summable ideals

The original motivation for this section comes from a question of Ilijas Farah about the number of ZFC-provably distinct Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is a 'definable' ideal [8]. Note that CH implies that all such Boolean algebras are isomorphic for F_σ ideals $\mathcal{I}$ [15]. The interpretation of 'definability' interesting in this context might be '$\mathrm{F}_{\sigma\delta}$,' 'Borel,' or 'analytic.' The basic question was answered by Oliver [18] by showing that there are 2^ω many $\mathrm{F}_{\sigma\delta}$ ideals for which the Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ are provably nonisomorphic. However, these constructions are not interesting from the forcing point of view, the constructed examples are locally isomorphic to $\mathcal{P}(\omega)/\mathrm{fin}$. On the other hand, Steprāns [21] showed that there are continuum many coanalytic ideals whose quotients are pairwise forcing not equivalent.

We are interested in (anti-)classification results about forcings of this form. The first result in this direction is due to Farah ad Solecki. They showed that the Boolean algebras $\mathcal{P}(\mathbb{Q})/\mathrm{nwd}_\mathbb{Q}$ and $\mathcal{P}(\mathbb{Q})/\mathrm{null}_\mathbb{Q}$ are nonisomorphic and homogeneous, see [9]. A systematic study of such forcing notions was done by Hrušák and Zapletal [14]. They provided several examples of forcings of this form. Their results imply that for each tall summable ideal $\mathcal{I}$ there is an $\mathrm{F}_{\sigma\delta}$ ideal denoted here $\mathrm{tr}_\mathcal{I}$ such that $\mathcal{P}(\omega)/\mathrm{tr}_\mathcal{I} = \mathbb{M}(\mathcal{I}^*) * \mathbb{Q}$ for some $\mathbb{Q}$, a name for a proper ω-distributive forcing notion. Therefore showing that the Mathias forcings $\mathbb{M}(\mathcal{I}^*)$ are different for various choices of summable ideals $\mathcal{I}$ seems to be a viable attempt to provide a spectrum of different forcings $\mathcal{P}(\omega)/\mathrm{tr}_\mathcal{I}$. However, the results of this section show that this approach is likely to fail, the Mathias forcings for tall summable ideals all mutually bi-embeddable.

Let us start with a general combinatorial characterization of Mathias generic reals.

Definition 3 Let $V \subseteq U$ be models of the set theory, $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter in V, and $x \in \mathcal{P}(\omega) \cap U$. We say that x is a *Mathias like real for $\mathcal{F}$ over V* if the following two conditions hold;

(1) $x \subset^* F$ for each $F \in \mathcal{F} \cap V$,
(2) $[x]^{<\omega} \cap H \neq \emptyset$ for each $H \in \mathcal{F}^{<\omega+} \cap V$.

Notice that an $\mathbb{M}(\mathcal{F})$ generic real is a Mathias like for $\mathcal{F}$. It was implicitly shown in [12] that Mathias like reals are already almost Mathias generic—it is sufficient to add a Cohen real to get the genericity. This explains why most results concerning the

Mathias forcing rely just on the fact that the generic reals are Mathias like. We provide the proof of this fact for reader's convenience.

Proposition 4 *Let $V \subseteq U$ be models of the set theory, $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter in V, and $x \in \mathcal{P}(\omega) \cap U$ be a Mathias like real for $\mathcal{F}$ over V. Let c be a $\mathbb{C}_x$ generic real over U. Then c is an $\mathbb{M}(\mathcal{F})$ generic real over V.*

Proof We need to prove that $G_c(\mathcal{F}) \cap \mathscr{D} \neq \emptyset$ for each dense subset $\mathscr{D} \in V$ of $\mathbb{M}(\mathcal{F})$, i.e. to show that the set of conditions forcing this fact is dense in $\mathbb{C}_x$. Choose any condition $s \in \mathbb{C}_x$. Denote

$$H = \{ t \setminus s \mid (\exists F \in \mathcal{F} \mid (t, F) \in \mathscr{D}), s \sqsubseteq t \}.$$

Note that $H \in \mathcal{F}^{<\omega+}$, otherwise there exists $F \in \mathcal{F}$ such that $[F]^{<\omega} \cap H = \emptyset$, and the condition (s, F) has no extension in $\mathscr{D}$. Condition (2) of Definition 3 now implies that there exists $(t, F_t) \in \mathscr{D}$ such that $s \sqsubseteq t$, and $t \setminus s \subset x$. Since $x \subset^* F_t$, there is $k \in x$, $t < k$ such that $x \setminus k \subset F_t$. Hence $t \cup \{k\} \in \mathbb{C}_x$, $t \cup \{k\} < s$, and $t \cup \{k\} \Vdash (t, F_t) \in G_c(\mathcal{F}) \cap \mathscr{D}$. □

For a poset P we denote by $\mathrm{RO}(P)$ the unique (up to isomorphism) complete Boolean algebra in which P densely embeds (while preserving incompatibility), and $\mathrm{RO}(P)^+$ denotes the set of non-zero elements of $\mathrm{RO}(P)$. The relation $\lessdot$ denotes complete embedding of Boolean algebras.

Corollary 5 *Let $\mathbb{P}$ be a forcing adding a Mathias like real for a filter $\mathcal{F}$.*

(1) $\mathrm{RO}(\mathbb{M}(\mathcal{F})) \lessdot \mathrm{RO}(\mathbb{P} \times \mathbb{C})$.
(2) If $\mathbb{Q}$ is a forcing adding a Cohen real, then $\mathrm{RO}(\mathbb{M}(\mathcal{F})) \lessdot \mathrm{RO}(\mathbb{P} \times \mathbb{Q})$.

Proof Proposition 4 implies that every generic extension via $\mathbb{P} * \dot{\mathbb{C}}$ contains a generic filter on $\mathbb{M}(\mathcal{F})$ over V. Hence there is $a \in \mathrm{RO}(\mathbb{M}(\mathcal{F}))^+$ such that

$$\mathrm{RO}(\mathbb{M}(\mathcal{F})) \restriction a \lessdot \mathrm{RO}\left(\mathbb{P} * \dot{\mathbb{C}}\right) = \mathrm{RO}(\mathbb{P} \times \mathbb{C}), \qquad \circledast$$

see e.g. [22]. For each $p \in \mathbb{M}(\mathcal{F})$, the poset $\mathbb{M}(\mathcal{F}) \restriction p$ is isomorphic to $\mathbb{M}(\mathcal{F} \restriction F)$ for some $F \in \mathcal{F}$. If x is a Mathias like real for $\mathcal{F}$, then it is also Mathias like for $\mathcal{F} \restriction F$ for each $F \in \mathcal{F}$, and we can deduce from Proposition 4 that the set of elements of $\mathrm{RO}(\mathbb{M}(\mathcal{F}))^+$ satisfying $\circledast$ is dense. Since $\mathbb{M}(\mathcal{F})$ is c.c.c. we can find A, a countable maximal antichain of such elements. Now

$$\begin{aligned} \mathrm{RO}(\mathbb{M}(\mathcal{F})) &\simeq \prod_{a \in A} \mathrm{RO}(\mathbb{M}(\mathcal{F})) \restriction a \lessdot \prod_{\omega} \mathrm{RO}(\mathbb{P} \times \mathbb{C}) \\ &\simeq \mathrm{RO}\left(\mathbb{P} \times \sum_{\omega} \mathbb{C}\right) \simeq \mathrm{RO}(\mathbb{P} \times \mathbb{C}). \end{aligned}$$

To justify the second last isomorphism, we construct a dense embedding e of the poset $\mathbb{P} \times \sum_{\omega} \mathbb{C}$ into the complete Boolean algebra $\prod_{\omega} \mathrm{RO}(\mathbb{P} \times \mathbb{C})$: If t is an element of the n-th copy of $\mathbb{C}$ in $\sum_\omega \mathbb{C}$, define $e(p,t)(i) = (p,t)$ if $i = n$, and $e(p,t)(i) = \mathbf{0}$ otherwise.

If $\mathbb{Q}$ adds a Cohen generic real, then there exists some $a \in \mathrm{RO}(\mathbb{C})^+$ such that $\mathrm{RO}(\mathbb{C}) \restriction a \lessdot \mathrm{RO}(\mathbb{Q})$. Since $\mathrm{RO}(\mathbb{C}) \restriction a$ is isomorphic to $\mathrm{RO}(\mathbb{C})$, the second statement follows from the first one. □

The next lemma states that Mathias like reals behave well with respect to the Rudin–Keisler ordering on filters.

Lemma 6 *Let $\mathcal{E}, \mathcal{F}$ be filters on ω, let $f: \omega \to \omega$ be a function witnessing $\mathcal{F} \leq_{\mathrm{RK}} \mathcal{E}$, and x be a Mathias like real for $\mathcal{E}$. Then $f[x]$ is a Mathias like real for $\mathcal{F}$.*

Proof It is obvious that $f[x] \subset^* F$ for each $F \in \mathcal{F}$, so we need to check only condition (2) of Definition 3. Define $f^*: \mathrm{fin} \to \mathrm{fin}$ by

$$f^*(h) = \{a \in \mathrm{fin} \mid f[a] = h\}.$$

Claim *If $H \in \mathcal{F}^{<\omega+}$, then $\bigcup f^*[H] \in \mathcal{E}^{<\omega+}$.*

For $E \in \mathcal{E}$ is $f[E] \in \mathcal{F}$, and there is $h \in H$ such that $h \subset f[E]$. Thus $f^*(h) \cap [E]^{<\omega} \neq \emptyset$. □

Choose any $H \in \mathcal{F}^{<\omega+}$. Since x is Mathias like for $\mathcal{E}$, there exists $a \in \bigcup f^*[H]$ such that $a \subset x$. Now $f[a] \subset f[x]$ and $f[a] \in H$. □

We focus now on summable ideals. The following simple observation appears in [7].

Lemma 7 *Let $\mathcal{I}, \mathcal{J}$ be tall summable ideals. There exists $A \in \mathcal{P}(\omega) \setminus \mathcal{J}^*$ such that $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \restriction A$.*

We are now equipped to prove the bi-embeddability result.

Theorem 8 *Let $\mathcal{I}, \mathcal{J}$ be tall summable ideals. Then* $\mathrm{RO}(\mathbb{M}(\mathcal{I}))$ *is completely embedded in* $\mathrm{RO}(\mathbb{M}(\mathcal{J}))$.

Proof Find A as in Lemma 7 and consider the decomposition

$$\mathbb{M}(\mathcal{J}^*) = \mathbb{M}(\mathcal{J}^* \restriction A) \times \mathbb{M}(\mathcal{J}^* \restriction (\omega \setminus A)).$$

The forcing $\mathbb{M}(\mathcal{J}^* \restriction A)$ adds a Mathias real for $\mathcal{J}^* \restriction A$. Lemma 6 implies that it also adds a Mathias like real for $\mathcal{I}^*$. Since $\mathcal{J}^* \restriction (\omega \setminus A)$ is not an ultrafilter, the forcing $\mathbb{M}(\mathcal{J}^* \restriction (\omega \setminus A))$ adds a Cohen real. The conclusion now follows from Corollary 5. □

This shows that the original plan of creating many essentially different forcings by using different summable ideals is likely to fail. However, we still do not know whether the Mathias forcing is the same for every tall summable ideal.

Question 9 Are $\mathbb{M}(\mathcal{J}^*)$ and $\mathbb{M}(\mathcal{I}^*)$ equivalent forcing notions if $\mathcal{I}$ and $\mathcal{J}$ are tall summable ideals?

To conclude this section let us mention a related result of Farah [7, Proposition 3.7.1].

Proposition 10 *Assume* $\mathsf{OCA} + \mathsf{MA}$. *If $\mathcal{I}$ is a summable ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ is weakly homogeneous iff $\mathcal{I} = \mathrm{fin}$.*

3 Mathias forcing with coideals

This section deals with the forcing $\mathbb{M}(\mathcal{F}^+)$ for $\mathcal{F}$ a filter on ω. We are mainly interested in the following question.

Question 11 When does $\mathbb{M}\left(\mathcal{F}^+\right)$ add dominating reals?

The following fact is well known.

Fact 12 *Let $\mathcal{I}$ be an ideal on ω. Then $\mathbb{M}\left(\mathcal{I}^+\right) = \mathcal{P}(\omega)/\mathcal{I} * \mathbb{M}(\dot{\mathcal{G}}^{\mathcal{I}}_{\text{gen}})$.*

Proposition 13 *If $\mathcal{I}$ is a Borel ideal and $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals, then $\mathbb{M}\left(\mathcal{I}^+\right)$ adds a dominating real.*

Proof First assume that $\mathcal{I}$ is an F_σ ideal. Let φ be a submeasure as in Proposition 2.

Let r be a $\mathbb{M}\left(\mathcal{I}^+\right)$ generic real and notice that $r \notin \text{fin}(\varphi)$. In $V[r]$ define an increasing function $g\colon \omega \to \omega$ by letting

$$g(n) = \min\left\{k \in \omega \mid 2^n \leq \varphi(r \cap k)\right\}.$$

We will show g is a dominating real. Let $(s, A) \in \mathbb{M}\left(\mathcal{I}^+\right)$ be a condition and $f : \omega \to \omega$ a function in V. We will extend (s, A) to a condition that forces that g dominates f. Pick $m \in \omega$ such that $\varphi(s) < 2^m$ and for every $i > m$ choose $t_i \subseteq A \setminus f(i)$ such that $\max(t_i) < \min(t_{i+1})$ and $2^i \leq \varphi(t_i) < 2^{i+1}$. This is possible since $\varphi(A) = \infty$ and the φ-mass of singletons is 1. Put $B = \bigcup_{m<i} t_i$, thus $\varphi(B) = \infty$ and $(s, B) \in \mathbb{M}\left(\mathcal{I}^+\right)$. Moreover $(s, B) \leq (s, A)$, and since $(s, B) \Vdash \dot{r} \subset s \cup B$ we have that $(s, B) \Vdash f(i) < g(i)$ for $i > m$.

For the general case let $\mathcal{I}$ be an analytic ideal such that $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals. If $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$ is not a P-point, then it is not a Canjar filter (see e.g. [4]), and $\mathbb{M}\left(\mathcal{I}^+\right) = \mathcal{P}(\omega)/\mathcal{I} * \mathbb{M}\left(\mathcal{G}^{\mathcal{I}}_{\text{gen}}\right)$ will add a dominating real. In case $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$ is a P-point, then by [13, Theorem 2.5] $\mathcal{I}$ is locally F_σ and $\mathbb{M}\left(\mathcal{I}^+\right)$ adds a dominating real as demonstrated in the first part of the proof. □

Question 14 Is there a Borel ideal $\mathcal{I}$ such that $\mathbb{M}\left(\mathcal{I}^+\right)$ does not add a dominating real?

It is easy to see that in every generic extension by $\mathbb{M}(\mathcal{F}^+)$ the ground model set of reals is meager, and thus $\mathbb{M}(\mathcal{F}^+)$ always adds an eventually different real. In [13] Michael Hrušák and Jonathan Verner asked the following question.

Question 15 Is there a Borel ideal $\mathcal{I}$ on ω such that $\mathcal{P}(\omega)/\mathcal{I}$ adds a Canjar ultrafilter?

We answer this question in negative.

Lemma 16 *If $\mathcal{I}$ is an ideal on ω such that $\mathcal{G}^{\mathcal{I}}_{\text{gen}}$ is a P-point, then $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals.*

Proof Let $A \in \mathcal{I}^+$ and r a name such that $A \Vdash \dot{r} \in \omega^\omega$. Let $\mathcal{G}_{\text{gen}}$ be a $\mathcal{P}(\omega)/\mathcal{I}$ generic filter such that $A \in \mathcal{G}_{\text{gen}}$ and for every $n \in \omega$ we can find $A_n \in \mathcal{G}_{\text{gen}}$ such that $A_n \leq A$ and A_n decides $\dot{r}(n)$. Since $\mathcal{G}_{\text{gen}}$ is a P-point, there is $B \in \mathcal{G}_{\text{gen}}$ such that $B \subseteq^* A_n$ for every $n \in \omega$ (note that we can assume B is a ground model set since $\mathcal{G}_{\text{gen}}$ is generated by ground model sets). Clearly $B \leq A$ and forces $\dot{r}$ to be a ground model real. □

Corollary 17 *If $\mathcal{I}$ is an analytic ideal then $\mathcal{G}^{\mathcal{I}}_{\mathrm{gen}}$ is not a Canjar ultrafilter.*

Proof By the previous proposition if $\mathcal{P}(\omega)/\mathcal{I}$ adds new reals then the generic filter is not a Canjar ultrafilter. Assume no new reals are added. By Proposition 13, $\mathbb{M}\left(\mathcal{I}^{+}\right)$ adds a dominating real and $\mathcal{G}_{\mathrm{gen}}$ is not Canjar. □

4 Mathias–Prikry forcing and eventually different reals

We turn our attention towards the forcing $\mathbb{M}(\mathcal{F})$ for a filter $\mathcal{F}$. Our goal is the characterization of filters for which this forcing does not add eventually different reals.

A filter $\mathcal{F}$ is $+$-*Ramsey* [16] if for each $\mathcal{F}^{+}$-tree T there is a branch $b \in [T]$ such that $b[\omega] \in \mathcal{F}^{+}$.

Definition 18 Let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{F}$ is $+$-*selective* if for every sequence $\{X_n \mid n \in \omega\} \subseteq \mathcal{F}^{+}$ there is a selector

$$S = \{a_n \in X_n \mid n \in \omega\} \in \mathcal{F}^{+}.$$

Every $+$-Ramsey filter is $+$-selective and every $+$-selective filter is a P^{+}-filter.

Let M be an extension of the universe of sets V. We say that $r \in \omega^{\omega} \cap M$ is an eventually different real over V if the set $\{n \in \omega \mid r(n) = f(n)\}$ is finite for each $f \in \omega^{\omega} \cap V$. We say that a forcing $\mathbb{P}$ does not add an eventually different real iff there is no eventually different real over V in any generic extension by forcing $\mathbb{P}$.

Theorem 19 *Let $\mathcal{F}$ be a filter. The following are equivalent;*

(1) Forcing $\mathbb{M}(\mathcal{F})$ does not add an eventually different real,
(2) $\mathcal{F}^{<\omega}$ is $+$-selective,
(3) $\mathcal{F}^{<\omega}$ is $+$-Ramsey.

Proof The implication (3) ⇒ (2) is clear. We start with (2) ⇒ (1).

Let $\mathcal{F}^{<\omega}$ be $+$-selective and x be an $\mathbb{M}(\mathcal{F})$ name for a function in ω^{ω}. Enumerate $\mathrm{fin} = \langle s_i \mid i \in \omega\rangle$ such that $\max s_i \leq i$ for each $i \in \omega$. Let $\{a_i \mid i \in \omega\}$ be a partition of ω into infinite sets, and denote by $a_i(k)$ the k-th element of a_i. For $k \in \omega$ let

$$\begin{aligned} X_k = \Big\{ t \in \mathrm{fin} \mid k < \min t \text{ and } \forall i < k \colon \exists h^t_i(k) \in \omega \colon \exists F \\ \in \mathcal{F} \colon (s_i \cup t, F) \Vdash \dot{x}(a_i(k)) = h^t_i(k) \Big\}. \end{aligned}$$

Claim *$X_k \in \mathcal{F}^{<\omega +}$ for each $k \in \omega$.*

Let $k \in \omega$. We need to show that for each $G \in \mathcal{F}$ there exists $t \in X_k$ such that $t \subset G$. Put $t_0 = \emptyset$, $F_0 = G \setminus (k+1)$, and for $i < k$ proceed with an inductive construction as follows.

Suppose t_i, F_i were defined, we will define t_{i+1}, F_{i+1}, $h^t_i(k)$. Find a condition $p = (s_i \cup t_{i+1}, F_{i+1}) < (s_i \cup t_i, F_i)$ and $h^t_i(k) \in \omega$ such that $p \Vdash \dot{x}(a_i(k)) = h^t_i(k)$. Finally put $t = t_k$, and notice that $t \in X_k$, $t \subset G$.

Let $S \in \mathcal{F}^{<\omega+}$ be a selector for $\langle X_k \mid k \in \omega \rangle$ guaranteed by the +selectivity of $\mathcal{F}^{<\omega}$. Define $g;\ \omega \to \omega$ by $g(a_i(k)) = h_i^t(k)$ if $t \in S$ and $i < k$. We claim that $\Vdash |\{ g(n) = \dot{x}(n) \mid n \in \omega \}| = \omega$.

Let (s_i, G) be a condition and n be an integer. There exists $k > n$, i and $t \in X_k \cap S$ such that $t \subset G$. Thus there is $F \in \mathcal{F}$ such that

$$(s_i \cup t, F) \Vdash \dot{x}(a_i(k)) = h_i^t(k) = g(a_i(k)).$$

Put $p = (s_i \cup t, F \cap G) < (s_i, G)$. Now $n < k \leq a_i(k)$ and $p \Vdash \dot{x}(a_i(k)) = g(a_i(k))$.

To prove (1) $\Rightarrow$ (3) assume $\mathbb{M}(\mathcal{F})$ does not add an eventually different real. Let T be an $\mathcal{F}^{<\omega+}$-tree and r be an $\mathbb{M}(\mathcal{F})$ generic real. For $n \in \omega$ let $O_n = \{ a \in [T] \mid \exists m > n : a(m) \subset r \setminus n \}$. Note that each such O_n is an open dense subset of $[T]$. Now $G = \bigcap \{ O_n \mid n \in \omega \}$ is a dense G_δ set, and Proposition 1 implies that there exists some $b \in G \cap V$. We claim that b is the desired branch for which $b[\omega] \in \mathcal{F}^{<\omega+}$. Otherwise there is $F \in \mathcal{F}$ such that $b[\omega] \cap F^{<\omega} = \emptyset$, which contradicts $r \subset^* F$ and $|[r]^{<\omega} \cap b[\omega]| = \omega$. □

The last part of the proof in fact demonstrated the following.

Theorem 20 *Let $V \subseteq U$ be models of the set theory, $\mathcal{F} \subset \mathcal{P}(\omega)$ be a filter in V. If U contains a Mathias like real for $\mathcal{F}$ but no eventually different real over V, then $\mathcal{F}$ is +-Ramsey.*

The implication (2) $\Rightarrow$ (3) of Theorem 19 can be proved directly with the same proof as is used in [19, Lemma 2]. Although this implication holds true for filters of the form $\mathcal{F}^{<\omega}$, this is not the case for filters in general. The filter on $2^{<\omega}$ generated by complements of $\subseteq$-chains and $\subseteq$-antichains is an F_σ +-selective filter which is not +-Ramsey.

The following proposition is a direct consequence of [16, Theorem 2.9].

Proposition 21 *Let $\mathcal{F}$ be a Borel filter. $\mathcal{F}$ is +-Ramsey if and only if $\mathcal{F}$ is countably generated.*

Corollary 22 *If $\mathcal{F}$ is a Borel filter on ω and $\mathbb{M}(\mathcal{F})$ does not add an eventually different real, then $\mathbb{M}(\mathcal{F})$ is forcing equivalent to the Cohen forcing.*

Proof If $\mathbb{M}(\mathcal{F})$ does not add an eventually different, then the Borel $\mathcal{F}^{<\omega}$ is +-Ramsey and hence countably generated. Thus $\mathcal{F}$ is also countably generated and $\mathbb{M}(\mathcal{F})$ has a countable dense subset. □

It is not hard to see that any forcing of size less than $\mathrm{cov}(\mathscr{M})$ can not add an eventually different real, so we have another proof of the following well known result,

Corollary 23 *If $\mathcal{I}$ is a Borel ideal which is not countably generated then* $\mathrm{cov}(\mathscr{M}) \leq \mathrm{cof}(\mathcal{I})$.

Corollary 22 can be derived directly from [20, Conclusion 9.16], which says that if a Suslin c.c.c. forcing adds a non-Cohen real, then it makes the set of ground model reals meager. See also [23, Corollary 3.5.7].

5 Laver type forcing

We will address the question of preserving hitting families with Laver type forcing. Since every forcing adding a real destroys some maximal almost disjoint family, it only makes sense to ask for survival of hitting families with some additional properties. Preservation of ω-hitting and ω-splitting families with Laver forcing $\mathbb{L}$ was studied in [6]. A characterization of the strong preservation of these properties with forcing $\mathbb{L}(\mathcal{F})$ for a filter $\mathcal{F}$ was given in [1]. Preservation of spliting familes with $\mathbb{L}(\mathcal{F})$ was also studied in [3]. We utilize methods used in [6] to characterize ω-hitting and ω-splitting families for which the Laver forcing $\mathbb{L}(\mathcal{F}^+)$ preserves the ω-hitting and the ω-splitting property.

Definition 24 Let $\mathcal{X} \subset \mathcal{P}(\omega)$ be a family of sets and let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{X}$ is $\mathcal{F}^+$-*ω-hitting* if for every countable set of functions $\{ f_n : \omega \to \omega \mid n \in \omega \}$ such that $f_n[\omega] \in \mathcal{F}^+$ for each $n \in \omega$, there exists $X \in \mathcal{X}$ such that $f_n[X] \in \mathcal{F}^+$ for each $n \in \omega$.

Obviously, every $\mathcal{F}^+$-ω-hitting family must be ω-hitting.

Proposition 25 *Let $\mathcal{F}$ be a filter on ω and let $\mathcal{X} \subset \mathcal{P}(\omega)$. The following are equivalent;*

(1) $\mathcal{X}$ is $\mathcal{F}^+$-ω-hitting,
(2) $\mathbb{L}(\mathcal{F}^+)$ preserves "$\check{\mathcal{X}}$ is ω-hitting."

Proof Start with (1) implies (2). For conditions $S, T \in \mathbb{L}(\mathcal{F}^+)$, where the stem of T is $r \in \omega^k$, we write $S <^n T$ if $S < T$ and $S \cap \omega^{k+n} = T \cap \omega^{k+n}$.

Let θ be a large enough cardinal and let $M \prec H_\theta$ be a countable elementary submodel containing $\mathcal{F}$. Let $X \in \mathcal{X}$ be such that $f[X] \in \mathcal{F}^+$ for each $f : \omega \to \omega$, $f \in M$ such that $f[\omega] \in \mathcal{F}^+$.

Claim A *Let $A \in M$ be an $\mathbb{L}(\mathcal{F}^+)$-name, and $S \in \mathbb{L}(\mathcal{F}^+) \cap M$ be a condition such that $S \Vdash \dot{A} \in [\omega]^\omega$. There exists $S' <^0 S$ such that for each $T' < S'$ there is $t \in T'$ such that $S'[t] \in M$ and $S'[t] \Vdash \check{X} \cap \dot{A} \neq \emptyset$.*

Since S is countable and A is a name for an infinite set, we can inductively build a sequence $\{ \langle t_n, k_n, R_n \rangle \mid n \in \omega \} \in M$ such that

- $t_n \in S$, $k_n \in \omega$, $R_n \in \mathbb{L}(\mathcal{F}^+)$,
- $R_n <^0 S[t_n]$,
- $R_n \Vdash k_n \in \dot{A}$,
- $k_n \neq k_m$ for $n \neq m$,
- $I = \{t_n \mid n \in \omega\}$ is a maximal antichain in S.

Put $S' = \bigcup \{ R_n \mid k_n \in X \}$. Let r be the stem of S. We only need to show that for each $s \in S$ such that $r \leq s < t$ for some $t \in I$, the set $\left\{ i \in \omega \mid s^\frown i \in S' \right\}$ is in $\mathcal{F}^+$. Define a function $f : \omega \to \omega$ in M by $f : k_n \mapsto i$ if $t_n \geq s^\frown i$ for $n \in \omega$, and $f : k \mapsto 0$ otherwise. Note that $\left\{ i \in \omega \mid s^\frown i \in S \right\} \subseteq f[\omega] \in \mathcal{F}^+$ since I is maximal bellow s. Thus $f[X] = \left\{ i \in \omega \mid s^\frown i \in S' \right\} \in \mathcal{F}^+$. □

Let $T \in \mathbb{L}(\mathcal{F}^+) \cap M$ be a condition with stem r. Enumerate $\{ A_n \mid n \in \omega \}$ all $\mathbb{L}(\mathcal{F}^+)$-names belonging to M such that $\Vdash \dot{A}_n \in [\omega]^\omega$ for each $n \in \omega$. We will inductively construct a fusion sequence of conditions $\{ T_n \mid n \in \omega \}$ starting with $T_0 = T$ such that

- $T_{n+1} <^n T_n$ for each $n \in \omega$,
- for each $T' < T_n$ there is $t \in T'$ such that $T_n[t] \in M$ and $T_n[t] \Vdash \dot{A}_n \cap \check{X} \neq \emptyset$.

Suppose that T_n is constructed and use the inductive hypothesis to find a maximal antichain $J \subset \{ t \in T_n \mid n + |r| < |t|, T_n[t] \in M \}$ in T_n. For each $t \in J$ use Claim A for $S = T_n[t]$ and $A = A_{n+1}$ to get $T_n'[t] <^0 T_n[t]$ as in the statement of the claim. Now $T_{n+1} = \bigcup \left\{ T_n'[t] \mid t \in J \right\}$ is as required.

Once this sequence is constructed put $R = \bigcap \{ T_n \mid n \in \omega \} \in \mathbb{L}(\mathcal{F}^+)$. Now $R \Vdash \dot{A}_n \cap \check{X} \neq \emptyset$ for each $n \in \omega$, and the implication is proved.

For the other direction, assume there are functions $\{ f_n : \omega \to \omega \mid n \in n \}$ such that $f_n[\omega] \in \mathcal{F}^+$, and for each $X \in \mathcal{X}$ there is $n \in \omega$ such that $f_n[X] \in \mathcal{F}^*$. Fix $\{ b_n \in [\omega]^\omega \mid n \in \omega \}$, a partition of ω into infinite sets. Let $\dot{\ell}$ be a name for the $\mathbb{L}(\mathcal{F}^+)$ generic real, and define a name for $\dot{A}_n^k \subset \omega$ by declaring $\dot{A}_n^k = f_n^{-1} \left[\dot{\ell}[b_n \backslash k] \right]$ for each $k, n \in \omega$. Inductively define $T \in \mathbb{L}(\mathcal{F}^+)$ such that $t^\frown i \in T$ iff $i \in f_n[\omega]$ for $t \in T^{[b_n]}$. Notice that T forces that $\dot{A}_n^k$ is infinite for each $k, n \in \omega$.

Take any $X \in \mathcal{X}$ and let $S < T$ be a condition with stem r. There is $n \in \omega$ such that $f_n[X] \in \mathcal{F}^*$. Put

$$S' = S \backslash \left\{ s \in T \mid \exists t \in T^{[b_n]}, r < t : \exists i \in f[X] : t^\frown i \subseteq s \in S \right\}.$$

Note that $S' \in \mathbb{L}(\mathcal{F}^+)$ since we removed only $\mathcal{F}^*$ many immediate successors of each splitting node of S. Also notice that $S' \Vdash X \cap \dot{A}_n^{|r|} = \emptyset$. Thus for each $X \in \mathcal{X}$ the condition T forces that X does not have infinite intersection with all sets $\dot{A}_n^k$, and $\mathcal{X}$ is not ω-hitting in the extension. $\square$

We can formulate the “splitting” version of the previous result. A similar result for $\mathbb{L}(\mathcal{F})$, where $\mathcal{F}$ is a filter, is contained in [3, Section 6].

Definition 26 Let $\mathcal{X} \subset \mathcal{P}(\omega)$ be a family of sets and let $\mathcal{F}$ be a filter on ω. We say that $\mathcal{X}$ is $\mathcal{F}^+$-*ω-splitting* if for every countable set of functions $\{ f_n : \omega \to \omega \mid n \in \omega \}$ such that $f_n[\omega] \in \mathcal{F}^+$ for each $n \in \omega$, there exists $X \in \mathcal{X}$ such that $f_n[X], f_n[\omega \backslash X] \in \mathcal{F}^+$ for each $n \in \omega$.

Again, every $\mathcal{F}^+$-ω-splitting family is ω-splitting. The same proof as before with the obvious adjustments gives us the following.

Proposition 27 *Let $\mathcal{F}$ be a filter on ω and let $\mathcal{X} \subset \mathcal{P}(\omega)$. The following are equivalent;*

(1) $\mathcal{X}$ is $\mathcal{F}^+$-ω-splitting,
(2) $\mathbb{L}(\mathcal{F}^+)$ preserves “$\check{\mathcal{X}}$ is ω-splitting.”

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