FRÉCHET-LIKE PROPERTIES AND ALMOST DISJOINT FAMILIES

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Abstract. We study the relationship between α_i properties and strong Fréchet-like properties in Ψ-spaces associated to almost disjoint families. In particular, we will prove that under some mild assumptions (e.g. $\mathfrak{c} \leq \aleph_2$) there is an almost disjoint family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is Fréchet, α_3 and not bisequential, answering a question of G. Gruenhage.

1. Introduction

Recall that a point x in a topological space X is a *Fréchet* point if whenever $x \in \overline{A} \subseteq X$, there is a sequence $\{x_n : n \in \omega\} \subseteq A$ such that $x_n \to x$. A space X is *Fréchet* if every point $x \in X$ is a Fréchet point.

Recall also ([1]) that a point $x \in X$ is an α_i*-point* $(i = 1, 2, 3, 4)$ if given a family $\{S_n : n \in \omega\}$ of sequences converging to x, there is a sequence $S \to x$ (we identify a convergent sequence with its range) such that

(α_1) $S \setminus S_n$ is finite for all $n \in \omega$,
(α_2) $S \cap S_n \neq \emptyset$ for all $n \in \omega$,
(α_3) $|S \cap S_n| = \omega$ for infinitely many $n \in \omega$,
(α_4) $S \cap S_n \neq \emptyset$ for infinitely many $n \in \omega$.

Notice that for an α_2-point, it is equivalent that $|S \cap S_n| = \omega$ for every $n \in \omega$. With this in mind, it should be obvious that the properties get progressively weaker. A space X is an α_i*-space* if every point $x \in X$ is an α_i-point. We say that a space X is α_i*-FU* if X is both Fréchet and α_i.

Definition 1.1. [1] A space X is *absolutely Fréchet* if every $x \in X$ is a Fréchet point in every (equiv. in some) compactification bX of X.

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Given a family $\mathcal{A} \subseteq \mathcal{P}(X)$ we will say that $x \in \overline{\mathcal{A}}$ if $x \in \overline{A}$ for every $A \in \mathcal{A}$. A filter base $\mathcal{G}$ converges to a point $x \in X$ if for every neighborhood U of x, there is a $G \in \mathcal{G}$ such that $G \subseteq U$. We then write $\mathcal{G} \to x$.

Definition 1.2. [5] X is *bisequential* at $x \in X$ if for every filter $\mathcal{F}$ in X such that $x \in \overline{\mathcal{F}}$ there is a decreasing sequence $\{G_n : n \in \omega\} \subseteq \mathcal{F}^+$ such that $G_n \to x$. A space X is bisequential if it is bisequential at every point.

These properties were introduced by A. Arhangel'skii [1] resp. E. Michael [5], in order to study when the product of Fréchet spaces is Fréchet.

All these concepts are related; every bisequential space is absolutely Fréchet and every absolutely Fréchet space is, of course, Fréchet [1]. Concerning the α_i-properties, every absolutely Fréchet space is α_4, and every bisequential space is α_3.

Most of the properties defined so far impose certain conditions such that the product is Fréchet. For instance, if X is bisequential and Y is α_4-FU, then $X \times Y$ is Fréchet [1].

A family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint (ad)* if $A \cap B$ is finite for every $A, B \in \mathcal{A}$. $\mathcal{A}$ is a *mad* family if it is an ad family and is maximal with respect to this property. The ideal $\mathcal{I}(\mathcal{A})$ generated by $\mathcal{A}$ is the set of all subsets of ω that can be covered by finitely many elements of $\mathcal{A}$ together with a finite subset of ω. Given an ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$, $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$, and given a family $\mathcal{W} \subseteq \mathcal{P}(\omega)$, $\mathcal{W}^\perp = \{X \subseteq \omega : \ \forall W \in \mathcal{W}\ |X \cap W| < \omega\}$.

Given an ad family $\mathcal{A}$ the *Mrówka-Isbell* space $\Psi(\mathcal{A})$ is the space $\omega \cup \mathcal{A}$ where ω is discrete and the basic open neighborhoods of an $A \in \mathcal{A}$ are of the form $\{A\} \cup A \setminus n$, i.e. the set $\{n \in \omega : n \in A\}$ converges to A for every $A \in \mathcal{A}$. This space is locally compact, $\psi(\mathcal{A})^* = \psi(\mathcal{A}) \cup \{\infty\}$ will denote its one-point compactification. Following [6], we will call the subspace $\omega \cup \{\infty\}$ of $\Psi(\mathcal{A})^*$ the *ad space generated by* $\mathcal{A}$. For more on ad families and Mrówka-Isbell spaces see [10, 9].

Notice that the study of α_i-spaces could be restricted to countable spaces since a space X is α_i if and only if every countable subset of X is. Indeed, in the class of Fréchet spaces, a meta-theorem in the area says that for every compact example illustrating a convergence property, there is one of the form $\psi(\mathcal{A})$, equivalently, there is an ad-space example. We will say that an ad family $\mathcal{A}$ *satisfies a topological property* $\mathcal{P}$ if the ad space associated $\omega \cup \{\infty\}$ does.

We will deal with G. Gruenhage's question of whether the properties of α_3-FU and bisequentiality are equivalent for ad spaces [6]. As a by-product we also solve some questions of Nyikos [16], and the construction gives new consistent examples of absolutely Fréchet spaces with strong α_i-properties which are not bisequential.

We will say that an ad family $\mathcal{A}$ is *hereditarily* α_3 if for every $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}$ is α_3. Since $\mathcal{B} \subseteq \mathcal{A}$ is Fréchet for every Fréchet $\mathcal{A}$, hereditarily α_3-FU is the same as Fréchet and hereditarily α_3.

Question. [6] *Is every α_3-FU (hereditarily α_3-FU) ad family $\mathcal{A}$ bisequential?*

Recall that if $\mathcal{A}$ is bisequential then it is hereditarily α_3-FU [6] and of course every hereditarily α_3-FU is α_3-FU.

$$\begin{tikzcd}
 & & \text{First countable} \arrow[d] \\
 & & \alpha_1 \arrow[d] \\
 & & \alpha_2 \arrow[d] \\
\text{bisequential} \arrow[r] \arrow[d] & \text{her. } \alpha_3 \arrow[r] & \alpha_3 \arrow[d] \\
\text{absolutely Fréchet} \arrow[rr] \arrow[d] & & \alpha_4 \\
\text{Fréchet} & &
\end{tikzcd}$$

FIGURE 1. Fréchet-like properties.

The diagram shows ZFC implication between these properties (of course, hereditarily α_3 only makes sense for almost disjoint families).

Our set theoretic notation is standard and follows e.g. [12]. The definitions and further information concerning cardinal invariants of the continuum can be found in [3] and [4].

2. AD spaces and bisequentiality

A large class of ad families is bisequential, namely, those that are $\mathbb{R}$-embeddable. Recall that an ad family $\mathcal{A}$ is $\mathbb{R}$-*embeddable* [8, 7] if there is a one-to-one function $f : \omega \to \mathbb{Q}$ which extends to a continuous one-to-one $\widehat{f} : \psi(\mathcal{A}) \to \mathbb{R}$. However, there are ZFC examples of bisequential ad families $\mathcal{A}$ which are not $\mathbb{R}$-embeddable.

On the other hand, under $\mathfrak{b} = \mathfrak{c}$, there is an ad family which is not even α_3 ([16]). We will prove that under the same assumption, the three concepts of $\mathcal{A}$ being bisequential, hereditarily α_3-FU and α_3-FU are different. We shall give combinatorial characterizations of these properties for ad families first. Since ω is a discrete subspace of the ad space of $\mathcal{A}$, the only point of interest is ∞. A sequence $X \subseteq \omega$ converges to ∞ if and only if $X \in \mathcal{A}^{\perp}$. Also, $\infty \in \overline{X}$ if and only if $X \in \mathcal{I}(\mathcal{A})^{+}$. Then, an ad family $\mathcal{A}$ is Fréchet iff it is *nowhere maximal* [19], i.e., for every $X \in \mathcal{I}(\mathcal{A})^{+}$, there exists $Y \in \mathcal{A}^{\perp}$ such that $|Y \cap X| = \omega$. $\mathcal{A}$ is α_3 iff for every sequence $\{X_n : n \in \omega\} \subseteq \mathcal{A}^{\perp}$ there is an $X \in \mathcal{A}^{\perp}$ which intersects infinitely many X_n's in an infinite set. We will need the following fact:

Theorem 2.1. [3] *The cardinal* $\mathsf{non}(\mathcal{M})$ *is the size of the smallest family* $\mathcal{F} \subseteq \omega^{\omega}$ *such that*

$$\forall g \in \omega^{\omega}\ \exists f \in \mathcal{F}\ \exists^{\infty} n \in \omega\ (f(n) = g(n)).$$

$\square$

So for every family $\mathcal{F} \subseteq \omega^{\omega}$ of size less than $\mathsf{non}(\mathcal{M})$, there is a function $g \in \omega^{\omega}$ which is eventually different from $\mathcal{F}$, i.e., for every $f \in \mathcal{F}$ and all but finitely many $n \in \omega$, $g(n) \neq f(n)$. Moreover, a slight modification to the proof shows that for every such family $\mathcal{F}$, the set of functions which are eventually different from $\mathcal{F}$ is not meager. Therefore we get the following corollary:

Corollary 2.2. *For every* $\mathcal{F} \subseteq \omega^{\omega}$ *of size less than* $\mathsf{non}(\mathcal{M})$ *and every dense* G_{δ} *subset* $G \subseteq \omega^{\omega}$, *there exists a function* $g \in G$ *which is eventually different from* $\mathcal{F}$. $\square$

Notation 2.3. A *column* in $\omega \times \omega$ will be a set of the form $\{n\} \times \omega$ for some $n \in \omega$. For an indexed set $\mathcal{H} = \{H_{\alpha} : \alpha < \kappa\}$ and $\eta < \kappa$ we denote the *restriction of* $\mathcal{H}$ *to* η by $\mathcal{H}_{\eta} = \{H_{\alpha} : \alpha < \eta\}$.

Theorem 2.4. ($\mathsf{non}(\mathcal{M}) = \mathfrak{c}$) *There is an α_3-FU ad family $\mathcal{A}$ such that $\mathcal{A}$ is not hereditarily α_3-FU.*

Proof. We will build recursively $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ as a family of subsets of $\omega \times \omega$. Moreover, A_α will be the graph of a function for every $\omega \leq \alpha < \mathfrak{c}$. For every function $f \in \omega^\omega$ we will use f both for the function and its graph as a subset of $\omega \times \omega$. Enumerate $([\omega \times \omega]^\omega)^\omega = \{D_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ and $[\omega \times \omega]^\omega = \{X_\alpha : \omega \leq \alpha < \mathfrak{c}\}$.

At step $\alpha \geq \omega$ we will define A_α together with two functions $Y_\alpha, Z_\alpha \in \omega^\omega$ so that $Y_\alpha, Z_\alpha \in \mathcal{A}_\alpha^\perp$ and $A_\alpha \in \mathcal{A}_\alpha^\perp \cap \mathcal{Y}_\alpha^\perp \cap \mathcal{Z}_\alpha^\perp$. For $n \in \omega$ define $A_n = \{n\} \times \omega$. Assume that we have defined A_β for $\beta < \alpha$ and Y_β, Z_β for $\omega \leq \beta < \alpha$.

If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$, the set

$$G = \{f \in \omega^\omega : |f \cap X_\alpha| = \omega\}$$

is G_δ and dense in ω^ω. Hence we can find $Y_\alpha \in \omega^\omega$ such that $Y_\alpha \in \mathcal{A}_\alpha^\perp$ and $|Y_\alpha \cap X_\alpha| = \omega$. Otherwise, define $Y_\alpha \in \mathcal{A}_\alpha^\perp$ arbitrarily.

If $D_\alpha \subseteq \mathcal{A}_\alpha^\perp$, then $D_\alpha(n)$ intersects infinitely many columns for every $n \in \omega$. Thus the set

$$G = \{f \in \omega^\omega : \forall n \in \omega \ (|f \cap D_\alpha(n)| = \omega)\}$$

is a dense G_δ subset of ω^ω. Applying Corollary 2.2, we can find a function $Z_\alpha \in \omega^\omega$ such that $Z_\alpha \in \mathcal{A}_\alpha^\perp$ and $|Z_\alpha \cap D_\alpha(n)| = \omega$ for all $n \in \omega$. If $D_\alpha \nsubseteq \mathcal{A}_\alpha^\perp$ define $Z_\alpha \in \omega^\omega$ such that $Z_\alpha \in \mathcal{A}_\alpha^\perp$ arbitrarily.

Finally, if there are infinitely many $n \in \omega$ such that $|X_\alpha \cap A_n| = \omega$, in particular $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and we already know that the set

$$G = \{f \in \omega^\omega : |f \cap X_\alpha| = \omega\}$$

is G_δ and dense. Then we can find $A_\alpha \in (\mathcal{A}_\alpha \cap \mathcal{Y}_\alpha \cap \mathcal{Z}_\alpha)^\perp$ such that $|A_\alpha \cap X_\alpha| = \omega$. Otherwise, chose $A_\alpha \in (\mathcal{A}_\alpha \cap \mathcal{Y}_\alpha \cap \mathcal{Z}_\alpha)^\perp$ arbitrarily.

From the definition it is clear that $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ is almost disjoint. Given $X \in \mathcal{I}(\mathcal{A})^+$ there exists $\alpha < \mathfrak{c}$ such that $X = X_\alpha$ and $|Y_\alpha \cap X| = \omega$ since $\mathcal{I}(\mathcal{A})^+ \subseteq \mathcal{I}(\mathcal{A}_\alpha)^+$. Moreover, since $A_\beta \in \mathcal{Y}_\beta^\perp$ for every $\beta > \alpha$, it follows that $Y_\alpha \in \mathcal{A}^\perp$. Hence $\mathcal{A}$ is Fréchet. The same idea shows that $\mathcal{A}$ is α_3 (even α_2) using Z_α as a witness for the sequence of convergent sequences $\mathcal{D}_\alpha$.

Now define $\mathcal{B} = \mathcal{A} \setminus \mathcal{A}_\omega$. Then $A_n \in \mathcal{B}^\perp$ for every $n \in \omega$ but for every possible witness $X \subseteq \omega \times \omega$ for the property α_3, i.e., for every X such that $|X \cap A_n| = \omega$ for infinitely many $n \in \omega$, there exists $\omega \leq \alpha < \mathfrak{c}$ such that $X = X_\alpha$ and then $A_\alpha \in \mathcal{B}$ satisfies that $|A_\alpha \cap X| = \omega$ and $X \notin \mathcal{B}^\perp$. Therefore $\mathcal{A}$ is not hereditarily α_3. $\square$

Let $\mathcal{A}$ be an α_3 ad family, assume $\mathcal{B} \subseteq \mathcal{A}$ is not α_3 and take $\{D_n : n \in \omega\} \subseteq \mathcal{B}^\perp$ witnessing this fact. We can assume that $D_n \notin \mathcal{A}^\perp$ and then there is $A(n) \in \mathcal{A} \setminus \mathcal{B}$ such that $D_n \subseteq A(n)$ for every $n \in \omega$ by shrinking D_n if necessary. Thus $\mathcal{A} \setminus \{A_n : n \in \omega\} \supseteq \mathcal{B}$ is not α_3. Hence $\mathcal{A}$ fails to be hereditarily α_3 iff there is a countable subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{B}$ is not α_3. Recall that $\mathcal{F} := (\mathbf{fin} \times \mathbf{fin})^*$ is the dual filter of the ideal

$$\mathbf{fin} \times \mathbf{fin} = \{A \subseteq \omega \times \omega : \exists n \in \omega \ \forall m > n \ (|\{j \in \omega : (m, j) \in A\}| < \omega)\}.$$

For every $X \subseteq \omega \times \omega$ and every family $\mathcal{W} \subseteq [\omega \times \omega]^\omega$ let

$$T(X, \mathcal{W}) = \{W \in \mathcal{W} : |X \cap W| = \omega\},$$

and define $\mathcal{T}(\mathcal{W}) = \{X \subseteq \omega \times \omega : |T(X, \mathcal{W})| \leq \omega\}$. The use of the set $\omega \times \omega$ is not essential in the previous definition and we will use the notation $T(\cdot, \cdot)$ and $\mathcal{T}(\cdot)$ for other countable sets which will be understood from the context.

Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ iff $\{n \in \omega : f(n) > g(n)\}$ is finite and $\mathfrak{b}$ is the least cardinality of a $\leq^*$-unbounded family in ω^ω.

Theorem 2.5. ($\mathfrak{b} = \mathfrak{c}$) *There is a hereditarily α_3-FU ad family $\mathcal{A}$ such that $\mathcal{A}$ is not bisequential.*

Proof. Similar to the proof of Theorem 2.4, enumerate

$$([\omega \times \omega]^\omega)^\omega = \{D_\alpha : \omega \leq \alpha \in \mathfrak{c}\},$$
$$[\omega \times \omega]^\omega = \{X_\alpha : \omega \leq \alpha \in \mathfrak{c}\}$$

and

$$\{\mathcal{G} \in (\mathcal{F}^+)^\omega : \mathcal{G} \text{ is decreasing}\} = \{\mathcal{G}_\alpha : \omega \leq \alpha \in \mathfrak{c}\}.$$

Fix a mad family $\mathcal{E} = \{e_\alpha : \alpha < \mathfrak{c}\}$. Again, we will construct our family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega \times \omega]^\omega$ recursively and for $\alpha \geq \omega$, A_α will be a partial function and will be accompanied by a function $Y_\alpha \in \omega^\omega$ and a set $Z_\alpha \subseteq \omega \times \omega$. In addition, these elements will satisfy that $A_\alpha \in (\mathcal{A}_\alpha \cap \mathcal{Y}_\alpha \cap \mathcal{Z}_\alpha)^\perp$ and $Y_\alpha, Z_\alpha \in \mathcal{A}_\alpha^\perp$. For every $n \in \omega$ define $A_n = \{n\} \times \omega$. Assume $\omega \leq \alpha < \mathfrak{c}$ and we have defined $\mathcal{A}_\alpha, \mathcal{Y}_\alpha$ and $\mathcal{Z}_\alpha$.

If $X_\alpha \notin \mathcal{I}(\mathcal{A}_\alpha)^+$ and since $\mathsf{non}(\mathcal{M}) \geq \mathfrak{b} = \mathfrak{c}$ apply theorem 2.1 and find $Y_\alpha \in \omega^\omega \cap \mathcal{A}_\alpha^\perp$ arbitrarily. Otherwise, if $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$, then the set $L = \{f \in \omega^\omega : |f \cap X| = \omega\}$ is G_δ and dense in ω^ω and applying corollary 2.2 there is a function $Y_\alpha \in L \cap \mathcal{A}_\alpha^\perp$.

If $\{D_\alpha(n) : n \in \omega\} \nsubseteq \mathcal{T}(\mathcal{A}_\alpha)$ define $Z_\alpha \in (e_\alpha \times \omega) \cap \mathcal{A}_\alpha^\perp$ arbitrarily. This is possible since $\mathfrak{a} \geq \mathfrak{b} = \mathfrak{c}$ and then $\mathcal{A}_\alpha$ is not maximal when restricted to $e_\alpha \times \omega$ (notice that this restriction is an infinite ad family since the elements of $\mathcal{A}_\alpha$ are partial functions). Otherwise we have two cases:

- *Case 1:* If $D_\alpha(i) \in \mathcal{A}_\omega^\perp$ for infinitely many $i \in \omega$, the set
$$M = \{f \in \omega^\omega : \exists^\infty i \in \omega(|f \cap D_\alpha(i)| = \omega)\}$$
is G_δ and dense in ω^ω and applying corollary 2.2 we can find a function $Z_\alpha \in M \cap \mathcal{A}_\alpha^\perp$.
- *Case 2:* If $D_\alpha(i) \notin \mathcal{A}_\omega^\perp$ for all but finitely many $i \in \omega$, we can find for all but finitely many $i \in \omega$ a $k(i)$ such that $|D_\alpha(i) \cap A_{k(i)}| = \omega$. We shall consider two subcases.
  - *Subcase 2.1:* If there exists $n \in \omega$ such that $\exists^\infty i \in \omega\ (k(i) = n)$ define $Z_\alpha = A_n$.
  - *Subcase 2.2* For every $n \in \omega$ there are only finitely many $i \in \omega$ such that $k(i) = n$. In this case let $\{m_i : i \in \omega\}$ be increasing and such that $\{k(m_i) : i \in \omega\}$ is also increasing. Define for every $\omega \leq \beta < \alpha$ a function $f_\beta \in \omega^\omega$ such that $f_\beta(n) = \max(A_\beta \cap A_n)$. Since $\mathfrak{b} = \mathfrak{c}$ there is a function $f \in \omega^\omega$ such that $f \geq^* f_\beta$ for all $\omega \leq \beta < \alpha$. Let $\gamma(\alpha)$ such that $\Gamma(\alpha) := e_{\gamma(\alpha)} \cap \{k(m_i) : i \in \omega\}$ is infinite. Thus define $Z_\alpha = \{(p, q) : p \in \Gamma(\alpha) \wedge q > f(p)\}$.

Finally suppose that $\mathcal{G}_\alpha \in (\mathcal{F}^+)^\omega$ is a decreasing sequence. Notice that $G \in \mathcal{F}^+$ if and only if G intersects infinitely many A_n's in an infinite set. Then the set $R = \{f \in \omega^\omega : \forall n \in \omega(f \cap \mathcal{G}_\alpha(n) \neq \emptyset)$ is G_δ and dense in ω^ω. Hence there is a function $f \in R \cap (\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1} \cup \mathcal{Z}_{\alpha+1}^*)^\perp$ where $\mathcal{Z}_\alpha^* = \{Z \in \mathcal{Z}_\alpha : Z \in \omega^\omega\}$. In other words, $\mathcal{Z}_\alpha^*$ is the subset of the elements of $\mathcal{Z}_\alpha$ defined by case 1. If $\mathcal{G}_\alpha$ is not decreasing define $f \in (\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1} \cup \mathcal{Z}_{\alpha+1}^*)^\perp$ arbitrarily. Now let $B = \{k_n\}$ be a set with its increasing enumeration such that $f(k_n) \in G_\alpha(n))$ for every $n \in \omega$ and take $d_\alpha \in [B]^\omega \cap \{e_{\gamma(\beta)} : \beta \leq \alpha\}^\perp$, assuming $e_{\gamma(\beta)} = \emptyset$ if it has not been defined. Set $A_\alpha := f \restriction d_\alpha$.

Then the elements of the set $\mathcal{Y} = \{Y_\alpha : \omega \leq \alpha < \mathfrak{c}\}$ witness that $\mathcal{A}$ is nowhere MAD like in the proof of theorem 2.4.

To see that $\mathcal{A}$ is hereditarily α_3 take $\mathcal{B} \subseteq \mathcal{A}$ and assume that $\mathcal{A} \setminus \mathcal{B}$ is countable. Then for every sequence $\mathcal{D} = \{D_n : n \in \omega\} \subseteq \mathcal{B}^\perp \subseteq \mathcal{T}(\mathcal{A})$

there exists $\alpha < \mathfrak{c}$ such that $\mathcal{D} = \mathcal{D}_\alpha$ and since $\mathcal{T}(\mathcal{A}) \subseteq \mathcal{T}(\mathcal{A}_\alpha)$ it follows from the definition that $|Z_\alpha \cap \mathcal{D}_\alpha(i)| = \omega$ for infinitely many $i \in \omega$. Also $Z_\alpha \in \mathcal{A}_\alpha^\perp$ and for $\beta \geq \alpha$, if Z_α was defined by case 1, $Z_\alpha \in \mathcal{Z}^*_{\alpha+1}$, if it was defined by case 2.1, Z_α is a column and if it was defined in case 2.2 $|dom(A_\beta) \cap e_{\gamma(\alpha)}| < \omega$, in any case, $|A_\beta \cap Z_\alpha| < \omega$ and then $Z_\alpha \in \mathcal{A}^\perp$.

On the other hand, since the elements of $\mathcal{A}$ are either functions or columns of the form $\{n\} \times \omega$, it follows that $\infty \in \overline{\mathcal{F}}$. However, if $\mathcal{G} = \{G_n : n \in \omega\} \subseteq \mathcal{F}^+$ is a decreasing sequence, there exists $\alpha < \mathfrak{c}$ such that $\mathcal{G} = \mathcal{G}_\alpha$ and A_α witnesses that $\mathcal{G}$ does not converge to ∞. □

Corollary 2.6. ($\mathfrak{b} = \mathfrak{c}$) *The three concepts of $\mathcal{A}$ being α_3, hereditarily α_3 and bisequential are each different from the others.*

We will now isolate the combinatorial properties of the almost disjoint family constructed in Theorem 2.4 for future constructions.

Remark 2.7. An almost disjoint family $\mathcal{A} = \langle A_\alpha : \alpha < \kappa \rangle \subseteq [\omega]^\omega$ is α_3-FU and non-hereditarily α_3 iff

(1) $\mathcal{A}$ is nowhere MAD.
(2) $\forall \langle D_n : n \in \omega \rangle \subseteq \mathcal{A}^\perp \exists Y \in \mathcal{A}^\perp (|\{n \in \omega : |Y \cap D_n| = \omega\}| = \omega)$.
(3) $\forall Y \in [\omega]^\omega \, ((|\{n \in \omega : |X \cap A_n| = \omega\}| = \omega) \Rightarrow (\exists A \in \mathcal{A} |A \cap X| = \omega))$

modulo a permutation of $\mathcal{A}$ in order to satisfy property 3.

3. The splitting and unbounding numbers

A mad family is said to be *completely separable* if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is an $A \in \mathcal{A}$ such that $A \subseteq X$. It was shown by Balcar and Simon (see [2]) that completely separable mad families exists under one of the following axioms: $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} \leq \mathfrak{a}$ and $\mathfrak{s} = \omega_1$. A more general theorem was proved by Shelah, who proved that completely separable mad families exists if either $\mathfrak{s} < \mathfrak{a}$ or if $\mathfrak{s} = \mathfrak{a}$ and a certain PCF-hypothesis holds or if $\mathfrak{s} > \mathfrak{a}$ and a stronger PCF-hypothesis holds. The method of Shelah is a powerful tool to construct almost disjoint families and was improved by Mildenberg, Raghavan and Steprāns in [14], eliminating the PCF-hypothesis in the case $\mathfrak{s} = \mathfrak{a}$.

Theorem 3.1. ([18],[14]) *Assume $\mathfrak{s} \leq \mathfrak{a}$. Then there is a completely separable mad family.* □

This improvement was the result of the introduction of a new cardinal invariant $\mathfrak{s}_{\omega,\omega}$ which turned out to be equal to $\mathfrak{s}$. Recall that a family

$\mathcal{S} \subseteq [\omega]^\omega$ is *splitting* if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that $|X \cap S| = |X \setminus S| = \omega$ and we will say that it is (ω, ω)-*splitting* if for every sequence $\langle X_n : n \in \omega \rangle \subseteq [\omega]^\omega$, there is $S \in \mathcal{S}$ such that the sets $\{n \in \omega : |X_n \cap S| = \omega\}$ and $\{n \in \omega : |X_n \setminus S| = \omega\}$ are both infinite. Thus, $\mathfrak{s}$ is the least size of a splitting family and $\mathfrak{s}_{\omega,\omega}$ is the least size of a (ω, ω)-splitting family. In [14], it is proved that $\mathfrak{s} = \mathfrak{s}_{\omega,\omega}$. The key feature of an (ω, ω)-splitting family $\mathcal{S}$ is that if $X \in \mathcal{I}(\mathcal{A})^+$ where $\mathcal{A}$ is an ad family, then there is $S \in \mathcal{S}$ such that $S \cap X \in \mathcal{I}(\mathcal{A})^+$ and $X \setminus S \in \mathcal{I}(\mathcal{A})^+$. The cardinal $\mathfrak{s}_{\omega,\omega}$ was introduced in [17] in order to construct a weakly tight mad family using the method of Shelah just mentioned. A mad family $\mathcal{A}$ is *tight* if for every family $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for every $n \in \omega$. It is shown in [11] that the existence of a tight mad family is equivalent to the existence of a Cohen-indestructible mad family and the notion of weakly tight mad family is introduced: A mad family $\mathcal{A}$ is *weakly tight* if for every collection $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for infinitely many $n \in \omega$.

It is an open problem whether weakly tight mad families exist in ZFC. Raghavan and Steprāns showed that they exist assuming $\mathfrak{s} \leq \mathfrak{b}$:

Theorem 3.2. [17] ($\mathfrak{s} \leq \mathfrak{b}$) *There is a weakly tight mad family.* □

The proof of their theorem actually shows that under $\mathfrak{s} \leq \mathfrak{b}$, there is a weakly tight mad family $\mathcal{A}$ such that for every countable collection $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there are $\mathfrak{c}$-many $A \in \mathcal{A}$ such that $|A \cap X_n| = \omega$ for infinitely many $n \in \omega$. We will take advantage of this fact in the next theorem.

Theorem 3.3. ($\mathfrak{s} \leq \mathfrak{b}$) *There is an α_3-FU ad family $\mathcal{A}$ which is not hereditarily α_3. In particular it is not bisequential.*

Proof. Let $\mathcal{E} = \{e_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ be a weakly tight mad family such that for every $\{X_n : n \in \omega\} \subseteq \mathcal{I}(\mathcal{E})^+$ there are $\mathfrak{c}$-many $e \in \mathcal{E}$ such that $|e \cap X_n| = \omega$ for infinitely many $n \in \omega$. We can assume that $\{e_n : n \in \omega\}$ forms a partition of ω. Enumerate $[\omega]^\omega = \{X_\alpha : \alpha < \mathfrak{c}\}$ and $([\omega]^\omega)^\omega = \{D_\alpha : \alpha < \mathfrak{c}\}$. Define recursively $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathcal{E}$ and $\{Y_{\alpha,i} : \alpha < \mathfrak{c} \wedge i \in 2\} \subseteq \mathcal{E}$ such that $A_\beta \neq A_\alpha \neq Y_{\eta,i}$ for all $\alpha, \beta \in \mathfrak{c}$ with $\alpha \neq \beta$, $\omega \leq \eta < \mathfrak{c}$ and $i \in 2$.

For $n \in \omega$ we start by defining $A_n = e_n$. Let $\omega \leq \alpha < \mathfrak{c}$. If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ there exists $Y_{\alpha,0} \in \mathcal{E} \setminus (\mathcal{A}_\alpha \cup \mathcal{Y}_\alpha)$ where $\mathcal{Y}_\alpha = \{Y_{\beta,i} : \beta < \alpha \wedge i \in 2\}$ such that $|Y_{\alpha,0} \cap X_\alpha| = \omega$. Similarly if $\{D_\alpha(n) : n \in \omega\} \subseteq \mathcal{A}^\perp \subseteq \mathcal{A}_\omega^\perp \subseteq \mathcal{I}(\mathcal{A}_\alpha)^+$, there exists $Y_{\alpha,1} \in \mathcal{E} \setminus (\mathcal{A}_\alpha \cup \mathcal{Y}_\alpha)$ such that $|Y_{\alpha,1} \cap D_\alpha(n)| = \omega$ for infinitely many $n \in \omega$. Finally, if $|X_\alpha \cap A_n| = \omega$ for infinitely many $n \in \omega$, then $X \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and there is $A_\alpha \in \mathcal{E} \setminus (\mathcal{A}_\alpha \cup \mathcal{Y}_{\alpha+1})$ such that $|A_\alpha \cap X_\alpha| = \omega$. Therefore, $\mathcal{A}$ is α_3-FU but not hereditarily α_3. □

Combining Theorems 2.4 and 3.3 and since $\mathfrak{s} \leq \mathsf{non}(\mathcal{M})$ we get the following corollary.

Corollary 3.4. ($\mathfrak{c} \leq \aleph_2$) *There is an α_3-FU ad family which is not bisequential.* □

4. Weak $\diamondsuit$ principles

The almost disjoint families constructed so far have all size $\mathfrak{c}$. We will use the parametrized diamond $\diamondsuit(\mathfrak{b})$ (see [15]) to construct counterexamples to Gruenhage's questions of size ω_1. Recall that this principle is defined as follows:

$$\diamondsuit(\mathfrak{b}) \equiv \forall F : 2^{<\omega_1} \to \omega^\omega \text{ Borel } \exists g : \omega_1 \to \omega^\omega \ \forall f \in 2^{\omega_1}$$
$$\{\alpha \in \omega_1 : g(\alpha) \not\leq^* F(f \restriction \alpha))\} \text{ is stationary.}$$

Theorem 4.1. $\diamondsuit(\mathfrak{b})$ *implies the existence of an α_3-FU non-hereditarily α_3 ad family.*

Proof. Let $\{A_n : n \in \omega\}$ be a partition of ω into infinite sets. For every infinite ordinal $\delta < \omega_1$ fix a bijection $e_\delta : \omega \to \delta$. We will define a Borel function F into the set ω^ω and such that its domain is the set of tuples $(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, X)$ where:

(1) δ is an infinite countable ordinal.
(2) $\mathcal{A}_{\delta+1} = \langle A_\alpha : \alpha \leq \delta \rangle$ is an almost disjoint family.
(3) $\mathcal{Y}_\delta = \langle Y_\alpha : \omega \leq \alpha < \delta \rangle \subseteq \mathcal{A}_\delta^\perp$.
(4) $X \in ([\omega]^\omega \times 2) \cup ([\omega]^\omega)^\omega$.
(5) If $X \in ([\omega]^\omega)^\omega$ then $X(n) \in \mathcal{A}_{\delta+1}^\perp$ for every $n \in \omega$.
(6) If $X = (x, i) \in ([\omega]^\omega \times 2)$ then $x \in \mathcal{I}(\mathcal{A}_{\delta+1})^+$. Moreover, if $i = 0$, then $|x \cap A_n| = \omega$ for infinitely many $n \in \omega$.

If $X = (x, 0)$, there are infinitely many $n \in \omega$ such that $x \cap A_{e_\delta(n)}$ is infinite. Let $\{n_k : k \in \omega\}$ be the increasing enumeration of this set and define

$$F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x, 0))(k) = \min\left(x \cap A_{e_\delta(n_k)} \setminus \bigcup_{i < n_k} \left[A_{e_\delta(i)} \cup Y_{e_\delta(i)}\right] \right).$$

Analogously, if $X = (x,1)$, there are infinitely many $n \in \omega$ such that the set $x \cap A_{e_{\delta+1}(n)}$ in nonempty. Redefine $\{n_k : n \in \omega\}$ as the increasing enumeration of this set and define

$$F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x,1))(k) = \min\left(x \cap A_{e_{\delta+1}(n_k)} \setminus \bigcup_{i<n_k} A_{e_{\delta+1}(i)})\right).$$

On the other hand if $X \in ([\omega]^\omega)^\omega$, $X(n) \in \mathcal{I}(\mathcal{A}_{\delta+1})^+$ for every $n \in \omega$. Define $f_n = F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (X(n),1))$. Take $g \in \omega^\omega$ such that $f_n \leq^* g$ for all $n \in \omega$ and define $F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, X) = g$.

Now suppose that $g : \omega_1 \to \omega^\omega$ is a $\diamondsuit(\mathfrak{b})$-sequence for F and assume that the entries of g form a $<^*$-strictly increasing sequence of increasing functions by making them larger if necessary.

We now construct our almost disjoint family $\mathcal{A} = \langle A_\alpha : \alpha < \omega_1\rangle$ together with a sequence $\mathcal{Y} = \langle Y_\alpha : \omega \leq \alpha < \omega_1\rangle \subseteq \mathcal{A}^\perp$. If $\langle A_\alpha : \alpha < \delta\rangle$ and $\langle Y_\alpha : \omega \leq \alpha < \delta\rangle$ have been defined for an infinite countable ordinal δ, set

$$A_\delta = \bigcup_{n\in\omega}\left(g(\delta)(n) \cap A_{e_\delta(n)} \setminus \bigcup_{i<n}\left[A_{e_\delta(i)} \cup Y_{e_\delta(i)}\right]\right)$$

and

$$Y_\delta = \bigcup_{n\in\omega}\left(g(\delta)(n) \cap A_{e_{\delta+1}(n)} \setminus \bigcup_{i<n} A_{e_{\delta+1}(i)}\right).$$

It is clear that $Y_\delta \in \mathcal{A}_{\delta+1}^\perp$ and that $A_\delta \in (\mathcal{A}_\delta \cup \mathcal{Y}_\delta)^\perp$. Then $\mathcal{A}$ is almost disjoint and $\mathcal{Y} \subseteq \mathcal{A}^\perp$. Let us prove that $\mathcal{A}$ satisfies the properties listed in remark 2.7.

Let us prove first that $\mathcal{A}$ is nowhere mad. Given $x \in \mathcal{I}(\mathcal{A})^+$ we have that $(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x,1))$ is in the domain of F for every infinite ordinal $\delta < \omega_1$. Then suppose that g guesses $F(\mathcal{A}, \mathcal{Y}, (x,1))$ at δ. Let $l \in \omega$, we shall find $m > l$ such that $m \in Y_\alpha \cap x$. Recall that in this case $\{n_k : k \in \omega\}$ is the increasing enumeration of the numbers n's such that $A_{e_{\delta+1}(n)}$ has nonempty intersection with x. Find $k \in \omega$ such that $[0,l] \subseteq \bigcup_{i<n_k} A_{e_{\delta+1}(i)}$ and $g(\delta)(k) > F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x,1))$. This is possible since $\{A_{e_{\delta+1}(n)} \setminus \bigcup_{i<n} A_{e_\delta(i)} : n \in \omega\}$ forms a partition of ω and $g \not\leq^* F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x,1))$. Then $m = F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x,1)) > l$ and since $m < g(\delta)(k) \leq g(\delta)(n_k)$ it follows that $m \in Y_\alpha \cap x$.

A similar argument shows that if x is like in point 3 of remark 2.7, then $(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, (x,0))$ is in the domain of F for every infinite ordinal $\delta < \omega_1$. Hence, if g guesses $F(\mathcal{A}, \mathcal{Y}, (x,0))$ at δ we have that $|A_\delta \cap x| = \omega$.

Finally suppose that $X \in ([\omega]^\omega)^\omega$ and let $\delta \in \omega_1$ such that g guesses $F(\mathcal{A}, \mathcal{Y}, X)$ at δ. Since $g(\delta) \not\leq^* F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, X)$ and $F(\mathcal{A}_{\delta+1}, \mathcal{Y}_\delta, X) \geq^* f_n$ for the associated functions f_n, it follows that $g \not\leq^* f_n$ for every $n \in \omega$. Using the same reasoning as below with f_n instead of F, we can prove that $Y_\delta \cap X(n)$ is infinite for every $n \in \omega$. Therefore we have proved not only that $\mathcal{A}$ is α_3 but α_2. $\square$

It is also possible to prove the equivalents of Theorem 2.5 and corollary 2.6 under $\diamondsuit(\mathfrak{b})$. Since no new ideas are used, we will not give a proof of these results. The interested reader can prove this by combining the ideas of the previous theorem and Theorem 2.5.

5. Further results

We have used different axioms for building the spaces/ad families studied so far, namely, $\mathsf{non}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{s} \leq \mathfrak{b}$, $\diamondsuit(\mathfrak{b})$ and in consequence, the results follow from $\mathfrak{c} \leq \aleph_2$ since $\mathfrak{s}, \mathfrak{b} \leq \mathsf{non}(\mathcal{M})$. For the remainder of this section let Φ be any of these axioms.

In [16], Nyikos built under $\mathfrak{b} = \mathfrak{c}$, an ad family $\mathcal{A} \subseteq [\omega \times \omega]^\omega$ consisting of graphs of functions which fails to be α_3 and asked whether it is possible to construct an α_3 non-bisequential ad family of this kind under the same assumption. Theorem 2.5 provides a positive answer to this question. He also asked the following:

- Is every compact α_3-FU space $\aleph_0$-bisequential?
- Is there a ZFC example of a compact space X that has Fréchet product with every regular countably compact Fréchet space, but is not $\aleph_0$-bisequential?

In [1], Arhangel'skii proved that a separable space is $\aleph_0$-bisequential iff it is bisequential. Since $\Psi(\mathcal{A})^*$ is compact and separable, we then get the following:

Corollary 5.1. (Φ) *There exists a compact α_3-FU space which is not $\aleph_0$-bisequential.*

As Nyikos pointed out, a (consistent) negative answer to the first problem gives an (consistent) affirmative one to the second question in view of the next theorem.

Theorem 5.2. [1] *If X is an α_3-FU space, then $X \times Y$ is Fréchet for every regular countably compact Fréchet space.*

Given a Fréchet ad family $\mathcal{A}$, the space $\Psi(\mathcal{A})^*$ is compact and Fréchet since every infinite subset of $\mathcal{A}$ converges to ∞. Then an ad family is Fréchet iff it is absolutely Fréchet. In [1], Arhangel'skii asked the following:

($*$) Is there a (countable) α_1-Fréchet space which is not bisequential?

Malyhin has constructed a consistent example for this question under $2^{\aleph_0} < 2^{\aleph_1}$ [13]. Here we will construct a countable absolutely Fréchet example under CH by strengthening our previous results in order to get α_1.

Theorem 5.3. (CH) *There exists a countable α_1 and absolutely Fréchet space which is not bisequential.*

Proof. We will construct an α_1 absolutely Fréchet ad family $\mathcal{A}$ which is not hereditarily α_3. For this purpose we will recursively define $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ and $\mathcal{B} = \{B_{\alpha,i} : \omega \leq \alpha < \mathfrak{c}\}$ such that $\mathcal{A}$ is almost disjoint and $\mathcal{B} \subseteq \mathcal{A}^\perp$. Enumerate $[\omega]^\omega = \{X_\alpha : \alpha < \omega_1\}$ and $([\omega]^\omega)^\omega = \{Y_\alpha : \alpha < \omega_1\}$. For every $\omega \leq \delta < \omega_1$ let $e_\delta : \omega \to \delta$ be a bijection. Suppose we have constructed $\mathcal{A}_\delta$ and $\mathcal{B}_\delta := \{B_{\alpha,i} : \alpha < \delta \wedge i \in 2\}$.

Define $X = X_\alpha$ if $X_\alpha \in \mathcal{I}(\mathcal{A}_\delta)^+$ and $X = X'$ for some $X' \in \mathcal{I}(\mathcal{A}_\delta)^+$ otherwise. Pick $x_n \in X \setminus (\bigcup_{i<n} A_{e_\delta(i)})$ and define $B_{\delta,0} = \{x_n : n \in \omega\}$.

Similarly define $Y = Y_\alpha$ if $\{Y_\alpha(n) : n \in \omega\} \subseteq \mathcal{A}_\delta^\perp$ and $Y = Y'$ for some $Y' \in (\mathcal{A}_\delta^\perp)^\omega$ otherwise. Define

$$B_{\delta,1} := \bigcup_{n\in\omega} \left[\left(\bigcup_{j \leq n} Y(j) \right) \cap \left(A_{e_\delta(n)} \setminus \bigcup_{i<n} A_{e_\delta(i)} \right) \right].$$

Notice that $Y(j) \setminus (\bigcup_{i<j} A_{e_\delta(i)}) \subseteq B_{\delta,1}$ and $\bigcup_{j \leq n} Y(j)$ has finite intersection with each $A_{e_\delta(n)}$. Hence $B_{\delta,1}$ is almost disjoint with $\mathcal{A}_\delta$ and almost contains each $Y(j)$.

Finally, define $Z = X_\alpha$ if $|X_\alpha \cap A_n| = \omega$ for infinitely many $n \in \omega$ and $Z = X'$ for some X' satisfying this property otherwise. Let $\{k_n : n \in \omega\} \subseteq \omega$ be an increasing sequence such that $|Z_\alpha \cap A_{e_\delta(k_n)}| = \omega$ for every $n \in \omega$. Pick $z_n \in Z \cap A_{e_\delta(k_n)} \setminus (\bigcup_{i<k_n} A_{e_\delta(i)})$ and define $A_\delta := \{z_n : n \in \omega\}$.

From the construction it is clear that $\mathcal{A}$ is α_1 because for every sequence $Y \in (\mathcal{A}^\perp)^\omega$ there exists an $\alpha < \omega_1$ such that $Y = Y_\alpha$ and $B_{\alpha,1}$ witnesses this property. With a similar argument we conclude that $\mathcal{A}$ is also Fréchet (hence absolutely Fréchet) and it is not bisequential.

□

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