COUNTABLY COMPACT GROUPS WITHOUT NON-TRIVIAL CONVERGENT SEQUENCES

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Abstract. We construct, in ZFC, a countably compact subgroup of $2^{\mathfrak{c}}$ without non-trivial convergent sequences, answering an old problem of van Douwen. As a consequence we also prove the existence of two countably compact groups $\mathbb{G}_0$ and $\mathbb{G}_1$ such that the product $\mathbb{G}_0 \times \mathbb{G}_1$ is not countably compact, thus answering a classical problem of Comfort.

1. Introduction

The celebrated Comfort-Ross theorem [7] states that any product of pseudo-compact topological groups is pseudo-compact, in stark contrast with the examples due to Novák [27] and Terasaka [33] who constructed pairs of countably compact spaces whose product is not even pseudo-compact. This motivated Comfort [9] (repeated in [8]) to ask:

Question 1.1 (Comfort [8]). *Are there countably compact groups $\mathbb{G}_0, \mathbb{G}_1$ such that $\mathbb{G}_0 \times \mathbb{G}_1$ is not countably compact?*

The first consistent positive answer was given by van Douwen [44] under MA, followed by Hart-van Mill [20] under MA_{ctble}. In his paper van Douwen showed that every boolean countably compact group without non-trivial convergent sequences contains two countably compact subgroups whose product is not countably compact, and asked:

Question 1.2 (van Douwen [44]). *Is there a countably compact group without non-trivial convergent sequences?*

In fact, the first example of such a group was constructed by Hajnal and Juhász [19] a few years before van Douwen's [44] assuming CH. Recall, that every compact topological group contains a non-trivial convergent sequence, as an easy consequence of the classical and highly non-trivial Ivanovskiĭ-Vilenkin-Kuz'minov theorem [24] that every compact topological group is *dyadic*, i.e. a continuous image of 2^κ for some cardinal number κ.

Both questions have been studied extensively in recent decades, providing a large variety of sufficient conditions for the existence of examples to these questions, much work being done by Tomita and collaborators [16, 17, 22, 29, 32, 39, 40, 41, 36, 42,

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43, 37], but also others [10, 12, 13, 26, 34]. The questions are considered central in the theory of topological groups [1, 2, 7, 8, 14, 31, 35].

Here we settle both problems by constructing in ZFC a countably compact subgroup of $2^{\mathfrak{c}}$ without non-trivial convergent sequences.

The paper is organized as follows: In Section 2 we fix notation and review basic facts concerning ulrapowers, Fubini products of ultrafilters and Bohr topology. In Section 3 we study van Douwen's problem in the realm of p-compact groups. We show how iterated ultrapowers can be used to give interesting partial solutions to the problem. In particular, we show that an iterated ultrapower of the countable Boolean group endowed with the Bohr topology via a selective ultrafilter p produces a p-compact subgroup of $2^{\mathfrak{c}}$ without non-trivial convergent sequences. This on one hand raises interesting questions about ultrafilters, amd on other hand serves as a warm up for Section 4, where the main result of the paper is proved by constructing a countably compact subgroup of $2^{\mathfrak{c}}$ without non-trivial convergent sequences using not a single ultrafilter, but rather a carefully constructed $\mathfrak{c}$-sized family of ultrafilters.

2. Notation and terminology

Recall that an infinite topological space X is *countably compact* if every infinite subset of X has an accumulation point. Given p a nonprincipal ultrafilter on ω (for short, $p \in \omega^*$), a point $x \in X$ and a sequence $\{x_n : n \in \omega\} \subseteq X$ we say (following [5]) that $x = p\text{-}\lim_{n\in\omega} x_n$ if for every open $U \subseteq X$ containing x the set $\{n \in \omega\colon x_n \in U\} \in p$. It follows that a space X is countably compact if and only if every sequence $\{x_n : n \in \omega\} \subseteq X$ has a p-limit in X for some ultrafilter $p \in \omega^*$. Given an ultrafilter $p \in \omega^*$, a space X is *p-compact* if for every sequence $\{x_n : n \in \omega\} \subseteq X$ there is an $x \in X$ such that $x = p\text{-}\lim_{n\in\omega} x_n$.

For introducing the following definition, we fix a bijection $\varphi : \omega \to \omega \times \omega$, and for a limit ordinal $\alpha < \omega_1$, we pick an increasing sequence $\{\alpha_n : n \in \omega\}$ of smaller ordinals with supremum α. Given an ultrafilter $p \in \omega^*$, the *iterated Fubini powers* or *Frolík sums* [15] of p are defined recursively as follows:

$$p^1 = p$$

$$p^{\alpha+1} = \{A \subseteq \omega : \{n : \{m : (n,m) \in \varphi(A)\} \in p^{\alpha}\} \in p\} \text{ and}$$

$$p^{\alpha} = \{A \subseteq \omega : \{n : \{m : (n,m) \in \varphi(A)\} \in p^{\alpha_n}\} \in p\} \text{ for } \alpha \text{ limit.}$$

The choice of the ultrafilter p^α depends on (the arbitrary) choice of φ and the choice of the sequence $\{\alpha_n : n \in \omega\}$, however, the *type* of p^α does not (see *e.g.*, [15, 18]).

For our purposes we give an alternative definition of the iterated Fubini powers of p: given $\alpha < \omega_1$ we fix a well-founded tree $T_\alpha \subset \omega^{<\omega}$ such that

(i) $\rho_{T_\alpha}(\varnothing) = \alpha$, where ρ_{T_α} denotes the rank function on $\langle T_\alpha, \subseteq\rangle$;
(ii) For every $t \in T_\alpha$, if $\rho_{T_\alpha}(t) > 0$ then $t^\frown n \in T_\alpha$ for all $n \in \omega$.

For $\beta \leqslant \alpha$, let $\Omega_\beta(T_\alpha) = \{t \in T_\alpha\colon \rho_{T_\alpha}(t) = \beta\}$ and $T_\alpha^+ = \{t \in T_\alpha\colon \rho_{T_\alpha}(t) > 0\}$.

If $p \in \omega^*$, then $\mathbb{L}_p(T_\alpha)$ will be used to denote the collection of all trees $T \subseteq T_\alpha$ such that for every $t \in T \cap T_\alpha^+$ the set $\mathrm{succ}_T(t) = \{n \in \omega\colon t^\frown n \in T\}$ belongs to p. Notice that each $T \in \mathbb{L}_p(T_\alpha)$ is also a well-founded tree with $\rho_T(\varnothing) = \alpha$. Moreover, the family $\{\Omega_0(T)\colon T \in \mathbb{L}_p(T_\alpha)\}$ forms a base for an ultrafilter on $\Omega_0(T_\alpha)$ which has the same type of p^α. If $T \in \mathbb{L}_p(T_\alpha)$ and $U \in p$, $T \upharpoonright U$ denotes the tree in $\mathbb{L}_p(T_\alpha)$ for which $\mathrm{succ}_{T\upharpoonright U}(t) = \mathrm{succ}_T(t) \cap U$ for all $t \in (T \upharpoonright U)^+$.

Next we recall the *ultrapower* construction from model theory and algebra. Given a group $\mathbb{G}$ and an ultrafilter $p \in \omega^*$, denote by

$$\mathsf{ult}_p(\mathbb{G}) = \mathbb{G}^\omega / \equiv \text{ , where } f \equiv g \text{ iff } \{n : f(n) = g(n)\} \in p.$$

The *Theorem of Lós* [25] states that for any formula ϕ with parameters $[f_0], [f_1], \dots [f_n]$, $\mathsf{ult}_p(\mathbb{G}) \models \phi([f_0], [f_1], \dots [f_n])$ if and only if $\{k : \mathbb{G} \models \phi(f_0(k), f_1(k), \dots f_n(k))\} \in p$. In particular, $\mathsf{ult}_p(\mathbb{G})$ is a group with the same first order properties as $\mathbb{G}$.

There is a natural embedding of $\mathbb{G}$ into $\mathsf{ult}_p(\mathbb{G})$ sending each $g \in \mathbb{G}$ to the equivalence class of the constant function with value g. We shall therefore consider $\mathbb{G}$ as a subgroup of $\mathsf{ult}_p(\mathbb{G})$. Also, without loss of generality, we can assume that $\mathrm{dom}(f) \in p$ for every $[f] \in \mathsf{ult}_p(\mathbb{G})$.

Recall that the *Bohr topology* on a group $\mathbb{G}$ is the weakest group topology making every homomorphism $\Phi \in \mathrm{Hom}(\mathbb{G}, \mathbb{T})$ continuous, where the circle group $\mathbb{T}$ carries the usual compact topology. We let $(\mathbb{G}, \tau_{\mathrm{Bohr}})$ denote $\mathbb{G}$ equipped with the Bohr topology.

Finally, our set-theoretic notation is mostly standard and follows [23]. In particular, recall that an ultrafilter $p \in \omega^*$ is a *P-point* if for every function on ω becomes finite-to-one or constant when restricted to some set in the ultrafilter and, an ultrafilter $p \in \omega^*$ is a *Q-point* if every finite-to-one function on ω becomes one-to-one when restricted to a suitable set in the ultrafilter. The ultrafilters $p \in \omega^*$ which are P-point and Q-point are called *selective* ultrafilters. For more background on set-theoretic aspects of ultrafilters see [6].

3. Iterated ultrapowers as p-compact groups

In this section we shall give a canonical construction of a p-compact group for every ultrafilter $p \in \omega^*$. This will be done by studying the iterated ultrapower construction.

Fix a group $\mathbb{G}$ and put $\mathsf{ult}_p^0(\mathbb{G}) = \mathbb{G}$. Given an ordinal α with $\alpha > 0$, let

$$\mathsf{ult}_p^\alpha(\mathbb{G}) = \mathsf{ult}_p\left(\varinjlim_{\beta<\alpha} \mathsf{ult}_p^\beta(\mathbb{G}))\right),$$

where $\varinjlim_{\beta<\alpha} \mathsf{ult}_p^\beta(\mathbb{G})$ denotes the direct limit of the direct system $\langle \mathsf{ult}_p^\beta(\mathbb{G}), \varphi_{\delta\beta} \colon \delta \leqslant \beta < \alpha\rangle$ with the following properties:

(1) $\varphi_{\delta\delta}$ is the identity of $\mathsf{ult}_p^\beta(\mathbb{G})$, and
(2) $\varphi_{\delta\beta} \colon \mathsf{ult}_p^\delta(\mathbb{G}) \to \mathsf{ult}_p^\beta(\mathbb{G})$ is the canonical embedding of $\mathsf{ult}_p^\delta(\mathbb{G})$ into $\mathsf{ult}_p^\beta(\mathbb{G})$.

In what follows, we will abbreviate $\mathsf{ult}_p^{\alpha^-}(\mathbb{G})$ for $\varinjlim_{\beta<\alpha} \mathsf{ult}_p^\beta(\mathbb{G})$. Moreover, we will treat $\mathsf{ult}_p^{\alpha^-}(\mathbb{G})$ as $\bigcup_{\beta<\alpha} \mathsf{ult}_p^\beta(\mathbb{G})$ and, in such case, we put $\mathrm{ht}(a) = \min\{\beta < \alpha \colon a \in \mathsf{ult}_p^\beta(\mathbb{G})\}$ for every $a \in \mathsf{ult}_p^{\alpha^-}(\mathbb{G})$. This is, of course, formally wrong, but is facilitated by our indentification of $\mathbb{G}$ with a subgroup of $\mathsf{ult}_p(\mathbb{G})$. In this way we can avoid talking about direct limit constructions.

We now consider $(\mathbb{G}, \tau_{\text{Bohr}})$. Having fixed an ultrafilter $p \in \omega^*$, this topology naturally *lifts* to a topology on $\mathsf{ult}_p(\mathbb{G})$ as follows: Every $\Phi \in \text{Hom}(\mathbb{G}, \mathbb{T})$ naturally extends to a homomorphism $\overline{\Phi} \in \text{Hom}(\mathsf{ult}_p(\mathbb{G}), \mathbb{T})$ by letting

$$\overline{\Phi}([f]) = p\text{-}\lim_{n\in\omega} \Phi(f(n)). \tag{3.1}$$

By Łós's theorem, $\overline{\Phi}$ is indeed a homomorphism from $\mathsf{ult}_p(\mathbb{G})$ to $\mathbb{T}$ and hence the weakest topology making every $\overline{\Phi}$ continuous, where $\Phi \in \text{Hom}(\mathbb{G}, \mathbb{T})$, is a group topology on $\mathsf{ult}_p(\mathbb{G})$. This topology will be denoted by $\tau_{\overline{\text{Bohr}}}$.

The following is a trivial, yet fundamental fact:

Lemma 3.1. *For every $f : \omega \to \mathbb{G}$, $[f] = p\text{-}\lim_{n\in\omega} f(n)$ in $\tau_{\overline{Bohr}}$.*

Proof. This follows directly from the definition of $\overline{\Phi}$. □

The group that will be relevant for us is the group $\mathsf{ult}_p^{\omega_1}(\mathbb{G})$, endowed with the topology $\tau_{\overline{\text{Bohr}}}$ induced by the homomorphisms in $\text{Hom}(\mathbb{G}, \mathbb{T})$ extended recursively all the way to $\mathsf{ult}_p^{\omega_1}(\mathbb{G})$ by the same formula (3.1).

The (iterated) ultrapower with this topology is usually not Hausdorff (see [11, 3]), so we identify the inseparable functions and denote by $(\mathsf{Ult}_p^{\omega_1}(\mathbb{G}), \tau_{\overline{\text{Bohr}}})$ these quotients. More explicitly,

$$\mathsf{Ult}_p^{\omega_1}(\mathbb{G}) = \mathsf{ult}_p^{\omega_1}(\mathbb{G})/K,$$

where $K = \bigcap_{\Phi\in\text{Hom}(\mathbb{G},\mathbb{T})} \text{Ker}(\overline{\Phi})$. The natural projection will be denoted by

$$\pi \colon \mathsf{ult}_p^{\omega_1}(\mathbb{G}) \to \mathsf{ult}_p^{\omega_1}(\mathbb{G})/K.$$

The main reason for considering the iterated Fubini powers here is the following simple and crucial fact:

Proposition 3.2. *Let $p \in \omega^*$ be an ultrafilter.*
(1) *$\mathsf{ult}_p^{\alpha}(\mathbb{G}) \simeq \mathsf{ult}_{p^\alpha}(\mathbb{G})$ for $\alpha < \omega_1$, and*
(2) *$(\mathsf{Ult}_p^{\omega_1}(\mathbb{G}), \tau_{\overline{Bohr}})$ is a Hausdorff p-compact topological group.*

Proof. To prove (1), fix an $\alpha < \omega_1$. For given $[f] \in \mathsf{ult}_p^{\alpha}(\mathbb{G})$, recursively define a tree $T_f \in \mathbb{L}_p(T_\alpha)$ and a function $\hat{f} \colon T_f \to \mathsf{ult}_p^{\alpha}(\mathbb{G})$ so that

- $\text{succ}_{T_f}(\varnothing) = \text{dom}(f_\varnothing)$ and $\hat{f}(\varnothing) = [f_\varnothing]$, where $f_\varnothing = f$;
- if $\hat{f}(t)$ is defined say $\hat{f}(t) = [f_t]$, then $\text{succ}_{T_f}(t) = \text{dom}(f_t)$ and $\hat{f}(t^\frown n) = f_t(n)$ for every $n \in \text{succ}_{T_f}(t)$.

We define $\varphi \colon \mathsf{ult}_p^{\alpha}(\mathbb{G}) \to \mathsf{ult}_{p^\alpha}(\mathbb{G})$ given by

$$\varphi([f]) = [\hat{f} \restriction \Omega_0(T_f)].$$

Claim 3.3. φ is an isomorphism.

Proof of the claim. To see that φ is a surjection, let $[f] \in \mathsf{ult}_{p^\alpha}(\mathbb{G})$ be such that $\text{dom}(f) = \Omega_0(T_f)$ for some $T_f \in \mathbb{L}_p(T_\alpha)$. Consider the function $\check{f} \colon T_f \to \mathsf{ult}_p^{\alpha}(\mathbb{G})$ defined recursively by

- $\check{f} \restriction \Omega_0(T_f) = f$ and,
- if $t \in T_\alpha^+$, then $\check{f}(t) = [\langle \check{f}(t^\frown n) \colon n \in \text{succ}_{T_f}(t)\rangle]$.

Notice that the function $\check{f}$ satisfies that $\check{f}(t) \in \mathsf{ult}_p^{\rho_{T_f}(t)}(\mathbb{G})$ for every $t \in T_f$. In particular, $\check{f}(\varnothing) \in \mathsf{ult}_p^{\alpha}(\mathbb{G})$ and, a routine calculation shows that $\varphi(\check{f}(\varnothing)) = [f]$.

To see that φ is injective, suppose that $\varphi([f]) = \varphi([g])$. Then there exists a tree $T \in \mathbb{L}_p(T_\alpha)$ such that

$$\hat{f} \upharpoonright \Omega_0(T) = \hat{g} \upharpoonright \Omega_0(T).$$

If set $h := \hat{f} \upharpoonright \Omega_0(T)$, then we can verify recursively that $\check{h}(\varnothing) = [f] = [g]$. Therefore, φ is a one-to-one function.

Finally, using again a recursive argument, one can check that φ preserves the group structure. ■

To prove (2) note that by definition $\mathsf{Ult}_p^{\omega_1}(\mathbb{G})$ is a Hausdorff topological group. To see that $\mathsf{Ult}_p^{\omega_1}(\mathbb{G})$ is p-compact, since $\mathsf{Ult}_p^{\omega_1}(\mathbb{G})$ is a continuous image of $\mathsf{ult}_p^{\omega_1}(\mathbb{G})$, so it remains only to check that $\mathsf{ult}_p^{\omega_1}(\mathbb{G})$ is p-compact. Let $f\colon \omega \to \mathsf{ult}_p^{\omega_1}(\mathbb{G})$ be a sequence and let $n \in \omega$. So $f(n) \in \mathsf{ult}_p(\mathsf{ult}_p^{\omega_1^-}(\mathbb{G}))$, that is, there exists $f_n\colon \omega \to \bigcup_{\alpha<\omega_1} \mathsf{ult}_p^{\alpha}(\mathbb{G})$ such that $f(n) = [f_n]$. Thus, for every $n \in \omega$ there exists $\alpha_n < \omega_1$ such that $f(n) \in \mathsf{ult}_p^{\alpha_n}(\mathbb{G})$ and hence $[f] \in \mathsf{ult}_p^{\alpha}(\mathbb{G})$ for some $\alpha < \omega_1$. Since by Lemma 3.1, $[f] = p\text{-}\lim_{n\in\omega} f(n)$ in $\tau_{\overline{\text{Bohr}}}$, this gives us the p-compactness of $\mathsf{ult}_p^{\omega_1}(\mathbb{G})$. □

The plan for our construction is as follows: fix an ultrafilter $p \in \omega^*$, find a suitable topological group $\mathbb{G}$ without convergent sequences and consider $(\mathsf{Ult}_p^{\omega_1}(\mathbb{G}), \tau_{\overline{\text{Bohr}}})$. The remaining issue is: Does $(\mathsf{Ult}_p^{\omega_1}(\mathbb{G}), \tau_{\overline{\text{Bohr}}})$ have non-trivial convergent sequences?

While our approach is applicable to an arbitrary group $\mathbb{G}$, in the remainder of this paper we will be dealing exclusively with *Boolean* groups, *i.e.,* groups where each element is its own inverse.[1] These groups are in every infinite cardinality κ isomorphic to the group $[\kappa]^{<\omega}$ with the symmetric difference $\triangle$ as the group operation and $\varnothing$ as the neutral element.

The following theorem is the main result of this section.

Theorem 3.4. *Let $p \in \omega^*$ be a selective ultrafilter. Then $(\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega}), \tau_{\overline{Bohr}})$ is a Hausdorff p-compact topological Boolean group without non-trivial convergent sequences.*

In order to prove this theorem, we apply the first step of our plan.

Proposition 3.5. *The group $[\omega]^{<\omega}$ endowed with the topology τ_{Bohr} is a non-discrete Hausdorff topological group without non-trivial convergent sequences.*

Proof. It is well-known and easy to see that τ_{Bohr} is a non-discrete Hausdorff group topology (*e.g.,* see [2] Section 9.9), to see that τ_{Bohr} has no non-trivial convergent sequences, assume that $f\colon \omega \to [\omega]^{<\omega}$ is a non-trivial sequence. Then $\operatorname{rng}(f)$ is an infinite set. Find an infinite linearly independent set $A \subseteq \operatorname{rng}(f)$ and split it into two infinite pieces A_0 and A_1, and take $\Phi \in \operatorname{Hom}([\omega]^{<\omega}, 2)$ such that $A_i \subseteq \Phi^{-1}(i)$ for every $i < 2$. Therefore, Φ is a witness that the sequence f does not converge. □

[1]The general case will be dealt with in a separate paper.

We say that a sequence $\langle [f_n] \colon n \in \omega\rangle \subset \mathsf{ult}_p([\omega]^{<\omega})$ is *p-separated* if for every $n \neq m \in \omega$ there is a $\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$ such that $\overline{\Phi}([f_n]) \neq \overline{\Phi}([f_m])$.

We next shows that, in general, the plan does not work for all $p \in \omega^*$.

Lemma 3.6. *The following are equivalent:*

(1) *There exists a $p \in \omega^*$ such that $(\mathsf{Ult}_p([\omega]^{<\omega}), \tau_{\overline{Bohr}})$ has non-trivial convergent sequences.*

(2) *There exist a sequence $\langle \Phi_n \colon n \in \omega\rangle \subset Hom([\omega]^{<\omega}, 2)$ and a mapping $H \colon Hom([\omega]^{<\omega}, 2) \to \omega$ such that for every $n \in \omega$ the family*
$$\{[\omega]^{<\omega} \setminus Ker(\Phi_n)\} \cup \{Ker(\Phi) \colon \ H(\Phi) \leqslant n\}$$
is centered.

Proof. Let us prove (1) implies (2). let $\tilde{f} \colon \omega \to \mathsf{Ult}_p([\omega]^{<\omega})$ be a non-trivial sequence, say $\tilde{f}(n) = \pi(f(n))$ $(n \in \omega)$ where $f \colon \omega \to \mathsf{ult}_p^{\omega_1}([\omega]^{<\omega})$. Without loss of generality we can assume that $\tilde{f}$ is an one-to-one function converging to $\pi([\langle\varnothing\rangle])$, here $\langle\varnothing\rangle$ denotes the constant sequence where each term is $\varnothing$. So $\langle [f_n] \colon n \in \omega\rangle$ is a p-separated sequence $\tau_{\overline{\mathrm{Bohr}}}$-converging to $[\langle\varnothing\rangle]$. By taking a subsequence if necessary, we may assume that for every $n \in \omega$ there is a $\Phi_n \in \mathrm{Hom}([\omega]^{<\omega}, 2)$ such that $\overline{\Phi}_n([f_n]) = 1$. Now, by $\tau_{\overline{\mathrm{Bohr}}}$-convergence of $\langle [f_n] \colon n \in \omega\rangle$, there is a mapping $H \colon \mathrm{Hom}([\omega]^{<\omega}, 2) \to \omega$ such that for each $\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$ and each $n \geqslant H(\Phi)$ it follows that $\overline{\Phi}([f_n]) = 0$. Now we will check that for every $n \in \omega$ the family $\{\mathrm{Ker}(\Phi_n)^c\} \cup \{\mathrm{Ker}(\Phi) \colon \ H(\Phi) \leqslant n\}$ is centered[2]. For this, since $([\omega]^{<\omega}, \tau_{\mathrm{Bohr}})$ is without non-trivial convergent sequences and $[f_n] \xrightarrow{\tau_{\overline{\mathrm{Bohr}}}} [\langle\varnothing\rangle]$, we may assume that $[f_n] \neq [\langle a\rangle]$ for every $\langle n, a\rangle \in \omega \times [\omega]^{<\omega}$, that is, $f_n[U]$ is infinite for all $\langle n, U\rangle \in \omega \times p$. Now, fix $n \in \omega$ and let $F \subset \mathrm{Hom}([\omega]^{<\omega}, 2)$ be a finite set such that $H(\Phi) \leqslant n$ for every $\Phi \in F$. Then $\overline{\Phi}([f_n]) = 0$ for every $\Phi \in F$ and hence there exists $U_F \in p$ such that $\Phi(f_n(k)) = 0$ for every $\langle k, \Phi\rangle \in U_F \times F$. Since $\overline{\Phi}_n([f_n]) = 1$, there exists $U_n \in p$ such that $\Phi_n(f_n(k)) = 1$ for every $k \in U_n$. Put $U = U_F \cap U_n \in p$. Then $f_n[U] \subset \mathrm{Ker}(\Phi_n)^c \cap \bigcap_{\Phi \in F} \mathrm{Ker}(\Phi)$, so we are done.

To prove (2) implies (1), first we observe that there is a sequence $\langle f_n \colon n \in \omega\rangle \subset ([\omega]^{<\omega})^\omega$ such that for each $F \in [\omega]^{<\omega}$ and every $\sigma \colon F \to [\omega]^{<\omega}$ there exists $k \in \omega$ such that $f_i(k) = \sigma(i)$ for all $i \in F$. Now, define $A^0_{\Phi,n} = \{k \in \omega \colon \Phi(f_n(k)) = 0\}$ and $A^1_{\Phi,n} = \{k \in \omega \colon \Phi(f_n(k)) = 1\}$ for all $(\Phi, n) \in \mathrm{Hom}([\omega]^{<\omega}, 2) \times \omega$.

Claim 3.7. The collection $\bigcup_{n\in\omega}\{A^1_{\Phi_n,n}\} \cup \{A^0_{\Phi,n} \colon H(\Phi) \leqslant n\}$ forms a centered family which generates a free filter $\mathcal{F}$.

Proof of the claim. To show that such family is centered, let $m > 0$ and for every $i < m$ fix a finite set $\{\Phi^j \colon j < m_i\} \subset H^{-1}[i+1]$. Then, considering all choice functions
$$\sigma \colon n \to \bigcup_{i<m} \left(\mathrm{Ker}(\Phi_i)^c \cap \bigcap_{j<m_i} \mathrm{Ker}(\Phi^j) \right),$$
we can ensure that
$$\bigcap_{i<m} \left(A^1_{\Phi_i,i} \cap \bigcap_{j<m_i} A^0_{\Phi^j,i} \right)$$

[2]For a subset A of the group $[\omega]^{<\omega}$, $A^c = [\omega]^{<\omega} \setminus A$.

is an infinite set.

To see that the filter $\mathcal{F}$ is free, let $k \in \omega$. If there is an $n \in \omega$ such that $f_n(k) = \varnothing$, then $k \notin A^1_{\Phi_n,n} \in \mathcal{F}$. In another case, since $\langle f_n(k) \colon n \in \omega\rangle$ does not τ_{Bohr}-converge to $\varnothing$, there exists $\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$ such that $\Phi(f_n(k)) = 1$ for infinitely many n. Then pick one of such n with $H(\Phi) \leqslant n$ and, hence $k \notin A^0_{\Phi,n} \in \mathcal{F}$. ■

Let $p \in \omega^*$ extending $\mathcal{F}$. By Claim 3.7, it follows that

(i) $\overline{\Phi}_n([f_n]) = 1$, for every $n \in \omega$.
(ii) The sequence $\langle \overline{\Phi}([f_n]) \colon n \in \omega\rangle$ converges to 0, for every $\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$, *i.e.*, $\langle [f_n] \colon n \in \omega\rangle$ is a $\tau_{\overline{\mathrm{Bohr}}}$-convergent sequence to $[\langle\varnothing\rangle]$.

Finally, taking a subsequence if necessary, we can assume that $\langle [f_n] \colon n \in \omega\rangle$ is p-separated and, hence $\langle \pi([f_n]) \colon n \in \omega\rangle$ is a non-trivial convergent sequence in $(\mathsf{Ult}_p([\omega]^{<\omega}), \tau_{\overline{\mathrm{Bohr}}})$. □

Remark 3.8. Note that the filter $\mathcal{F}$ is actually an F_σ-filter.

Theorem 3.9. *There exists a $p \in \omega^*$ such that $(\mathsf{Ult}_p([\omega]^{<\omega}), \tau_{\overline{Bohr}})$ has non-trivial convergent sequences.*

Proof. We will show that the second clause of the Lemma 3.6 holds. To see this, choose any countable linearly independent set $\{\Phi_n \colon n \in \omega\} \subset \mathrm{Hom}([\omega]^{<\omega}, 2)$. Let W be a vector subspace of $\mathrm{Hom}([\omega]^{<\omega}, 2)$ such that $\mathrm{Hom}([\omega]^{<\omega}, 2) = \mathrm{span}\{\Phi_n \colon n \in \omega\} \oplus W$. We define the mapping $H \colon \mathrm{Hom}([\omega]^{<\omega}, 2) \to \omega$ as follows:

$$H(\Phi) = \min\{n \colon \Phi \in \mathrm{span}\{\Phi_i \colon i < n\} \oplus W\}.$$

Now, let $n \in \omega$ and fix a finite set $\{\Phi^j \colon j < m\} \subset H^{-1}[n+1]$. In order to show that

$$\mathrm{Ker}(\Phi_n)^c \cap \bigcap_{j<m} \mathrm{Ker}(\Phi^j)$$

is infinite, we shall need a fact concerning linear functionals on a vector space.

Fact 3.10 ([30], p. 124)**.** Let V be a vector space and $\Phi, \Phi^0, \ldots, \Phi^{m-1}$ linear functionals on V. Then the following statements are equivalent:

(1) $\bigcap_{j<m} \mathrm{Ker}(\Phi^j) \subset \mathrm{Ker}(\Phi)$.
(2) $\Phi \in \mathrm{span}\{\Phi^j \colon j < m\}$. □

Using this fact, and noting that $\Phi_n \notin \mathrm{span}\{\Phi^j \colon j < m\}$, one sees that

$$\mathrm{Ker}(\Phi_n)^c \cap \bigcap_{j<m} \mathrm{Ker}(\Phi^j) \neq \emptyset.$$

Pick an arbitrary $a \in \mathrm{Ker}(\Phi_n)^c \cap \bigcap_{j<m} \mathrm{Ker}(\Phi^j)$ and put

$$K = \mathrm{Ker}(\Phi_n) \cap \bigcap_{j<m} \mathrm{Ker}(\Phi^j).$$

Then K is an infinite set, and hence $a + K$ is an infinite set too. But

$$a + K \subset \mathrm{Ker}(\Phi_n)^c \cap \bigcap_{j<m} \mathrm{Ker}(\Phi^j),$$

so we are done. □

Corollary 3.11 (CH). *There is a P-point $p \in \omega^*$ such that $(\mathsf{Ult}_p([\omega]^{<\omega}), \tau_{\overline{Bohr}})$ has non-trivial convergent sequences.*

Proof. It is well-known (*e.g.*, see [21]) that assuming CH every F_σ-filter can be extend to a P-point. □

Selective ultrafilters and Q-points, have immediate combinatorial reformulations relevant in our context. Given a non-empty set I and $\mathbb{G}$ a Boolean group, we shall call a set $\{f_i : i \in I\}$ of functions $f_i : \omega \to \mathbb{G}$ *p-independent* if

$$\left\{n : a + \sum_{i \in E} f_i(n) = \varnothing\right\} \notin p$$

for every non-empty finite set $E \subset I$ and every $a \in \mathbb{G}$. Note that, in particular, a function $f : \omega \to \mathbb{G}$ is not constant on an element of p if and only if $\{f\}$ is p-independent. Now, we will say that a function $f : I \to \mathbb{G}$ is *linearly independent* if f is one-to-one and $\{f(i) : i \in I\}$ is a linearly independent set and, a function $f : I \to \mathsf{ult}_p(\mathbb{G})$ is *p-independent* if f is one-to-one and $\{f_i : i \in I\}$ is a p-independent set, where $f(i) = [f_i]$ for $i \in I$.

Proposition 3.12. *Let $p \in \omega^*$ be an ultrafilter. Then:*

(1) *p is a Q-point if and only if for every finite-to-one function $f : \omega \to [\omega]^{<\omega}$ there is a set $U \in p$ such that $f \restriction U$ is linearly independent.*
(2) *The following are equivalent*
 (a) *p is selective;*
 (b) *for every function $f : \omega \to [\omega]^{<\omega}$ which is not constant on an element of p there is a set $U \in p$ such that $f \restriction U$ is linearly independent;*
 (c) *for every p-independent set $\{f_n : n \in \omega\}$ of functions $f_n : \omega \to [\omega]^{<\omega}$, there is a set $U \in p$ and a function $g : \omega \to \omega$ so that $f_n \restriction U \setminus g(n)$ is one-to-one for $n \in \omega$, $f_n[U \setminus g(n)] \cap f_m[U \setminus g(m)] = \varnothing$ if $n \neq m$, and*

$$\bigsqcup_{n \in \omega} f_n[U \setminus g(n)]$$

 is linearly independent.[3]

Proof. Let us prove (1). Suppose first that p is a Q-point. Let $f : \omega \to [\omega]^{<\omega}$ be a finite-to-one function. Recursively define a strictly increasing sequence $\langle n_k : k \in \omega \rangle$ of elements of ω and a strictly increasing sequence of finite subgroups $\langle H_n : n \in \omega \rangle$ of $[\omega]^{<\omega}$ so that

(i) $H_n \cap \operatorname{rng}(f) \neq \emptyset$ for all $n \in \omega$, and
(ii) $n_k = \max f^{-1}[H_k]$ & $f''[0, n_k] \subset H_{k+1}$, for all $k \in \omega$.

Then partitioning ω into the union of even intervals, and the union of odd intervals, one of them is in p, say

$$A = \bigcup_{i \in \omega} [n_{2i}, n_{2i+1}) \in p.$$

Applying Q-pointness we can assume that there exists an $U \in p$ such that

$$|[n_{2i}, n_{2i+1}) \cap U| = 1 \text{ for every } i \in \omega,$$

[3]Here $\sqcup$ denotes the disjoint union.

and $U \subseteq A$. By item (ii) and since $\langle H_n : n \in \omega \rangle$ is a strictly increasing sequence, it follows that $f \restriction U$ is one-to-one and $\{f(n) : n \in U\}$ is linearly independent.

Suppose now that for every finite-to-one function $f : \omega \to [\omega]^{<\omega}$ there is an $U \in p$ such that $f \restriction U$ is one-to-one and $\{f(n) : n \in U\}$ is linearly independent. Let $\langle I_n : n \in \omega \rangle$ be a partition of ω into finite sets. Define a finite-to-one function $f : \omega \to [\omega]^{<\omega}$ by putting $f(k) = \{n\}$ for each $k \in I_n$. Then there is an $U \in p$ such that $f \restriction U$ is one-to-one and $\{f(n) : n \in U\}$ is linearly independent. Note that necessarily $|I_n \cap U| \leqslant 1$ for every $n \in \omega$ and therefore p is a Q-point.

(2) To see (a) implies (b), let $f : \omega \to [\omega]^{<\omega}$ be a function which is not constant on an element of p. Using P-pointness, we may assume without loss of generality that f is a finite-to-one function. So, by item (1), there is an $U \in p$ such that $f \restriction U$ is one-to-one and $\{f(n) : n \in U\}$ is linearly independent.

To see (b) implies (a), let $f : \omega \to [\omega]^{<\omega}$ be a function which is not constant on an element of p. By item (b), there is an $U \in p$ such that $f \restriction U$ is one-to-one and $\{f(n) : n \in U\}$ is linearly independent, and hence p is a P-point. To verify that p is a Q-point, notice that every finite-to-one function $f : \omega \to [\omega]^{<\omega}$ is not constant on an element of p. Thus, by clause (1) we get the desired conclusion.

To prove (a) implies (c), first note the following simple fact about p-independence.

Fact 3.13. Let $\{f_i : i < n\}$ be a finite p-independent set, and let $A \subset [\omega]^{<\omega}$ be a finite linearly independent set. Then, the set of all $m \in \omega$ such that $A \sqcup \{f_i(m) : i < n\}$ is linearly independent, belongs to p. □

Assume now that $\{f_n : n \in \omega\}$ is a p-independent set of functions $f_n : \omega \to [\omega]^{<\omega}$. Using Fact 3.13, we can recursively construct a p-branching tree $T \subset \omega^{<\omega}$ such that for every $t \in T$, it follows that

$$\mathrm{succ}_T(t) = \{m : A_t \sqcup \{f_i(m) : i \leqslant |t|\} \text{ is linearly independent}\},$$

where $A_t = \{f_i(t(j)) : i < |t| \;\&\; j \in [i, |t|)\}$.

By Galvin-Shelah's theorem ([4, Theorem 4.5.3]), let $x \in [T]$ be a branch such that $\mathrm{rng}(x) \in p$. Thus, if we put $U = \mathrm{rng}(x)$ and $g(n) = \max(x \restriction n)$ for $n \in \omega$, we get the required.

Finally, notice that (b) is a particular instance of (c) when $\{f_n : n \in \omega\} = \{f\}$. Therefore, (c) implies (b). □

Remark 3.14. In the previous theorem, it is possible to change the group $[\omega]^{<\omega}$ by any arbitrary Boolean group and, the conclusions of the theorem remain true.

For technical reasons, it will be necessary reformulate the notion of p-independen-ce.

Lemma 3.15. *Let $\mathbb{G}$ be a Boolean group and $0 < \alpha < \omega_1$. Then:*

(1) *A set $\{f_i : i \in I\}$ of functions $f_i : \omega \to \mathbb{G}$ is p-independent if and only if the function*
$$\tilde{f} : I \to \mathsf{ult}_p^1(\mathbb{G})/\mathsf{ult}_p^0(\mathbb{G})$$
defined by $\tilde{f}(i) = \pi_0^1([f_i])$ for $i \in I$ is linearly independent, where $\pi_0^1 : \mathsf{ult}_p^1(\mathbb{G}) \to \mathsf{ult}_p^1(\mathbb{G})/\mathsf{ult}_p^0(\mathbb{G})$ denotes the natural projection.

(2) *A set* $\{f_i \colon i \in I\}$ *of functions* $f_i \colon \omega \to \mathsf{ult}_p^\alpha(\mathbb{G})$ *is* p*-independent if and only if the set* $\{\tilde{f}_i \colon i \in I\}$ *of functions* $\tilde{f}_i \colon \omega \to \mathsf{ult}_p^\alpha(\mathbb{G})/\mathsf{ult}_p^{\alpha^-}(\mathbb{G})$ *is a* p*-independent set, where each* $\tilde{f}_i$ *is defined by* $\tilde{f}_i(n) = \pi^\alpha_{\alpha^-}(f_i(n))$ *for* $n \in \omega$ *and*

$$\pi^\alpha_{\alpha^-} \colon \mathsf{ult}_p^\alpha(\mathbb{G}) \to \mathsf{ult}_p^\alpha(\mathbb{G})/\mathsf{ult}_p^{\alpha^-}(\mathbb{G})$$

denotes the natural projection.

Proof. To see (1), note that

$$\sum_{i\in E}[f_i] = [\langle a\rangle]$$

iff

$$\left\{n \colon a + \sum_{i\in E} f_i(n) = \varnothing\right\} \in p,$$

for every non-empty finite set $E \subset I$ and every $a \in \mathbb{G}$.

To see (2). Let $E \subseteq I$ be a non-empty finite set and $a \in \mathsf{ult}_p^\alpha(\mathbb{G})$ and, notice that

$$\left\{n \colon \sum_{i\in E} \tilde{f}_i(n) = \pi^\alpha_{\alpha^-}(a)\right\} \in p$$

iff

$$\left\{n \colon a + \sum_{i\in E} f_i(n) \in \mathsf{ult}_p^{\alpha^-}(\mathbb{G})\right\} \in p$$

iff

$$\left\{n \colon (a + [f]) + \sum_{i\in E} f_i(n) = \varnothing\right\} \in p,$$

where for some $U \in p$ we have that $f(n) = a + \sum_{i\in E} f_i(n) \in \mathsf{ult}_p^{\alpha^-}(\mathbb{G})$ for $n \in U$. $\square$

Note also that if $\mathrm{ht}([f]) = \alpha$ for $\alpha > 0$, then f is not constant on an element of p (equivalently, $\{f\}$ is p-independent).

Lemma 3.16. *Let* $0 < \alpha < \omega_1$, $[f] \in \mathsf{ult}_p^\alpha([\omega]^{<\omega})$ *and* p *a selective ultrafilter. If* f *is not constant on an element of* p*, then there is a tree* $T \in \mathbb{L}_p(T_\alpha)$ *with* $T \subseteq T_f$ *such that* $\hat{f} \restriction \Omega_0(T)$ *is linearly independent.*[4]

Proof. First, if $\alpha = 1$, then the conclusion of the lemma follows from Proposition 3.12 (2) (b). Thus, we may assume that $\alpha \geqslant 2$.

We plan to construct a tree $T \in \mathbb{L}_p(T_\alpha)$ with $T \subseteq T_f$, so that the following hold for any $\beta \leqslant \alpha$:

- if $\beta > 0$, then $\langle \hat{f}(t) \colon t \in \Omega_\beta(T)\rangle$ forms a p-independence sequence;
- if $\beta = 0$, then $\langle \hat{f}(t) \colon t \in \Omega_0(T)\rangle$ forms a linearly independent sequence.

In order to do this, first, we recursively construct a tree $T^* \in \mathbb{L}_p(T_\alpha)$ with $T^* \subseteq T_f$, so that the following hold for any $t \in T^*$ with $\rho_{T^*}(t) \geqslant 1$:

- if $\mathrm{ht}(\hat{f}(t)) = 1$, then $\langle \hat{f}(t^\frown n) \colon n \in \mathrm{succ}_{T^*}(t)\rangle \subset [\omega]^{<\omega}$ forms a linearly independent sequence;

[4]Here, we are using the notation from the proof of Proposition 3.2 (1).

- if $\mathrm{ht}(\hat{f}(t)) = \beta + 1$ with $\beta \geqslant 1$, then $\langle \hat{f}(t^\frown n) \colon n \in \mathrm{succ}_{T^*}(t)\rangle \subset \mathsf{ult}_p^\beta([\omega]^{<\omega})$ forms a p-independent sequence;
- if $\mathrm{ht}(\hat{f}(t))$ is a limit ordinal, then $\langle \mathrm{ht}(\hat{f}(t^\frown n)) \colon n \in \mathrm{succ}_{T^*}(t)\rangle$ is a strictly increasing sequence of non-zero ordinals.

At step t. If $\mathrm{ht}(\hat{f}(t)) = 1$ and $\langle \hat{f}(t^\frown n) \colon n \in \mathrm{succ}_{T_f}(t)\rangle$ is not constant on an element of p, then $\rho_{T_f}(t) = 1$ and applying Proposition 3.12 (2) (b) there exists $U \in p$ with $U \subseteq \mathrm{succ}_{T_f}(t)$ such that $\langle \hat{f}(t^\frown n) \colon n \in U\rangle$ is linearly independent. Therefore, in this case we put $\mathrm{succ}_{T^*}(t) = U$.

If $\mathrm{ht}(\hat{f}(t)) = \beta + 1$ with $\beta \geqslant 1$ and $\langle \hat{f}(t^\frown n) \colon n \in \mathrm{succ}_{T_f}(t)\rangle$ is not constant on an element of p, then consider the sequence

$$\tilde{f}_t \colon \mathrm{succ}_{T_f}(t) \to \mathsf{ult}_p^\beta([\omega]^{<\omega})/\mathsf{ult}_p^{-\beta}([\omega]^{<\omega})$$

defined by $\tilde{f}_t(n) = \pi^\beta_{\beta^-}(\hat{f}(t^\frown n))$ for $n \in \mathrm{succ}_{T_f}(t)$. Since $\langle \hat{f}(t^\frown n) \colon n \in \mathrm{succ}_{T_f}(t)\rangle$ is not constant on an element of p, by Lemma 3.15 (2), the sequence $\tilde{f}_t$ is not constant on an element of p. Therefore, applying Proposition 3.12 (2) (b) and Remark 3.14, we can find an element $U \in p$ with $U \subseteq \mathrm{succ}_{T_f}(t)$ such that $\tilde{f}_t \restriction U$ is linearly independent. Thus, by Lemma 3.15 (1), putting $\mathrm{succ}_{T^*}(t) = U$ we can conclude that $\langle \hat{f}(t^\frown n) \colon n \in \mathrm{succ}_{T^*}(t)\rangle$ forms a p-independent sequence.

If $\mathrm{ht}(\hat{f}(t)) = \beta$ is a limit ordinal, then for every $\delta < \beta$ we set $U_\delta = \{n \in \mathrm{succ}_{T_f}(t) \colon \mathrm{ht}(\hat{f}(t^\frown n)) = \delta\}$. Then

$$\bigsqcup_{\delta<\beta} U_\delta = \mathrm{succ}_{T_f}(t),$$

where each $U_\delta \notin p$. The selectiveness of p implies that there is an $U \in p$ such that $|U \cap U_\delta| \leqslant 1$ for every $\delta < \beta$. Thus, in this case put $\mathrm{succ}_{T^*}(t) = U \setminus U_0$. This concludes recursive construction of T^*.

Notice that $\rho_{T^*}(t) = \mathrm{ht}(\hat{f}(t))$ for every $t \in T^*$. Now given a tree $T' \in \mathbb{L}_p(T_\alpha)$ with $T' \subseteq T^*$, we can canonically list its members $t' \in T'$ as $\{t_k^{T'} : k < \omega\}$ so that

- $t_k^{T'} \subset t_l^{T'}$ entails $k < l$;
- $t_k^{T'} = t^\frown n$, $t_l^{T'} = t^\frown m$, $\mathrm{ht}(\hat{f}(t))$ is a limit ordinal, and $\mathrm{ht}(\hat{f}(t^\frown n)) < \mathrm{ht}(\hat{f}(t^\frown m))$ entails $k < l$;
- $t_k^{T'} = t^\frown n$, $t_l^{T'} = t^\frown m$, $\mathrm{ht}(\hat{f}(t))$ is a successor ordinal, and $n < m$ entails $k < l$.

Choose a sufficiently large regular cardinal θ and a countable elementary submodel M of $\langle H(\theta), \in\rangle$ containing all the relevant objects as p and T^*. Fix $U \in p$ so that U is a pseudo-intersection of $p \cap M$. Put $T^{**} = T^* \restriction U$ and $V_t = \mathrm{succ}_{T^{**}}(t)$ for $t \in (T^{**})^+$.

We unfix t, and construct by recursion on k the required condition $T = \{t_k^T \colon k \in \omega\} \in \mathbb{L}_p(T_\alpha)$ with $T \subseteq T^{**}$, as well as an auxiliary function $g \colon T^+ \to \omega$ and sets $W_t \subseteq V_t$ for $t \in T^+$ such that the following are satisfied:

(a) $W_t = V_t \setminus g(t) = \mathrm{succ}_T(t)$ for all $t \in T^+$ (by definition).

(b) For all k,

- if $\rho_T(t_k^T) = 1$, then

$$\Big\langle \hat{f}(t_l^{T\frown} n) \colon \exists\, l \leqslant k \,\Big(n \in W_{t_l^T} \;\&\; \rho_T(t_l^{T\frown} n) = 0\Big)\Big\rangle \subseteq [\omega]^{<\omega}$$

forms a linearly independent sequence;

- if $\rho_T(t_k^T) = \beta + 1$ with $\beta \geqslant 1$, then
$$\left\langle \hat{f}(t_l^T{}^\frown n) \colon \exists\, l \leqslant k \left(n \in W_{t_l^T} \;\&\; \rho_T(t_l^T{}^\frown n) = \beta\right)\right\rangle \subset \mathsf{ult}_p^{\beta}([\omega]^{<\omega})$$
forms a p-independence sequence;
- if $\rho_T(t_k^T) = \beta$ is a limit ordinal, then
$$\left\langle \mathrm{ht}(\hat{f}(t_l^T{}^\frown n)) \colon \exists\, l \leqslant k \left(n \in W_{t_l^T} \;\&\; \rho_T(t_l^T) = \beta\right)\right\rangle$$
forms an one-to-one sequence, and
$$\sup\left\{\mathrm{ht}(\hat{f}(t_l^T{}^\frown n)) \colon \exists\, l < k \left(\rho_T(t_l^T) \neq \beta \;\&\; n \in W_{t_l^T} \;\&\; \rho_T(t_l^T{}^\frown n) < \beta\right)\right\}$$
$$< \min\left\{\mathrm{ht}(\hat{f}(t_k^T{}^\frown n)) \colon n \in W_{t_k^T}\right\}.$$

Before describing the construction let us recall a simple fact from linear algebra:

Fact 3.17. Let A and B be linearly independent sets in a Boolean group with A a finite set. Then there is $A' \subseteq B$ such that $|A'| \leq |A|$ and $A \cup (B \setminus A')$ is linearly independent. □

Basic step $k = 0$. So $t_0^T = \varnothing$. We put $g(t_0^T) = 0$ and hence $W_{t_0^T} = V_{t_0^T}$. The conditions (a) and (b) are immediate.

Recursion step $k > 0$. Assume $W_{t_l^T}$ (for $l < k$) as well as $g \restriction k$ have been defined so as to satisfy (a) and (b). In particular, we know already t_k^T, for it is of the form $t_l^T{}^\frown n$ for some $n \in W_{t_l^T}$ where $l < k$. Put $\rho_T(t_k^T) = \gamma$ and assume $\gamma \geqslant 1$. Note that, since (b) is satisfied for l, we must have $\rho_T(t_l^T) = \gamma + 1$ and
$$\left\langle \hat{f}(t_j^T{}^\frown m) \colon \exists\, j \leqslant l \left(m \in W_{t_j^T} \;\&\; \rho_T(t_j^T{}^\frown m) = \gamma\right)\right\rangle \subset \mathsf{ult}_p^{\gamma}([\omega]^{<\omega})$$
is a p-independent sequence. Put
$$\begin{aligned} A_l &= \{t_{l'}^T \colon l' \leqslant k \;\&\; \rho_T(t_{l'}^T) = \gamma\} \\ &\subset \left\{t_j^T{}^\frown m \colon \exists\, j \leqslant l \left(m \in W_{t_j^T} \;\&\; \rho_T(t_j^T{}^\frown m) = \gamma\right)\right\} \end{aligned}$$
and $A_l^- = A_l \setminus \{t_k^T\}$.

If $\gamma = 1$, then applying Proposition 3.12 (2) (c) there exists $V \in p$ and a function $g_l \colon A_l \to \omega$ such that
$$\left\langle \hat{f}(t^\frown m) \colon t \in A_l \;\&\; m \in V \setminus g_l(t)\right\rangle \subseteq [\omega]^{<\omega}$$
is a linearly independent sequence. Using the elementarity of M and our assumption about U we conclude that there exists a function $g_{l,U} \colon A_l \to \omega$ such that
$$\left\langle \hat{f}(t^\frown m) \colon t \in A_l \;\&\; m \in U \setminus g_{l,U}(t)\right\rangle \subseteq [\omega]^{<\omega}$$
is a linearly independent sequence. Note that $V_{t_k^T} \setminus g_{l,U}(t_k^T) \subseteq U \setminus g_{l,U}(t_k^T)$ and $W_t \setminus g_{l,U}(t) \subseteq U \setminus g_{l,U}(t)$ for $t \in A_l^-$. Since A_l is a finite set, using Fact 3.17, we can find a natural number $g(t_k^T) \geqslant g_{l,U}(t_k^T)$ so that
$$\left\langle \hat{f}(t^\frown m) \colon t \in A_l^- \;\&\; m \in W_t\right\rangle \cup \left\langle \hat{f}(t_k^T{}^\frown m) \colon m \in V_{t_k^T} \setminus g(t_k^T)\right\rangle$$
forms a linearly independent sequence, as required.

For the case $\gamma = \beta + 1$ with $\beta \geqslant 1$, we will proceed in a similar way as the previous case. Given $t \in A_l$, let

$$\tilde{f}_t : V_t \to \mathsf{ult}_p^{\beta}([\omega]^{<\omega})/\mathsf{ult}_p^{\beta^-}([\omega]^{<\omega})$$

be defined by $\tilde{f}_t(m) = \pi_\beta^{\beta^-}(\hat{f}(t^\frown m))$ for $m \in V_t$. By Lemma 3.15 (2), $\{\tilde{f}_t : t \in A_l\}$ is a p-independent set. Thus, applying Proposition 3.12 (2) (c) and Remark 3.14, we can find an element $V \in p$ and a function $g_l : A_l \to \omega$ such that

$$\left\langle \tilde{f}_t(m) : t \in A_l \ \&\ m \in V \setminus g_l(t) \right\rangle \subseteq \mathsf{ult}_p^{\beta}([\omega]^{<\omega})/\mathsf{ult}_p^{\beta^-}([\omega]^{<\omega})$$

is a linearly independent sequence. By elementarity of M and the property of U we have that there exists a function $g_{l,U} : A_l \to \omega$ such that

$$\left\langle \tilde{f}_t(m) : t \in A_l \ \&\ m \in U \setminus g_{l,U}(t) \right\rangle$$

is a linearly independent sequence. Since A_l is a finite set, $V_{t_k^T} \setminus g_{l,U}(t_k^T) \subseteq U \setminus g_{l,U}(t_k^T)$ and $W_t \setminus g_{l,U}(t) \subseteq U \setminus g_{l,U}(t)$ for $t \in A_l^-$, using Fact 3.17, we can find a natural number $g(t_k^T) \geqslant g_{l,U}(t_k^T)$ so that

$$\left\langle \tilde{f}_t(m) : t \in A_l^- \ \&\ m \in W_t \right\rangle \cup \left\langle \tilde{f}_{t_k^T}(m) : m \in V_{t_k^T} \setminus g(t_k^T) \right\rangle$$

forms a linearly independent sequence and, by Lemma 3.15 (1), this means that

$$\left\langle \hat{f}(t^\frown m) : t \in A_l^- \ \&\ m \in W_t \right\rangle \cup \left\langle \hat{f}(t_k^{T\frown} m) : m \in V_{t_k^T} \setminus g(t_k^T) \right\rangle \subset \mathsf{ult}_p^{\beta}([\omega]^{<\omega})$$

forms a p-independent sequence, as required.

If γ is a limit ordinal, then applying Proposition 3.12 (2) (c) there exists $V \in p$ and a function $g_l : A_l \to \omega$ such that

$$\left\langle \hat{f}(t^\frown m) : t \in A_l \ \&\ m \in V \setminus g_l(t) \right\rangle \subset \mathsf{ult}_p^{\gamma^-}([\omega]^{<\omega})$$

is a linearly independent sequence. Thus, proceeding as previous cases, it is possible to find a function $g_{l,U} : A_l \to \omega$ and a natural number $g(t_k^T) \geqslant g_{l,U}(t_k^T)$ so that

$$\left\langle \hat{f}(t^\frown m) : t \in A_l^- \ \&\ m \in W_t \right\rangle \cup \left\langle \hat{f}(t_k^{T\frown} m) : m \in V_{t_k^T} \setminus g(t_k^T) \right\rangle$$

forms a linearly independent sequence. In particular,

$$\left\langle \mathrm{ht}(\hat{f}(t^\frown m)) : t \in A_l^- \ \&\ m \in W_t \right\rangle \cup \left\langle \mathrm{ht}(\hat{f}(t_k^{T\frown} m)) : m \in V_{t_k^T} \setminus g(t_k^T) \right\rangle$$

forms an one-to-one sequence and, since γ is a limit ordinal, one sees that without loss of generality, we may assume that

$$\begin{aligned} &\sup \left\{ \mathrm{ht}(\hat{f}(t_l^{T\frown} m)) : \exists\, l < k \left(\rho_T(t_l^T) \neq \gamma \ \&\ m \in W_{t_l^T} \ \&\ \rho_T(t_l^{T\frown} m) < \gamma \right) \right\} \\ &< \min \left\{ \mathrm{ht}(\hat{f}(t_k^{T\frown} m)) : m \in V_{t_k^T} \setminus g(t_k^T) \right\}, \end{aligned}$$

as required. □

Now we are ready to prove the main theorem of this section.

Proof of the Theorem 3.4. According to Proposition 3.2, $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ is a Hausdorff p-compact topological group. It remains therefore only to show that $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains no non-trivial convergent sequences to $\pi([\langle\varnothing\rangle])$. To see this, let $\tilde{f} : \omega \to \mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ be a non-trivial sequence, say $\tilde{f}(n) = \pi(f(n))$ $(n \in \omega)$

where $f\colon \omega \to \mathsf{ult}_p^{\omega_1}([\omega]^{<\omega})$. Without loss of generality we can assume that $\tilde{f}$ is an one-to-one function. Thus, since

$$\mathsf{ult}_p^{\omega_1}([\omega]^{<\omega}) = \mathsf{ult}_p\left(\bigcup_{\alpha<\omega_1} \mathsf{ult}_p^{\alpha}([\omega]^{<\omega})\right),$$

there exists $0 < \alpha < \omega_1$ so that $[f] \in \mathsf{ult}_p^{\alpha}([\omega]^{<\omega})$ and f is not constant on an element of p. By Lemma 3.16, there is a tree $T \in \mathbb{L}_p(T_\alpha)$ with $T \subseteq T_f$ such that $\hat{f} \restriction \Omega_0(T)$ is linearly independent. Note that $\hat{f}[\Omega_0(T)] \subseteq [\omega]^{<\omega}$. Take $\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$ so that $\hat{f}[\Omega_0(T)] \subseteq \Phi^{-1}(1)$. So $\overline{\Phi}([\hat{f}]) = 1$ and hence $\overline{\Phi}([f]) = 1$. Thus, $\overline{\Phi}$ is a witness that the sequence f does not $\tau_{\overline{\mathrm{Bohr}}}$-converge to $[\langle\varnothing\rangle]$ and, since $\tilde{f}$ is one-to-one, in fact $\tilde{f}$ does not converge to $\pi([\langle\varnothing\rangle])$. ■

4. Countably compact group without convergent sequences

In this section we develop the ideas introduced in the previous section into a ZFC construction of a countably compact subgroup of $2^{\mathfrak{c}}$ without non-trivial convergent sequences. Similarly to the ultrapower construction, we shall extend the Bohr topology τ_{Bohr} on $[\omega]^{<\omega}$ to a group topology $\tau_{\overline{\mathrm{Bohr}}}$ on $[\mathfrak{c}]^{<\omega}$ to obtain the result. The difference is that rather than using a single ultrafilter, we shall use a carefully constructed $\mathfrak{c}$-sized family of ultrafilters.

Theorem 4.1. *There is a Hausdorff countably compact topological Boolean group without non-trivial convergent sequences.*

Proof. We shall construct a countably compact topology on $[\mathfrak{c}]^{<\omega}$ starting from $([\omega]^{<\omega}, \tau_{\mathrm{Bohr}})$ as follows:

Fix an indexed family $\{f_\alpha\colon \alpha \in [\omega, \mathfrak{c})\} \subset ([\mathfrak{c}]^{<\omega})^\omega$ of one-to-one sequences such that

(1) for every infinite $X \subseteq [\mathfrak{c}]^{<\omega}$ there is an $\alpha \in [\omega, \mathfrak{c})$ with $\mathrm{rng}(f_\alpha) \subseteq X$,
(2) each f_α is a sequence of linearly independent elements, and
(3) $\mathrm{rng}(f_\alpha) \subset [\alpha]^{<\omega}$ for every $\alpha \in [\omega, \mathfrak{c})$.

Given a sequence $\{p_\alpha\colon \alpha \in [\omega, \mathfrak{c})\} \subset \omega^*$ define for every $\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$ its extension $\overline{\Phi} \in \mathrm{Hom}([\mathfrak{c}]^{<\omega}, 2)$ recursively by putting

$$\overline{\Phi}(\{\alpha\}) = p_\alpha\text{-}\lim_{n\in\omega} \overline{\Phi}(f_\alpha(n)).$$

Note that doing this indeed defines unique extension of Φ to a homomorphism on $[\mathfrak{c}]^{<\omega}$ to 2, which, moreover, has the property that $\overline{\Phi}(\{\alpha\}) = p_\alpha\text{-}\lim_{n\in\omega} \overline{\Phi}(f_\alpha(n))$ for every Φ and every $\alpha \in [\omega, \mathfrak{c})$.

This allows us to define the topology $\tau_{\overline{\mathrm{Bohr}}}$ induced by $\{\overline{\Phi} : \Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)\}$ on $[\mathfrak{c}]^{<\omega}$ as the weakest topology making all $\overline{\Phi}$ continuous ($\Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)$), or equivalently, the group topology having $\{\mathrm{Ker}(\overline{\Phi}) : \Phi \in \mathrm{Hom}([\omega]^{<\omega}, 2)\}$ as a subbasis of the filter of neighbourhoods of the neutral element $\varnothing$. It follows directly from the above observation that independently of the choice of the ultrafilters the topology is a countably compact group topology on $[\mathfrak{c}]^{<\omega}$. Indeed, $\{\alpha\} \in \overline{\{f_\alpha(n)\colon n \in \omega\}}^{\tau_{\overline{\mathrm{Bohr}}}}$ for every $\alpha \in [\omega, \mathfrak{c})$, in fact $\{\alpha\} = p_\alpha\text{-}\lim_{n\in\omega} f_\alpha(n)$.

Call a set $D \in [\mathfrak{c}]^\omega$ *suitably closed* if $\omega \subseteq D$ and $\bigcup_{n\in\omega} f_\alpha(n) \subseteq D$ for every $\alpha \in D$. The following claim shows that the construction is locally countable.

Claim 4.2. The topology $\tau_{\overline{\text{Bohr}}}$ contains no non-trivial convergent sequences if and only if $\forall D \in [\mathfrak{c}]^\omega$ suitably closed $\exists \Psi \in Hom([D]^{<\omega}, 2)$ such that

(1) $\forall \alpha \in D \setminus \omega \ \Psi(\{\alpha\}) = p_\alpha\text{-}\lim_{n\in\omega} \Psi(f_\alpha(n))$;
(2) $\forall i \in 2 \ |\{n : \Psi(f_\alpha(n)) = i\}| = \omega$.

Proof of the claim. Given an infinite $X \subseteq [\mathfrak{c}]^{<\omega}$ there is an $\alpha \in [\omega, \mathfrak{c})$ such that $\text{rng}(f_\alpha) \subseteq X$. Let D be suitably closed with $\alpha \in D$, and let Ψ be the given homomorphism. It follows directly from the definition, and property (1) of Ψ, that, if $\Phi = \Psi \restriction [\omega]^{<\omega}$ then in turn $\Psi = \overline{\Phi} \restriction [D]^{<\omega}$, which implies that $\langle f_\alpha(n) \colon n \in \omega\rangle$ (and hence also X) is not a convergent sequence as $\overline{\Phi}$ takes both values 0 and 1 infinitely often on the set $\{f_\alpha(n) \colon n \in \omega\}$.

The reverse implication is even more trivial (and not really necessary for the proof). ∎

Note that if this happens then, in particular,

$$K = \bigcap_{\Phi \in \text{Hom}([\omega]^{<\omega}, 2)} \text{Ker}(\overline{\Phi})$$

is finite, and $[\mathfrak{c}]^{<\omega}/K$ with the quotient topology is the Hausdorff countably compact group without non-trivial convergent sequences we want.

Hence to finish the proof it suffices to produce a suitable family of ultrafilters:

Claim 4.3. There is a family $\{p_\alpha \colon \alpha < \mathfrak{c}\}$ of free ultrafilters on ω such that for every $D \in [\mathfrak{c}]^\omega$ and $\{f_\alpha : \alpha \in D\}$ such that each f_α is an one-to-one enumeration of linearly independent elements of $[\mathfrak{c}]^{<\omega}$ there is a sequence $\langle U_\alpha : \alpha \in D\rangle$ such that

(1) $\{U_\alpha : \alpha \in D\}$ is a family of pairwise disjoint subsets of ω,
(2) $U_\alpha \in p_\alpha$ for every $\alpha \in D$, and
(3) $\{f_\alpha(n) : \alpha \in D \ \& \ n \in U_\alpha\}$ is a linearly independent subset of $[\mathfrak{c}]^{<\omega}$.

Proof of the claim. Fix $\{I_n : n \in \omega\}$ a partition of ω into finite sets such that

$$|I_n| > n \cdot \sum_{m<n} |I_m|,$$

and let

$$\mathcal{B} = \{B \subseteq \omega : \ \forall n \in \omega \ |I_n \setminus B| > \sum_{m<n} |I_m|\}.$$

Note that $\mathcal{B}$ is a centered family, and denote by $\mathcal{F}$ the filter it generates. Note also, that if A is an infinite subset of ω then $\bigcup_{n\in A} I_n \in \mathcal{F}^+$.

Let $\{A_\alpha : \alpha \in \omega\}$ be any almost disjoint family of size $\mathfrak{c}$ of infinite subsets of ω, and let, for every $\alpha < \mathfrak{c}$, p_α be any ultrafilter on ω extending $\mathcal{F} \restriction \bigcup_{n\in A_\alpha} I_n$.

To see that this works, let $D = \{\alpha_n : n \in \omega\}$ and a family $\{f_\alpha : \alpha \in D\}$ of one-to-one sequences of linearly independent elements of $[\mathfrak{c}]^{<\omega}$ be given. Let $\{B_n : n \in \omega\}$ be a partition of ω such that for $B_n =^* A_{\alpha_n}$ for every $n \in \omega$, and recursively define a set B such that, $I_0 \subseteq B$,

$$|I_n \setminus B| > \sum_{m<n} |I_m|$$

for every $n > 0$, and

$$\{f_{\alpha_n}(m) : m \in B \cap I_l, l \in B_n\} \text{ is linearly independent.}$$

This is easy to do using Fact 3.17. Then $B \in \mathcal{B}$ and letting $U_n = \bigcup_{l\in B_n} I_l$ gives the sequence required. ∎

Now, use this family of ultrafilters as the parameter in the construction of the topology described above. By Claim 4.2 it suffices to show that given a suitably closed $D \subseteq \mathfrak{c}$ and $\alpha \in D \setminus \omega$ there is a homomorphism $\Psi : [D]^{<\omega} \to 2$ such that

(1) $\forall \alpha \in D \setminus \omega\ \Psi(\{\alpha\}) = p_\alpha\text{-}\lim_{n\in\omega} \Psi(f_\alpha(n))$
(2) $\forall i \in 2\ |\{n : \Psi(f_\alpha(n)) = i\}| = \omega$.

By Claim 4.3, there is a sequence $\langle U_\alpha : \alpha \in D \setminus \omega\rangle$ such that

(1) $\{U_\alpha : \alpha \in D \setminus \omega\}$ is a family of pairwise disjoint subsets of ω,
(2) $U_\alpha \in p_\alpha$ for every $\alpha \in D \setminus \omega$, and
(3) $\{f_\alpha(n) : \alpha \in D \setminus \omega\ \&\ n \in U_\alpha\}$ is a linearly independent subset of $[\mathfrak{c}]^{<\omega}$.

Enumerate $D \setminus \omega$ as $\{\alpha_n : n \in \omega\}$ so that $\alpha = \alpha_0$. Recursively define a function $h : \{f_\alpha(n) : \alpha \in D \setminus \omega\ \&\ n \in U_\alpha\} \to 2$ so that

(1) h takes both values 0 and 1 infinitely often on $\{f_{\alpha_0}(n) :\ n \in U_{\alpha_0} \setminus \{\alpha_0\}\}$,
(2) $\Psi_0(\{\alpha_0\}) = p_{\alpha_0}\text{-}\lim_{k\in U_{\alpha_0}} \Psi_0(f_{\alpha_0}(k))$, and
(3) if $\{\alpha_n\}$ is in the subgroup generated by $\{f_{\alpha_m}(n) : m < n\ \&\ n \in U_{\alpha_m}\}$ then $\Psi_n(\{\alpha_n\}) = p_{\alpha_n}\text{-}\lim_{k\in U_{\alpha_n}} \Psi_n(f_{\alpha_n}(k))$, and making sure that
(4) $\Psi_n(\{\alpha_n\}) = p_{\alpha_n}\text{-}\lim_{k\in U_{\alpha_n}} \Psi_n(f_\alpha(k))$.

where Ψ_n is a homomorphism defined on the subgroup generated by

$$\{f_{\alpha_m}(n) : m < n\ \&\ n \in U_{\alpha_m}\} \cup \{\{\alpha_m\} : m < n\}$$

extending $h \restriction \{f_{\alpha_m}(n) : m < n\ \&\ n \in U_{\alpha_m}\}$. Then let Ψ be any homomorphism extending $\bigcup_{m\in\omega} \Psi_m$. Doing this is straightforward given that the set

$$\{f_\alpha(n) : \alpha \in D \setminus \omega\ \&\ n \in U_\alpha\}$$

is linearly independent.

Finally, note that if we, for $a \in [\mathfrak{c}]^{<\omega}$, let

$$H(a)(\Phi) = \overline{\Phi}(a)$$

then H is a continuous homomorphism from $[\mathfrak{c}]^{<\omega}$ to $2^{\mathrm{Hom}([\omega]^{<\omega})}$ whose kernel is the same group $K = \bigcap_{\Phi\in\mathrm{Hom}([\omega]^{<\omega})} \mathrm{Ker}(\overline{\Phi})$, which defines a homeomorphism (and isomorphism) of $[\mathfrak{c}]^{<\omega}/K$ onto a subgroup of $2^{\mathrm{Hom}([\omega]^{<\omega})} \simeq 2^{\mathfrak{c}}$. □

5. Concluding remarks and questions

Even though the results of the paper solve longstanding open problems, they also open up very interesting new research possibilities. In Theorem 3.4 we showed that if p is a selective ultrafilter then $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ is a p-compact group without non-trivial convergent sequences. This raises the following two interesting questions, the first of which is the equivalent of van Douwen's problem for p-compact groups.

Question 5.1. *Is there in* ZFC *a Hausdorff p-compact topological group without a non-trivial convergent sequence?*

A closely related problem asks how much can the property of being selective be weakened in Theorem 3.4. Recall that by Corollary 3.11 it is consistent that there is a P-point p for which $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ does contain a non-trivial convergent sequence. On the other hand, $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega}) \simeq \mathsf{Ult}_{p^\alpha}^{\omega_1}([\omega]^{<\omega})$ for every $\alpha < \omega_1$, so there are consistently non-P-points for which $(\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains no non-trivial convergent sequences.

Question 5.2. *Is the existence of an ultrafilter p such that $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains no non-trivial convergent sequences equivalent to the existence of a selective ultrafilter?*

Question 5.3. *Is it consistent with* ZFC *that $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains a non-trivial convergent sequence for every ultrafilter $p \in \omega^*$?*

Assuming $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$ contains no non-trivial convergent sequences, it is easy to construct for every $n \in \omega$ a subgroup $\mathbb{H}$ of $\mathsf{Ult}_p^{\omega_1}([\omega]^{<\omega})$, such that $\mathbb{H}^n$ is countably compact while $\mathbb{H}^{n+1}$ is not. It should be possible to modify the construction in Theorem 4.1 to construct such groups in ZFC. These issues will be dealt with in a separate paper.

Another interesting question is:

Question 5.4. *Is it consistent with* ZFC *that for some ultrafilter $p \in \omega^*$ there is a Hausdorff p-compact topological group without non-trivial convergent sequences of weight $< \mathfrak{c}$?*

Finally, let us recall a 1955 problem of Wallace:

Question 5.5 (Wallace [45])**.** *Is every both-sided cancellative countably compact topological semigroup necessarilly a group?*

It is well known that a counterexample can be recursively constructed inside of any non-torsion countably compact topological group without non-trivial convergent sequences [28, 38]. The fact that we do not know how to modify (in ZFC) the construction in Theorem 4.1 to get a non-torsion example of a countably compact group seems surprising. Also the proof of Theorem 3.4 does not seem to easily generalize to non-torsion groups. Hence:

Question 5.6. *Is there, in* ZFC *, a non-torsion countably compact topological group without non-trivial convergent sequences?*

Question 5.7. *Assume $p \in \omega^*$ is a selective ultrafilter. Does $(\mathsf{Ult}_p^{\omega_1}(\mathbb{Z}), \tau_{\overline{Bohr}})$ contain no non-trivial convergent sequence?*

Here the $\tau_{\overline{\mathrm{Bohr}}}$ is defined as before as the weakest topology on $\mathsf{ult}_p^{\omega_1}(\mathbb{Z})$ which makes all extensions of homomorphisms from $\mathbb{Z}$ to $\mathbb{T}$ continuous, and the group $\mathsf{Ult}_p^{\omega_1}(\mathbb{Z}) = \mathsf{ult}_p^{\omega_1}(\mathbb{Z})/K$ with K being the intersection of all kernels of the extended homomorphisms.

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