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GENERIC EXISTENCE OF MAD FAMILIES

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Abstract. In this note we study generic existence of maximal almost disjoint (MAD) families. Among other results we prove that Cohen-indestructible families exist generically if and only if $\mathfrak{b} = \mathfrak{c}$. We obtain analogous results for other combinatorial properties of MAD families, including Sacks-indestructibility and being $+$-Ramsey.

§1. Introduction. An infinite family $\mathscr{A}$ of infinite subsets of ω is *almost disjoint* (AD) if the intersection of any two distinct elements of $\mathscr{A}$ is finite. It is *maximal almost disjoint* (MAD) if it is not properly included in any larger AD family or, equivalently, if given an infinite $X \subseteq \omega$ there is an $A \in \mathscr{A}$ such that $|A \cap X| = \omega$.

Many MAD families with special combinatorial or topological properties can be constructed using set-theoretic assumptions like CH, MA, or $\mathfrak{b} = \mathfrak{c}$. However, special MAD families are notoriously difficult to construct in ZFC alone. The reason being the lack of a device ensuring that a recursive construction of a MAD family would not prematurely terminate, an object that would serve a similar purpose as independent linked families do for the construction of special ultrafilters (see [16]). There is also a definite lack of negative (i.e., consistency) results. The following problem due to J. Steprāns presents one the basic open test problems for understanding the behaviour of MAD families in forcing extensions.

Problem 1.1 ([25]). *Is there a Cohen-indestructible MAD family in* ZFC?

As we mentioned before, the main difficulty lies in ensuring that a recursive construction of a MAD family does not terminate prematurely. This can be done typically either by means of cardinality considerations alone or by using an ad hoc construction for the problem at hand. In this paper we focus on the former.

The following is one of the most important definitions in this note.

Definition 1.2. Let P be a property of MAD families. We say *MAD families with property P exist generically* if every AD family of size less than $\mathfrak{c}$ can be extended to a MAD family with property P.

We begin with a simple example. Recall that a MAD family $\mathcal{A}$ is *completely separable* if every subset of ω which can not be almost covered by finitely many

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elements of $\mathcal{A}$ contains an element of $\mathcal{A}$. It is not known, whether completely separable MAD families exist in ZFC [24]. However, it is easy to see that completely separable MAD families exist generically if and only if $\mathfrak{a} = \mathfrak{c}$. On the one hand, assuming $\mathfrak{a} = \mathfrak{c}$, a straightforward and well-known recursive construction permits to extend any AD family of size less than $\mathfrak{c}$ to a completely separable MAD family, while if $\mathfrak{a} < \mathfrak{c}$, then there is a MAD family of size less than $\mathfrak{c}$, which is not completely separable[1] and since it is already maximal can not be extended to a completely separable MAD family.

One of the main results in this paper is the following theorem which gives a partial answer to the Problem 1.1.

THEOREM 1.3. *Cohen-indestructible families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

Extensions of AD families to maximal ones have been previously investigated by Leathrum in [19] and by Fuchino, Geschke, and Soukup in [8]. Generic existence of ultrafilters has been introduced by Canjar in [7] and was recently investigated by Brendle and Flašková in [5].

Given a forcing notion $\mathbb{P}$, a MAD family $\mathscr{A}$ is $\mathbb{P}$-*indestructible* if $\mathscr{A}$ remains maximal after forcing with $\mathbb{P}$. It follows from the proof of $\mathfrak{b} \leq \mathfrak{a}$ (see [2]) that if $\mathbb{P}$ adds a dominating real then it destroys every MAD family from the ground model, so the definition is only interesting when $\mathbb{P}$ does not add dominating reals. Our main focus is on Sacks and Cohen indestructible MAD families.

If $\mathscr{A}$ is an AD family on ω (or any countable set) we denote by $\mathscr{A}^{\perp}$ the set of all infinite $X \subseteq \omega$ that are almost disjoint with every element of $\mathscr{A}$. If $\mathscr{I}$ is an ideal on ω, we denote by $\mathscr{I}^{+}$ as those subsets of ω that are not in $\mathscr{I}$. We shall only consider ideals which extend the ideal of finite sets. If $X \in \mathscr{I}^{+}$ then by $\mathscr{I} \restriction X$ we will denote the restriction of $\mathscr{I}$ to X, that is, $\mathscr{I} \restriction X = \{I \cap X : I \in \mathscr{I}\}$ which is an ideal on X. We say $\mathscr{I}$ is *tall* if for every infinite $X \subseteq \omega$ there is an infinite $A \in \mathscr{I}$ such that $A \subseteq X$. There is a close relationship between MAD families and definable ideals (typically Borel of low complexity). We shall also investigate the connection here.

DEFINITION 1.4. Let $\mathscr{I}$ be an ideal (on a countable set). Then:

(1) We define $\mathsf{cov}^{*}(\mathscr{I})$ as the least size of a family $\mathscr{B} \subseteq \mathscr{I}$ such that for every infinite $X \in \mathscr{I}$ there is $B \in \mathscr{B}$ for which $B \cap X$ is infinite.
(2) If $\mathscr{I}$ is tall, we define $\mathsf{cov}^{+}(\mathscr{I})$ as the least size of a family $\mathscr{B} \subseteq \mathscr{I}$ such that for every $Y \in \mathscr{I}^{+}$ there is $B \in \mathscr{B}$ for which $B \cap Y$ is infinite.
(3) We say an AD family $\mathscr{A} \subseteq \mathscr{I}$ is a *MAD family restricted to* $\mathscr{I}$ if for every infinite $X \in \mathscr{I}$ there is $A \in \mathscr{A}$ such that $|X \cap A| = \omega$.
(4) $\mathfrak{a}(\mathscr{I})$ is the least size of a MAD family restricted to $\mathscr{I}$.
(5) For tall ideals, we define $\mathfrak{a}^{+}(\mathscr{I})$ as the least size of an AD family $\mathscr{A}$ such that $\mathscr{A} \cup \mathscr{A}^{\perp} \subseteq \mathscr{I}$ (or in other words, if $Y \in \mathscr{I}^{+}$ then there is $A \in \mathscr{A}$ such that $|Y \cap A| = \omega$).

Note that if $\mathscr{I}$ is tall, then $\mathsf{cov}^{*}(\mathscr{I})$ is just the least size of a family $\mathscr{B} \subseteq \mathscr{I}$ such that for every infinite $X \subseteq \omega$ there is $B \in \mathscr{B}$ for which $B \cap X$ is infinite, also $\mathsf{cov}^{+}(\mathscr{I}) \leq \mathsf{cov}^{*}(\mathscr{I})$. In general, $\mathsf{cov}^{*}(\mathscr{I}) \leq \mathfrak{a}(\mathscr{I})$ and for tall ideals $\mathsf{cov}^{+}(\mathscr{I}) \leq \mathfrak{a}^{+}(\mathscr{I})$ and $\mathfrak{a}^{+}(\mathscr{I}) \leq \mathfrak{a}(\mathscr{I})$. For the definitions of the classical invariants of the continuum see [2].

[1] Every completely separable MAD family has size $\mathfrak{c}$.

§2. Destructibility of MAD families. Let $\mathscr{I}$ be a tall ideal on ω and $\mathbb{P}$ be a forcing notion. Recall that $\mathscr{I}$ is $\mathbb{P}$-*indestructible* if $\mathscr{I}$ remains tall after forcing with $\mathbb{P}$ and otherwise it is $\mathbb{P}$-*destructible*. It is easy to see that a MAD family $\mathscr{A}$ is $\mathbb{P}$-indestructible if and only if $\mathscr{I}(\mathscr{A})$ is $\mathbb{P}$-indestructible. For the sake of the reader, we will now quote some results about destructibility. The key is to associate to every σ-ideal on the Baire space (Cantor space) an ideal on $\omega^{<\omega}$ $(2^{<\omega})$ and this is done by the notion of trace ideal.

Definition 2.1 ([6]). Given $a \subseteq \omega^{<\omega}$ we define $\pi(a) = \{b \in \omega^{\omega} : \exists^{\infty} n(b \restriction n \in a)\}$. If I is a σ-ideal on ω^{ω} define its *trace ideal* $tr(I) = \{a \subseteq \omega^{<\omega} : \pi(a) \in I\}$.

Clearly $\pi(a)$ is a G_{δ} set for every $a \subseteq \omega^{<\omega}$. The Katětov order plays a crucial role when studying destructibility of ideals and MAD families.

Definition 2.2. Let A, B be two countable sets and $\mathscr{I}$, $\mathscr{J}$ be two ideals on A and B, respectively.

(1) We say that $\mathscr{I}$ is *Katětov below* $\mathscr{J}$ (denoted by $\mathscr{I} \leq_K \mathscr{J}$) if there is a function $f : B \longrightarrow A$ such that $f^{-1}[I] \in \mathscr{J}$, for all $I \in \mathscr{I}$. The function f is called a *Katětov function*. We say that $\mathscr{I}$ is a *Katětov-Blass* below $\mathscr{J}$ $(\mathscr{I} \leq_{KB} \mathscr{J})$ if the function f may be taken finite-to-one (in this case f is called a Katětov-Blass function).
(2) We say that $\mathscr{I}$ is *Katětov equivalent to* $\mathscr{J}$ if $\mathscr{I} \leq_K \mathscr{J}$ and $\mathscr{J} \leq_K \mathscr{I}$ and we denote it by $\mathscr{I} \simeq_K \mathscr{J}$, the analogous definition holds for the Katětov-Blass order.

Observe that if a forcing notion $\mathbb{P}$ destroys an ideal $\mathscr{J}$ and $\mathscr{I} \leq_K \mathscr{J}$, then $\mathbb{P}$ also destroys $\mathscr{I}$. In fact, suppose that $\mathbb{P}$ destroys $\mathscr{J}$ and that $f : B \longrightarrow A$ witness $\mathscr{I} \leq_K \mathscr{J}$. Find a $\mathbb{P}$-name $\dot{X}$ for an infinite subset of B such that $\Vdash$ "$|\dot{X} \cap J| < \omega$ for all $J \in \mathscr{J}$." Note that $\Vdash$ "$f''\dot{X} \notin \mathscr{I}$," so in particular is infinite and it also witness that $\mathscr{I}$ is not tall in the extension. It is also immediate to see that if $X \in \mathscr{I}^{+}$ then $\mathscr{I} \leq_{KB} \mathscr{I} \restriction X$. An ideal $\mathscr{I}$ is called *Katětov-Blass uniform* if $\mathscr{I}$ is Katětov-Blass equivalent to all its restrictions (equivalently, if $X \in \mathscr{I}^{+}$, then $\mathscr{I} \restriction X \leq_{KB} \mathscr{I}$).

Given a σ-ideal I on ω^{ω}, $\mathbb{P}_I$ denotes the collection of all Borel sets in I^{+} ordered by the I-almost inclusion. The ideal I has the *continuous reading of names* [26] if for all $B \in \mathbb{P}_I$ and each Borel function $f : B \longrightarrow \omega^{\omega}$, there is a Borel set $C \in I^{+}$ such that $C \subseteq B$ and $f \restriction C$ is continuous. We shall need the following result of Hrušák and Zapletal.

Proposition 2.3 ([12]). *Let I be a σ-ideal on ω^{ω} such that $\mathbb{P}_I$ is proper with continuous reading of names and $\mathscr{J}$ be an ideal on ω. Then the following are equivalent*:

(1) *There is a $B \in \mathbb{P}_I$ such that $B \Vdash$ "$\mathscr{J}$ is not tall."*
(2) *There is an $a \in tr(I)^{+}$ such that $\mathscr{J} \leq_K tr(I) \restriction a$.*

In many cases occurring in practice, particularly for the forcing notions discussed in this article, the previous items are also equivalent to there is an $a \in tr(I)^{+}$ such that $\mathscr{J} \leq_{KB} tr(I) \restriction a$.

Let I be a σ-ideal on ω^{ω}, recall that I is *continuously homogeneous* if for every $B \in \mathbb{P}_I$ there is a continuous function $f : \omega^{\omega} \longrightarrow B$ such that $f^{-1}[A] \in I$ for every $A \in I \restriction B$.

Corollary 2.4 ([12]). *Let I be continuously homogeneous σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper with continuous reading of names such that $tr(I)$ is Katětov-Blass uniform and $\mathscr{J}$ be an ideal on ω. Then the following are equivalent*:

(1) *$\mathscr{J}$ is $\mathbb{P}_I$-destructible,*
(2) *$\mathscr{J} \leq_K tr(I)$,*
(3) *$\mathscr{J} \leq_{KB} tr(I)$.*

Let nwd be the ideal of all nowhere dense sets of the rational numbers, ctbl be the σ-ideal of all countable sets in the Baire space, and K_σ be the ideal generated by all σ-compact sets of the Baire space. It can be shown that $tr(\mathcal{M})$ is Katětov equivalent to nwd and it is easy to see that both $\mathcal{M}$ and ctbl are continuously homogeneous. Therefore, we can conclude the following.

Corollary 2.5 ([10] and [6]). *Let $\mathscr{I}$ be an ideal on ω, then the following holds*:

(1) *$\mathscr{I}$ is Cohen-destructible if and only if $\mathscr{I} \leq_{KB} \mathsf{nwd}$,*
(2) *$\mathscr{I}$ is Sacks-destructible if and only if $\mathscr{I} \leq_{KB} tr(\mathsf{ctbl})$,*
(3) *$\mathscr{I}$ is Miller-destructible if and only if $\mathscr{I} \leq_{KB} tr(K_\sigma)$.*

Sacks destructibility is particularly interesting due to the following result.

Proposition 2.6 ([10]). *If $\mathbb{P}$ adds a new real, then $\mathbb{P}$ destroys $tr(\mathsf{ctble})$. Therefore, if $\mathscr{I}$ is Sacks-destructible, then it is $\mathbb{Q}$-destructible by any forcing $\mathbb{Q}$ that adds a new real.*

Proof. Let $r \in V[G]$ be a new real and set $\widehat{r} = \{r \restriction n : n \in \omega\}$. Note that if $a \in (tr(\mathsf{ctbl})) \cap V$, then $r \notin \pi(a)$. This implies that $\widehat{r} \cap a =^* \emptyset$, hence $tr(\mathsf{ctbl})^V$ was destroyed. ⊣

By a similar argument, we can show the following:

Proposition 2.7 ([6]). *If $\mathbb{P}$ adds an unbounded real, then $\mathbb{P}$ destroys $tr(K_\sigma)$. Therefore, if $\mathscr{I}$ is Miller-destructible, then it is $\mathbb{Q}$-destructible by any forcing $\mathbb{Q}$ that adds an unbounded real.*

It is easy to show that if $\mathscr{I} \leq_K \mathscr{J}$, then $\mathsf{cov}^*(\mathscr{J}) \leq \mathsf{cov}^*(\mathscr{I})$. It is a result of Keremedis [15] (see also [1]) that $\mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$, it is not hard to see that $\mathsf{cov}^*(tr(\mathsf{ctbl})) = \mathfrak{c}$ and $\mathsf{cov}^*(tr(K_\sigma)) = \mathfrak{d}$. If $\mathscr{A}$ is a MAD family, then it is straight forward to check that $\mathsf{cov}^*(\mathscr{I}(\mathscr{A})) = |\mathscr{A}|$.

Corollary 2.8. *Let $\mathscr{A}$ be a MAD family. Then*:

(1) *If $\mathscr{A}$ is Cohen-destructible, then $\mathsf{cov}(\mathcal{M}) \leq |\mathscr{A}|$,*
(2) *If $\mathscr{A}$ is Sacks-destructible, then $|\mathscr{A}| = \mathfrak{c}$,*
(3) *If $\mathscr{A}$ is Miller-destructible, then $\mathfrak{d} \leq |\mathscr{A}|$.*

It follows.

Corollary 2.9 ([10] and [6]). (1) *If $\mathfrak{a} < \mathsf{cov}(\mathcal{M})$, then there is a Cohen-indestructible MAD family,*
(2) *If $\mathfrak{a} < \mathfrak{c}$, then there is a Sacks-indestructible MAD family,*
(3) *If $\mathfrak{a} < \mathfrak{d}$, then there is a Miller-indestructible MAD family.*

Given an AD family $\mathscr{A}$ we say that it is $\mathscr{I}$-MAD if $\mathscr{I}(\mathscr{A}) \nleq_{KB} \mathscr{I}$.

Proposition 2.10. *$\mathscr{I}$-MAD families exist generically if and only if $\mathfrak{a}^+(\mathscr{I}) = \mathfrak{c}$.*

Proof. First assume $\mathfrak{a}^+(\mathscr{I}) = \mathfrak{c}$. We will show that $\mathscr{I}$-MAD families exist generically. Let $\mathscr{A}$ be an AD family of size less than $\mathfrak{c}$ we will show how to extend it to a MAD family $\mathscr{B}$ so that $\mathscr{I}(\mathscr{B}) \nleq_{KB} \mathscr{I}$. Let $\langle f_\alpha : \alpha \in \mathfrak{c}\rangle$ be an enumeration of the set of all finite to one functions from $\omega^{<\omega}$ to ω. We will construct recursively an increasing sequence $\langle \mathscr{B}_\alpha : \alpha \in \mathfrak{c}\rangle$ of AD families such that:

(1) $\mathscr{B}_0 = \mathscr{A}$,
(2) $|\mathscr{B}_\alpha| < \mathfrak{c}$,
(3) $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ is not a Katětov function from $(\omega^{<\omega}, \mathscr{I})$ into $(\omega, \mathscr{I}(\mathscr{B}_{\alpha+1}))$.

Assume we are at step α and $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ is a Katětov function from $(\omega^{<\omega}, \mathscr{I})$ into $(\omega, \mathscr{I}(\mathscr{B}_\alpha))$. Let $\mathscr{D} = f_\alpha^{-1}[\mathscr{B}_\alpha]$ and note that, since f_α is finite to one, $\mathscr{D}$ is an almost disjoint family and it is contained in $\mathscr{I}$. Since $\mathscr{D}$ has size less than $\mathfrak{c}$, it follows that $\mathscr{D}^\perp$ is not contained in $\mathscr{I}$ so there is $X \notin \mathscr{I}$ that is almost disjoint with $\mathscr{D}$. Let $Y = f_\alpha[X]$ and note that $\mathscr{B}_\alpha = \mathscr{B} \cup \{Y\}$ is almost disjoint and $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ is no longer a Katětov function.

For the forward implication, note that if $\mathfrak{a}^+(\mathscr{I}) < \mathfrak{c}$, then there is an AD family $\mathscr{A}$ of size strictly less than continuum such that $\mathscr{A} \cup \mathscr{A}^\perp \subseteq \mathscr{I}$, and obviously $\mathscr{A}$ can not be extended to a MAD family not K-below $\mathscr{I}$. ⊣

We now address the question of when $\mathbb{P}_I$-indestructible MAD families exist generically.

Corollary 2.11. *Let I be a σ-ideal with continuous reading of names such that $tr(I)$ is KB-uniform. Then $\mathbb{P}_I$-indestructible MAD families exist generically if and only if $\mathfrak{a}^+(tr(I)) = \mathfrak{c}$.*

§3. Cohen-indestructibility. We will now show that $\mathfrak{b} = \mathfrak{a}^+(\mathsf{nwd})$. First we give several formulations of $\mathfrak{b}$ that might be of independent interest.

Proposition 3.1. *Let κ be an infinite cardinal, then the following are equivalent*:

(1) $\kappa < \mathfrak{b}$.
(2) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $C_n \subseteq^* X$ for every $n \in \omega$, i.e., $\mathscr{A}^\perp$ is a P-ideal.*
(3) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $|X \cap C_n| = \omega$ for every $n \in \omega$.*
(4) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $X \cap C_n \neq \emptyset$ for every $n \in \omega$.*
(5) *If $\mathscr{A}$ is an AD family of size κ and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$ is a pairwise disjoint family, then there is $X \in \mathscr{A}^\perp$ such that $|X \cap C_n| = \omega$ for every $n \in \omega$.*
(6) *If $\mathscr{A}$ is an AD family of size κ and for every pairwise disjoint family $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$, then there is $X \in \mathscr{A}^\perp$ such that $X \cap C_n \neq \varnothing$ for every $n \in \omega$.*

Proof. Obviously 2 implies 3, and 3 implies 5. It is easy to see that 4 implies 3 by splitting each C_n in countably many disjoint parts. By a similar reasoning it follows that 5 and 6 are equivalent. We now show that 1 implies 2. Let $\kappa < \mathfrak{b}$, $\mathscr{A} = \{A_\alpha : \alpha \in \kappa\}$ be an AD family in ω and $\{C_n : n \in \omega\} \subseteq \mathscr{A}^\perp$. Enumerate each $C_n = \{c_n^i : i \in \omega\}$ and for every $\alpha < \kappa$, define $f_\alpha : \omega \longrightarrow \omega$ in such a way that $A_\alpha \cap C_n \subseteq \{c_n^i : i < f_\alpha(n)\}$, for every $n \in \omega$. Since $\kappa < \mathfrak{b}$, we can find an increasing function $g : \omega \longrightarrow \omega$ that dominates each f_α, and define $X = \{c_n^{g(n)} : n \in \omega\}$. It is clear that X has the desired properties.

We now show that 5 implies 1 by contrapositive. Let $\mathscr{B} \subseteq \omega^\omega$ be an unbounded family of size κ, define the function $H : \omega^\omega \longrightarrow 2^\omega$ by $H\left(\langle x_n\rangle_{n\in\omega}\right) = 0^{x_0\frown}1^\frown 0^{x_1\frown}1^\frown 0^{x_2\frown}\cdots$ (where 0^{x_n} is the sequence of x_n consecutive 0's). Let $\mathbb{Q} \subseteq 2^\omega$ be the set of all sequences that are eventually zero, it is not hard to see that H is an homeomorphism between ω^ω and $2^\omega \setminus \mathbb{Q}$. Given $b \in 2^\omega$ define $\widehat{b} = \{b \restriction n : n \in \omega\}$ and let $\mathscr{A} = \{\widehat{H(f)} : f \in \mathscr{B}\}$ which clearly is an AD family and note that $\{\widehat{q} : q \in \mathbb{Q}\} \in \mathscr{A}^\perp$. We must now show there is no $X \in \mathscr{A}^\perp$ such that $|X \cap \widehat{q}| = \omega$ for every $q \in \mathbb{Q}$.

Suppose this is not the case. For each $n \in \omega$, define $U_n = \{b \in 2^\omega : |\widehat{b} \cap X| \geq n\}$ and note that each U_n is an open set, hence $G = \bigcap U_n$ is a G_δ set and $\mathbb{Q} \subseteq G$ while $G \cap H(\mathscr{B}) = \emptyset$. Let $K = 2^\omega \setminus G$. It is clear that $H[\mathscr{B}] \subseteq K$ and K is σ-compact. It follows that $H^{-1}[K]$ is σ-compact and contains $\mathscr{B}$, which is a contradiction. ⊣

We are now ready to prove the main theorem of the section.

Theorem 3.2. *Cohen-indestructible MAD families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

Proof. It suffices to show that $\mathfrak{b} = \mathfrak{a}^+(\mathsf{nwd})$. First, let $\kappa < \mathfrak{b}$ be given. Let us show that $\kappa < \mathfrak{a}^+(\mathsf{nwd})$. Let $\mathscr{A}$ be an AD family in $\mathbb{Q}$ such that $\mathscr{A} \subseteq \mathsf{nwd}$, we must prove that $\mathscr{A}^\perp$ is not contained in nwd. Let $\langle U_n : n \in \omega\rangle$ be a base for the topology of the rational numbers. Since $\kappa < \mathfrak{b}$, then it is also smaller than $\mathfrak{a}$, so $\mathscr{A}_n = \mathscr{A} \restriction U_n$ is not a MAD family in U_n (note that $U_n \notin \mathscr{I}(\mathscr{A})$ as the elements of $\mathscr{A}$ are nowhere dense) so we can find an infinite $C_n \subseteq U_n$ that is almost disjoint from every element of $\mathscr{A}_n$. Using $\kappa < \mathfrak{b}$ and the previous proposition, we can find an $X \in \mathscr{A}^\perp$ that intersects every C_n, and hence it is dense.

In order to show that $\mathfrak{a}^+(\mathsf{nwd}) \leq \mathfrak{b}$, we will construct an AD family $\mathscr{A}$ of size $\mathfrak{b}$ such that both $\mathscr{A}$ and $\mathscr{A}^\perp$ are contained in nwd. Recursively, we construct families $\{\mathscr{A}_s : s \in \omega^{<\omega}\}$ and $\{\overline{C}_s : s \in \omega^{<\omega}\}$ such that:

(1) $\mathscr{A}_\emptyset$ is an AD family on ω of size $\mathfrak{b}$ which is not maximal,
(2) $\overline{C}_s = \{C_s(n) : n \in \omega\}$ are pairwise disjoint infinite sets,
(3) $\mathscr{A}_{s\frown\langle n\rangle}$ is an AD family on $C_s(n)$ of size $\mathfrak{b}$ which is not maximal,
(4) $\overline{C}_\emptyset$ is a partition of ω and $\overline{C}_{s\frown\langle n\rangle}$ is a partition of $C_s(n)$,
(5) $\overline{C}_s \subseteq \mathscr{A}_s^\perp$ and if $Y \in [\omega]^\omega$ intersects infinitely every $C_s(n)$ then $Y \notin \mathscr{A}_s^\perp$,
(6) For every $a, b \in \omega$ there are s and n such that $|\{a, b\} \cap C_s(n)| = 1$.

In order to do this, fix an enumeration $\{\{a_k, b_k\} : k \in \omega\}$ of $[\omega]^2$. Using Proposition 3.1(5), there exists an AD family $\mathscr{A}$ of size $\mathfrak{b}$ on ω and a pairwise disjoint family $\{C(n) : n \in \omega\} \subseteq \mathscr{A}^\perp$ such that if $Y \in [\omega]^\omega$ intersects infinitely every $C(n)$ then $Y \notin \mathscr{A}^\perp$. Put $\mathscr{A}_\emptyset = \mathscr{A}$ and by a adding finitely many points to each $C(n)$ we may assume $\{C(n) : n \in \omega\}$ forms a partition of ω. Moreover, by making finite changes we can assume that $|\{a_0, b_0\} \cap C(0)| = 1$. Now set $\overline{C}_\emptyset = \{C(n) : n \in \omega\}$.

Suppose that we have constructed $\mathscr{A}_s$ and $\overline{C}_s$ for $s \in \omega^{\leq m}$. Again using Proposition 3.1(5), for every $n \in \omega$ and $s \in \omega^m$, there exists an AD family $\mathscr{A}$ of size $\mathfrak{b}$ on $C_s(n)$ and a pairwise disjoint family $\{C(k) : k \in \omega\} \subseteq \mathscr{A}^\perp$ such that if $Y \in [\omega]^\omega$ intersects infinitely every $C(k)$ then $Y \notin \mathscr{A}^\perp$. Put $\mathscr{A}_{s\frown\langle n\rangle} = \mathscr{A}$ and by a adding finitely many points to each $C(k)$ we may assume $\{C(k) : k \in \omega\}$ forms a partition of $C_s(n)$. Moreover, by making finite changes, we can assume that if

$a_m, b_m \in C_s(n)$ then $|\{a_m, b_m\} \cap C(0)| = 1$. Now set $\overline{C}_{s^\frown\langle n\rangle} = \{C(k) \colon k \in \omega\}$. This concludes the construction.

Let τ be the topology on ω generated by declaring each $C_s(n)$ clopen. It follows from a result of Sierpiński (see [17] or [14]) that (ω, τ) is homeomorphic to the rational numbers with the usual topology. Let $\mathscr{A} = \bigcup_{s\in\omega^{<\omega}} \mathscr{A}_s$, it suffices to show that $\mathscr{A} \cup \mathscr{A}^{\perp} \subseteq \mathsf{nwd}$. Let $A \in \mathscr{A}_t$ and $\overline{C}_s(n)$ first note that if $A \cap \overline{C}_s(n) \neq \varnothing$ then t and s are incompatible, by further extending s if necessary we may assume that s extends t. By (5) we extend s even further to s' so that $A \cap \overline{C}(n) \neq \varnothing$ is finite and then using (6) we can find an open subset of $\overline{C}_s(n)$ disjoint from A. Thus, $\mathscr{A} \subseteq \mathsf{nwd}$. The argument for $\mathscr{A}^{\perp}$ is analogous. ⊣

A closely related notion is that of a tight MAD family [20].

Definition 3.3. We say a MAD family $\mathscr{A}$ is *tight* if for every $\langle X_n \colon n \in \omega\rangle \subseteq \mathscr{I}(\mathscr{A})^{+}$ there is $B \in \mathscr{I}(\mathscr{A})$ such that $|B \cap X_n| = \omega$ for every $n \in \omega$.

Every tight family is Cohen indestructible and every Cohen indestructible family has a restriction that is tight (see [11, 18]). In particular, the existence of a tight MAD family is equivalent to the existence of a Cohen indestructible MAD family.

Corollary 3.4. *Tight families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

Proof. If tight families exist generically then obviously there exist Cohen-indestructible MAD families, therefore $\mathfrak{b}$ must be equal to $\mathfrak{c}$. The other implication follows from standard recursive construction. ⊣

We will now show that there are also tight families in many models where $\mathfrak{b}$ equals to ω_1. The following guessing principle was defined in [22].

$\diamondsuit(\mathfrak{b})$: For every Borel coloring $C : 2^{<\omega_1} \longrightarrow \omega^\omega$ there is a $G : \omega_1 \longrightarrow \omega^\omega$ such that for every $R \in 2^{\omega_1}$ the set $\{\alpha \colon C(R \restriction \alpha) \ {}^{*}\!\ngeq G(\alpha)\}$ is a stationary set (such G is called a *guessing sequence* for C).

A coloring $C : 2^{<\omega_1} \longrightarrow \omega^\omega$ is *Borel* if for every α, the function $C \restriction 2^\alpha$ is Borel in the classical sense. It is easy to see that $\diamondsuit(\mathfrak{b})$ implies that $\mathfrak{b} = \omega_1$ and in [22] it was proved that it implies $\mathfrak{a} = \omega_1$. The following answers a question of Hrušák and García-Ferreira from [11].

Proposition 3.5. *Assuming* $\diamondsuit(\mathfrak{b})$, *there is a tight MAD family.*

Proof. For every $\alpha < \omega_1$, fix an enumeration $\alpha = \{\alpha_n \colon n \in \omega\}$. Using a suitable coding, the coloring C will be defined on pairs $t = (\mathscr{A}_t, X_t)$, where $\mathscr{A}_t = \langle A_\xi \colon \xi < \alpha\rangle$ and $X_t = \langle X_n \colon n \in \omega\rangle$. We define $C(t)$ to be the constant 0 function in case $\mathscr{A}_t$ is not an almost disjoint family or X_t is not a sequence of elements in $\mathscr{I}(\mathscr{A}_t)^{+}$. In the other case, define an increasing function $C(t) : \omega \longrightarrow \omega$ such that for every $n \in \omega$ and $i \leq n$ the set $X_i \cap (C(t-1), C(t)) \setminus A_{\alpha_0} \cup \cdots \cup A_{\alpha_n}$ is not empty (where $C(t)(-1) = 0$).

By $\diamondsuit(\mathfrak{b})$ there is a guessing sequence $G : \omega_1 \longrightarrow \omega^\omega$ for C, changing G if necessary, we may assume that all the $G(\alpha)$ are increasing and if $\alpha < \beta$ then $G(\alpha) <^{*} G(\beta)$. We will now construct our MAD family by recursion on ω_1: Let $\{A_n \colon n \in \omega\}$ be a partition of ω. Suppose we have defined A_ξ for all $\xi < \alpha$, we put $A_\alpha = \bigcup_{n\in\omega} (G(\alpha)(n) \setminus A_{\alpha_0} \cup \cdots \cup A_{\alpha_n})$ in case this is an infinite set, otherwise just take any A_α that is almost disjoint with $\mathscr{A}_\alpha$.

We will now show that $\mathscr{A}$ is a tight family. Let $X = \langle X_n : n \in \omega \rangle$ where each $X_n \in \mathscr{I}(\mathscr{A})^+$. Consider the branch $R = (\langle A_\xi : \xi < \omega_1 \rangle, X)$ and pick $\beta > \omega$ such that $C(R \restriction \beta) \ ^* \not\geq G(\beta)$. It follows from the construction that A_β intersects infinitely every X_n. ⊣

With the aid of a result of [22] we can conclude the following.

Proposition 3.6 ([22]). *Let $\langle \mathbb{Q}_\alpha : \alpha \in \omega_2 \rangle$ be a sequence of Borel proper partial orders where each $\mathbb{Q}_\alpha$ is forcing equivalent to $\mathscr{P}(2)^+ \times \mathbb{Q}_\alpha$ and let $\mathbb{P}_{\omega_2}$ be the countable support iteration of this sequence. Then there is a tight family in any forcing extension by $\mathbb{P}_{\omega_2}$.*

Proof. If $\mathbb{P}_{\omega_2}$ forces $\mathfrak{b} = \mathfrak{c}$ then tight families exist generically, otherwise, it follows from [22] that $\mathbb{P}_{\omega_2}$ forces $\diamondsuit(\mathfrak{b})$ and, hence forces the existence of tight families. ⊣

The following weakening of tightness was introduced in [11].

Definition 3.7. We say that $\mathscr{A}$ is *weakly tight* if for every $\{B_n : n \in \omega\} \subseteq \mathscr{I}(\mathscr{A})^+$ there is $A \in \mathscr{A}$ such that $|A \cap B_n| = \omega$ for infinitely many $n \in \omega$.

Obviously every tight family is weakly tight. The proof of the next proposition is virtually identical to the proof of Corollary 3.4.

Proposition 3.8. *Weakly tight families exist generically if and only if* $\mathfrak{b} = \mathfrak{c}$.

Mildenberger, Raghavan, and Steprāns (see [23] and [21]) proved that if $\mathfrak{s} \leq \mathfrak{b}$ then there is a weakly tight family. However, it is still open if it is possible to construct a weakly tight family without any additional axioms beyond ZFC.

The following invariant was introduced by Shelah in [24].

Definition 3.9. We define $\mathfrak{a}_*$ as the minimal size of an AD family $\mathscr{A}$ such that there are (disjoint) $C_0, C_1, C_2, \ldots \in \mathscr{A}^\perp$ such that for every $B \in [\omega]^\omega$ if $C_n \cap B \neq^* \emptyset$ for infinitely many C_n then there is $A \in \mathscr{A}$ for which $A \cap B \neq^* \emptyset$.

The relation of $\mathfrak{a}_*$ with the other cardinal invariants is the following.

Proposition 3.10 ([24]). $\mathfrak{b} \leq \mathfrak{a}_* \leq \mathfrak{a}$.

Proof. By the characterization of $\mathfrak{b}$, it is clear that $\mathfrak{b} \leq \mathfrak{a}_*$. In order to prove that $\mathfrak{a}_* \leq \mathfrak{a}$, let $\mathscr{A}$ be a MAD family of minimum size. Choose $C_0, C_1, C_2, \ldots \in \mathscr{A}$ and let $\mathscr{A}_1 = \mathscr{A} \setminus \{C_n : n \in \omega\}$ we will show that $\mathscr{A}_1$ is a witness for $\mathfrak{a}_*$. Assume $C_n \cap B \neq^* \emptyset$ for infinitely many C_n, then it follows that $B \setminus C_0 \cup \cdots \cup C_n$ is infinite for every $n \in \omega$, so we may find $B' \subseteq B$ that is almost disjoint from every C_n. Since $\mathscr{A}$ is MAD, it follows that there is an $A \in \mathscr{A}$ such that $A \cap B \neq^* \emptyset$. ⊣

Proposition 3.11. $\mathfrak{a}_*$ *has uncountable cofinality.*

Proof. Assume $\mathrm{cof}(\mathfrak{a}_*) = \omega$ and let $\mathcal{A}$ be an AD family of size $\mathfrak{a}_*$ and $\{C_n \mid n \in \omega\} \subseteq \mathcal{A}^\perp$ such that for every B with the property that there are infinitely many $n \in \omega$ such that $|C_n \cap B| = \omega$ then there is $A \in \mathcal{A}$ for which $|A \cap B| = \omega$. Since $\mathfrak{a}_*$ has countable cofinality then we can find an increasing chain $\{\mathcal{A}_n \mid n \in \omega\} \subseteq \wp(\mathcal{A})$ such that $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n$ and $|\mathcal{A}_n| < \mathfrak{a}_*$ for every $n \in \omega$.

(1) Since $|\mathcal{A}_0| < \mathfrak{a}_*$ then we can find $B_0 \in \mathcal{A}_0^\perp$ such that $D_0 = \{n : |C_n \cap B_0| = \omega\}$ is infinite. Let $m_0 = min(D_0)$.
(2) Let $\mathcal{A}_1' = \mathcal{A}_1 \restriction B_0$ since $|\mathcal{A}_1'| < \mathfrak{a}_*$ then we can find $B_1 \subseteq B_0$ such that $B_1 \in \mathcal{A}_1^\perp$ and $D_1 = \{n > m_0 \mid |(C_n \cap B_0) \cap B_1| = \omega\}$ is infinite. Let $m_1 = min(D_1)$.

(3) Let $\mathcal{A}_2' = \mathcal{A}_2 \restriction B_1$ since $|\mathcal{A}_2'| < \mathfrak{a}_*$ then we can find $B_2 \subseteq B_1$ such that $B_2 \in \mathcal{A}_2^{\perp}$ and $D_2 = \{n > m_1 \mid |(C_n \cap B_1) \cap B_2| = \omega\}$ is infinite. Let $m_2 = min\,(D_2)$.

$\vdots$

Finally, let $X = \bigcup_{i\in\omega}(B_i \cap C_{m_i})$ then X intersects infinitely every C_{m_i} and $X \in \mathcal{A}^{\perp}$ which is a contradiction. ⊣

With the previous proposition we can conclude the following,

Corollary 3.12. *There is a model where* $\mathfrak{a}_* < \mathfrak{a}$.

Proof. In [4] Brendle constructed a model where $\mathfrak{a}$ has countable cofinality. By the previous proposition, it is clear that $\mathfrak{a}_* < \mathfrak{a}$ holds in that model. ⊣

§4. Sacks-indestructibility. For simplicity, call $\mathfrak{a}_{Sacks} = \mathfrak{a}^{+}(tr\,(\mathsf{ctbl}))$ which is the least size of an AD family $\mathscr{A} \subseteq tr\,(\mathsf{ctbl})$ such that for every $X \in tr\,(\mathsf{ctbl})^{+}$ there is $A \in \mathscr{A}$ such that $A \cap X$ is infinite. Recall that Sacks-indestructible MAD families exist generically if and only if $\mathfrak{a}_{Sacks} = \mathfrak{c}$. Likewise, call $\mathfrak{a}^{+}_{Miller} = \mathfrak{a}\,(tr\,(K_\sigma))$ and as before, Miller indestructible MAD families exist generically if and only if $\mathfrak{a}_{Miller} = \mathfrak{c}$. Since $\mathsf{ctbl} \subseteq K_\sigma \subseteq \mathcal{M}$ then $\mathfrak{b} = \mathfrak{a}\,(\mathsf{nwd})^{+} \leq \mathfrak{a}_{Miller} \leq \mathfrak{a}_{Sacks}$.

The following result is an easy one but is very important.

Corollary 4.1. *If* $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$ *then there is a Sacks-indestructible family.*

Proof. Assume $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$. On the one hand if $\mathfrak{a} < \mathfrak{c}$, then any MAD family of size $\mathfrak{a}$ is Sacks indestructible. On the other hand, if $\mathfrak{a} = \mathfrak{c}$, then $\mathfrak{a}_{Sacks} = \mathfrak{c}$ and so there are also Sacks indestructible MAD families. ⊣

We do not know if $\mathfrak{a}$ can consistently be bigger than $\mathfrak{a}_{Sacks}$.

Problem 4.2. *Is* $\mathfrak{a} \leq \mathfrak{a}_{Sacks}$?

Given $s \in 2^{<\omega}$ we define $\langle s\rangle_{<\omega} = \{t \in 2^{<\omega} : s \sqsubseteq t\}$, it is clear that if $X \cap \langle s\rangle_{<\omega} \neq \emptyset$ for every $s \in 2^{<\omega}$ then $X \notin tr\,(\mathsf{ctbl})$. Let $\mathscr{BR}$ be the ideal of $2^{<\omega}$ generated by branches, in this way $\mathscr{BR}^{\perp}$ is the ideal of all well-founded subsets of $2^{<\omega}$, its elements are called *off-branch*, it is clear that $\mathscr{BR}^{\perp} \subseteq tr\,(\mathsf{ctbl})$. We have the following simpler characterization of $\mathsf{cov}^{+}(tr\,(\mathsf{ctbl}))$:

Lemma 4.3. $\mathsf{cov}^{+}(tr\,(\mathsf{ctbl}))$ *is the minimum size of a family* $\mathscr{B} \subseteq \mathscr{BR}^{\perp}$ *such that for every* $A \in tr\,(\mathsf{ctbl})^{+}$ *there is* $B \in \mathscr{B}$ *such that* $|A \cap B| = \omega$.

Proof. Call μ the minimum size a family $\mathscr{B} \subseteq \mathscr{BR}^{\perp}$ such that for every $A \in tr\,(\mathsf{ctbl})^{+}$ there is $B \in \mathscr{B}$ such that $|A \cap B| = \omega$. It is clear that $\mathsf{cov}^{+}(tr\,(\mathsf{ctbl})) \leq \mu$, we shall now prove the other inequality. We split the proof in two cases: if $\mathsf{cov}^{+}(tr\,(\mathsf{ctbl})) = \mathfrak{c}$ then there is nothing to prove, so assume $\mathsf{cov}^{+}(tr\,(\mathsf{ctbl}))$ is less than size of the continuum and let $\mathscr{B} \subseteq tr\,(\mathsf{ctbl})$ witness this fact. Since $2^{\omega\times\omega} \cong 2^{\omega}$ we may find a partition $\{[T_\alpha] : \alpha < \mathfrak{c}\}$ of 2^{ω} where each T_α is a Sacks tree. Since $\mathscr{B} \subseteq tr\,(\mathsf{ctbl})$ and has size less than $\mathfrak{c}$, then there is T_α such that $\pi\,(B) \cap [T_\alpha] = \emptyset$ for every $B \in \mathscr{B}$. The splitting nodes of T_α is isomorphic to $2^{<\omega}$ and for every $B \in \mathscr{B}$ it is the case that $B \cap T_\alpha$ is off-branch in T_α. ⊣

Using an analogous argument, we can prove the following.

Lemma 4.4. $\mathfrak{a}_{Sacks}$ *is the smallest size of an almost disjoint family* $\mathscr{A} \subseteq \mathscr{BR}^{\perp}$ *such that* $\mathscr{A} \cup \mathscr{A}^{\perp} \subseteq tr\,(\mathsf{ctbl})$.

We call a maximal AD family restricted to $\mathscr{BR}^{\perp}$ a *MOB* family. In [19] Leathrum defined $\mathfrak{o}$ as the smallest size of a MOB family and he showed that $\mathfrak{a} \leq \mathfrak{o}$.

LEMMA 4.5. $\mathfrak{a}_{Sacks} \leq \mathfrak{o}$.

PROOF. Let $\mathscr{O}$ be a MOB family of size $\mathfrak{o}$, then $\mathscr{O} \subseteq \mathscr{BR}^{\perp}$ and any $A \in \mathscr{O}^{\perp}$ must be contained in a union of finitely many branches, therefore $A \in tr\,(\mathsf{ctbl})$. ⊣

We have the following inequalities:

LEMMA 4.6. $\mathsf{cov}^{+}\,(tr\,(\mathsf{ctbl})) \leq min\left\{\mathfrak{a}_{Sacks}, \mathsf{cov}^{*}\left(\mathscr{BR}^{\perp}\right)\right\}$.

PROOF. The inequality $\mathsf{cov}^{+}\,(tr\,(\mathsf{ctbl})) \leq \mathfrak{a}_{Sacks}$ follows by definition, and for the other it is enough to recall that any $B \in tr\,(\mathsf{ctbl})^{+}$ contains an infinite antichain. ⊣

Now we compare them with some of the category related cardinal invariants.

PROPOSITION 4.7. $\mathsf{cov}\,(\mathcal{M}) \leq \mathsf{cov}^{+}\,(tr\,(\mathsf{ctbl}))$.

PROOF. Let $\kappa < \mathsf{cov}\,(\mathcal{M})$ and $\mathscr{A} = \{A_\alpha : \alpha \in \kappa\} \subseteq \mathscr{BR}^{\perp}$, we ought to find $B \in tr\,(\mathsf{ctbl})^{+}$ that is AD with $\mathscr{A}$. Let $\mathbb{P}$ be the partial order of all finite trees contained in $2^{<\omega}$ ordered by end extension. Obviously, $\mathbb{P}$ is isomorphic to Cohen forcing. Let $\dot{T}_{gen}$ be the name for the generic tree, clearly $\dot{T}_{gen}$ is forced to be a Sacks tree. For every $\alpha < \kappa$ define the set D_α of all $T \in \mathbb{P}$ such that if $s \in T$ is a maximal node, then $\langle s \rangle_{\leq\omega} \cap A_a = \emptyset$. It is straightforward to see that D_α is dense. Since $\kappa < \mathsf{cov}\,(\mathcal{M})$ then we can find, in the ground model, a filter that intersects every D_α and the result follows. ⊣

We recursively define $S = \{t_s : s \in 2^{<\omega}\}$ as follows:

We will now compare the Miller related invariants with the unbounding number. Recall that $\mathsf{cov}^{*}\,(tr\,(K_\sigma)) = \mathfrak{d}$.

PROPOSITION 4.8. $\mathsf{cov}^{+}\,(tr\,(K_\sigma)) = \mathfrak{b}$.

PROOF. We first show that $\mathsf{cov}^{+}\,(tr\,(K_\sigma)) \leq \mathfrak{b}$. Let $\{f_\alpha : \alpha \in \mathfrak{b}\}$ be an unbounded family of strictly increasing functions. For every $s \in \omega^{<\omega}$ and $\alpha < \mathfrak{b}$, we define $T_\alpha\,(s)$ as the downward closed subtree of the set consisting of nodes of the form $s^\frown \langle n \rangle^\frown t$, where $n \in \omega$ and $t \in \omega^{f_\alpha(n)}$.

Note that each $T_\alpha\,(s)$ is in $tr\,(K_\sigma)$, as $\pi(T_\alpha\,(s))$. Now let $A \subseteq 2^{<\omega}$ be such that $\pi\,(A)$ is unbounded. Find $s \in \omega^{<\omega}$ such that for infinitely many $n \in \omega$, $s^\frown \langle n \rangle$ has a successor in A. For each $n \in \omega$, let $g(n)$ be the minimum integer k so that there is a $t \in \omega^k$ with $s \frown \langle n \rangle t \in A$. Using that $\{f_\alpha : \alpha \in \mathfrak{b}\}$ is an unbounded family, we can find $\alpha < \mathfrak{b}$ so that $f_\alpha * \nleq g$. It follows that $A \cap T_\alpha\,(s)$ is infinite.

Now, let $\kappa < \mathfrak{b}$ and $\{A_\alpha : \alpha \in \kappa\} \subseteq tr\,(K_\sigma)$, we must show it is not a covering family. Since $\kappa < \mathfrak{b}$ we can find f that bounds $\pi\,(A_\alpha)$ for every $\alpha < \kappa$. Let T be the tree such that every branch though T is bigger or equal than f, we may assume $T = \omega^{<\omega}$.

For every $s \in \omega^{<\omega}$ choose $b_s \in \langle s \rangle$ and given $\alpha < \kappa$ define $f_\alpha : \omega^{<\omega} \longrightarrow \omega$ be such that if $m \geq f_\alpha\,(s)$ then $b_s \restriction m \notin A_\alpha$ (recall A_α is off-branch in $T = \omega^{<\omega}$) since κ is less than $\mathfrak{b}$, we may find $g : \omega^{<\omega} \longrightarrow \omega$ that dominates every f_α. We define a Miller tree S in the following way:

(1) The stem of S is $b_\emptyset \restriction g\,(\emptyset)$,
(2) If $s \in S$ is a splitting node, then $\mathrm{succ}_S\,(s) = \omega$,
(3) If s is a splitting node, then the next splitting node below $s^\frown \langle n \rangle$ is $b_{s^\frown\langle n \rangle} \restriction g\,(s^\frown \langle n \rangle)$.

Let $B = \text{split}(S)$ which obviously is $tr(K_\sigma)$ positive, we now claim $B \cap A_\alpha$ is finite for every α but this is clear. ⊣

Likewise, $\mathsf{cov}^+(\mathsf{nwd})$ is smaller than $\mathsf{cov}^*(\mathsf{nwd})$.

Proposition 4.9. $\mathsf{cov}^+(\mathsf{nwd}) = \mathsf{add}(\mathcal{M})$.

Proof. On the one hand, $\mathsf{cov}^+(\mathsf{nwd}) \leq \mathsf{cov}^*(\mathsf{nwd}) = \mathsf{cov}(\mathcal{M})$ and on the other hand, $\mathsf{cov}^+(\mathsf{nwd}) \leq \mathsf{cov}^+(K_\sigma) = \mathfrak{b}$. Now we proceed to prove $\mathsf{add}(\mathcal{M}) \leq \mathsf{cov}^+(\mathsf{nwd})$. Let $\kappa < \mathsf{add}(\mathcal{M})$ and $\{N_\alpha : \alpha \in \kappa\} \subseteq \mathsf{nwd}$. Let $\{U_n : n \in \omega\}$ be a base for $\mathbb{Q}$ and note that since $\kappa < \mathsf{cov}^*(\mathsf{nwd})$, then we may find an infinite $B_n \subseteq U_n$ almost disjoint from every N_α. Define $h_\alpha : \omega \longrightarrow [\mathbb{Q}]^{<\omega}$ where $h_\alpha(n) = B_n \cap A_\alpha$. Since $\kappa < \mathfrak{b}$ then there is $g : \omega \longrightarrow [\mathbb{Q}]^{<\omega}$ such that for every α it is the case that $h_\alpha(n) \subseteq g(n)$ for almost all $n \in \omega$, we may further assume that $g(n) \subseteq B_n$. Define $B = \bigcup_{n\in\omega}(B_n \setminus g(n))$ then B is dense and almost disjoint with each A_α. ⊣

In [13] Kamburelis and Węglorz introduced the following definitions,

Definition 4.10. (1) $\mathfrak{s}(\mathscr{B}_0)$ is the smallest size of a family of open sets $\mathscr{U} \subseteq \mathscr{P}(2^\omega)$ such that for every infinite antichain $\{s_n : n \in \omega\} \subseteq 2^{<\omega}$ there is $U \in \mathscr{U}$ such that both sets $\{n : \langle s_n \rangle \subseteq U\}$ and $\{n : \langle s_n \rangle \cap U = \emptyset\}$ are infinite.

(2) Given $x \in 2^\omega$ and $n \in \omega$, let $r(x, n)$ be the sequence of length $n + 1$ that agrees with x in the first n places but disagrees in the last one.

(3) Let $x \in 2^\omega$, $A \in [\omega]^\omega$, and $U \subseteq 2^\omega$ an open set. We say U *separates* (x, A) if $x \notin U$ and there are infinitely many $n \in A$ such that $\langle r(x, n) \rangle \subseteq U$.

(4) $\mathfrak{sep}$ is the smallest size of a family of open sets $\mathscr{U}$ such that for every (x, A) there is $U \in \mathscr{U}$ that separates (x, A).

It was later proved by Brendle in [3] that the two previous invariants are actually equal.

Proposition 4.11. $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right) = \mathfrak{sep}$.

Proof. We first show that $\mathfrak{sep} \leq \mathsf{cov}^*\left(\mathscr{BR}^\perp\right)$. Let $\mathscr{B} \subseteq \mathscr{BR}^\perp$ be a witness for $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right)$, we might assume it is closed under finite changes. For every $B \in \mathscr{B}$, let $\mathscr{U}_B = \bigcup\{\langle s \rangle : s \in B\}$. We will show that $\{\mathscr{U}_B : B \in \mathscr{B}\}$ witness $\mathfrak{sep}$. Let $x \in 2^\omega$, $A \in [\omega]^\omega$ and define the off-branch family $Y = \{r(x, n) : n \in A\}$ then find $B \in \mathscr{B}$ such that $B \cap Y$ is infinite. Since B is off-branch, by taking a finite subset of it we may assume no restriction of x is in B, it then follows that $\mathscr{U}_B$ separates (x, A).

We will now show $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right) \leq \mathfrak{s}(\mathscr{B}_0)$. Let $\left\{U_\beta : \beta < \mathfrak{s}(\mathscr{B}_0)\right\}$ be a witness for $\mathfrak{s}(\mathscr{B}_0)$ and $\{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded set of functions where each $f_\alpha : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ and if $n < m$ then $f_\alpha(n) \subseteq f_\alpha(m)$. For every $\beta < \mathfrak{s}(\mathscr{B}_0)$, let $A_\beta = \left\{s_n^\beta : n \in \omega\right\} \subseteq 2^{<\omega}$ be the set of all minimal nodes of $\left\{s : \langle s \rangle \subseteq U_\beta\right\}$, note that they form an antichain. For every $\alpha < \mathfrak{b}$ and $\beta < \mathfrak{s}(\mathscr{B}_0)$, define $B(\alpha, \beta) = A_\beta \cup \bigcup\limits_{n\in\omega}\left(f_\alpha(n) \cap \left\langle s_n^\beta \right\rangle_{<\omega}\right)$, observe that this is an off-branch set. We will show that for every infinite off branch Y there are $\alpha < \mathfrak{b}$ and $\beta < \mathfrak{s}(\mathscr{B}_0)$ such that $B(\alpha, \beta)$ intersects Y infinitely.

Let $A \subseteq Y$ be an infinite antichain, first find $\beta < \mathfrak{s}(\mathscr{B}_0)$ such that the set $X = \left\{t \in A \colon \langle t \rangle \subseteq U_\beta\right\}$ is infinite. Define $g : \omega \longrightarrow [2^{<\omega}]^{<\omega}$ such that for every $n \in \omega$ there is $m > n$ such that there is $t \in A$ and t extends s_m^β. Let $\alpha < \mathfrak{b}$ such that $g(n) \subseteq f_\alpha(n)$ for infinitely many $n \in \omega$. It is then clear that $B(\alpha, \beta)$ intersects A infinitely. Finally since $\mathfrak{b} \leq \mathfrak{s}(\mathscr{B}_0)$ (see [3]) we get the desired result. ⊣

It then follows by a result of Kamburelis and Węglorz (see [13]) that $\mathsf{cov}^*\left(\mathscr{BR}^\perp\right) \leq \mathsf{cof}(\mathcal{N})$.

§5. +-Ramsey MAD families. Let $\mathscr{I}$ be an ideal, we say $T \subseteq \omega^{<\omega}$ is $\mathscr{I}^+$-*branching* if for every $s \in T$, the set $\operatorname{succ}_T(s) = \{n \colon s^\frown \langle n \rangle\} \in \mathscr{I}^+$. We say $\mathscr{I}$ is +-*Ramsey* if for every $\mathscr{I}^+$-branching tree T, there is $b \in [T]$ such that $\operatorname{ran}(b) \in \mathscr{I}^+$. An AD family $\mathscr{A}$ is called +-Ramsey if $\mathscr{I}(\mathscr{A})$ is +-Ramsey. The following was introduced in [9].

Definition 5.1. $\mathfrak{ra}$ is the minimum size of an AD family that is not +-Ramsey.

In respect to the generic existence of +-Ramsey families, we have the following:

Proposition 5.2. $\mathfrak{ra} = \mathfrak{c}$ *if and only if +-Ramsey families exist generically.*

Proof. First assume $\mathfrak{ra} = \mathfrak{c}$ and let $\mathscr{A}$ be an AD family of size less than the continuum. Enumerate $\{T_\alpha \colon \alpha < \mathfrak{c}\}$ all the trees in $\omega^{<\omega}$. Recursively, we shall construct a sequence $\langle \mathscr{A}_\alpha \colon \alpha < \mathfrak{c}\rangle$ of AD families such that:

(1) $\mathscr{A}_0 = \mathscr{A}$,
(2) If $\alpha < \beta$ then $\mathscr{A}_\alpha \subseteq \mathscr{A}_\beta$ and if γ is limit then $\mathscr{A}_\gamma = \bigcup_{\delta<\gamma} \mathscr{A}_\delta$,
(3) Every $\mathscr{A}_\alpha$ has size less than $\mathfrak{c}$,
(4) For every $\alpha < \mathfrak{c}$ either T_α is not a $\mathscr{I}^+(\mathscr{A}_{\alpha+1})$-branching tree or there is $b \in [T_\alpha]$ such that $\operatorname{ran}(b) \in \mathscr{I}(\mathscr{A}_{\alpha+1})^{++}$.

It is clear that if the construction can be carried out, we just extend $\bigcup_{\alpha<\mathfrak{c}} \mathscr{A}_\alpha$ to a MAD family and this will be a +-Ramsey MAD family. Assume $\mathscr{A}_\alpha$ has been defined, we will see how to define $\mathscr{A}_{\alpha+1}$. First consider the case where there is $s \in T_\alpha$ such that $\operatorname{succ}_{T_\alpha}(s) \notin \mathscr{I}(\mathscr{A}_\alpha)^{++}$. If $\operatorname{succ}_{T_\alpha}(s) \in \mathscr{I}(\mathscr{A}_\alpha)$ then we just define $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha$ otherwise we can find an infinite $A \subseteq \operatorname{succ}_{T_\alpha}(s)$ that is AD with $\mathscr{A}_\alpha$, so we just define $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha \cup \{A\}$. Now assume $\operatorname{succ}_{T_\alpha}(s) \in \mathscr{I}(\mathscr{A}_\alpha)^{++}$ for every $s \in T_\alpha$. Since $\mathscr{A}_\alpha$ has size less than $\mathfrak{ra}$ then we know there is $b \in [T_\alpha]$ such that $\operatorname{ran}(b) \in \mathscr{I}(\mathscr{A}_\alpha)^+$. In case $\operatorname{ran}(b) \in \mathscr{I}(\mathscr{A}_\alpha)^{++}$ then we can just define $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha$, in the other case as before, we can find pairwise disjoint $\{A_n \colon n \in \omega\} \subseteq [b]^\omega \cap \mathscr{A}_\alpha^+$ and let $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha \cup \{A_n \colon n \in \omega\}$.

Now assume $\mathfrak{ra} \leq \mathfrak{c}$, let $\mathscr{A}$ be an non +-Ramsey AD family of size less than $\mathfrak{c}$. In this way, we know there is T a $\mathscr{I}(\mathscr{A}_\alpha)^+$ branching tree such that if $b \in [T]$ then $\operatorname{ran}(b) \in \mathscr{I}(\mathscr{A}_\alpha)$. ⊣

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