MALYKHIN'S PROBLEM

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Abstract. We construct a model of ZFC where every separable Fréchet group is metrizable. This solves a 1978 problem of V. I. Malykhin.

1. Introduction

Metrization theorems have been central to topology since its beginnings. Among the classical metrization theorems are those of Urysohn [63] for separable spaces, Bing [13] and Nagata-Smirnov [44, 58] for general topological spaces, Birkhoff-Kakutani [14, 35] for topological groups, and Katětov [36] for compact spaces.

In more recent decades, several important metrization problems were solved using special set-theoretic assumptions, *i.e.,* were proved to be independent of the usual axioms of set theory (ZFC). Many such results, including the solution to Suslin's problem [59], were solved appealing to some form of a forcing axiom (Martin's Axiom MA, Proper Forcing Axiom PFA). A recent example is the solution to a problem of von Neumann (*The Scottish book* [40][Problem 163]) about weakly distributive Boolean algebras (a metrization problem in disguise) using the P-Ideal Dichotomy (PID) [8, 7], a consequence of PFA.

Then there are metrization problems, the solution of which requires a special forcing construction as they have provably negative solutions both in the constructible universe (assuming $\mathsf{V} = \mathsf{L}$) and assuming a forcing axiom such as PFA: The famous Normal Moore Space Conjecture [33, 34] was solved in the late 70's and early 80's by appealing to forcing techniques combined with large cardinals [23, 45, 24], the problem of Katětov's [36] about metrizability of compact spaces with hereditarily normal squares was solved by forcing with a Suslin tree over a model of a large fragment of the Proper Forcing Axiom (PFA) [38], hence creating a model which shares several properties of mutually contradictory theories $\mathsf{V} = \mathsf{L}$ and PFA.

Following this line of research, we present a consistent metrization theorem for separable Fréchet topological groups.

The classical metrization theorem of Birkhoff and Kakutani states that a T_1 topological group is metrizable if and only if it is first countable. The natural question as to what extent can the first countability be weakened was asked by Malykhin (see [4, 43]):

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Problem 1.1 (Malykhin 1978)**.** *Is there a separable (or, equivalently, countable) Fréchet-Urysohn group that is not metrizable?*

Recall that a topological space X is *Fréchet-Urysohn* or just *Fréchet* if for every $A \subseteq X$ and every x in the closure of A there is a sequence of elements of A converging to x. All spaces and groups are assumed to be T_1 and completely regular.

Malykhin knew that assuming $\mathfrak{p} > \omega_1$ every countable dense subgroup of 2^{ω_1} is Fréchet and not metrizable. There are known consistent positive solutions to Malykhin's problem also under either of the following assumptions: the existence of an uncountable γ-set (Gerlits-Nagy [25]) and $\mathfrak{p} = \mathfrak{b}$ (Nyikos [48]). In fact, by a recent result of Ohrenstein and Tsaban [50] the existence of an uncountable γ-sets is the weakest of the assumptions.

On the other hand, Todorčević and Uzcátegui [61] recently showed that there are no definible examples of non-metrizable countable Fréchet groups.

One of the first consistency results in the "negative direction" was proved by Brendle and the first author. Recall that a filter $\mathcal{F}$ on ω is a *FUF-filter* [52, 27, 28] if given a family $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ such that every element of $\mathcal{F}$ contains an element of X there is a sequence $\langle a_n : n \in \omega \rangle \subseteq X$ such that every element of $\mathcal{F}$ contains all but finitely many a_n's. Every FUF filter $\mathcal{F}$ induces a T_1 group topology on the Boolean group $[\omega]^{<\omega}$ by declaring the family $\{[F]^{<\omega} : F \in \mathcal{F}\}$ basis for open neighbourhoods of the neutral element $\emptyset$. Moreover, the weight of the induced topology coincides with the character of the filter $\mathcal{F}$ (see [52] or [27]).

Theorem 1.2 ([18])**.** *It is consistent with the continuum arbitrarily large that no uncountably generated filter of character less than $\mathfrak{c}$ is a FUF-filter.*

The method of the proof of our main theorem is largely based on the proof of this theorem.

Another recent theorem of Barman and Dow [9] hinted in the direction of a consistency result.

Theorem 1.3 ([9])**.** *It is consistent with* ZFC *that every countable Fréchet space has π-weight at most ω_1.*

Corollary 1.4. *It is consistent with* ZFC *that every separable Fréchet group has weight at most ω_1.*

This paper is dedicated to showing that

Theorem 1.5. *It is consistent with* ZFC *that every countable Fréchet topological group is metrizable.*

The basic forcing used is the Laver-Mathias-Prikry forcing with a suitable filter (the filter of dense open subsets of a Fréchet space or a group). The main technical notion involved is that of an ω-hitting family and its variations.

We say that a family $\mathcal{H} \subseteq [\omega]^\omega$ is *ω-hitting* [20] if given $\langle A_n : n \in \omega \rangle \subset [\omega]^\omega$ there is an $H \in \mathcal{H}$ such that $H \cap A_n$ is infinite for all n. An important property of ω-hitting families, which will be used several times, is that if an ω-hitting family is partitioned into countably many pieces, then at least one of the pieces is ω-hitting.

The outline of the proof is as follows. Put $S^2_1 = \{\alpha < \omega_2 : \mathrm{cof}(\alpha) = \omega_1\}$. Start with a model of CH and $\diamondsuit(S^2_1)$ and construct a finite support iteration $\mathbb{P}_{\omega_2} =$

$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ so that each $\dot{\mathbb{Q}}_\alpha$ is forced to be the Laver-Mathias-Prikry forcing with the filter of dense open subsets of a Fréchet space or a group and use $\diamondsuit(S^2_1)$ to trap all "future" countable Fréchet spaces/groups.

At stage $\alpha < \omega_2$ we deal, say, with a countable Fréchet group $\mathbb{G}$ with the neutral element $1_\mathbb{G}$ endowed with a group topology τ. We show that, assuming the weight of τ is uncountable, forcing with the Laver-Mathias-Prikry forcing $\mathbb{L}_{\mathsf{nwd}^*(\tau)}$ generically adds a set A such that $1_\mathbb{G}$ is in the closure of A yet there is no sequence in A convergent to $1_\mathbb{G}$. This procedure will be called *sealing*. Then we have to make sure that the rest of the iteration preserves

(1) there is no sequence in A convergent to $1_\mathbb{G}$, and
(2) $1_\mathbb{G}$ is in the closure of A (in all future group topologies extending τ).

The first item is taken care of by preservation of ω-hitting, and is a direct generalization of the results in [18]. The second item required new ideas. It is here where the relationship between the topological and algebraic structure of the group comes into play and, ultimately, a variant of ω-hitting, called ω-hitting w.r.t. A is introduced and used here.

Malykhin's problem was considered one of the principal open problems in Set-theoretic topology [43, 30, 41] and its solution is a major contribution to the study of convergence properties in topological groups [4, 46, 53, 48, 5, 56, 54, 60, 27, 28, 6] as well as a contribution to the study of structural properties of Fréchet spaces in general [1, 26, 2, 57, 47, 22].

Our set-theoretic notation is standard and follows [37]. For more background on forcing see [37, 11, 10] and on cardinal invariants of the continuum see [15, 11].

2. Laver-Mathias-Prikry forcing

The *Laver-Mathias-Prikry forcing* $\mathbb{L}_\mathcal{F}$ associated to a free filter $\mathcal{F}$ on ω, is defined as the set of those trees $T \subseteq \omega^{<\omega}$ for which there is $s_T \in T$ (the *stem of* T) such that for all $s \in T$, $s \subseteq s_T$ or $s_T \subseteq s$ and such that for all $s \in T$, with $s \supseteq s_T$ the set $\mathrm{succ}_T(s) = \{n \in \omega : s^\frown n \in T\} \in \mathcal{F}$ ordered by inclusion.

$\mathbb{L}_\mathcal{F}$ is a σ-centered forcing notion which adds generically a dominating function $\dot{\ell}_\mathcal{F} : \omega \to \omega$. The range $\dot{A}_{gen} = \mathrm{ran}(\dot{\ell}_\mathcal{F})$ of which *separates* the filter $\mathcal{F}$ (that is, $\dot{A}_{gen}$ is almost contained in all members of $\mathcal{F}$ and has infinite intersection with each $\mathcal{F}$-positive set).

For a tree $T \subseteq \omega^{<\omega}$ and $s \in T$ with $s \supseteq s_T$, let $T_s = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$ be the *full subtree of* T *below* s.

Names for reals in forcings of the type $\mathbb{L}_\mathcal{F}$ can be analyzed using *ranks* as introduced by Baumgartner and Dordal in [12] (and further developed by Brendle [16, 17]). Given a formula φ in the forcing lenguage and $s \in \omega^{<\omega}$, we say that s *favors* φ if there is no condition $T \in \mathbb{L}_\mathcal{F}$ with stem s such that $T \Vdash$ "$\neg\varphi$", or equivalently, every condition $T \in \mathbb{L}_\mathcal{F}$ with stem s has an extension T' such that $T' \Vdash$ "φ".

3. Preservation of ω-hitting

As already mentioned earlier ω-hitting families and their preservation by forcing are crutial in our arguments and appeared already in [20] and [18].

Definition 3.1 ([18])**.** A forcing notion $\mathbb{P}$ *strongly preserves ω-hitting* if for every sequence $\langle \dot{A}_n \colon n \in \omega\rangle$ of $\mathbb{P}$-names for infinite subsets of ω there is a sequence $\langle B_n \colon n \in \omega\rangle$ of infinite subsets of ω such that for any $B \in [\omega]^\omega$, if $B \cap B_n$ is infinite for all n then $\Vdash_{\mathbb{P}}$ "$B \cap \dot{A}_n$ is infinite for all n".

Clearly, every forcing notion that strongly preserves ω-hitting preserves that all ground model ω-hitting families remain ω-hitting in the extension.

Lemma 3.2 ([18])**.** *Finite support iteration of ccc forcings strongly preserving ω-hitting strongly preserves ω-hitting.*

In [18] preservation of ω-hitting for the forcing notions of type $\mathbb{L}_{\mathcal{F}}$ was characterized.

Proposition 3.3 ([18])**.** *Let $\mathcal{I}$ be an ideal on ω and let $\mathcal{F} = \mathcal{I}^*$ be the dual filter. Then the following are equivalent:*

(1) *For every $X \in \mathcal{I}^+$ and every $\mathcal{J} \leqslant_K \mathcal{I} \restriction X$ the ideal $\mathcal{J}$ is not ω-hitting.*
(2) *$\mathbb{L}_{\mathcal{F}}$ strongly preserves ω-hitting.*
(3) *$\mathbb{L}_{\mathcal{F}}$ preserves ω-hitting.*

Recall the definition of *Katětov order* (see [29, 32]): Given two ideals $\mathcal{I}$, $\mathcal{J}$ on ω, we say that $\mathcal{I} \leqslant_K \mathcal{J}$ if there is a function $f \colon \omega \to \omega$ such that $f^{-1}[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$.

4. Sealing by $\mathbb{L}_{\mathcal{F}}$

Definition 4.1. Let $\mathcal{I}$ be an ideal on ω, $\mathbb{P}$ a forcing notion and let $\dot{A}$ be a $\mathbb{P}$-name for a subset of ω. We say that $\mathbb{P}$ *seals* the ideal $\mathcal{I}$ via $\dot{A}$ if $\Vdash_{\mathbb{P}}$ "$\dot{A} \in \mathcal{I}^+ \wedge \mathcal{I} \restriction \dot{A}$ is ω-hitting".

The following variation of ω-hitting is used to characterize when $\mathbb{L}_{\mathcal{F}}$ seals an ideal $\mathcal{I}$ via $\dot{A}_{gen}$.

Definition 4.2. Given an ideal $\mathcal{I}$ and a free filter $\mathcal{F}$ both on ω, we say that $\mathcal{I}$ is *ω-hitting mod $\mathcal{F}$*, if $\mathcal{I} \cap \mathcal{F} = \emptyset$ and for every countable family $\mathcal{H} \subset \mathcal{F}^+$ there is an $I \in \mathcal{I}$ such that $H \cap I \in \mathcal{F}^+$ for all $H \in \mathcal{H}$.

Note that an ideal $\mathcal{I}$ is ω-hitting if and only if $\mathcal{I}$ is ω-hitting mod Fr, where Fr is the Fréchet filter (the filter of co-finite sets of ω).

Lemma 4.3. *The forcing $\mathbb{L}_{\mathcal{F}}$ seals the ideal $\mathcal{I}$ via $\dot{A}_{gen}$ if and only if $\mathcal{I}$ is ω-hitting mod $\mathcal{F}$.*

Proof. Suppose that $\mathbb{L}_{\mathcal{F}}$ seals the ideal $\mathcal{I}$ via $\dot{A}_{gen}$. Since $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\dot{A}_{gen} \in \mathcal{I}^+$", it follows that $\mathcal{I} \cap \mathcal{F} = \emptyset$. Assume that there is a countable family $\mathcal{H} \subset \mathcal{F}^+$ such that for every $I \in \mathcal{I}$ there is a $H \in \mathcal{H}$ with $H \cap I \in \mathcal{F}^*$. The forcing $\mathbb{L}_{\mathcal{F}}$ separates the filter $\mathcal{F}$. Then $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\dot{A}_H := \dot{A}_{gen} \cap H$ is infinite for every $H \in \mathcal{H}$" and $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\dot{A}_{gen}$ is almost disjoint from each member of $\mathcal{F}^*$". Therefore, $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\mathcal{I} \restriction \dot{A}_{gen}$ is not ω-hitting".

Conversely, suppose that $\mathcal{I}$ is ω-hitting mod $\mathcal{F}$. As $\mathcal{I} \cap \mathcal{F} = \emptyset$, it follows that $\mathcal{I}^* \subseteq \mathcal{F}^+$ and by genericity $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$X \cap \dot{A}_{gen}$ is infinite for all $X \in \mathcal{F}^+$". Thus, $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\dot{A}_{gen} \in \mathcal{I}^+$".

We now show that $\Vdash_{\mathbb{L}_{\mathcal{F}}}$ "$\mathcal{I} \restriction \dot{A}_{gen}$ is ω-hitting". This is a rank argument based on [18]. Let $\langle \dot{A}_n \colon n \in \omega\rangle$ be a sequence of $\mathbb{L}_{\mathcal{F}}$-names for infinite subsets of $\dot{A}_{gen}$.

Aiming towards a contradiction, assume that for all $I \in \mathcal{I}$ there are $T_I \in \mathbb{L}_{\mathcal{F}}$, and natural numbers n_I, m_I such that

$$T_I \Vdash \text{“}\dot{A}_{n_I} \cap I \subseteq m_I\text{”}. \qquad (\star)$$

Define the *rank* $\mathrm{rk}_n(s)$ by recursion on the ordinals by

$$\mathrm{rk}_n(s) = 0 \Leftrightarrow \exists B \in \mathcal{F}^+ \, \forall b \in B (s^\frown b \text{ favors } b \in \dot{A}_n)$$
$$\mathrm{rk}_n(s) \leqslant \alpha \Leftrightarrow \exists B \in \mathcal{F}^+ \, \forall b \in B (\mathrm{rk}_n(s^\frown b) < \alpha)$$

for $\alpha > 0$.

Claim 4.4. $\mathrm{rk}_n(s) < \infty$ for all s and n.

Proof of the claim. Fix n. Let $k \in \omega$. Define an *auxiliary rank* $\rho_k(s)$ by recursion such that

$$\rho_k(s) = 0 \Leftrightarrow \exists b > k (s \text{ favors } b \in \dot{A}_n)$$

and $\rho_k(s) \leqslant \alpha$ is defined as for rk_n, for $\alpha > 0$. First, notice that $\rho_k(s) < \infty$ for all s and k. Indeed, suppose that $\rho_k(s) = \infty$ for some s and k. Recursively build condition T with $s_T = s$ such that $\rho_k(t) = \infty$ for every $t \in T$ with $t \supseteq s$. As $\dot{A}_n$ is forced to be infinite, there are $T' \leqslant T$ and $b > k$ such that $T' \Vdash$ "$b \in \dot{A}_n$". In particular, $s_{T'} \supseteq s$ and $s_{T'}$ favors $b \in \dot{A}_n$. Hence, $\rho_k(s_{T'}) = 0$, a contradiction. Now, also note that since $\dot{A}_n$ is forced to be a subset of $\dot{A}_{gen}$, any s can favor only elements of $\mathrm{ran}(s)$.

If $\rho_k(s) = 1$, then there is a $\mathcal{F}$-positive set of b such that $s^\frown b$ favors $a \in \dot{A}_n$ for some $a = a_b$ with $a > k$. If on a $\mathcal{F}$-positive set, the same $a \in \mathrm{ran}(s)$ works, we get $\rho_k(s) = 0$, a contradiction. Since $a_b \in \mathrm{ran}(s) \cup \{b\}$, it follows that on a $\mathcal{F}$-positive set, $a_b = b$. This, however, means that $\mathrm{rk}_n(s) = 0$.

Now, let $k > \max(\mathrm{ran}(s))$. Then $\rho_k(s) \geqslant 1$. By the preceding paragraph and induction, we see that $\mathrm{rk}_n(s) < \infty$, as required. ∎

Let s_I be the stem of T_I. By strengthening the T_I, if necessary, by Claim 4.4 and using induction on the rank of the stem, we may assume that $\mathrm{rk}_{n_I}(s_I) = 0$ for all $I \in \mathcal{I}$. According to the definition of rk_n, for every $I \in \mathcal{I}$ there is a $B_{s_I, n_I} \in \mathcal{F}^+$ such that $s_I^\frown b$ favors $b \in \dot{A}_{n_I}$ for all $b \in B_{s_I, n_I}$. Since $\mathcal{I}$ is ω-hitting mod $\mathcal{F}$, there is an I which intersects all the $B_{s,n}$'s in a $\mathcal{F}$-positive set. In particular, $I \cap B_{s_I, n_I} \in \mathcal{F}^+$ and hence there is a $b > m_I$ with $b \in I \cap B_{s_I, n_I} \cap \mathrm{succ}_{T_I}(s_I)$. As $s_I^\frown b$ favors $b \in \dot{A}_{n_I}$, there is a $T \leqslant T_I$ whose stem extends $s_I^\frown b$ such that $T \Vdash$ "$b \in \dot{A}_{n_I}$", a contradiction to the initial assumption $(\star)$. □

5. Topology

Given a topological space X and a point $x \in X$, we denote by $\mathcal{I}_x$ the dual ideal to the neighourhood filter of x and by $\mathsf{nwd}(X)$ the ideal of nowhere dense subsets of X. Then $\mathsf{nwd}^*(X)$ is the filter generated by the dense open subset of X.

The following simple yet important topological fact due to Barman and Dow [9] is used several times in what follows:

Lemma 5.1 ([9])**.** *Let X be a Fréchet space without isolated points, let $x \in X$ and let $\mathcal{M}$ be a countable family of nowhere dense subsets of X. Then there is an infinite sequence C_x converging to x with only finite intersection with every element of $\mathcal{M}$.*

Proof. Fix $\langle M_n \colon n \in \omega\rangle$ an enumeration of $\mathcal{M}$. Since the space X has no isolated points, there is a sequence $\langle x_n \colon n \in \omega\rangle \subseteq X \setminus \{x\}$ converging to x. Let $X_n = X \setminus \big(\{x\} \cup \bigcup_{i<n} M_i\big)$. Then X_n is dense in X for every n. Since X is Fréchet, there is a sequence $\langle x^n_k \colon k \in \omega\rangle \subseteq X_n$ converging to x_n for each $n \in \omega$. Put $X' = \{x^n_k \colon k, n \in \omega\}$. Then $x \in \overline{X'}$, and hence there is a sequence $C_x \subseteq X'$ converging to x. By the construction, the sequence C_x is as required. □

Recall that $\pi w(X)$, the *π-weight* of a space X is the minimal size of a π-base, *i.e.,* a family of non-empty open sets such that every non-empty open set contains an element of the family.

Theorem 5.2 ([3])**.** *Let $\mathbb{G}$ be a topological group. Then $w(\mathbb{G}) = \pi w(\mathbb{G})$.*

Recall that the *π-character* $\pi\chi(x, X)$ of a point x in a space X is the minimal size of a local π-base of x that is the minimal size of a family of non-empty open subsets of X such that any neighborhood of x contains one of them. In a countable group $\mathbb{G}$ the weight is equal to the π-character of any of its points, and a countable space X has uncountable π-weight if and only if there is an $x \in X$ such that $\pi\chi(x, X) > \omega$.

Recall that we wish to introduce a set $A \subseteq X$ such that x is in the closure of A yet no sequence from A converges to x. This is easily seen to be equivalent to A being $\mathcal{I}_x$-positive and such that the ideal $\mathcal{I}_x \restriction A$ is tall. However, tallness of ideals is in general not preserved by any forcing adding a real. However, a slight strengthening - countable tallness, is preserved as seen in Section 3.

Proposition 5.3. *Let X be a countable regular space and let $x \in X$.*

(a) *If $\pi\chi(x, X) > \omega$, then $\mathbb{L}_{\mathsf{nwd}^*(X)}$ seals the ideal $\mathcal{I}_x$ via $\dot{A}_{gen}$.*
(b) *If X is a T_1 Fréchet space with no isolated points, then $\mathbb{L}_{\mathsf{nwd}^*(X)}$ strongly preserves ω-hitting.*

Proof. To see (a), by Lemma 4.3, it is enough to show that the ideal $\mathcal{I}_x$ is ω-hitting mod $\mathsf{nwd}^*(X)$. Clearly $\mathcal{I}_x \cap \mathsf{nwd}^*(X) = \emptyset$. For the other part, suppose that there is a countable family $\mathcal{H} \subset \mathsf{nwd}^+(X)$ such that for every $I \in \mathcal{I}_x$ there is a $H \in \mathcal{H}$ with $H \cap I \in \mathsf{nwd}(X)$. For each $H \in \mathcal{H}$, put $U_H = \mathrm{Int}(\overline{H}) \neq \emptyset$.

Claim 5.4. The family $\{U_H \colon H \in \mathcal{H}\}$ forms a π-base at x.

Proof of the claim. Let U be an arbitrary neighbourhood of x. By regularity of X, there exists a neighbourhood V of x such that $\overline{V} \subseteq U$. There exists a $H \in \mathcal{H}$ such that $H \setminus V \in \mathsf{nwd}(X)$, and hence $\mathrm{Int}(\overline{H \setminus V}) = \emptyset$. We claim that $U_H = \mathrm{Int}(\overline{H}) \subseteq \overline{V}$. Indeed, if $y \in U_H \setminus \overline{V}$ and W is an arbitrary open neighbourhood of y, then $H \cap (W \cap (U_H \setminus \overline{V})) \neq \emptyset$. So $W \cap (H \setminus V) \neq \emptyset$, and therefore $y \in \overline{H \setminus V}$, *i.e.,* $U_H \setminus \overline{V} \subseteq \overline{H \setminus V}$, which contradicts the fact that $\mathrm{Int}(\overline{H \setminus V}) = \emptyset$. ■

Thus, $\pi\chi(x, X) = \omega$, a contradiction to the initial assumption. Therefore, $\mathbb{L}_{\mathsf{nwd}^*(X)}$ seals the ideal $\mathcal{I}_x$ via $\dot{A}_{gen}$.

To see (b), by Proposition 3.3, it suffices to show that for every $Y \in \mathsf{nwd}(X)^+$ and every $\mathcal{J} \leqslant_K \mathsf{nwd}(X) \restriction Y$ the ideal $\mathcal{J}$ is not ω-hitting. Aiming towards a contradiction, assume that there is a ω-hitting ideal $\mathcal{J}$ Katětov below $\mathsf{nwd}(X) \restriction Y$, for some $Y \in \mathsf{nwd}(X)^+$, witnessed by a function $f \colon Y \to \omega$. Put $U = \mathrm{Int}(\overline{Y}) \neq \emptyset$ and $Z = U \cap Y$.

Claim 5.5. For every $x \in Z$ there is an infinite sequence $C_x \subseteq Z$ converging to x such that $f \restriction C_x$ is finite to one.

Proof of the claim. Note that $f^{-1}(n) \in \mathsf{nwd}(X)$ for all $n \in \omega$. Apply Lemma 5.1 to Z with $\mathcal{M} = \{f^{-1}(n) \cap Z : n \in \omega\}$. ■

Since $\mathcal{J}$ is ω-hitting there is a $J \in \mathcal{J}$ such that $J \cap f[C_x]$ is infinite for every $x \in Z$. Then $f^{-1}[J]$ is dense in Z, but Z is dense in U, a contradiction. □

Using the results mentioned so far, we can prove that it is relatively consistent with the continuum arbitrarily large, that every separable Fréchet space of weight less than $\mathfrak{c}$ has countable π-weight, in particular, every separable Fréchet group of weight less than $\mathfrak{c}$ is metrizable.

6. Algebra

So far we could get by with only topology. However, now we turn our attention to the problem of preserving that $1_{\mathbb{G}}$ is in the closure of the generically added set A. Here is where algebra comes into play, and necessarily so, as Dow [21] has recently showed that, assuming $\mathfrak{b} = \mathfrak{c}$ there is a countable Fréchet space of uncountable π-weight.

Let us elaborate a bit more: In particular, in our model there is such a space. However, a fragment of such a space has been trapped by our bookkeeping device and "killed" by adding a set A which has a point x in its closure and contains no converging sequences, in fact, $\mathcal{I}_x \restriction A$ is ω-hitting. Moreover, as the iteration preserves ω-hitting, A will never contain any sequence converging to x. It means that there is an extension of the original topology in which x is no longer in the closure of A, *i.e.,* there is a new open set containing x which is disjoint from A.

We will show that this does not happen in the case of topological groups.

Definition 6.1. Let $(\mathbb{G}, \cdot)$ be an abstract group and let $X \subseteq \mathbb{G} \setminus \{1_{\mathbb{G}}\}$. A subset A of $\mathbb{G}$ is called *X-large* if for every $b \in X$ and $a \in \mathbb{G}$, either $a \in A$ or $b \cdot a^{-1} \in A$. By X-large we will denote the collection of all subsets of $\mathbb{G}$ which are X-large.

The intention is as follows: Let X be the generically added set. If there were a finer group topology in which $1_{\mathbb{G}}$ is not in the closure of X, then there would be a set $U \subseteq \mathbb{G}$ containing $1_{\mathbb{G}}$ and such that $U \cdot U \cap X = \emptyset$, a new open neighbourhood of $1_{\mathbb{G}}$ disjoint from X. A set A is X-large if it contains the complement of such a U.

Definition 6.2. A family $\mathcal{C}$ of subsets of an abstract group $\mathbb{G}$ is *ω-hitting with respect to* (or just ω-hitting w.r.t.) X if given $\langle A_n : n \in \omega\rangle \subset X$-$\mathsf{large}$ there is a $C \in \mathcal{C}$ such that $C \cap A_n$ is infinite for all n.

Let us continue with the intended meaning of this. Again, suppose that a set U is a new open neighbourhood of $1_{\mathbb{G}}$ disjoint from X (such that $U \cdot U \cap X = \emptyset$). The information about the future group topologies extending our topology τ, includes information about convergent sequences: If C converges to $1_{\mathbb{G}}$ in τ it will also converge in all future topologies, in particular, C would have to be almost contained in U. However, if we manage to prove that for every candidate for U there is a C which is not almost contained in U (equivalently, has an infinite intersection with the X-large set which is the complement of U) and also manage to preserve this, we will be able to show that $1_{\mathbb{G}}$ will be in the closure of X in all future group topologies extending τ. This is, indeed, what we shall do.

Definition 6.3. We say that a relation $R \subseteq \mathbb{G} \times \mathbb{G}$ is *large* if for every $b, a \in \mathbb{G}$, either $\langle b, a\rangle \in R$ or $\langle b, b \cdot a^{-1}\rangle \in R$. By large we will denote the family of relations which are large.

The following is the crutial topological lemma, which makes the proof work:

Lemma 6.4. *Let $\mathbb{G}$ be a countable Fréchet group and let $\langle R_n : n \in \omega\rangle$ be a sequence of large relations. Then there is a sequence C convergent to $1_{\mathbb{G}}$ such that $R_n^{-1}[C \setminus F] \in \mathsf{nwd}(\mathbb{G})^+$ for every $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$.*

Proof. For every $n \in \omega$ let $A_n = \{a \in \mathbb{G} : R_n^{-1}(a) \in \mathsf{nwd}(\mathbb{G})\}$ and put

$$\mathcal{M} = \{R_n^{-1}(a) \cdot a^{-1} : a \in A_n, n \in \omega\} \cup \{A_n : A_n \in \mathsf{nwd}(\mathbb{G})\}.$$

By Lemma 5.1, there is a sequence C converging to $1_{\mathbb{G}}$ such that $C \cap M$ is finite for every $M \in \mathcal{M}$. We claim that $R_n^{-1}[C \setminus F] \in \mathsf{nwd}(\mathbb{G})^+$ for every $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$. To see this, let $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$ are given. Consider two cases.

Case 1. $A_n \in \mathsf{nwd}(\mathbb{G})$.

Then there is an $a \in C \setminus F$ such that $R_n^{-1}(a) \in \mathsf{nwd}(\mathbb{G})^+$.

Case 2. $A_n \in \mathsf{nwd}(\mathbb{G})^+$.

Fix $a \in A_n$. Then $b \notin R_n^{-1}(a) \cdot a^{-1}$ (or equivalently $b \cdot a \notin R_n^{-1}(a)$) for all but finitely many $b \in C$. Since R_n is a large relation, we must have $\langle b \cdot a, b\rangle \in R_n$ for all but finitely many $b \in C$. In particular, $\{b \cdot a : b \in C\} \subseteq^* R_n^{-1}[C \setminus F]$ and $\{b \cdot a : b \in C\}$ converges to a. Thus, $A_n \subseteq \overline{R_n^{-1}[C \setminus F]}$ and hence also $R_n^{-1}[C \setminus F] \in \mathsf{nwd}(\mathbb{G})^+$. □

Lemma 6.4 allows us to prove the following:

Lemma 6.5. *Let $\mathbb{G}$ be a countable Fréchet topological group. Then*

$$\Vdash_{\mathbb{L}_{\mathsf{nwd}^*(\mathbb{G})}} \text{“}\mathcal{C} \textit{ is } \omega\textit{-hitting w.r.t. } \dot{A}_{gen}\text{”},$$

where $\mathcal{C} = \mathcal{I}_{1_{\mathbb{G}}}^{\perp}$ is the ideal consisting of sequences converging to $1_{\mathbb{G}}$.

Proof. Aiming towards a contradiction, assume that there are a sequence $\langle \dot{A}_n : n \in \omega\rangle$ of $\mathbb{L}_{\mathsf{nwd}^*(\mathbb{G})}$-names and a condition $T^* \in \mathbb{L}_{\mathsf{nwd}^*(\mathbb{G})}$ such that $T^* \Vdash$ “$\forall n \in \omega\, (\dot{A}_n \in \dot{A}_{gen}\text{-}\mathsf{large})$” and for every $C \in \mathcal{C}$ there are a condition $T_C \in \mathbb{L}_{\mathsf{nwd}^*(\mathbb{G})}$, a natural number n_C and F_C a finite subset of $\mathbb{G}$ such that

$$T_C \Vdash \text{“}C \cap \dot{A}_{n_C} \subseteq F_C\text{”}. \qquad (\star)$$

For each $s \in T^*$ with $s \supseteq s_{T^*}$ and each natural number n, put

$$R_{s,n} = \{\langle b, a\rangle : b \in \mathrm{succ}_{T^*}(s) \Rightarrow s^\frown b \text{ favors } a \in \dot{A}_n\}.$$

Claim 6.6. The relation $R_{s,n}$ is large.

Proof of the claim. Let b and a be two arbitrary elements of $\mathbb{G}$. Assume that $\langle b, a\rangle \notin R_{s,n}$. We have to show that $\langle b, b \cdot a^{-1}\rangle \in R_{s,n}$. Let T be an arbitrary condition with $s_T = s^\frown b$. By the assumption, necessarily $b \in \mathrm{succ}_{T^*}(s)$ and there is a condition T' with $s_{T'} = s^\frown b$ such that $T' \Vdash$ “$a \notin \dot{A}_n$”. Put $T'' = T^*_{s^\frown b} \cap T \cap T'$. Then $T'' \Vdash$ “$b \in \dot{A}_{gen}$ and $a \notin \dot{A}_n$”, but also $T'' \Vdash$ “$\dot{A}_n \in \dot{A}_{gen}\text{-}\mathsf{large}$”, hence there is a condition $S \leqslant T''$ such that $S \Vdash$ “$b \cdot a^{-1} \in \dot{A}_n$”. ■

By Lemma 6.4, there is a $C \in \mathcal{C}$ such that $R_{s,n}^{-1}(C \setminus F) \in \mathsf{nwd}(\mathbb{G})^+$ for every $s \in T^*$ with $s \supseteq s_{T^*}$, $n \in \omega$ and $F \in [\mathbb{G}]^{<\omega}$. In particular, $R_{s_C,n_C}^{-1}(C \setminus F_C) \in \mathsf{nwd}(\mathbb{G})^+$, where $s_C = s_{T_C}$. Pick a $b \in \mathrm{succ}_{T_C}(s_C) \cap R_{s_C,n_C}^{-1}(C \setminus F_C)$. Then, there is an

$a \in C \setminus F_C$ such that $s_C^\frown b$ favors $a \in \dot{A}_{n_C}$, and hence there is a condition $T \leqslant T_C$ whose stem extends $s_C^\frown b$ such that $T \Vdash$ "$a \in \dot{A}_{n_C}$", a contradiction to the initial assumption ($\star$). $\square$

We now turn to the preservation of ω-hitting w.r.t. X in iterations. In order to do this, we introduce the corresponding iterable property: We say that a forcing notion $\mathbb{P}$ *strongly preserves ω-hitting w.r.t. X* if for every $\dot{A}$ a $\mathbb{P}$-name for a subset X-large of a group $\mathbb{G}$ there is a sequence $\langle A_n : n \in \omega\rangle \subset X\text{-}\mathsf{large}$ such that for any $C \subseteq \mathbb{G}$, if $C \cap A_n$ is infinite for all n then $\Vdash_{\mathbb{P}}$ "$C \cap \dot{A}$ is infinite". Clearly, every forcing notion that strongly preserves ω-hitting w.r.t. X preserves ω-hitting w.r.t. X.

Lemma 6.7. *Let $\mathbb{P}$ be a σ-centered forcing notion. Then $\mathbb{P}$ strongly preserves ω-hitting w.r.t. X.*

Proof. Without loss of generality, $\mathbb{P}$ is a complete Boolean algebra, $\mathbb{P}^+ = \bigcup_{n\in\omega} \mathcal{U}_n$, all $\mathcal{U}_n$ being ultrafilters. Let $\dot{A}$ be a $\mathbb{P}$-name for a subset X-large of a group $\mathbb{G}$. Put $A_n = \{a \in \mathbb{G} : [\![a \in \dot{A}]\!] \in \mathcal{U}_n\}$ for every $n \in \omega$. It is easy to see that each A_n is X-large. Now, let C be a subset of $\mathbb{G}$ such that $C \cap A_n$ is infinite for all n. We claim, $\Vdash_{\mathbb{P}}$ "$C \cap \dot{A}$ is infinite". Indeed, let $p \in \mathbb{P}$ and let F be a finite subset of $\mathbb{G}$. Then there exists an n with $p \in \mathcal{U}_n$ and $a \in C \cap A_n \setminus F$. Thus, $[\![a \in \dot{A}]\!] \in \mathcal{U}_n$ and hence there is a condition $q \leqslant p$ such that $q \Vdash$ "$a \in \dot{A}$". $\square$

Lemma 6.8. *Finite support iteration of ccc forcings strongly preserving ω-hitting w.r.t. X strongly preserves ω-hitting w.r.t. X.*

Proof. This is a standard argument. Obviously, it suffices to consider only limit stages of cofinality ω.

Let $\langle \mathbb{P}_k, \dot{\mathbb{Q}}_k : k \in \omega\rangle$ be a finite support iteration of ccc forcing such that

$$\Vdash_{\mathbb{P}_k} \text{“}\dot{\mathbb{Q}}_k \text{ strongly preserves } \omega\text{-hitting w.r.t. } X\text{”},$$

for each $k \in \omega$.

Let $\dot{A}$ be a $\mathbb{P}_\omega$-name for a subset X-large of a countable group $\mathbb{G}$. In the intermediate extension $\mathbf{V}[G_k]$ find a decreasing sequence of conditions $\langle p_{n,k} : n \in \omega\rangle \subset \mathbb{P}_{[k,\omega)}$ and subsets X-large $A_{n,k}$ of $\mathbb{G}$ such that

$$p_{n,k} \Vdash_{\mathbb{P}_{[k,\omega)}} \text{“the first } n \text{ elements of } A_{m,k} \text{ and } \dot{A} \text{ agree for } m \leq n\text{”},$$

where $\mathbb{P}_\omega = \mathbb{P}_k * \mathbb{P}_{[k,\omega)}$. The $A_{n,k}$ are approximations to $\dot{A}$.

Now, as each $\mathbb{P}_k$ strongly preserves ω-hitting w.r.t. X, there is a $\langle A^m_{n,k} : m \in \omega\rangle \subset X\text{-}\mathsf{large}$ such that for any $C \subseteq \mathbb{G}$, if $C \cap A^m_{n,k}$ is infinite for all m then

$$\Vdash_{\mathbb{P}_k} \text{“}C \cap \dot{A}_{n,k} \text{ is infinite”}.$$

Consider $\langle A^m_{n,k} : n, k, m \in \omega\rangle$ and let $C \subseteq \mathbb{G}$ be such that $C \cap A^m_{n,k}$ is infinite for all n, k and m. To finish the proof, it suffices to show that

$$\Vdash_{\mathbb{P}_\omega} \text{“}C \cap \dot{A} \text{ is infinite”}.$$

If not, then there are a $q \in \mathbb{P}_\omega$ and $F \in [\mathbb{G}]^{<\omega}$ such that $q \Vdash_{\mathbb{P}_\omega}$ "$C \cap \dot{A} \subseteq F$". Let k be such that $q \in \mathbb{P}_k$. Let G_k be a $\mathbb{P}_k$-generic such that $q \in G_k$. As $C \cap A_{m,k}$ is infinite, let $a \notin F$ with $a \in C \cap A_{m,k}$. For large enough n,

$$p_{n,k} \Vdash_{\mathbb{P}_{[k,\omega)}} \text{“}a \in \dot{A}\text{”}.$$

Since $q \in G_k$, this contradicts the initial assumption about q. $\square$

7. The main result

We are now in position to give the proof of the main theorem.

Proof of the Theorem 1.5. Assume that the ground model $\mathbf{V}$ satisfies CH, and suppose $\langle A_\alpha \colon \alpha \in S^2_1\rangle$ witnesses that $\diamondsuit(S^2_1)$ holds in it. We construct a finite support iteration $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\rangle$ so that at a stage $\alpha \in S^2_1$, if A_α codes a $\mathbb{P}_\alpha$-name for a regular Fréchet topology τ with no insolated points on ω with a point $n \in \omega$ such that $\pi\chi(n,\tau) > \omega$, we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathbb{L}_{\mathsf{nwd}^*(\tau)}$. If α is not of this form, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for, say, $\mathbb{L}_{\mathsf{nwd}^*(\mathbb{Q})}$. Let G_{ω_2} be a $\mathbb{P}_{\omega_2}$-generic over $\mathbf{V}$. We shall show that, in $\mathbf{V}[G_{\omega_2}]$, every countable Fréchet topological group is metrizable. Aiming towards a contradiction, assume that in $\mathbf{V}[G_{\omega_2}]$ there is a non-metrizable Fréchet group topology τ on a group $\mathbb{G} = (\omega, \cdot)$ with the neutral element 0.

By Theorem 5.2, $\pi\chi(0,\tau) > \omega$. Now, by a standard argument, there is a set $C \subset S^2_1$ which is a club relative to S^2_1 such that for all $\alpha \in C$, $\mathbf{V}[G_\alpha] \models \tau_\alpha$ is Fréchet and $\pi\chi(0,\tau_\alpha) > \omega$, where $\tau_\alpha = \tau \cap \mathbf{V}[G_\alpha]$. Therefore, at some stage $\alpha \in C$, we would have added a set A_{gen} such that $\mathbf{V}[G_\alpha] \models A_{gen} \in \mathcal{I}^+_0(\tau_\alpha)$ and the ideal $\mathcal{I}_0(\tau_\alpha) \restriction A_{gen}$ is ω-hitting (Proposition 5.3 (a)).

We claim that in $\mathbf{V}[G_{\omega_2}]$ also $A_{gen} \in \mathcal{I}^+_0(\tau)$. That is A_{gen} has the point 0 in its closure. If not, then there is a open neighbourhood U of 0 disjoint from A_{gen} such that $U \cdot U \cap A_{gen} = \emptyset$. Then $A = \mathbb{G} \setminus U$ is A_{gen}-large. By Lemma 6.5, in $\mathbf{V}[G_\alpha]$ the ideal $\mathcal{I}^\perp_0(\tau_\alpha)$ is ω-hitting w.r.t. A_{gen}, and by Lemmata 6.7 and 6.8, it follows that in $\mathbf{V}[G_{\omega_2}]$ the ideal $\mathcal{I}^\perp_0(\tau_\alpha)$ is also ω-hitting w.r.t. A_{gen}. In particular, there is a $C \in [A]^\omega$ such that $C \in \mathcal{I}^\perp_0(\tau_\alpha)$, *i.e.*, C is a sequence in A τ_α-converging to 0. But $\mathcal{I}^\perp_0(\tau_\alpha) \subset \mathcal{I}^\perp_0(\tau)$, then also C is a sequence in A τ-converging to 0, a contradiction ($A \cap U = \emptyset$).

Thus, by the assumption that τ is Fréchet, in $\mathbf{V}[G_{\omega_2}]$ there is a $C \in [A_{gen}]^\omega$ such that $C \in \mathcal{I}^\perp_0(\tau)$, *i.e.*, C is a sequence in A_{gen} converging to 0. By Proposition 5.3 (b) and Lemma 3.2, in $\mathbf{V}[G_{\omega_2}]$ the ideal $\mathcal{I}_0(\tau_\alpha) \restriction A_{gen}$ is ω-hitting (in particular tall), and hence there is a $I \in \mathcal{I}_0(\tau_\alpha)$ such that $I \cap C$ is infinite, which contradicts the fact that C converged to 0 ($\mathcal{I}_0(\tau_\alpha) \subseteq \mathcal{I}_0(\tau)$). ■

8. Concluding remarks

The question as to what extent is algebra involved in the problem was asked by Juhasz:

Question 8.1 (Juhász)**.** *Is there in* ZFC *a countable Fréchet space of uncountable* π*-weight?*

Dow [21] has recently shown that

Theorem 8.2 (Dow)**.** *There is a countable Fréchet space of uncountable* π*-weight, assuming* $\mathfrak{b} = \mathfrak{c}$.

In particular, there is a countable Fréchet space of π-weight $\mathfrak{c}$ in our model. Also, Dow's result together with the result of Nyikos [48] that there are countable non-metrizable Fréchet groups assuming $\mathfrak{p} = \mathfrak{b}$ imply that there is a countable Fréchet space of π-weight if $\mathfrak{c} \leq \omega_2$.

In [31] it was noted that if $\mathfrak{p} > \omega_1$ then any countable group which admits a non-discrete group topology, *i.e.*, is *topologizable* (see [49]), admits a non-metrizable Fréchet group topology. It would be interesting to know:

Question 8.3. *Is it consistent with* ZFC *that some topologizable group admits a non-metrizable Fréchet group topology while another does not?*

Up to now, uncountable γ-sets[1] existed in every known model with an example of a non-metrizable separable Fréchet group. It has therefore been asked [31, 62] whether the existence of a non-metrizable separable Fréchet group implies the existence of an uncountable γ-set.

We will show that it is not the case. In fact, we will prove that there is an ω_1-generated FUF filter, hence a simple example of a countable non-metrizable group, in virtually any model obtained by iterating definable forcing over a model of CH. In particular, there is an ω_1-generated FUF filter in the Laver model for Borel's Conjecture [39], in which there are no uncountable γ-sets.

Our construction uses a parametrized diamond principle, in fact, the Borel version of the "weak diamond principle" introduced by Devlin and Shelah in [19].

Definition 8.4 ([42])**.** $\diamondsuit(2,=)$ is the following statement:

$\diamondsuit(2,=)$ For every Borel map $F\colon 2^{<\omega_1} \to 2$ there is a $g\colon \omega_1 \to 2$ such that for every $f\colon \omega_1 \to 2$ the set $\{\alpha < \omega_1 \colon F(f \restriction \alpha) = g(\alpha)\}$ is stationary.

A map $F\colon 2^{<\omega_1} \to 2$ is *Borel* if for every α the restriction of F to 2^α is a Borel map. The witness g for a given F in this statement will be called a $\diamondsuit(2,=)$*-sequence for* F. If $F(f \restriction \alpha) = g(\alpha)$, then we will say that g *guesses* f (via F) at α.

Parametrized diamond principles were introduced in [42]. They have a similar relationship to Jensen's $\diamondsuit$ as cardinal invariants of the continuum have to CH. The parametrized $\diamondsuit$-principles are useful tools for topological and combinatorial constructions. It is well known that the guessing principle $\diamondsuit(2,=)$ holds if $2^\omega < 2^{\omega_1}$ [19], after forcing with a Suslin tree [42], and in models obtained by iterating definable forcing over a model of CH [42].

Theorem 8.5. $\diamondsuit(2,=)$ *implies the existence of an ω_1-generated FUF filter.*

Proof. We will first define a Borel function F into the set 2 as follows. The domain of F (using a suitable coding) is the set of all fifths $t = \langle X, \langle A_\beta \colon \beta < \alpha\rangle, \langle B_\beta \colon \beta < \alpha\rangle, A, B\rangle$ such that:

(1) $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$.
(2) α is an infinite countable ordinal.
(3) $\langle A_\beta \colon \beta < \alpha\rangle$, $\langle B_\beta \colon \beta < \alpha\rangle$ is a pair of mutually orthogonal, almost increasing sequences of infinite subsets of ω.
(4) A and B are disjoint sets such that A almost contains all A_β, $\beta < \alpha$, while B almost contains all B_β, $\beta < \alpha$.

Define
$$F(t) = \begin{cases} 0 & \text{if } \exists n \in \omega\, \forall a \in X (a \cap (A \cup n) \neq \emptyset); \\ 1 & \text{if } \forall n \in \omega\, \exists a_n \in X (a \cap (A \cup n) = \emptyset). \end{cases}$$
Note that $a \cap (A \cup n) = \emptyset$ is equivalent to $\min a_n \geqslant n$ and $a_n \subset B$.

Now suppose that $g\colon \omega_1 \to 2$ is a $\diamondsuit(2,=)$-sequence for F. By recursion, we will construct a pair of mutually orthogonal, almost increasing sequences of infinite

[1] A separable metric space X is called a γ*-set* if every ω-cover of X has a γ-subcover. An open cover $\mathcal{U} = \{U_n : n \in \omega\}$ is an ω*-cover* if for every finite $F \subseteq X$ there is a $n \in \omega$ such that $F \subseteq U_n$; $\mathcal{U} = \{U_n : n \in \omega\}$ is a γ*-cover* if for every $x \in X$ and for all but finitely many $n \in \omega$, $x \in U_n$.

subsets of ω (in fact, a *Hausdorff gap*) $\langle A_\alpha : \alpha < \omega_1 \rangle$, $\langle B_\alpha : \alpha < \omega_1 \rangle$ so that for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ there exists an $\alpha < \omega_1$ such that either

(a) there is an $n \in \omega$ such that $a \cap (A_\alpha \cup n) \neq \emptyset$ for every $a \in X$, or
(b) for every $n \in \omega$ there is an $a_n \in X$ such that $\min a_n \geqslant n$ and $a_n \subset B_\alpha$.

Let $\langle A_n : n < \omega \rangle$, $\langle B_n : n < \omega \rangle$ be any pair of mutually orthogonal, almost increasing sequences of infinite subsets of ω. If $\langle A_\beta : \beta < \alpha \rangle$, $\langle B_\beta : \beta < \alpha \rangle$ has been defined, consider a partition $\omega = A \cup B$ such that A almost contains all A_β, $\beta < \alpha$, while B almost contains all B_β, $\beta < \alpha$. If $g(\alpha) = 0$, then we let $A_\alpha = A$ and let B_α be a co-infinite subset of B still almost containing all B_β, $\beta < \alpha$. If it were not the case (*i.e.*, $g(\alpha) = 1$), then let $B_\alpha = B$ and let A_α be a co-infinite subset of A almost containing all A_β, $\beta < \alpha$. Let $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$. Clearly, if g guesses $\langle t_\beta : \beta < \omega_1 \rangle$ at α where $t_\beta = \langle X, \langle A_\xi : \xi < \beta \rangle, \langle B_\xi : \xi < \beta \rangle, A_\beta, B_\beta \rangle$, then either clause (a) or (b) is satisfied.

Let $\mathcal{F}$ be the dual filter to the ideal generated by $\langle A_\alpha : \alpha < \omega_1 \rangle$. We claim that $\mathcal{F}$ is a FUF filter. Indeed, let $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ so that every element of $\mathcal{F}$ contains an element of X. Now, there is an $\alpha < \omega_1$ such that either clause (a) or (b) is satisfied. By election of X, clause (a) fails. Then there is a sequence $\langle a_n : n \in \omega \rangle \subset X$ such that $\min a_n \geqslant n$ and $a_n \subset B_\alpha$ for every $n \in \omega$. Since B_α is almost disjoint with $\langle A_\alpha : \alpha < \omega_1 \rangle$, it follows that every element of $\mathcal{F}$ contains all but finitely many a_n's. □

One of the major obstacles in proving the main theorem of this paper was the apparent lack of a "reflection theorem"

Question 8.6. *Is it consistent with* ZFC *that there is no Fréchet group of weight* ω_1 *while there is one of weight* ω_2*?*

In particular,

Question 8.7. *Is it consistent with* ZFC *that there is no* γ*-set of size* ω_1 *while there is one of size* ω_2*?*

In more technical terms, the problem was to ensure that the generically added set A_{gen} remained $\mathcal{I}_0$-positive. Here the problem was solved using algebra. While proving the consistency of non-existence of *strongly separable MAD families* in [51] (answering a question of Shelah and Steprāns [55] concerning masas in the Calkin algebra) Raghavan faced an analogous problem. There it was solved using a clever Ramsey theoretic argument. Even though both problems seem related, it appears that neither his approach works here nor our aproach could be used in the context of strongly separable MAD families.

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