WEAK PARTITION PROPERTIES ON TREES

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Abstract. We investigate the following weak Ramsey property of a cardinal κ: If χ is coloring of nodes of the tree $\kappa^{<\omega}$ by countably many colors, call a tree $T \subseteq \kappa^{<\omega}$ χ-homogeneous if the number of colors no each level of T is finite. Write $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\omega}$ to denote that for any such coloring there is a χ-homogeneous λ-branching tree of height ω.

1. Introduction

Let us quote a classical theorem of Hurewicz [7] (see also [9]):

Theorem 1.1. *Let X be an analytic space. Then one and only one of the following holds.*

(i) *X is σ-compact,*
(ii) *X has a closed subset homeomorphic to the set of irrationals.*

One may ask if this theorem can be extended some way to nonseparable spaces. Of course, both "analytic" and "σ-compact" have to be replaced by appropriate notions.

Recall that a metrizable space X is *Čech-analytic* if there is a completely metrizable space $Y \subseteq \omega^\omega \times X$ that projects onto X. Equivalently, if X is a Suslin set in a completion. In particular, each analytic or completely metrizable space is Čech–analytic.

Given a property $\mathcal{P}$, a metrizable space is σ-$\mathcal{P}$ if it admits a countable cover by subspaces with property $\mathcal{P}$. Thus we have notions of σ-locally compact space, σ-locally separable space etc.

It turns out that substituting Čech-analytic for analytic and σ-locally compact for σ-compact, Hurewicz's theorem still holds. In this introductory section we first prove a combinatorial theorem and then we show how to use it to derive the extended Hurewicz's theorem and two more results. The combinatorial principle will be generalized and investigated in the subsequent sections.

In the sequel ω and ω_1 denote the first infinite and uncountable cardinal, respectively, and $\omega_1^{<\omega}$ the tree of all finite sequences in ω_1, i.e. $\omega_1^{<\omega} = \bigcup_{n\in\omega} \omega_1^n$. A set $T \subseteq \omega_1^{<\omega}$ is a *tree* if $q \subseteq p \in T$ implies $q \in T$. A *successor* of $p \in T$ is $q \supseteq p$ such that $|q| = |p| + 1$. The *height* of T (denoted $\operatorname{ht} T$) is the least $n \in \omega$ such that

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$|p| < n$ for all $p \in T$. If κ is a cardinal, T is called *κ-branching* if each $p \in T$ such that $|p| < \operatorname{ht} T - 1$ has at least κ successors. T is a *κ-tree* if it is a κ-branching tree of height ω.

Proposition 1.2. *For any mapping $\chi : \omega_1^{<\omega} \to (0, \infty)$ there is an ω-tree $T \subseteq \omega_1^{<\omega}$ such that*

$$\inf\{\chi(p) : p \in T, |p| = n\} > 0 \textit{ for each } n \in \omega.$$

Proof. We shall inductively construct, for each $n \in \omega$, trees W_n and S_n and a real number $\varepsilon(n) > 0$ subject to the following conditions.

(i) $\operatorname{ht} W_n = \operatorname{ht} S_n = n$,
(ii) W_n is n-branching,
(iii) S_n is ω_1-branching,
(iv) $\chi(p) \geqslant \varepsilon(|p|)$ for each $p \in S_n$,
(v) $W_n \subseteq S_n \cap W_{n+1}$,
(vi) $\{p \in S_{n+1} : |p| < n\} \subseteq S_n$.

Put $W_0 = S_0 = \emptyset$. At stage $n+1$ the construction of W_{n+1}, S_{n+1} and $\varepsilon(n)$ goes as follows. Put

$$\widehat{S}_n = S_n \cup \{p^\frown\alpha : p \in S_n, |p| = n - 1\}.$$

For each $p \in \widehat{S}_n$ define $\delta(p)$: If $|p| = n$, let $\delta(p) = \chi(p)$ and proceed by induction down to the root. If $|p| < n$ and $\delta(q)$ is already set up for each $q \in \widehat{S}_n$ such that $|q| > |p|$, let

$$A(p, \xi) = \{q \in \widehat{S}_n : |q| = |p| + 1 \wedge \delta(q) > \xi\}, \quad \xi > 0. \tag{1.1}$$

As U is ω_1-branching, there is $\xi > 0$ such that $|A(p, \xi)| = \omega_1$. Put

$$\delta(p) = \min\{\xi, \varepsilon(0), \dots, \varepsilon(n)\}. \tag{1.2}$$

The tree S_{n+1} is defined by: $\emptyset \in S_{n+1}$ and when $p \in S_{n+1}$ and $0 < |p| \leqslant n$, then the set of successors is $A(p, \delta(p))$. Definitions (1.1) and (1.2) yield $W_n \subseteq S_{n+1}$. Thus W_{n+1} is easily constructed by adding new nodes to W_n within S_{n+1} subject to (i) and (ii). Finally put $\varepsilon(n) = \delta(\emptyset)$. Conditions (i)–(vi) are easily verified. The induction step is complete.

The tree T is now easily defined by $T = \bigcup_{n \in \omega} W_n$. Conditions (ii) and (iv) ensure that T is ω-branching and that $\chi(p) \geqslant \varepsilon(n)$ whenever $p \in T$ and $|p| = n$, as required. $\square$

We shall see the basic idea of this proof reoccurring at several occasions.

Uniform embedding. We now derive an embedding result that is of independent interest. Consider the set of irrationals ω^ω and provide it with the usual least difference metric given by $d(f, g) = 1/(1 + \min\{n \in \omega : f(n) \neq g(n)\})$. The metric space (ω^ω, d) is sometimes referred to as the *Baire space*.

Theorem 1.3. *Let (X, ρ) be a Čech–analytic metric space. If X is not σ-locally separable, then it contains a uniform copy of (ω^ω, d).*

Proof. By a theorem of A. H. Stone [13, 3.4], a Čech–analytic space X that is not σ-locally separable contains a homeomorphic copy of ω_1^ω, where ω_1 is given the discrete topology. We may thus assume without loss of generality that (X, ρ) is a completely metrizable space none of whose nonempty open subset is separable. Let

γ be a complete metric on X. For a set $A \subseteq$ denote $\operatorname{diam}_\rho A$ the diameter of A in the metric ρ, and likewise for any other metric.

Construct inductively a tree of nonempty open subsets $\{V_p : p \in \omega_1^{<\omega}\}$ such that for each $p \in \omega_1^{<\omega}$

(i) $V_q \subseteq V_p$ whenever $p \subseteq q \in \omega_1^{<\omega}$,
(ii) $\operatorname{diam}_\gamma V_p + \operatorname{diam}_\rho V_p < \frac{1}{1+|p|}$,
(iii) $\inf\{\rho(x,y) : x \in V_p, y \in V_q, q \in \omega_1^{<\omega}, |q| = |p|, q \neq p\} > 0$

as follows: Put $V_\emptyset = X$. When V_p is constructed, let $A = \{q \in \omega_1^{<\omega} : |q| = |p| + 1\}$ and choose an uncountable disjoint family $\{W_q : q \in A\}$ of nonempty open subsets of V_p. For each $q \in A$ choose an open set $V_q \subseteq W_q$ subject to (ii) such that

$$\inf\{\rho(x,y) : x \in V_q, y \notin W_q\} > 0,$$

what ensures (iii). This completes the induction step of the construction.

For each $p \in \omega_1^{<\omega}$ put

$$\chi(p) = \inf\{\rho(x,y); x \in V_p, y \in V_q, q \in \omega_1^{<\omega}, |q| = |p|, q \neq p\}. \tag{1.3}$$

(iii) yields $\chi(p) > 0$. Therefore Proposition 1.2 yields an ω-tree $T \subseteq \omega_1^{<\omega}$ and a sequence $\langle \varepsilon(n) : n \in \omega\rangle$ such that $\chi(p) > \varepsilon(|p|)$ for each $p \in T$. We may assume without loss of generality that $T = \omega^{<\omega}$. It follows from (i) (remind that γ is a complete metric) that for each $f \in \omega^\omega$ there is a unique point $x_f \in X$ such that $\{x_f\} = \bigcap_{n\in\omega} \overline{V}_{f\restriction n}$.

We show that the mapping $f \mapsto x_f$ is a uniform embedding of (ω^ω, d) into (X, ρ). Let $f, g \in \omega^\omega$, $n = \inf\{i \in \omega : f(i) \neq g(i)\}$ and $p = f\restriction n$. Infer from (i) that

$$\rho(x_f, x_g) \leqslant \operatorname{diam}_\rho V_p \leqslant \frac{1}{1+|p|}. \tag{1.4}$$

On the other hand, $V_{f\restriction n+1} \neq V_{g\restriction n+1}$, hence (1.3), (ii) and (iii) imply

$$\rho(x_f, x_g) \geqslant \chi(f\restriction n+1) \geqslant \varepsilon(n+1). \tag{1.5}$$

Inequalities (1.4) and (1.5) show that $f \mapsto x_f$ is a uniform embedding. The proof is complete. □

Call a space *nowhere separable* if it has no nonempty open separable subset.

Corollary 1.4. *If X is nowhere separable completely metrizable metric space, then it contains a uniform copy of (ω^ω, d).*

Generalized Hurewicz' theorem. We now draw the promised generalization of Hurewicz' theorem.

Theorem 1.5. *Let X be a Čech-analytic space. Then one and only one of the following holds.*

(i) *X is σ-locally compact,*
(ii) *X has a closed subset homeomorphic to the set of irrationals.*

Proof. A straightforward Baire category argument shows that ω^ω is not σ-locally compact. Therefore if ω^ω embeds into X as a closed subspace, then X is not σ-locally compact.

Now suppose that X is σ-locally separable but not σ-locally compact. It is easy to show that X actually admits a σ-discrete cover by closed separable subsets. One of these sets, F, say, is not σ-compact. Therefore Hurewicz theorem applies: F contains a closed copy of ω^ω. *A fortiori* X contains a closed copy of ω^ω.

Now suppose that X is not σ-locally compact. Apply Theorem 1.3: X contains a uniform copy of the complete space (ω^ω, d). Since any complete subspace is closed, we are done. $\square$

Hausdorff dimension. The combinatorial principle 1.2 was originally motivated by the following (rather vague) question: Is there any reasonable way of extending the notion of Hausdorff dimension to nonseparable metric spaces? The minimal requirement for such an extension would be monotonicity. Thus the minimal Hausdorff dimension would be defined by

$$\dim_{\mathsf{H}} X = \sup\{\dim_{\mathsf{H}} Y : Y \subseteq X \text{ is separable}\}.$$

We will prove that if a completely metrizable metric space X is nowhere separable, then $\dim_{\mathsf{H}} X = \infty$.

Let us now recall the notions of Hausdorff measure and dimension. Let $\mathbb{H}$ denote the set of all functions $h : [0,\infty) \to [0,\infty)$ that are nondecreasing, right-continuous, and satisfy $h(r) = 0$ iff $r = 0$. Elements of $\mathbb{H}$ are called *Hausdorff functions*. The following is the common ordering of $\mathbb{H}$:

$$g \prec h \quad \stackrel{\text{def}}{\equiv} \quad \lim_{r\to 0+} \frac{h(r)}{g(r)} = 0.$$

Given $s > 0$, we shall write $h \prec s$ to abbreviate that $h \prec g_s$, where $g_s(r) = r^s$.

Let (X, ρ) be a separable metric space. If $\delta > 0$, a cover $\mathcal{A}$ of a set $E \subseteq X$ is termed a *δ-cover* if $\operatorname{diam} A \leqslant \delta$ for all $A \in \mathcal{A}$. Given $g \in \mathbb{H}$, the *g-dimensional Hausdorff measure*, $\mathcal{H}^g(E)$, of a set E in a space X is defined thus: For each $\delta > 0$ set

$$\mathcal{H}^g_\delta(E) = \inf\Big\{\sum_{n\in\omega} h(\operatorname{diam} E_n) : \{E_n\} \text{ is a countable } \delta\text{-cover of } E\Big\}$$

and put $\mathcal{H}^g(E) = \sup_{\delta>0} \mathcal{H}^g_\delta(E)$. In the common case when $g(r) = r^s$ for some $s > 0$ we write $\mathcal{H}^s$ for $\mathcal{H}^g$ and call the measure *s-dimensional. Hausdorff dimension of* X is defined by

$$\dim_{\mathsf{H}} X = \sup\{s > 0 : \mathcal{H}^s(X) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(X) = 0\}.$$

General references: [11], [4].

Theorem 1.6. *Let (X, ρ) be a Čech–analytic metric space. If X is not σ-locally separable, then for each Hausdorff function g there exists a compact set $C \subseteq X$ such that $\mathcal{H}^g(C) = \infty$.*

In particular, there is a compact set $C \subseteq X$ such that $\dim_{\mathsf{H}} C = \infty$.

Proof. Theorem 1.3 yields a subset $Y \subseteq X$ that is uniformly equivalent to the Baire space (ω^ω, d). We may thus assume that $X = \omega^\omega$ and that ρ is uniformly equivalent with d. Therefore there is a Hausdorff function h such that $d(x,y) \leqslant h(\rho(x,y))$ for all x, y. Thus $\operatorname{diam}_d E \leqslant h(\operatorname{diam}_\rho E)$ for every $E \subseteq \omega^\omega$. For each $k \in \omega$ put $r_k = \inf\{r > 0 : h(r) \geqslant \frac{1}{k+2}\}$ and find $n(k) \in \omega$ such that $n(k) \geqslant 1/g(r_k)$. Consider the subspace of ω^ω

$$C = \{f \in \omega^\omega : \forall k\ f(k) < n(k)\} = \prod_{k\in\omega} n(k).$$

It is a compact topological group. Let μ be its Haar measure. Let $E \subseteq C$. Put $r = \operatorname{diam}_\rho E$. Then $\operatorname{diam}_d E \leqslant h(r)$ and therefore there is a (unique) $k \in \omega$ such that

$r_k \leqslant r$ and $\operatorname{diam}_d E = \frac{1}{k+2}$. It follows that there is a sequence $\langle p(0), p(1), \dots, p(k)\rangle$ such that $x(i) = p(i)$ for each $i \leqslant k$ and $x \in E$. Consequently

$$\mu(E) \leqslant \frac{1}{n(0)}\frac{1}{n(1)}\cdots\frac{1}{n(k)} \leqslant \frac{1}{n(k)} \leqslant g(r_k) \leqslant g(r).$$

Now suppose that $\{E_n\}$ is a cover of C. Then

$$\sum_n g(\operatorname{diam}_\rho E) \geqslant \sum_n \mu(E_n) \geqslant \mu\left(\bigcup\nolimits_n E_n\right) \geqslant \mu(C) = 1,$$

which is enough for $\mathcal{H}^g(C) \geqslant 1$. For the second assertion it is enough to pick any g such that $g \succ s$ for all $s > 0$. $\square$

This theorem has a (little esoteric) partial converse. The family $\mathbb{H}$ of all Hausdorff functions with the order $\prec$ is a poset. Consider its usual completion $\mathbb{H}^*$ and denote its largest element by $\mathbb{1}$. We already described how $(0,\infty)$ embeds into $\mathbb{H}$: Each $s > 0$ is identified with (the equivalence class of) the function $g_s \in \mathbb{H}$ given by $g_s(r) = r^s$. We can extend this embedding by identifying ∞ with the least upper bound $\vee(0,\infty)$. It is clear that $\infty = \vee(0,\infty)$ is much smaller than $\mathbb{1}$, there is a huge gap between ∞ and $\mathbb{1}$.

Now suppose that X is a metrizable space and define

$$\delta(X) = \wedge_\rho \vee \{g \in \mathbb{H} : \mathcal{H}^g(Y,\rho) > 0 \text{ for some separable } Y \subseteq X\},$$

where the outer meet is over all compatible metrics on X. Theorem 1.6 can be rephrased as follows.

Corollary 1.7. *Let X be a Čech–analytic metrizable space. If X is not σ-locally separable, then $\delta(X) = \mathbb{1}$.*

Here is the partial converse. For locally separable metrizable spaces all major topological dimensions (small and large inductive dimensions and the covering one) coincide. They are denoted $\dim X$.

Theorem 1.8. *If X is a locally separable metrizable space, then $\delta(X) = \dim X$. In particular, $\delta(X) \leqslant \infty < \mathbb{1}$.*

Let us explain the statement in more detail: If $\dim X = n < \infty$, then $\delta(X) = n$, i.e. the element of $\mathbb{H}^*$ corresponding to the Hausdorff function $r \mapsto r^n$. If $\dim X = \infty$, then $\delta(X) = \infty = \vee(0,\infty)$.

Proof. We shall make use of several classical topological theorems. By [1] every locally separable metrizable space is a free sum of separable spaces, i.e. there is a disjoint clopen cover $\mathcal{X}$ of X.

Suppose $\dim X \geqslant n \in \omega$. Then there is $Y \in \mathcal{X}$ such that $\dim Y = n$. Let ρ be any compatible metric on Y. By [8] $\mathcal{H}^n(Y,\rho) > 0$. It follows that $\delta(X) \geqslant \delta(Y) \geqslant n$. Conclude that $\delta(X) \geqslant \dim X$.

Now suppose that $\dim X = n < \infty$. Let $g \in \mathbb{H}$, $g \succ n$. Fix $Y \in \mathcal{X}$. By a slight improvement of [10] there is a compatible metric ρ_Y on Y such that (Y,ρ_Y) can be, for any $r > 0$, covered by $\frac{1}{g(r)}$ many sets of diameters not exceeding r. A routine argument shows that consequently $\mathcal{H}^g(Y,\rho_Y) \leqslant 1$. Mutatis mutandis we may actually assume $\mathcal{H}^g(Y,\rho_Y) = 0$. Find such a metric for every $Y \in \mathcal{X}$ and using the fact that all Y are clopen set up a compatible metric ρ on X that extends simultaneously all ρ_Y's. Since any separable subset of X is covered by countably many $Y \in \mathcal{X}$, it follows that

$$\vee\{g \in \mathbb{H} : \mathcal{H}^g(Y,\rho) > 0 \text{ for some separable } Y \subseteq X\} \leqslant g.$$

Therefore

$$\delta(X) \leqslant \wedge\{g \in \mathbb{H} : g \succ n\} = n = \dim X.$$

The proof for $\dim X = \infty$ is similar. We only have to show that if X is separable and $g \succ s$ for all $s > 0$, then there is a compatible metric ρ on X such that $\mathcal{H}^g(X) = 0$. Since every separable metric space embeds into $[0,1]^\omega$, we may assume $X = [0,1]^\omega$. There is a sequence $r_n \searrow 0$ such that if $r_{n+1} \leqslant r < r_n$, then $g(r) \leqslant r^{n+1}$. Define a metric on $[0,1]^\omega$ by

$$\rho(x,y) = \sup_{n\in\omega} r_n \cdot |x(n) - y(n)|.$$

It is clear that ρ is a compatible metric. We count the number $N(r)$ of sets of diameter at most r needed to cover $[0,1]^\omega$. Assume without loss of generality that $r_0 \leqslant \frac{1}{2}$. If $r_{n+1} \leqslant r < r_n$, then obviously

$$N(r) \leqslant \prod_{j=0}^{n} \left(\tfrac{r_j}{r} + 1\right) = r^{-(n+1)} \prod_{j=0}^{n} (r_j + r) \leqslant r^{-(n+1)} \leqslant \frac{1}{g(r)}.$$

A routine argument shows that this is enough for $\mathcal{H}^g([0,1]^\omega, \rho) \leqslant 1$. Proceed as above. □

2. The arrow

In this section we isolate the combinatorial core of Proposition 1.2 in a slightly more general setting.

We use the following notation and terminology: κ, λ, μ, ν and τ denote cardinals (often, but not always, infinite). For a set A (A is usually a cardinal), $A^{<\omega} = \bigcup_{n\in\omega} A^n$ is assumed to be given a tree ordering by inclusion.

A set $T \subseteq A^{<\omega}$ is a *tree* if $q \subseteq p \in T$ implies $q \in T$. For a tree $T \subseteq A^{<\omega}$ and $n \in \omega$, $(T)_n = T \cap A^n = \{p \in T : |p| = n$ denotes the n–th level of T and $T{\restriction}n = \{p \in T : |p| < n\}$. The height of T is denoted and defined by $\operatorname{ht} T = \min\{n : (T)_n = \emptyset\}$.

For $p \in T$, we define $\operatorname{succ}_T p = \{a \in A : q^\frown a \in T\}$ and $\operatorname{dg}_T p = |\operatorname{succ}_T p|$. A tree T is *λ-branching* if each $p \in T$ such that $|p| < \operatorname{ht} T - 1$ has at least λ successors. T is a *λ-tree* if it is a κ-branching tree of height ω.

Suppose that $\chi : A^{<\omega} \to \mu$ is a coloring. A tree $T \subseteq A^{<\omega}$ is *χ-homogeneous* (or just *homogeneous*) if it has finitely many colors on each level, i.e. $\forall n\ |\chi''(T)_n| < \omega$.

Similar license is used for other types of homogeneity: If $g \in \omega^\omega$ and $\forall n\ |\chi''(T)_n| \leqslant g(n)$, the tree T is termed *(χ, g)-homogeneous* or just *g-homogeneous*; and if ν is a cardinal and $\forall n\ |\chi''(T)_n| \leqslant \nu$, the tree T is termed *(χ,ν)-homogeneous* or just *ν-homogeneous.* Clearly a tree T is homogeneous iff there is $g \in \omega^\omega$ such that T is g-homogeneous.

Definition 2.1. If for each coloring $\chi : \kappa^{<\omega} \to \mu$ there exists a χ-homogeneous λ-tree $T \subseteq \kappa^{<\omega}$, then we write

$$\kappa \rightsquigarrow (\lambda)^{<\omega}_\mu.$$

We also consider some variations that obtain by altering the notion of "homogeneous tree". Given $g \in \omega^\omega$, if for each coloring $\chi : \kappa^{<\omega} \to \mu$ there exists a (χ, g)-homogeneous λ-tree $T \subseteq \kappa^{<\omega}$, then we write

$$\kappa \rightsquigarrow (\lambda)^{<\omega}_{\mu,g}.$$

Given a cardinal ν, if for each coloring $\chi : \kappa^{<\omega} \to \mu$ there exists a (χ, ν)-homogeneous λ-tree $T \subseteq \kappa^{<\omega}$, then we write

$$\kappa \rightsquigarrow (\lambda)^{<\omega}_{\mu,\nu}.$$

If $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\nu}$ for each $\nu < \mu$, then we write

$$\kappa \rightsquigarrow (\lambda)^{<\omega}_{<\mu}.$$

It is clear that Proposition 1.2 can be rephrased as follows: $\omega_1 \rightsquigarrow (\omega)^{<\omega}_{\omega}$. We now prove a little stronger and more general statement. Here and later on,

$$\boldsymbol{\mathcal{C}} = \{g \in \omega^\omega : \lim_{n\to\infty} g(n) = \infty \wedge \forall n\; g(n) \geqslant 1\}.$$

Theorem 2.2. *If κ is an infinite cardinal, then $\kappa \rightsquigarrow (\omega)^{<\omega}_{<\kappa,g}$ for all $g \in \boldsymbol{\mathcal{C}}$.*

Proof. Let $\mu < \kappa$ and $\chi : \kappa^{<\omega} \to \mu$. The case $\kappa = \omega$ is easy, so assume that κ is uncountable. Let $g \in \boldsymbol{\mathcal{C}}$.

If κ is singular, then there is a regular cardinal λ such that $\mu < \lambda < \kappa$. We may thus suppose that κ is regular.

Define a function $G \in \omega^{\omega\times\omega}$ by ($\lfloor x \rfloor$ denotes the integer part of x)

$$G(i,n) = \begin{cases} \lfloor g(n)^{2^{-i-1}} - 1 \rfloor & \text{if } i < n-1, \\ 0 & \text{if } i \geqslant n-1, \end{cases} \tag{2.1}$$

We shall inductively construct, for each $n \in \omega$, trees W_n and S_n and a function $s_n : n \to [\mu]^{<\omega}$ subject to the following conditions.

(i) If $G(0,n) > 0$, then $W_n \neq \emptyset$ and $\forall p \in W_n\ \ \mathrm{dg}_W\, p = G(|p|, n)$,
(ii) S_n is κ-branching of height n,
(iii) $W_n \subseteq S_n \cap W_{n+1}$,
(iv) $S_{n+1}{\restriction}n \subseteq S_n$,
(v) $s_n \subseteq s_{n+1}$,
(vi) $\forall i < n\ |s_n(i)| \leqslant g(i)$,
(vii) $\forall p \in S_n\ \chi(p) \in s_n(|p|)$.

Put $W_0 = S_0 = s_0 = \emptyset$. At stage $n+1$ the construction of W_{n+1}, S_{n+1} and s_{n+1} goes as follows. Suppose that W_n, S_n and s_n are constructed. First recklessly extend S_n by letting

$$S'_{n+1} = \{p^\frown\alpha : p \in S_n, |p| = n-1, \alpha < \kappa\}.$$

For each $p \in S'_{n+1}$ we now define $\delta(p) \subseteq \mu$ so that the following conditions are met:

(viii) $p \subseteq q \in W_n \Rightarrow \delta(p) \supseteq \delta(q)$,
(ix) $|\delta(p)| \leqslant \prod_{i=|p|}^{\infty}(1 + G(i,n))$.

If $|p| = n$, let $\delta(p) = \{\chi(p)\}$ and proceed by induction down to the root. Suppose $j = |p| < n$ and $\delta(q)$ is already set up for each $q \in U$ such that $|q| > |p|$. Since $\mu < \kappa$, there is a set $F_p \in [\mu]^{<\omega}$ such that the set

$$A(p, F_p) = \{q \in U : |q| = |p| + 1 \wedge \delta(q) = F_p\} \tag{2.2}$$

is of cardinality κ. Put

$$\delta(p) = F_p \cup \{\delta(p^\frown\alpha) : \alpha \in \operatorname{succ}_{W_n} p\}. \tag{2.3}$$

Condition (viii) is obviously met. If (ix) is satisfied by all $p^\frown\alpha$, $\alpha \in \operatorname{succ}_{W_n} p$, then

$$\begin{aligned} |\delta(p)| &\leqslant |F_p| + \operatorname{dg}_{W_n} p \prod_{i=|p|+1}^{\infty} (1+G(i,n)) \leqslant (1+\operatorname{dg}_{W_n} p) \prod_{i=|p|+1}^{\infty} (1+G(i,n)) \\ &\leqslant (1+G(|p|,n)) \prod_{i=|p|+1}^{\infty} (1+G(i,n)) = \prod_{i=|p|}^{\infty} (1+G(i,n)). \end{aligned}$$

Since clearly $|\delta(p)| = 1$ if $|p| = n$, we verified (ix).

When δ is constructed, define s_{n+1} by $s_{n+1}{\restriction}n = s_n$ and $s(n) = \delta(\emptyset)$.

The tree S_{n+1} is defined by: $\emptyset \in S_{n+1}$ and when $p \in S_{n+1}$ and $0 < |p| \leqslant n$, then the set of successors is $A(p, F_p)$. Definitions (1.1) and (1.2) yield $W_n \subseteq S_{n+1}$. Thus W_{n+1} is easily constructed by adding new nodes to W_n within S_{n+1} subject to (i). Condition (vi) follows from (ix) and the definition of G:

$$|s_{n+1}(n)| = |\delta(\emptyset)| \leqslant \prod_{i=0}^{\infty}(1+G(i,n)) \leqslant \prod_{i=0}^{\infty} g(n)^{2^{-i-1}} = g(n)^{\sum_{i=0}^{\infty} 2^{-i-1}} = g(n).$$

All other conditions are easily verified. The induction step is complete.

Now define the tree T by $T = \bigcup_{n\in\omega} W_n$. Define also $s : \omega \to [\mu]^{<\omega}$ by $s = \bigcup_{n\in\omega} s_n$. Condition (i) ensures that T is nonempty and $\forall p \in T\ \operatorname{dg}_T p = \lim_{n\to\infty} G(|p|, n) = \infty$. Hence T is an ω-tree. Condition (vii) ensures that $\forall p \in T\ \chi(p) \in s(|p|)$, i.e. $\forall n\ \chi''(T)_n \subseteq s(n)$. Therefore (vi) yields $|\chi''(T)_n| \leqslant g(n)$, as required. □

3. Game

Cichoń's diagram. Since several small cardinals will get into play soon, we now recall the relevant material.

Denote by $\mathcal{M}, \mathcal{N}$, respectively, the ideals of meager and Lebesgue null subsets of 2^ω. The following are the usual cardinal invariants of $\mathcal{N}$:

$$\begin{aligned} \mathsf{add}(\mathcal{N}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \wedge \textstyle\bigcup\mathcal{A} \notin \mathcal{N}\}, \\ \mathsf{cov}(\mathcal{N}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \wedge \textstyle\bigcup\mathcal{A} = 2^\omega\}, \\ \mathsf{cof}(\mathcal{N}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{N} \wedge (\forall N \in \mathcal{N})(\exists A \in \mathcal{A})(N \subseteq A)\}, \\ \mathsf{non}(\mathcal{N}) &= \min\{|Y| : Y \subseteq 2^\omega \wedge Y \notin \mathcal{N}\}. \end{aligned}$$

The cardinal invariants of $\mathcal{M}$ are defined likewise.

For $f, g \in \omega^\omega$, the *modulo finite* order is defined by $f \leqslant^* g$ if $f(n) \leqslant g(n)$ for all but finitely many $n \in \omega$. A set $F \subseteq \omega^\omega$ is *bounded* if $\exists h \in 2^\omega\ \forall f \in F\ f \leqslant^* h$, and F is *dominating* if $\forall g \in \omega^\omega\ \exists f \in F\ g \leqslant^* f$. The associated cardinal invariants are $\mathfrak{b}$, the minimal cardinality of an unbounded set, and $\mathfrak{d}$, the minimal cardinality of a dominating set.

We shall also make use of two *Martin numbers*, $\mathfrak{m}_{\sigma\text{-centered}}$ and $\mathfrak{m}_{\sigma\text{-linked}}$. Let $\mathbb{P}$ be a poset. A set $A \subseteq \mathbb{P}$ is *centered* (*linked*, respectively) if for any $p, q \in A$ there is $r \in A$ ($r \in \mathbb{P}$) such that $r \leqslant p$ and $r \leqslant q$. A poset $\mathbb{P}$ is called *σ-centered* or *σ-linked*, respectively, if there exists a cover $\{P_i : i \in \omega\}$ of $\mathbb{P}$ such that each P_i is centered or linked.

For a cardinal κ, $\mathrm{MA}_{\sigma\text{-centered}}(\kappa)$ is the statement: For any σ-centered poset $\mathbb{P}$ and any family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, with $|\mathcal{D}| \leqslant \kappa$, there is a filter that meets every member of $\mathcal{D}$, and $\mathrm{MA}_{\sigma\text{-linked}}(\kappa)$ is defined likewise. The corresponding Martin numbers are defined by

$$\begin{aligned} \mathfrak{m}_{\sigma\text{-centered}} &= \min\{\kappa : \mathrm{MA}_{\sigma\text{-centered}}(\kappa) \text{ fails}\}, \\ \mathfrak{m}_{\sigma\text{-linked}} &= \min\{\kappa : \mathrm{MA}_{\sigma\text{-linked}}(\kappa) \text{ fails}\}^{1}. \end{aligned}$$

The provable inequalities between these cardinals are summarized in the following diagram[2].

$$\begin{tikzcd}
\mathsf{cov}(\mathcal{N}) \arrow[r] & \mathsf{non}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[r] \arrow[u] & \mathfrak{d} \arrow[u] & \\
\mathsf{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathsf{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathsf{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathsf{non}(\mathcal{N}) \arrow[uu] \\
\mathfrak{m}_{\sigma\text{-linked}} \arrow[r] \arrow[u] & \mathfrak{m}_{\sigma\text{-centered}} \arrow[u] & &
\end{tikzcd}$$

Six of these numbers will get into play, namely $\mathfrak{d}, \mathfrak{b}, \mathsf{cof}(\mathcal{N}), \mathfrak{m}_{\sigma\text{-centered}}, \mathfrak{m}_{\sigma\text{-linked}}$ and $\mathsf{add}(\mathcal{N})$.

In this section we provide a game-theoretic characterization of the arrow $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$.

Let κ and μ be cardinals. Let $\mathcal{G} \subseteq \mathcal{P}(\mu)^{\omega}$. Consider the following two-player game $\Gamma_{\mathcal{G}}$: Player I plays first a coloring $\chi : \kappa^{<\omega} \to \mu$ and Player II responds with a $G \in \mathcal{G}$. Then Player I starts playing sets $J_n \in [\kappa]^{<\kappa}$ one at a time and Player II responds with ordinals $\alpha_n \notin J_n$:

I	χ		J_0		J_1		$\dots$	
II		$G \in \mathcal{G}$		$\alpha_0 \notin J_0$		$\alpha_1 \notin J_1$	$\dots$	$\chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in G(n)$

Player II wins if $\chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in G(n)$ for each $n \in \omega$, otherwise Player I wins.

Theorem 3.1. *Player* II *has a winning strategy in* $\Gamma_{\mathcal{G}}$ *if and only if*

$$\forall \chi : \kappa^{<\omega} \to \mu \ \exists G \in \mathcal{G} \ \exists \kappa\text{-tree } T \subseteq \kappa^{<\omega} \quad \forall n \ \chi''(T)_n \subseteq G(|p|). \tag{3.1}$$

Proof. Suppose that Player II has a winning strategy. Denote by $\mathcal{T}$ the family of all runs of $\Gamma_{\mathcal{G}}$ that II plays according to her winning strategy. For $f = \langle J_0, \alpha_0, J_1, \alpha_1, J_2, \alpha_2, \dots\rangle \in \mathcal{T}$ put $\widetilde{f} = \langle \alpha_0, \alpha_1, \alpha_2, \dots\rangle$. The tree we look for is $T = \{\widetilde{f}{\restriction}n : f \in \mathcal{T}, n \in \omega\}$. Indeed, as II wins each run in $\mathcal{T}$, $\chi(\widetilde{f}{\restriction}n) \in G(n)$ for all $f \in \mathcal{T}$ and $n \in \omega$. Hence $\chi''(T)_n \subseteq G(n)$. If $p \in T$, then for any $J \in [\kappa]{<}\,\kappa$ II can play $\beta \notin J$ such that $p^\frown\beta \in T$. Hence $\operatorname{succ}_T p$ is not contained in any set of cardinality below κ, i.e. $\operatorname{dg}_T p = |\operatorname{succ}_T p| = \kappa$, which proves that T is a κ-tree.

[1]By Bell's theorem, $\mathfrak{m}_{\sigma\text{-centered}}$ is equal to the *pseudointersection number* $\mathfrak{p}$.

[2]As usual, the arrows in the diagram point from the smaller to the larger cardinal.

On the other hand, suppose (3.1) is satisfied. The winning strategy for Player II is to respond to χ with G given by (3.1) and then choose $\alpha_n \notin J_n$ such that $\langle \alpha_0, \alpha_1, \dots, \alpha_{n-1}, \alpha_n \rangle \in T$, which is possible, since $\mathrm{dg}_T \langle \alpha_0, \alpha_1, \dots, \alpha_{n-1} \rangle = \kappa$. $\square$

The particular choice $\mathcal{G} = \{ \langle [0, g(n)] : n \in \omega \rangle : g \in \omega^\omega \}$ gives a game theoretic characterization of $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$. The corresponding game Γ: Player I plays a coloring $\chi : \kappa^{<\omega} \to \omega$ and Player II responds with a $g \in \omega^\omega$. Then Player I starts playing sets $J_n \in [\kappa]^{<\kappa}$ one at a time and Player II responds with ordinals $\alpha_n \notin J_n$:

I	χ		J_0		J_1		$\dots$	
II		g		$\alpha_0 \notin J_0$		$\alpha_1 \notin J_1$	$\dots$	$\chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \leqslant g(n)$

Player II wins if $\chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \leqslant g(n)$ for each $n \in \omega$, otherwise Player I wins.

Note that if κ is regular, the game simplifies even more: Player I plays a coloring $\chi : \kappa^{<\omega} \to \omega$ and Player II responds with a $g \in \omega^\omega$. Then Player I starts playing ordinals $\beta_n < \kappa$ and Player II responds with ordinals $\alpha_n > \beta_n$:

I	χ		β_0		β_1		$\dots$	
II		g		$\alpha_0 > \beta_0$		$\alpha_1 > \beta_1$	$\dots$	$\chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \leqslant g(n)$

Theorem 3.2. *Player* II *has a winning strategy in* Γ *if and only if* $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$.

The next lemma is the key to the main results of this section.

Lemma 3.3. *Suppose that* $\forall g \in \mu^\omega\ \exists G \in \mathcal{G}\ \forall n\ g(n) \in G(n)$. *If* $|\mathcal{G}| < \mathsf{cf}\,\kappa$, *then Player* II *has a winning strategy for* $\Gamma_{\mathcal{G}}$.

Proof. It is easy to check that the game Γ_{G} is closed. By the Gale–Steward theorem it is therefore determined. Suppose that II does not have a winning strategy. Then I has a winning strategy. Thus there is $\chi : \kappa^{<\omega} \to \mu$ such that for each $G \in \mathcal{G}$ I has a winning strategy τ_G for the rest of the game that began with χ and G. Construct a sequence

$$\begin{aligned}
&A_0 = \bigcup \{ \tau_G(\emptyset) : G \in \mathcal{G} \} \\
&\alpha_0 \notin A_0 \\
&A_1 = \bigcup \{ \tau_G(\tau_G(\emptyset), \alpha_0) : G \in \mathcal{G} \} \\
&\alpha_1 \notin A_1 \\
&A_2 = \bigcup \{ \tau_G(\tau_G(\emptyset), \alpha_0, \tau_G(\tau_G(\emptyset), \alpha_0), \alpha_1) : G \in \mathcal{G} \} \\
&\alpha_2 \notin A_2 \\
&\quad \vdots
\end{aligned}$$

Each A_n is a union of $|\mathcal{G}|$ many sets from $[\kappa]^{<\kappa}$ and thus does not cover κ. Therefore it is possible to choose $\alpha_n \notin A_n$. For each $\in \omega$ put $f(n) = \chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$. By assumption, there is $G \in \mathcal{G}$ such that $f(n) \in G(n)$ for all $n \in \omega$. Let I play his winning strategy for G and II the ordinals α_n. Then obviously II follows the rules and $\chi(\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in G(n)$, so II wins, a contradiction. $\square$

We now derive three theorems that say that if κ is sufficiently large, then it satisfies $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$ or even more. The following lemma lets us consider situations with an uncountable number of colors.

Lemma 3.4. *If* $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$ *and* $\mathsf{cf}\,\kappa > \omega_n$*, then* $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega_n}$.

Proof. It is easy to verify that there exists a family $\mathcal{H} \subseteq [\omega_n]^\omega$ of cardinality ω_n that is cofinal in $[\omega_n]^\omega$. Let $\mathcal{G}$ be the family of all $\mathcal{H}$–valued constant sequences. Then obviously $|\mathcal{G}| = \omega_n < \mathsf{cf}\,\kappa$. For each $f \in \omega_n^\omega$ there is $H \in \mathcal{H}$ such that $f''\omega \subseteq H$. Therefore both conditions of Lemma 3.3 are met and we can infer from Theorem 3.1 that $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega_n,\omega}$. Combine with $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$ to conclude that $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega_n}$. □

Theorem 3.5. *If* $\mathsf{cf}\,\kappa > \mathfrak{d}$*, then* $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega_1}$.

Proof. According to Lemma 3.4 we only have to prove $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$. Let $\{g_\alpha : \alpha < \mathfrak{d}\} \subseteq \omega^\omega$ be a dominating family. Define $G_\alpha : n \mapsto [0, g_\alpha(n)]$ and set $\mathcal{G} = \{G_\alpha : \alpha < \mathfrak{d}\}$. Apply the lemma once more and use condition (3.1). □

This theorem is, in a sense, sharp: By the next proposition that follows at once from Corollary 5.6(ii), *infra* $\mathfrak{d} \rightsquigarrow (\mathfrak{d})^{<\omega}_{\omega}$ fails.

Proposition 3.6. *If* $\mathsf{cf}\,\kappa = \mathsf{cf}\,\mathfrak{d}$*, then* $\kappa \not\rightsquigarrow (\kappa)^{<\omega}_{\omega}$*. In particular,* $\mathfrak{d} \not\rightsquigarrow (\mathfrak{d})^{<\omega}_{\omega}$.

We now show that strengthening the condition $\mathsf{cf}\,\kappa > \mathfrak{d}$ in Theorem 3.5 a bit yields more.

Theorem 3.7. *If* $\mathsf{cf}\,\kappa > \mathsf{cof}(\mathcal{N})$*, then* $\forall g \in \boldsymbol{\mathcal{C}}\ \kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega_1,g}$.

Proof. Let $g \in \boldsymbol{\mathcal{C}}$. Recall that a mapping $S : \omega \to [\omega]^{<\omega}$ is called a *g-slalom* if $\forall n \in \omega\ |S(n)| \leqslant g(n)$. By T. Bartoszyński's results [2] (or see [3, 2.3.9]), $\mathsf{cof}(\mathcal{N})$ is the size of the minimal family $\mathcal{S}$ of g-slaloms such that $\forall f \in \omega^\omega\ \exists S \in \mathcal{S}\ \forall n\ f(n) \in S(n)$. Hence Lemma 3.3 and (3.1) yield $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,g}$. To get $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega_1,g}$, argue the same way as in the proof of 3.5. □

Theorem 3.8. *If* $\mathsf{cf}\,\kappa > \mathfrak{c}$*, then the following are equivalent.*

(i) $\mu^\omega < \mathsf{cf}\,\kappa$,
(ii) $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\mu}$,
(iii) $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\mu,1}$,
(iv) $\mathsf{cf}(\kappa) \rightsquigarrow (\mathfrak{c}^+)^{<\omega}_{\mu,\mathfrak{c}}$.

We shall need two lemmas.

Lemma 3.9. *If* $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\mu,\nu}$*, then* $\mathsf{cf}\,\kappa \rightsquigarrow (\mathsf{cf}\,\kappa)^{<\omega}_{\mu,\nu}$*. The analogous statement holds for* $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\mu}$.

Proof. Let $\langle \kappa_\alpha : \alpha < \mathsf{cf}\,\kappa \rangle$ be a sequence of regular cardinals converging to κ. For $\beta < \kappa$ let $\bar\beta = \min\{\alpha : \beta < \kappa_\alpha\}$ and for $p = \langle \beta_0\beta_1 \dots \beta_n \rangle \in \kappa^{<\omega}$ let $\bar p = \langle \bar\beta_0 \bar\beta_1 \dots \bar\beta_n \rangle$. Let $\chi : \mathsf{cf}\,\kappa^{<\omega} \to \mu$ be a coloring. Define $\bar\chi : \kappa^{<\omega} \to \mu$ by $\bar\chi(p) = \chi(\bar p)$. By assumption there is a κ-tree $T \subseteq \kappa^{<\omega}$ such that $\mathrm{dg}_T(p) = \kappa$ for each $p \in T$ and $|\bar\chi''(T)_n| \leqslant \nu$ for all $n \in \omega$. The tree we are looking for is $\{\bar p : p \in T\} \subseteq \mathsf{cf}\,\kappa^{<\omega}$. □

Lemma 3.10. *If* $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\mu,\mathfrak{c}}$ *and* $\lambda \rightsquigarrow (\tau)^{<\omega}_{\mathfrak{c}}$*, then* $\kappa \rightsquigarrow (\tau)^{<\omega}_{\mu^\omega}$.

Proof. Let $\chi : \kappa^{<\omega} \to \mu^\omega$. Define $f : \kappa^{<\omega} \to \mu$ as follows. Let $n \in \omega$ and $p = \langle \alpha_1\alpha_2 \dots \alpha_n \rangle \in \kappa^{<\omega}$. Let $i, j \in \omega$ be the unique numbers satisfying $2^i(2j+1) = n$. Set $\xi(p) = \chi(\langle \alpha_1\alpha_2 \dots \alpha_i \rangle)(j)$. Since $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\mu,\mathfrak{c}}$, there is a λ-tree $T \subseteq \kappa^{<\omega}$ such that $|\xi''T| \leqslant \mathfrak{c}$. Let $p = \langle \alpha_1\alpha_2 \dots \alpha_i \rangle \in T$ and $j \in \omega$. There are $\alpha_{i+1}, \dots, \alpha_{2^i(2j+1)} < \kappa$ such that $\langle \alpha_1\alpha_2 \dots \alpha_{2^i(2j+1)} \rangle \in T$, hence

$$\chi\langle \alpha_1\alpha_2 \dots \alpha_i \rangle(j) = \xi\langle \alpha_1\alpha_2 \dots \alpha_{2^i(2j+1)} \rangle \in \xi''T.$$

Therefore $\chi(p) \in (\xi''T)^\omega$. Since p was arbitrary, it follows that $\chi''T \subseteq (\xi''T)^\omega$. Since T is a λ-tree and $|(\xi''T)^\omega| \leqslant \mathfrak{c}^\omega = \mathfrak{c}$, we can apply $\lambda \rightsquigarrow (\tau)^{<\omega}_{\mathfrak{c}}$ to $\chi\restriction T$ and $\kappa \rightsquigarrow (\tau)^{<\omega}_{\mu^\omega}$ follows. $\square$

Proof of Theorem 3.8. (i)$\Rightarrow$(iii): For each $g \in \mu^\omega$ let $\widehat{g} : n \mapsto \{g(n)\}$, put $\mathcal{G} = \{\widehat{g} : g \in \mu^\omega\}$. Apply Lemma 3.3 and (3.1). (iii)$\Rightarrow$(ii) is trivial. (ii)$\Rightarrow$(iv) follows at once from $\mathsf{cf}\,\kappa > \mathfrak{c}$ and Lemma 3.9. (iv)$\Rightarrow$(i): By the assumption $\mathsf{cf}(\kappa) \rightsquigarrow (\mathfrak{c}^+)^{<\omega}_{\mu,\mathfrak{c}}$. By Theorem 2.2 $\mathfrak{c}^+ \rightsquigarrow (\omega)^{<\omega}_{\mathfrak{c}}$. Apply Lemma 3.10 to conclude that $\mathsf{cf}\,\kappa \rightsquigarrow (\omega)^{<\omega}_{\mu^\omega}$. The latter arrow can hold only if $\mathsf{cf}\,\kappa > \mu^\omega$. $\square$

Proposition 3.11. (i) $\mathfrak{c} \not\rightsquigarrow (2)^{<\omega}_{2,1}$.

(ii) *There is a coloring $\chi : \mathfrak{c} \to \omega$ such that any well-pruned $(\chi, 1)$-homogeneous tree $T \subseteq \mathfrak{c}^{<\omega}$ consists of a single branch.*

Proof. (i) For $p \in \mathfrak{c}^{<\omega}$ define $\chi(p) = p(0)(|p|)$. Suppose $T \subseteq \mathfrak{c}^{<\omega}$ is a 1-homogeneous tree such that $|(T)_1| \geqslant 2$ and let $p \neq q \in (T)_1$. Let f and g be infinite branches passing through p and q, respectively. For each n, $\chi(f\restriction n) = p(n)$ and likewise $\chi(g\restriction n) = q(n)$. Since T is 1-homogeneous, it follows that $p(n) = q(n)$ for all n, i.e. $p = q$, a contradiction.

(ii) For $p \in \mathfrak{c}^{<\omega}$ define $\chi(p) = \langle p(i)\restriction|p| : i < |p|\rangle$. Suppose there is a 1-homogeneous tree $T \subseteq \mathfrak{c}^{<\omega}$ that has at least two branches $f, g \in \mathfrak{c}^\omega$. Then there is $p \in T$ and $x \neq y \in \mathfrak{c}$ such that $p^\frown x \subseteq f$ and $p^\frown y \subseteq g$. Let $n \in \omega$ be such that $x(n) \neq y(n)$. Consider $s = f\restriction(n+1)$ and $t = g\restriction(n+1)$. Then $\chi(s)(|p|) = x\restriction(n+1)$ and $\chi(t)(|p|) = y\restriction(n+1)$ and since $x(n) \neq y(n)$, $\chi(s)(|p|) \neq \chi(t)(|p|)$ and *a fortiori* $\chi(s) \neq \chi(t)$. Thus $|\chi''(T)_n| \geqslant 2$. $\square$

Corollary 3.12. *The following are equivalent.*

(i) $\mathsf{cf}\,\kappa > \mathfrak{c}$,
(ii) $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,1}$,
(iii) $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\mathfrak{c},1}$.

Proof. By the above Proposition 3.11(i), if $\mathsf{cf}\,\kappa \leqslant \mathfrak{c}$, then $\mathsf{cf}\,\kappa \not\rightsquigarrow (\mathsf{cf}\,\kappa)^{<\omega}_{\omega,1}$ and thus by Lemma 3.9 $\kappa \not\rightsquigarrow (\kappa)^{<\omega}_{\omega,1}$. On the other hand, if $\mathsf{cf}\,\kappa > \mathfrak{c}$, then by Theorem 3.8 $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\mathfrak{c},1}$. $\square$

Theorem 3.8 yields one more interesting fact.

Corollary 3.13. *For each cardinal λ there is a regular cardinal $\kappa > \lambda$ such that $\kappa \not\rightsquigarrow (\mathfrak{c}^+)^{<\omega}_{<\kappa}$.*

Proof. Let $\mu > \lambda + \mathfrak{c}$ be a cardinal of countable cofinality and $\kappa = \mu^+$. Then $\kappa \leqslant \mu^\omega$ and therefore by Theorem 3.8 (iv) $\kappa \not\rightsquigarrow (\mathfrak{c}^+)^{<\omega}_{\mu}$. $\square$

4. Martin Axiom

In the previous section we established the arrows $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$, $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,g}$ and $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,1}$ for sufficiently big cardinals κ. In this section we prove that if, on the other hand, κ is small enough, then $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$ or even $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,g}$. Note that according to Corollary 3.12, there is no corresponding result for $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,1}$. The proofs are based on the two versions of Martin axiom behind the numbers $\mathfrak{m}_{\sigma\text{-centered}}$ and $\mathfrak{m}_{\sigma\text{-linked}}$.

Theorem 4.1. *If $\kappa < \mathfrak{m}_{\sigma\text{-centered}}$ is regular uncountable, then $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$.*

Proof. Fix a coloring $\chi : \kappa^{<\omega} \to \omega$. Define a poset as follows: $(W, s) \in \mathbb{P}$ if

(i) $W \subseteq \kappa^{<\omega}$ is a finite tree,
(ii) $s \in \omega^{<\omega}$, $|s| \geqslant \operatorname{ht} W$,
(iii) there is a κ-branching tree $S \supseteq W$ of height $|s|$ such that $\forall p \in S\ \chi(p) \leqslant s(|p|)$.

A tree S of (iii) is said to *control* (W, s). For a finite tree $W \subseteq \omega^{<\omega}$ define

$$\mathsf{r}(W) = \min\{\max \operatorname{succ}_W p : p \in W \wedge |p| + 1 < \operatorname{ht} W\} \tag{4.1}$$

and order $\mathbb{P}$ by: $(W, s) \leqslant (W', s')$ if $W \supseteq W'$, $\mathsf{r}(W) \geqslant \mathsf{r}(W')$ and $s \supseteq s'$.

Claim. *$\mathbb{P}$ is a nonempty σ-centered poset.*

Proof. $\mathbb{P}$ is a nonempty because it contains $(\emptyset, \emptyset)$. To prove that it is σ-centered, we show that for each $t \in \omega^{<\omega}$ the family $\{(W, s) \in \mathbb{P} : s = t\}$ is centered. Let $(W_0, t), (W_1, t) \in \mathbb{P}$. Let S_0 and S_1 control, respectively, (W_0, t) and (W_1, t). The tree $S = S_0 \cup S_1$ obviously controls any $(W, t) \in \mathbb{P}$ such that $W \subseteq S$. Hence it is enough to find a finite tree $W \subseteq S$ such that $W \supseteq W_0 \cup W_1$ and $\mathsf{r}(W) \geqslant \max(\mathsf{r}(W_0), \mathsf{r}(W_1))$, what can be easily achieved by adding finitely many branches to the tree $W_0 \cup W_1$ within S. □

Claim. *For each $\alpha < \kappa$ and $n \in \omega$, the following set is dense:*

$$H_{\alpha,n} = \{(W, s) \in \mathbb{P} : \operatorname{ht} W = |s| \geqslant n \wedge \mathsf{r}(W) > \alpha\}$$

Proof. Fix n and α. Let $(W, s) \in \mathbb{P}$ and let S control (W, s). If we succeed to extend s by one level, then we can extend it by induction to level n and adding finitely many branches to the tree W within the new controlling tree we can easily achieve that the new tree W' satisfies $\mathsf{r}(W') > \alpha$. Thus it is enough to find $s' \supseteq s$ such that $|s|' = |s| + 1$ and S' that controls (W, s'). This is done much like in the proof of Proposition 1.2.

Put $m = |s| = \operatorname{ht} S$. Let $\widehat{S} = S \cup \{p^\frown\alpha : \alpha < \kappa \wedge p \in (S)_{m-1}\}$. Define inductively $h : \widehat{S} \to \omega$ as follows: If $p \notin S$, set $h(p) = \chi(p)$. If $p \in (S)_j$, $j < m$, choose $\delta(p) \in \omega$ so that $|\{\alpha \in \operatorname{succ}_{\widehat{S}} p : h(p^\frown\alpha) \leqslant \delta(p)\}| = \kappa$. Such a choice is possible since $\mathsf{cf}\,\kappa > \omega$. Set $h(p) = \delta(p) + \max h''(W)_{j+1}$.

Extend s by letting $s' \restriction m = s$ and $s'(m) = h(\emptyset)$. The new controlling tree is defined recursively by $\emptyset \in S'$ and $\operatorname{succ}_{S'} p = \{\alpha \in \operatorname{succ}_{\widehat{S}} p : h(p^\frown\alpha) \leqslant h(p)\}$.

Since $\operatorname{succ}_{S'} p \supseteq \{\alpha \in \operatorname{succ}_{\widehat{S}} p : h(p^\frown\alpha) \leqslant \delta(p)\}$, the tree S' is by the choice of δ a κ-tree. In order to prove that S' controls (W, s') it is enough to observe that (a) $W \subseteq S'$ and (b) that if $p \in S'$, then $\chi(p) \leqslant s'(|p|)$. To prove (a) we have to verify that $p \in W$ and $\alpha \in \operatorname{succ}_W p$, then $h(p^\frown\alpha) \leqslant h(p)$, which obviously follows from $h(p) \geqslant \max h''(W)_{|p|+1}$. (b) holds, by the induction hypothesis, for $|p| < m$, and if $|p| = m$ then clearly $\chi(p) = h(p) \leqslant h(\emptyset)$. □

Since $\kappa < \mathfrak{m}_{\sigma\text{-centered}}$, there is a filter $\mathsf{G} \subseteq \mathbb{P}$ that intersects all $H_{\alpha,n}$. Put

$$T = \bigcup\{W : (W, s) \in \mathsf{G}\},$$
$$g = \bigcup\{s : (W, s) \in \mathsf{G}\}.$$

It is clear that T is a nonempty tree. We show that it is a κ-tree. Suppose $p \in T$. Then $p \in W$ for some $(W, s) \in \mathsf{G}$. Let $\alpha < \kappa$ be arbitrary and let $n = |p| + 1$. There is $(W', s') \in \mathsf{G} \cap H_{\alpha,n}$. Let $(W'', s'') \in \mathsf{G}$ be the common continuation of (W, s) and (W', s'). Then obviously $p \in W''$ and since $\mathsf{r}(W'') \geqslant \mathsf{r}(W') > \alpha$, there is $\beta > \alpha$ such that $p^\frown\beta \in W'' \subseteq T$. Hence $\operatorname{succ}_T p$ is cofinal in κ. Since κ is regular, $\operatorname{dg}_T p = \kappa$. Conclude that T is a κ-tree.

It remains to show that T is homogeneous. Let $p \in W$, $(W, s) \in \mathsf{G}$. Then $\chi(p) \leqslant s(|p|) = f(|p|)$. Thus $|\chi''(T)_n| \leqslant f(n)$. $\square$

The following theorem obtains by merging the ideas of theorems 4.1 and 2.2.

Theorem 4.2. *If $\kappa < \mathfrak{m}_{\sigma\text{-linked}}$ is regular uncountable, then $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega,g}$ for all $g \in \boldsymbol{\mathcal{C}}$.*

Proof. Fix a coloring $\chi : \kappa^{<\omega} \to \omega$. Define a poset $\mathbb{P}$ as follows: $(W, s) \in \mathbb{P}$ if

(i) $W \subseteq \kappa^{<\omega}$ is a finite tree,
(ii) $s \in ([\omega]^{<\omega})^{<\omega}$, $\forall i < |s|\ |s(i)| \leqslant g(i)$,
(iii) there is a κ-branching tree $S \supseteq W$ of height $|s|$ such that $\forall p \in S\ \chi(p) \in s(|p|)$.

A tree S of (iii) is said to *control* (W, s). For a finite tree $W \subseteq \omega^{<\omega}$ define $\mathsf{r}(W)$ by

$$\mathsf{r}(W) = \min\{\max \operatorname{succ}_W p : p \in W \wedge \operatorname{succ}_W p \neq \emptyset\}$$

and order $\mathbb{P}$ by: $(W, s) \leqslant (W', s')$ if $W \supseteq W'$, $\mathsf{r}(W) \geqslant \mathsf{r}(W')$ and $s \supseteq s'$.

Fix $g \in \boldsymbol{\mathcal{C}}$ and define a function $F \in \omega^{\omega\times\omega}$ by ($\lfloor x \rfloor$ denotes the integer part of x)

$$F(i, n) = \begin{cases} \lfloor \frac{1}{2}(g(n)^{2^{-i-1}} - 1)\rfloor & \text{if } i < n - 1, \\ 0 & \text{if } i \geqslant n - 1, \end{cases}$$

and define

$$\mathbb{Q}_1 = \{(W, s) \in \mathbb{P} : \forall p \in W\ \ \operatorname{dg}_W p = F(|p|, |s|)\},$$
$$\mathbb{Q}_2 = \{(W, s) \in \mathbb{P} : \forall p \in W\ \ \operatorname{dg}_W p \leqslant 2F(|p|, |s|)\}.$$

Claim 1. *For each $(W, s) \in \mathbb{Q}_2$ there exists $s' \supseteq s$ such that $|s|' = |s| + 1$ and $(W, s') \in \mathbb{P}$.*

Proof. Notice that the function F satisfies $2F \leqslant G$, where G is the function defined by (2.1). Thus (i) is proved exactly the same way as the induction step in the proof of Theorem 2.2. $\square$

Claim 2. *For each $(W, s) \in \mathbb{Q}_2$, every $n \in \omega$ and every $\alpha < \kappa$ there is $(W', s') \in \mathbb{Q}_1$ such that $(W', s') \leqslant (W, s)$, $\mathsf{r}(W') > \alpha$ and $\operatorname{ht} W' > n$.*

Proof. First choose $m \in \omega$ such that

$$\forall i < |s| - 1 \quad 2F(i, |s|) < F(i, m), \tag{4.2}$$
$$F(n, m) > 0. \tag{4.3}$$

Then use repeatedly Claim 1 to extend s to level m, i.e. to find $s' \supseteq s$ such that $|s|' = n$ and $(W, s') \in \mathbb{P}$. Since (4.2) yields $\operatorname{dg}_W p < F(|p|, |s'|)$ for all $p \in W$, there is enough room to extend W within the tree S that controls (W, s') to get $W' \supseteq W$ such that first, $\mathsf{r}(W') > \max(\mathsf{r}(W), \alpha)$ and second, $\forall p \in W'\ \ \operatorname{dg}_{W'} p = F(|p|, |s'|)$. Since W' is constructed within S, we get $(W', s') \in \mathbb{Q}_1$. Moreover (4.3) ensures that $\operatorname{dg}_{W'} p > 0$ whenever $p \in (W')_n$, and in particular $\operatorname{ht} W' > n$. $\square$

Since clearly $\mathbb{Q}_1 \subseteq \mathbb{Q}_2$, Claim 2 yields:

Claim 3. *For each $\alpha < \kappa$ and $n \in \omega$, the following set is dense in $\mathbb{Q}_1$:*

$$H_{\alpha,n} = \{(W, s) \in \mathbb{Q}_1 : \operatorname{ht} W > n \wedge \mathsf{r}(W) > \alpha\}$$

Claim 4. $\mathbb{Q}_1$ *is* σ*-linked.*

Proof. We show that for each s satisfying (iii) the family $\{(W,t) \in \mathbb{Q}_1 : t = s\}$ is linked. Let $(W_0, s), (W_1, s) \in \mathbb{Q}_1$. Let S_0 and S_1 control, respectively, (W_0, s) and (W_1, s). Put $W = W_0 \cup W_1$ and $S = S_0 \cup S_1$. Clearly S controls (W, s). Thus $(W, s) \in \mathbb{P}$. Since $\mathrm{dg}_W\, p \leqslant \mathrm{dg}_{W_0}\, p + \mathrm{dg}_{W_1}\, p = 2F(|p|, |s|)$ for all $p \in W$, we have $(W, s) \in \mathbb{Q}_2$. Let $\alpha = \max(\mathsf{r}(W_0), \mathsf{r}(W_1))$. Use Claim 1 to find $(W', s') \in \mathbb{Q}_1$ such that $(W', s') \leqslant (W, s)$ and $\mathsf{r}(W') \leqslant \alpha$. The latter ensures that $(W', s') \leqslant (W_0, s)$ and $(W', s') \leqslant (W_1, s)$, as required. □

The rest of the proof is very much like that of Theorem 4.1. Since $\kappa < \mathfrak{m}_{\sigma\text{-linked}}$, there is a filter $\mathsf{G} \subseteq \mathbb{Q}_1$ that intersects all $H_{\alpha,n}$. Put

$$T = \bigcup\{W : (W, s) \in \mathsf{G}\},$$
$$f = \bigcup\{s : (W, s) \in \mathsf{G}\}.$$

The tree T is a κ-tree. This is proved the same way as the corresponding statement in the proof of Theorem 4.1. It remains to show that T is g-homogeneous. Let $p \in W$, $(W, s) \in \mathsf{G}$. Then $\chi(p) \in s(|p|) = f(|p|)$. Condition (ii) yields $|f(n)| \leqslant g(n)$ for all n. Thus $|\chi''(T)_n| \leqslant g(n)$. The proof is complete. □

We do not know if $\mathfrak{m}_{\sigma\text{-centered}} \rightsquigarrow (\mathfrak{m}_{\sigma\text{-centered}})^{<\omega}_{\omega}$ is consistent, but we do know from Corollary 5.6(ii) *infra* that $\mathfrak{b} \rightsquigarrow (\mathfrak{b})^{<\omega}_{\omega}$ is not:

Proposition 4.3. *If* $\mathsf{cf}\,\kappa = \mathfrak{b}$*, then* $\kappa \not\rightsquigarrow (\kappa)^{<\omega}_{\omega}$*. In particular,* $\mathfrak{b} \not\rightsquigarrow (\mathfrak{b})^{<\omega}_{\omega}$*.*

5. The boundedness property

The arrow $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\omega}$, roughly speaking, requires an existence of a large tree that is homogeneous. The arrows $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\mu}$ and $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\omega,g}$ are modifications of this arrow obtained by altering the notion of "homogeneous". Here we discuss two other variations of the arrow $\kappa \rightsquigarrow (\lambda)^{<\omega}_{\omega}$ that obtain from a weakening of the notion of "large tree".

Call a tree $L \subseteq \kappa^{<\omega}$ a *Laver* λ*-tree* if there is $s \in \kappa^{<\omega}$ (called the *stem* of L) and a λ-tree $T \subseteq \kappa^{<\omega}$ such that $L = \{s^\frown t : t \in T\}$.

Definition 5.1. (i) Write $\kappa \rightsquigarrow_{\mathsf{w}} (\lambda)^{<\omega}_{\omega}$ to abbreviate: For every coloring $\kappa^{<\omega} \to \omega$ there is a χ-homogeneous well-pruned tree of cardinality λ. This arrow will be called the *weak arrow.*

(ii) Write $\kappa \rightsquigarrow_{\mathsf{L}} (\lambda)^{<\omega}_{\omega}$ to abbreviate: For every coloring $\kappa^{<\omega} \to \omega$ there is a χ-homogeneous Laver λ-tree. This arrow will be called the *Laver arrow.*

Obviously

$$\kappa \rightsquigarrow (\lambda)^{<\omega}_{\omega} \Rightarrow \kappa \rightsquigarrow_{\mathsf{L}} (\lambda)^{<\omega}_{\omega} \Rightarrow \kappa \rightsquigarrow_{\mathsf{w}} (\lambda)^{<\omega}_{\omega}.$$

It turns out that the weak arrow is related to the following combinatorial property of a cardinal. We shall need to distinguish between two orders on ω^ω: The *pointwise order* $\leqslant$ defined by $f \leqslant g$ iff $\forall n \in \omega\ f(n) \leqslant g(n)$, and the *modulo finite order* $\leqslant^*$ defined above.

Definition 5.2. Let $\kappa \leqslant \mathfrak{c}$ be a cardinal. Say that κ has the *boundedness property* if

$$\forall F \in [\omega^\omega]^\kappa\ \exists F' \in [F]^\kappa\ \exists g \in \omega^\omega\ \forall f \in F'\ f \leqslant g,$$

i.e. if every set $F \subseteq \omega^\omega$ of size κ has a pointwise bounded subset of size κ.

Lemma 5.3. (i) *If $\kappa \leqslant \mathfrak{c}$ has the boundedness property, then so does $\mathsf{cf}\,\kappa$.*
(ii) *If $\mathsf{cf}\,\kappa = \omega$, then κ does not have the boundedness property.*
(iii) *The cardinals $\mathfrak{b}$, $\mathfrak{d}$ and $\mathsf{cf}\,\mathfrak{d}$ do not have the boundedness property.*
(iv) *Every cardinal $\kappa < \mathfrak{b}$ of uncountable cofinality has the boundedness property.*
(v) *Every regular cardinal $\kappa \in (\mathfrak{d}, \mathfrak{c}]$ has the boundedness property.*

Proof. (i) Write $\lambda = \mathsf{cf}\,\kappa$. We may suppose $\lambda < \kappa$. Let $F = \{g_\alpha : \alpha < \lambda\} \in [\omega^\omega]^\lambda$. Let $\langle \kappa_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of regular cardinals converging to κ. For each $\alpha < \lambda$ choose a set $G_\alpha \in [\omega^\omega]^{\kappa_\alpha}$ bounded pointwise from below by g_α in such a way that the sets F_α are disjoint. Consider the set $G = \bigcup_{\alpha<\lambda} G_\alpha \in [\omega^\omega]^\kappa$. By the boundedness property of κ there is a bounded set $G' \in [G]^\kappa$. Let $F' = \{g_\alpha : G_\alpha \cap G \neq \emptyset\}$. Clearly $|F'| = \kappa$ and since G' is bounded, so is F'.

(ii) In view of (i) it is enough to show that ω does not have the boundedness property. For $n \in \omega$ let $g_n \in \omega^\omega$ be the constant function attaining at each point value n. It is clear that every bounded subset of $\{g_n : n \in \omega\}$ is finite.

(iii) Consider first the case $\kappa = \mathfrak{b}$. Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be a $\leqslant^*$-unbounded sequence arranged so that $\alpha < \beta \Rightarrow f_\alpha \leqslant^* f_\beta$. Suppose for the contradiction that there are $g \in \omega^\omega$ and $I \in [\mathfrak{b}]^\mathfrak{b}$ such that $\{f_\alpha : \alpha \in I\}$ is $\leqslant$-bounded by g. There is $\alpha < \mathfrak{b}$ such that $f_\alpha \not\leqslant^* g$, and since I is cofinal in $\mathfrak{b}$, there is $\beta \in I$, $\beta > \alpha$. Thus $f_\beta \geqslant^* f_\alpha \not\leqslant^* g$. It follows that $f_\beta(n) > g(n)$ for infinitely many n, which contradicts $f_\beta \leqslant g$.

Now consider the case $\kappa = \mathsf{cf}\,\mathfrak{d}$. Let $\{f_\alpha : \alpha < \mathfrak{b}\}$ be a $\leqslant^*$-dominating sequence arranged so that $\alpha < \beta \Rightarrow f_\alpha \not\geqslant^* f_\beta$. Let $\alpha_\delta \nearrow \kappa$ be a cofinal sequence of order type κ. Suppose for the contradiction that there are $g \in \omega^\omega$ and $I \in [\kappa]^\kappa$ such that $\{f_{\alpha_\delta} : \delta \in I\}$ is $\leqslant$-bounded by g. There is $\alpha < \mathfrak{d}$ such that $g \leqslant^* f_\alpha$. Since $\{\alpha_\delta : \delta \in I\}$ is cofinal in $\mathfrak{d}$, there is $\delta \in I$ such that $\alpha < \alpha_\delta$. Thus $g \leqslant^* f_\alpha \not\geqslant^* f_{\alpha_\delta}$. It follows that $g(n) < f_{\alpha_\delta}(n)$ for infinitely many n, which contradicts $f_{\alpha_\delta} \leqslant g$.

The case $\kappa = \mathfrak{d}$ follows at once from the above and (i).

(iv) Any set of cardinality $\kappa < \mathfrak{b}$ is $\leqslant^*$-bounded and therefore is a countable union of $\leqslant$-bounded sets. Since $\mathsf{cf}\,\kappa > \omega$, one of these sets must be of cardinality κ.

(v) By Theorem 3.5, $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$ and *a fortiori* $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$. The rest follows from the following Theorem 5.4(i). $\square$

Theorem 5.4. *Let κ be an uncountable cardinal.*
(i) *If $\kappa \leqslant \mathfrak{c}$, then $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$ if and only if κ has the boundedness property.*
(ii) *If $\kappa \geqslant \mathfrak{d}$, then $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$ if and only if either $\mathsf{cf}\,\kappa \geqslant \mathfrak{c}$ or $\mathsf{cf}\,\kappa \leqslant \mathfrak{c}$ has the boundedness property.*

Proof. (i)$\Rightarrow$: Let $\chi : \kappa \to \omega$ be a coloring. For each $s \in \kappa^{<\omega}$ choose a cofinal branch $\widehat{s} \supseteq s$ and $g_s \in \omega^\omega$ such that $g_s(n) \geqslant \chi(\widehat{s}\restriction n)$ for all $n \in \omega$ in such a way that $s \neq s' \Rightarrow g_s \neq g_{s'}$. Consider the set $F = \{g_s : s \in \kappa^{<\omega}\}$. Obviously $|F| = \kappa$, therefore, by the boundedness property, there is $S \in [\kappa^{<\omega}]^\kappa$ such that $F' = \{g_s : s \in S\}$ is bounded, i.e. there is $g \in \omega^\omega$ such that $g_s \leqslant g$ for all $s \in S$. Let $T = \{\widehat{s}\restriction n : s \in S \wedge n \in \omega\}$. The tree T is obviously well-pruned and $|T| = \kappa$. If $p = \widehat{s}\restriction n \in T$, then $\chi(p) = \chi(\widehat{s}\restriction n) \leqslant g_s(n) \leqslant g(n) = g(|p|)$. Thus T is g-homogeneous.

(i)$\Leftarrow$: Suppose that κ does not have the boundedness property and let $F \in [\omega^\omega]^\kappa$ witness that. Define a coloring $\chi : F^{<\omega} \to \omega$ by

$$\chi(s) = \sum_{i<|s|} \sum_{j \leqslant |s|} s(i)(j).$$

Suppose that $T \subseteq F^{<\omega}$ is a well-pruned χ-homogeneous tree. We shall show that $|T| < \kappa$. Let $g \in \omega^\omega$ be such that T is (χ, g)-homogeneous. Let

$$F' = \{f \in F : \exists s \in T\ \exists i < |s|\ s(i) = f\} = \{s(i) : s \in T \wedge i < |s|\}.$$

Let $f = s(i) \in F'$ and $n \in \omega$. Since T is well-pruned, there is $t \supseteq s$ such that $|t| = n$. Hence $f(n) = s(i)(n) = t(i)(n) \leqslant \chi(t) \leqslant g(|t|) = g(n)$. It follows that F' is bounded (by g) and thus $|F'| < \kappa$. Since clearly $T \subseteq F'^{<\omega}$, we also have $|T| < \kappa$, as required.

(ii)$\Rightarrow$: Suppose $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$. Then also $\mathsf{cf}\,\kappa \rightsquigarrow_{\mathsf{w}} (\mathsf{cf}\,\kappa)^{<\omega}_\omega$. This is proved the same way as Lemma 3.9. So if $\mathsf{cf}\,\kappa < \mathfrak{c}$, then by (i) $\mathsf{cf}\,\kappa$ has the boundedness property.

(ii)$\Leftarrow$: If $\mathsf{cf}\,\kappa \geqslant \mathfrak{c}$, then, by Theorem 3.5, $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$ and *a fortiori* $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$. The case $\kappa = \mathfrak{d}$ is covered by Lemma 5.3, so suppose that $\kappa > \mathfrak{d}$ and that $\mathsf{cf}\,\kappa \leqslant \mathfrak{c}$ has the boundedness property. Denote $\lambda = \mathsf{cf}\,\kappa$. Let $\langle \kappa_\alpha : \alpha < \lambda \rangle$ be an increasing sequence of regular cardinals converging to κ. We may suppose $\kappa_0 > \mathfrak{d}$. Let $\chi : \kappa \to \omega$ be a coloring. Theorem 3.5 yields for each $\alpha < \lambda$ a sequence $g_\alpha \in \omega^\omega$ and a (χ, g_α)-homogeneous κ_α-tree $T_\alpha \subseteq \kappa^{<\omega}$. Consider the set $F = \{g_\alpha : \alpha < \lambda\}$. Since λ has the boundedness property, there is $g \in \omega^\omega$ and a set $I \in [\lambda]^\lambda$ such that $\{g_\alpha : \alpha \in I\}$ is bounded by g. Consider the tree $T = \bigcup_{\alpha \in I} T_\alpha$. It is obviously well-pruned and $|T| \geqslant \sup_{\alpha \in I} |T_\alpha| = \kappa$. If $p \in T$, then there is $\alpha \in I$ such that $p \in T_\alpha$, hence $\chi(p) \leqslant g_\alpha(|p|) \leqslant g(|p|)$. Thus T is g-homogeneous and $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$ follows. □

Corollary 5.5. (i) *If $\kappa < \mathfrak{b}$, then $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$ if and only if $\mathsf{cf}\,\kappa > \omega$.*
(ii) *If $\mathsf{cf}\,\kappa > \mathfrak{d}$, then $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$.*
(iii) *If $\mathsf{cf}\,\kappa = \omega$ or $\mathsf{cf}\,\kappa = \mathfrak{b}$ or $\mathsf{cf}\,\kappa = \mathsf{cf}\,\mathfrak{d}$, then $\kappa \not\rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$.*

The latter is particularly simple when κ is assumed regular:

Corollary 5.6. (i) *If $\kappa \notin [\mathfrak{b}, \mathfrak{d}]$ is regular uncountable, then $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$,*
(ii) *$\mathfrak{b} \not\rightsquigarrow_{\mathsf{w}} (\mathfrak{b})^{<\omega}_\omega$, $\mathfrak{d} \not\rightsquigarrow_{\mathsf{w}} (\mathfrak{d})^{<\omega}_\omega$.*

The boundedness property and the weak arrow yield the following equivalence of the basic arrow $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$ and the Laver arrow.

Proposition 5.7. *Let $\kappa \leqslant \mathfrak{c}$. Then $\kappa \rightsquigarrow_{\mathsf{L}} (\kappa)^{<\omega}_\omega$ if and only if $\kappa \rightsquigarrow (\kappa)^{<\omega}_\omega$.*

Proof. First of all note that since $\kappa \rightsquigarrow_{\mathsf{L}} (\kappa)^{<\omega}_\omega$, then $\kappa > \omega$ and we also have $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_\omega$. Thus κ has by Theorem 5.4(i) the boundedness property.

Let $\chi : \kappa^{<\omega} \to \omega$ be a coloring. Define a rank function $\mathsf{rk} : \kappa^{<\omega} \to \mathrm{On}$ as follows:

- $\mathsf{rk}(s) = 0$ if there is a χ-homogeneous Laver κ-tree with stem s,
- $\mathsf{rk}(s) \leqslant \alpha$ if $|\{\beta < \kappa : \mathsf{rk}(s^\frown\beta) < \alpha\}| = \kappa$,
- $\mathsf{rk}(s) = \infty$ otherwise.

We first show that $\mathsf{rk}(s) < \infty$ for all $s \in \kappa^{<\omega}$. Suppose the contrary. Then one recursively constructs a Laver κ-tree T with stem t such that $\mathsf{rk}(p) = \infty$ for every $p \in T$ with $p \supseteq s$. Since $\kappa \rightsquigarrow_{\mathsf{L}} (\kappa)^{<\omega}_\omega$, there is a Laver κ-tree $L \subseteq T$ which is homogeneous. Thus $\mathsf{rk}(s) = 0$, where s is the stem of S: a contradiction.

To finish the proof, it suffices to observe that $\mathsf{rk}(\emptyset) = 0$. In fact, $\mathsf{rk}(t) = 0$ for all $t \in \kappa^{<\omega}$. Indeed, if not, then there is $t \in \kappa^{<\omega}$ such that $\mathsf{rk}(t) = 1$, i.e. there is $Y \in [\kappa]^\kappa$ such that for every $\beta \in Y$ there is $g_\beta \in \omega^\omega$ and a Laver κ-tree T_β with stem $s_\beta = t^\frown\beta$ which is g_β-homogeneous. Since κ has the boundedness property, there is $g \in \omega^\omega$ and $Y' \in [Y]^\kappa$ such that $g_\beta \leqslant g$ for every $\beta \in Y'$. This, however, means that the tree $T = \bigcup_{\beta \in Y'} T_\beta$ is a Laver κ-tree with stem t which is g-homogeneous, i.e. $\mathsf{rk}(t) = 0$: a contradiction. $\square$

Corollary 5.8. *Let κ be regular. Then $\kappa \rightsquigarrow_{\mathsf{L}} (\kappa)^{<\omega}_{\omega}$ if and only if $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$.*

6. Remarks and questions

Consistency results.

Theorem 6.1. *Each of the following is relatively consistent with $\mathfrak{c}$ being arbitrarily large:*

(i) $\omega_1 \not\rightsquigarrow (\omega_1)^{<\omega}_{\omega} \wedge \forall \kappa > \omega_1\ \kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$,
(ii) $\mathfrak{c} \not\rightsquigarrow (\mathfrak{c})^{<\omega}_{\omega} \wedge \forall \kappa \neq \mathfrak{c}\ \kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$,
(iii) $\forall \kappa \leqslant \mathfrak{c}\ \kappa \not\rightsquigarrow (\kappa)^{<\omega}_{\omega}$.

Proof. (i) holds, by Theorem 3.5, in any model of $\mathfrak{d} = \omega_1$.

(ii) holds, by Theorems 3.5 and 4.1, in any model of Martin's Axiom.

(iii) Let λ be a regular cardinal. Let G be $\mathrm{Fn}(\lambda)$-generic over a model of CH. Then G naturally codes a λ-sized set F of functions from ω^ω, such that $V[G] \vDash$ "$\forall g \in \omega^\omega\ |\{f \in F : f \leqslant g\}| \leqslant \omega$". Of course $\lambda = \mathfrak{c}$ in $V[G]$. It follows that every $\kappa \leqslant \mathfrak{c}$ fails to have the boundedness property. Thus $\kappa \not\rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_{\omega}$ by Theorem 5.4 and *a fortiori* $\kappa \not\rightsquigarrow (\kappa)^{<\omega}_{\omega}$ for all $\kappa \leqslant \mathfrak{c}$ in the model. $\square$

We also know that consistently there is $\kappa \in (\mathfrak{b}, \mathfrak{d})$ such that $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$ and that, also consistently, there is $\kappa \in (\mathfrak{m}_{\sigma\text{-centered}}, \mathfrak{b})$ such that $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega}$:

Theorem 6.2. *Each of the following is relatively consistent:*

(i) $\mathfrak{b} = \omega_1$, $\mathfrak{d} = \omega_3$ *and* $\omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega}$,
(ii) $\mathfrak{m}_{\sigma\text{-centered}} = \omega_1$, $\mathfrak{b} = \omega_3$ *and* $\omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega}$.

Proof. (i) Let V be a model of $\mathfrak{m}_{\sigma\text{-centered}} = \mathfrak{c} = \omega_3$. Note that by Theorem 4.1 and Lemma 3.4 $\omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega_1}$ holds in V. Let G be $\mathrm{Fn}(\omega_1, 2)$-generic over V (i.e. add ω_1 many Cohen reals over V). It is standard that $\mathfrak{b} = \omega_1$ and $\mathfrak{d} = \omega_3$ in $V[G]$, so the only thing we need to check is $\omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega}$.

Fix a $\mathrm{Fn}(\omega_1, 2)$-name $\dot\chi$ for a coloring of finite ω_2-sequences. As $\mathrm{Fn}(\omega_1, 2)$ is *ccc*, there is, for every $s \in \omega_2^{<\omega}$, an $\alpha_s \in \omega_1$ such that a maximal antichain in $\mathrm{Fn}(\alpha_s, 2)$ decides $\dot\chi(s)$. Define $\xi(s) = \alpha_s$. By the $\omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega_1}$ arrow which holds in V, there is a ξ-homogeneous ω_2-tree. In particular, there is an ω_2-tree in V and $\alpha < \omega_1$ such that $\chi \restriction T$ is in $V[G_\alpha]$. However, $V[G_\alpha]$ is a model of $\mathfrak{m}_{\sigma\text{-centered}} = \omega_3$ by a theorem of J. Roitman [12] (or see [3, Theorem 3.3.8] and hence (in $V[G_\alpha]$) there is a χ-homogeneous ω_2-subtree of T.

(ii) Start with a model V of $\mathfrak{m}_{\sigma\text{-centered}} = \omega_3$ in which there is well-pruned Suslin tree $\mathbb{T}$ on ω_1 (see [3]). Force with $\mathbb{T}$ with reverse order. Then, in $V^{\mathbb{T}}$, $\mathfrak{b} = \omega_3$, $\mathfrak{m}_{\sigma\text{-centered}} = \omega_1$ (see [5]). All we need to show is that $\omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega}$ in $V^{\mathbb{T}}$. As $V \vDash \mathfrak{m}_{\sigma\text{-centered}} > \omega_2$, $V \vDash \omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega_1}$ by the above lemma. Let $\dot\chi$ be a $\mathbb{T}$-name such that $\mathbb{1} \Vdash$ "$\dot\chi : \omega_2^{<\omega} \to \omega$". For each $s \in \omega_2^{<\omega}$ there is $\alpha_s < \omega_1$ and $f_s : \mathbb{T}_{\alpha_s} \to \omega$ such that for every $t \in \mathbb{T}_{\alpha_s}$, $t \Vdash$ "$\dot\chi(s) = f_s(t)$". Note that this is also true for any $\alpha \geqslant \alpha_s$.

Consider $\chi_0 : \omega_2^{<\omega} \to \omega_1$ defined by $\chi_0(s) = \alpha_s$. Since $\mathfrak{m}_{\sigma\text{-centered}} > \omega_2$, using Lemma 3.4, $V \vDash \omega_2 \rightsquigarrow (\omega_2)^{<\omega}_{\omega_1}$, so there is a ω_2-tree T that is χ_0-homogeneous. Let $\alpha < \omega_1$ be such that $\alpha > \alpha_s$ for all $s \in T$. Then for every $s \in \omega_2^{<\omega}$ there is a function $g_s : \mathbb{T}_\alpha \to \omega$ so that $t \Vdash$ "$\dot\chi(s) = g_s(t)$" for every $t \in \mathbb{T}_\alpha$. Now, for $t \in \mathbb{T}_\alpha$

let $\chi_t : T \to \omega$ be defined by $\chi_t(s) = g_s(t)$. For every $t \in \mathbb{T}_\alpha$ there is an ω_2-tree $T_t \subseteq T$ that is χ_t-homogeneous. Then $t \Vdash$ "T_t is χ-homogeneous" and since $\mathbb{T}_\alpha$ is a maximal antichain in $\mathbb{T}$, $\mathbb{1} \Vdash$"There is a $\dot\chi$-homogeneous ω_2-tree". $\square$

Questions 6.3. (i) Is $\mathfrak{b} = \min\{\kappa \text{ regular} : \kappa \not\rightsquigarrow (\kappa)^{<\omega}_{\omega}\}$?
(ii) Is it consistent that $\mathfrak{m}_{\sigma\text{-centered}} \rightsquigarrow (\mathfrak{m}_{\sigma\text{-centered}})^{<\omega}_{\omega}$?

Questions 6.4. (i) Does $\operatorname{\mathsf{cof}}\mathcal{N} \not\rightsquigarrow (\operatorname{\mathsf{cof}}\mathcal{N})^{<\omega}_{\omega,g}$ for some (every) $g \in \boldsymbol{\mathcal{C}}$?
(ii) Does $\operatorname{\mathsf{add}}\mathcal{N} \not\rightsquigarrow (\operatorname{\mathsf{add}}\mathcal{N})^{<\omega}_{\omega,g}$ for some (every) $g \in \boldsymbol{\mathcal{C}}$?
(iii) Is it consistent that $\mathfrak{m}_{\sigma\text{-centered}} \rightsquigarrow (\mathfrak{m}_{\sigma\text{-centered}})^{<\omega}_{\omega,g}$ for some (every) $g \in \boldsymbol{\mathcal{C}}$?

Real–valued measurable cardinal. Recall that a cardinal κ is real–valued measurable if there is a κ-additive probability measure m on κ such that $m(\{\alpha\}) = 0$ for all $\alpha < \kappa$ and every subset of κ is m-measurable. A real-valued measurable cardinal, if it exists, may be smaller than $\mathfrak{c}$. Reference: [6].

Theorem 6.5. *If κ is real–valued measurable, then $\kappa \rightsquigarrow (\kappa)^{<\omega}_{<\kappa}$.*

Proof. Let m be a κ-additive probability measure m over κ. For a tree $T \subseteq \kappa^{<\omega}$ put

$$\widehat{m}(T) = \inf_{p \in T} m\big(\operatorname{succ}_T(p)\big).$$

Obviously, if $\widehat{m}(T) > 0$, then T is a κ-tree. Though $\widehat{m}$ is far from being a measure, it satisfies the following subadditive property that can be verified by straightforward computation. For each countable family $\{T_n : n \in \omega\}$ of trees,

$$1 - \widehat{m}\Big(\bigcap_{n\in\omega} T_n\Big) \leqslant \sum_{n\in\omega} \big(1 - \widehat{m}(T_n)\big). \tag{6.1}$$

Let $\mu < \kappa$ and let $\chi : \kappa^{<\omega} \to \mu$ be a coloring.

Claim. *For each $n \in \omega$, $\varepsilon > 0$ there is a tree $T_{\varepsilon,n} \subseteq \kappa^{<\omega}$ such that $\widehat{m}\big(T_{\varepsilon,n}\big) > 1-\varepsilon$ and $\chi''(T_{\varepsilon,n}{\restriction}n)$ is finite.*

Proof. Induction on n: Put $T_{\varepsilon,0} = \kappa^{<\omega}$. Assume that for $n \in \omega$ the tree $T_{\varepsilon,n}$ is constructed. Consider the tree

$$T = \{p \in \kappa^{<\omega} : |p| \leqslant n \wedge p{\restriction}n-1 \in T_{\varepsilon,n}\}$$

and for $p \in T$ define inductively $h(p) \in [\mu]^{<\omega}$ as follows: $h(p) = \{\chi(p)\}$ if $p \in (T)_n$, and if $j < n$ and $h{\restriction}(T)_{j+1}$ is constructed, for $p \in (T)_j$, $F \in [\mu]^{<\omega}$ set

$$A(p, F) = \{\alpha < \kappa : p^\frown\alpha \in T \wedge h(p^\frown\alpha) \subseteq F\}. \tag{6.2}$$

Since $|[\mu]^{<\omega}| = \mu < \kappa$ and m is κ-additive, according to the induction hypothesis there is $F_p \in [\mu]^{<\omega}$ such that $\widehat{m}(A(p, F_p)) > 1 - \varepsilon$. Put $h(p) = F_p$.

When h is constructed, define a tree S by $(S)_0 = \{\emptyset\}$ and for $j < n$

$$(S)_{j+1} = \{p^\frown\alpha : p \in (S)_j \wedge \alpha \in A(p, h(p))\}. \tag{6.3}$$

To see that $\chi''S$ is finite note that (6.2) and (6.3) imply that if $p, q \in S$ and $p \subseteq q$, then $h(p) \supseteq h(q)$, whence $\chi''(S)_n \subseteq h(\emptyset) \in [\mu]^{<\omega}$, and that $S{\restriction}n \subseteq T_{n,\varepsilon}$. Eventually set $T_{\varepsilon,n+1} = \{p \in \kappa^{<\omega} : p{\restriction}n \in S\}$ and note that $\widehat{m}T_{\varepsilon,n+1} > 1 - \varepsilon$. $\square$

Now use the claim construct, for each $n \in \omega$, the tree $T_{2^{-n-2},n}$ and set $T = \bigcap_{n\in\omega} T_{2^{-n-2},n}$. Then (6.1) ensures that $\widehat{m}(T) \geqslant 1 - 2\cdot 2^{-2} > 0$, hence T is a κ-tree, and $|\chi''T{\restriction}n| \leqslant |\chi''T_{\varepsilon\cdot 2^{-n},n}{\restriction}n| < \omega$, as required. $\square$

Combining this theorem with Propositions 3.6 and 4.3 yields the following facts known also to D. H. Fremlin [6].

Corollary 6.6 ([6])**.** *$\mathfrak{b}$ and $\mathsf{cf}\,\mathfrak{d}$ are not real-valued measurable.*

Weaker arrows. Besides the weak arrow $\kappa \rightsquigarrow_{\mathsf{w}} (\lambda)^{<\omega}_{\mu}$ and Laver arrow $\kappa \rightsquigarrow_{\mathsf{L}} (\lambda)^{<\omega}_{\mu}$ that were introduced in Definition 5.1 we also consider the *Miller arrow* $\kappa \rightsquigarrow_{\mathsf{M}} (\lambda)^{<\omega}_{\mu}$ that obtains by defining a tree to be large if it is a Miller tree. Call a nonempty tree $T \subseteq \kappa^{<\omega}$ a *Miller λ-tree* if $\forall s \in T\ \exists t \supseteq s\ \operatorname{dg}_T t = \lambda$. Write $\kappa \rightsquigarrow_{\mathsf{M}} (\lambda)^{<\omega}_{\omega}$ to abbreviate: For every coloring $\kappa^{<\omega} \to \omega$ there is a χ-homogeneous Miller λ-tree.

Miller arrow is justified by the following: It is enough for a uniform embedding of λ^ω into a sufficiently ample completely metrizable space. This follows by analysis of the proof of Theorem 1.3. Let d denote the usual least distance metric on λ^ω.

Proposition 6.7. *Let (X,ρ) be a completely metrizable metric space such that each nonempty open subset of X has weight at least κ. If $\kappa \rightsquigarrow_{\mathsf{M}} (\lambda)^{<\omega}_{\omega}$, then (X,ρ) contains a closed uniform copy of the space (λ^ω, d).*

As to the weak arrow, we know from Corollary 5.6 that if $\kappa \notin [\mathfrak{b},\mathfrak{d}]$ is regular uncountable, then $\kappa \rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_{\omega}$ and that $\mathfrak{b} \not\rightsquigarrow_{\mathsf{w}} (\mathfrak{b})^{<\omega}_{\omega}$ and $\mathfrak{d} \not\rightsquigarrow_{\mathsf{w}} (\mathfrak{d})^{<\omega}_{\omega}$. The following is a consequence of the proof of Theorem 6.1(iii).

Proposition 6.8. *It is relatively consistent with $\mathfrak{c}$ being arbitrarily large that $\kappa \not\rightsquigarrow_{\mathsf{w}} (\kappa)^{<\omega}_{\omega}$ for each uncountable $\kappa \leqslant \mathfrak{c}$.*

Question 6.9. It is clear that

$$\kappa \rightsquigarrow (\lambda)^{<\omega}_{\omega} \Rightarrow \kappa \rightsquigarrow_{\mathsf{L}} (\lambda)^{<\omega}_{\omega} \Rightarrow \kappa \rightsquigarrow_{\mathsf{M}} (\lambda)^{<\omega}_{\omega} \Rightarrow \kappa \rightsquigarrow_{\mathsf{w}} (\lambda)^{<\omega}_{\omega}.$$

Which of these implications are reversible (or consistently not reversible)?

A partial answer is given in Proposition 5.7 and Corollary 5.8: If $\kappa \leqslant \mathfrak{c}$ or if κ is regular, then $\kappa \rightsquigarrow (\kappa)^{<\omega}_{\omega} \Leftrightarrow \kappa \rightsquigarrow_{\mathsf{L}} (\kappa)^{<\omega}_{\omega}$.

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