UNIVERSAL SUBMEASURES AND IDEALS

M. HRUŠÁK AND D. MEZA-ALCÁNTARA

Abstract. The motivation for this work comes from the following general question: Given a class $\mathcal{M}$ of ideals on ω, is there $\mathsf{I} \in \mathcal{M}$ such that for every $\mathsf{J} \in \mathcal{M}$, J is isomorphic to $\mathsf{I} \restriction X$ for some I-positive set X? We show that for the classes of F_σ-ideals and analytic P-ideals there are such "universal" ideals, by using well-known results from Mazur [3] and Solecki [4] which characterize ideals of these classes in terms of lower semicontinuous submeasures. The key fact is that for $\mathbb{Z}$-valued and $\mathbb{Q}$-valued submeasures on $[\omega]^{<\aleph_0}$ there are universal submeasures.

Introduction

By an *ideal on* ω we mean a family I of subsets of the first infinite ordinal ω which satisfies (1) $\emptyset \in \mathsf{I}$, $\omega \notin \mathsf{I}$, (2) if $B \in \mathsf{I}$ and $A \subseteq B$ then $A \in \mathsf{I}$, and (3) if $A, B \in \mathsf{I}$ then $A \cup B \in \mathsf{I}$. Every ideal on ω can be considered as a subspace of the Cantor space 2^ω. When we say that an ideal is F_σ, Borel, analytic, etc, we mean it is with respect to the product topology of the Cantor space.

A *submeasure* on a set X is a real-valued function φ whose domain is a family of subsets of X and satisfies $\varphi(\emptyset) = 0$ and $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$. A submeasure φ is *lower semicontinuos* (lsc) if for any set A in $dom(\varphi)$, any F finite subset of A, $F \in dom(\varphi)$ and $\varphi(A) = \sup\{\varphi(F) : F \in [A]^{<\omega}\}$.

Note that if $dom(\varphi) = [X]^{<\aleph_0}$ then there is a unique lsc submeasure $\overline{\varphi}$ whose domain is $\mathcal{P}(X)$ and $\overline{\varphi} \restriction [X]^{<\aleph_0} = \varphi$. There are two ideals naturaly associated with any lsc submeasure φ on ω:

$$Fin(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}, \text{ and}$$

$$Exh(\varphi) = \{A \subseteq \omega : \lim_{n\to\infty} \varphi(A \setminus n) = 0\}.$$

2000 *Mathematics Subject Classification.* Primary and Secondary: 03E15, 03E05.

Key words and phrases. Submeasures, F_σ-ideals, P-ideals.

The research of first and second authors was partially supported by PAPIIT grant IN102311 and CONACYT grant 177758.

Second author was supported by CIC-UMSNH 9.30 and PROMEP-CA-9382.

K. Mazur in [3] proved that for every F_σ-ideal I, $\mathsf{I} = Fin(\varphi)$ for some lsc submeasure φ.

An ideal I on ω is a *P-ideal* if for every countable subfamily $\{I_n : n < \omega\}$ of I, there is $I \in \mathsf{I}$ such that $|I_n \setminus I| < \infty$ for all $n < \omega$. S. Solecki [4] proved that for each analytic P-ideal I on ω, $\mathsf{I} = Exh(\varphi)$ for some lsc submeasure φ. In particular, all the analytic P-ideals are $F_{\sigma\delta}$. We remark that, in Mazur's (respectively, Solecki's) proof, the construction of a such a lsc submeasure was done by extending an integer-valued (resp. rational-valued) submeasure on $[\omega]^{<\aleph_0}$.

Set theoretic notation we use is standard and follows [2]. In particular, a natural number is identified with the set of all smaller natural numbers. .

1. Universal submeasures

We construct two submeasures on $[\omega]^{<\aleph_0}$. The first, ρ, integer-valued, and the other, ρ', rational-valued as Fraïssé limits. We present a detailed construction of ρ, while ρ' can be constructed by a simple modifications to the cosntruction of ρ.

Theorem 1.1. *There is an integer-valued submeasure ρ (respectively, rational-valued submeasure ρ') on $[\omega]^{<\aleph_0}$ such that:*

For every $a \in [\omega]^{<\aleph_0}$, every $z \notin a$ and every integer-valued (resp. rational-valued) submeasure φ on $\mathcal{P}(a \cup \{z\})$, if $\varphi \restriction a = \rho \restriction a$ (resp $\varphi \restriction a = \rho' \restriction a$), then there is $l \in \omega$ such that $id_a \cup \{(l, z)\}$ is an isomorphism from $\langle a \cup \{l\}, \rho \restriction a \cup \{l\}\rangle$ (resp. $\langle a \cup \{l\}, \rho' \restriction a \cup \{l\}\rangle$) onto $\langle a \cup \{z\}, \varphi\rangle$.

Proof. Let $\{\langle s_n, \varphi_n\rangle : n \in \omega\}$ be an enumeration of the family of all pairs $\langle s, \varphi\rangle$, where $s \in \omega \setminus \{0\}$ and φ is an integer-valued submeasure on $\mathcal{P}(s)$. We can assume that this enumeration satisfies the following conditions for all n and m:

(1) if $\max\{s_n, \varphi_n(s_n)\} < \max\{s_m, \varphi_m(s_m)\}$ then $n < m$, and
(2) if $\max\{s_n, \varphi_n(s_n)\} = \max\{s_m, \varphi_m(s_m)\}$ and $s_n < s_m$ then $n < m$.

Recursively, we define:

- an increasing sequence $\langle M_n : n < \omega\rangle$ of natural numbers, and
- an $\subseteq$-increasing sequence $\langle \rho_n : n < \omega\rangle$ of submeasures on each respective $\mathcal{P}(M_n)$;

satisfying that for every $n < \omega$, every $a \subseteq M_n$ and every $j \leq n$, if $\langle a, \rho_n \restriction a\rangle \cong \langle s_j \setminus \{s_j - 1\}, \varphi_j \restriction (s_j \setminus \{s_j - 1\})\rangle$ then there is $k < M_{n+1}$ such that $\langle a \cup \{k\}, \rho_{n+1} \restriction a \cup \{k\}\rangle \cong \langle s_j, \varphi_j\rangle$.

Define $M_0 = 0$, $\rho_0(\emptyset) = 0$; and for every n, let $\{\langle a_l, m_l, f_l\rangle : l < p_n\}$ an enumeration of the finite set of 3-tuples $\langle a, m, f\rangle$ so that $a \subseteq M_n$, $m \leq n$ and f is an isomorphism from $\langle s_m - 1, \varphi_m \restriction s_m - 1\rangle$ onto $\langle a, \rho_n \restriction a\rangle$. Now we define $M_{n+1} = M_n + p_n$ and $\rho_{n+1} = \bigcup_{l=0}^{p_n} \rho_n^l$ where ρ_n^l is defined on $\mathcal{P}(M_n + l + 1)$ as follows:

(a) If $l = 0$, extend f_0 to an isomorphism f'_0 from $\langle s_{m_0}, \varphi_{m_0}\rangle$ onto $a_0 \cup \{M_n\}$ and define $\rho_n^0(b) = \max\{\varphi_{m_0}(f_0'^{-1}[b]), \rho_n(b \setminus a_0)\}$ for all $b \subseteq M_n + 1$.
(b) If $0 < l < p_n$, extend f_l to an isomorphism f'_l from $\langle s_{m_l}, \varphi_{m_l}\rangle$ onto $a_l \cup \{M_n + l\}$ and define $\rho_n^l(b) = \max\{\varphi_{m_l}(f_l'^{-1}[b]), \rho_n^{l-1}(b \setminus a_l)\}$ for all $b \subseteq M_n + l$.

Let us check that $\rho = \bigcup_n \rho_n$ works. Let a be a finite subset of ω, and suppose $z \notin a$ and φ a submeasure on $\mathcal{P}(a \cup \{z\})$ so that $\rho \restriction a = \varphi \restriction a$. Let m be so that $\langle a, \rho \restriction a\rangle \cong \langle s_m, \varphi_m\rangle$, witnessed by a function h. Clearly $h' = h \cup \{(z, s_m)\}$ induces a submeasure ψ on s_m, which makes h' an isomorphism. By (2), there is $k > m$ so that $\langle s_m + 1, \psi\rangle = \langle s_k, \varphi_k\rangle$. Take $N = \max(a \cup \{k\}) + 1$. Then $a \subseteq M_N$ and consequently, there is $l < p_N$ such that $id_a \cup \{(l, z)\}$ is an isomorphism from $\langle a \cup \{l\}, \rho \restriction a \cup \{l\}\rangle$ onto $\langle a \cup \{z\}, \varphi\rangle$.

An easy modification to the construction of ρ enables us to construct ρ': In conditions (1) and (2) for the ordering on submeasures, replace $\varphi_n(s_n)$ for $\max\{j : (\exists a \subseteq s_n)(\varphi_n(a) = q_j\}$, where $\{q_j : j \in \omega\}$ is a fixed enumeration of the non-negative rational numbers with $q_0 = 0$. This modification works because again, $\langle s_n - 1, \rho_n \restriction s_n - 1\rangle = \langle s_k, \rho_k\rangle$ for some $k < n$. □

Theorem 1.2. *There is a lsc submeasure $\overline{\rho}$ on $\mathcal{P}(\omega)$ such that for all lsc submeasures φ, if $\varphi(a) \in \mathbb{N}$ for all $a \in [\omega]^{<\aleph_0}$ then there is $X \subseteq \omega$ such that $\langle \omega, \varphi\rangle \cong \langle X, \overline{\rho} \restriction X\rangle$.*

Analogously, there is a lsc submeasure $\overline{\rho'}$ on ω such that for all lsc submeasures φ, if $\varphi(a) \in \mathbb{Q}$ for all $a \in [\omega]^{<\aleph_0}$ then exists $X \subseteq \omega$ such that $\langle \omega, \varphi\rangle \cong \langle X, \overline{\rho'} \restriction X\rangle$.

Proof. Consider $\overline{\rho}$ and $\overline{\rho'}$ as the unique lower semicontinuous extensions to $\mathcal{P}(\omega)$ of the submeasures ρ and ρ' from the previous lemma. □

Remark 1.3. From the proof of Lemma 1.1, it is easy to see that the class $\mathcal{K}$ (respectively, $\mathcal{K}'$) of all the integer (resp, rational)-valued submeasures on finite sets is a *Fräissé class* [1], i.e., satisfies:

(1) *hereditarity*: If φ is a submeasure on a and $b \subseteq a$ then $\varphi \restriction b$ is a submeasure on b,

(2) *joint embedding property*: If φ, ψ are submeasures on finite sets a and b, then there is a submeasure χ on a set c and embeddings f and g from $\langle a, \varphi \rangle$ and $\langle b, \psi \rangle$ into $\langle c, \chi \rangle$,
(3) *amalgamation property*: If $f : a \to c$ and $g : a \to d$ are embeddings of $\langle a, \varphi \rangle$ in $\langle c, \chi \rangle$ and $\langle d, \rho \rangle$ respectively, then there is a submeasure ψ on a set b and embeddings f' and g' from c and d to b, respectively, so that $f' \circ f = g' \circ g$, and
(4) $\mathcal{K}$ contains, up to isomorphism, only countably many submeasures and contains submeasures of arbitrarily large finite cardinalities.

In particular, ρ and ρ' are *Fraïssé structures*, i.e. they are countable, *locally finite* (finitely generated substructures are finite) and *ultrahomogeneous*: If f is an isomorphism from $\langle a, \rho \restriction a \rangle$ onto $\langle b, \rho \restriction b \rangle$ (resp, replacing ρ with ρ'), then f is extendable to an automorphism of $\langle \omega, \rho \rangle$ (resp, ρ'). Moreover, ρ (resp, ρ') is the *Fraïssé limit* of $\mathcal{K}$ (resp, $\mathcal{K}'$), i.e., each submeasure in $\mathcal{K}$ (resp, $\mathcal{K}'$) is embedded in ρ (resp, ρ'). Fraïssé limits are unique up to isomorphims, and satisfy the following Ramsey property (see [1]):

Theorem 1.4. *For all $A \subseteq \omega$, either $\langle A, \rho \restriction A \rangle \cong \langle \omega, \rho \rangle$ or $\langle \omega \setminus A, \rho \restriction \omega \setminus A \rangle \cong \langle \omega, \rho \rangle$.* □

2. Universal ideals

Let $\mathcal{M}$ be a class of ideals on ω. We say that an ideal $\mathsf{I} \in \mathcal{M}$ is *universal* for $\mathcal{M}$ if for every ideal $\mathsf{J} \in \mathcal{M}$ there is an I-positive set X such that $\mathsf{J} \cong \mathsf{I} \restriction X$. We say that $\mathsf{I} \in \mathfrak{M}$ is *Fraïssé-universal* for $\mathcal{M}$ if, moreover, for every $A \subseteq \omega$, either $\mathsf{I} \restriction A \cong \mathsf{I}$ or $\mathsf{I} \restriction (\omega \setminus A) \cong \mathsf{I}$.

An immediate consequence of the last theorem is that there are Fraïssé-universal ideals for the class of F_σ-ideals and the class of analytic P-ideals.

Theorem 2.1. *(1) There is a Fraïssé-universal F_σ-ideal.*
(2) There is an Fraïssé-universal analytic P-ideal.

Proof. Let J be an F_σ-ideal. By Mazur's theorem, there is a lsc submeasure φ such that $\mathsf{J} = Fin(\varphi)$. From Mazur's proof, we can assume $\varphi \restriction [\omega]^{<\aleph_0}$ only takes integer values. Then there is $X \in Fin(\rho)^+$ so that $\varphi \restriction [\omega]^{<\aleph_0} \cong \rho \restriction [X]^{<\aleph_0}$. Finally, the unique lower semicontinuous extension of $\rho \restriction [X]^{<\aleph_0}$ is $\rho \restriction \mathcal{P}(X)$ and is isomorphic to φ. Hence, $\mathsf{J} \cong Fin(\rho \restriction [X]^{<\aleph_0})$. The proof of the second part is analogous. □

We pose the following general question:

Question 2.2. *For which families $\mathcal{M}$ of definable ideals on ω is there a (Fraïssé)-universal ideal I? In particular:*

(1) Is there a (Fraïssé)-universal $F_{\sigma\delta}$-ideal?
(2) Is there a (Fraïssé)-universal analytic ideal?

Let us remark that there is no universal ideal I for the class of all Borel ideals: If a Borel ideal I is say Σ^0_α then all restrictions are at most Σ^0_α. As there are Borel ideals of arbitrarily high Borel complexity the claim follows.

Our last question is motivated by [1]:

Question 2.3. *Is the automorphism group of $\langle\omega,\rho\rangle$ (resp. $\langle\omega,\rho'\rangle$) extremely amenable?*

References

[1] Alexander Kechris, Vladimir Pestov and Stevo Todorčević, *Fraïssé limits, Ramsey theory and topological dynamics of isomorphism groups*, Geometric and Fucntional Analysis **15** (2005), no. 1, 106–189.
[2] Kenneth Kunen, *Set theory: An introduction to independence proofs*, North Holland, 1983.
[3] Krzysztof Mazur, *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$*, Fundamenta Mathematicae, **138** (1991), no. 2, 103–111.
[4] Sławomir Solecki, *Analytic Ideals and their Applications*, Annals of Pure and Applied Logic, **99**(1999), 51–72.

Centro de Ciencias Matemáticas, UNAM, Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México.
E-mail address: michael@matmor.unam.mx

Facultad de Ciencias Físico-Matemáticas, UMSNH, Edificio B, Ciudad Universitaria, Morelia, Michoacán, México, 58060.
E-mail address: dmeza@fismat.umich.mx