UNIVERSAL SUBMEASURES AND IDEALS

M. HRUŠÁK AND D. MEZA-ALCÁNTARA

Abstract. The motivation for this work comes from the following general question: Given a class $\mathcal{M}$ of ideals on $\omega$, is there $I \in \mathcal{M}$ such that for every $J \in \mathcal{M}$, $J$ is isomorphic to $I \upharpoonright X$ for some $I$-positive set $X$? We show that for the classes of $F_\sigma$-ideals and analytic P-ideals there are such “universal” ideals, by using well-known results from Mazur [3] and Solecki [4] which characterize ideals of these classes in terms of lower semicontinuous submeasures. The key fact is that for $\mathbb{Z}$-valued and $\mathbb{Q}$-valued submeasures on $[\omega]<^{\aleph_0}$ there are universal submeasures.

Introduction

By an **ideal on $\omega$** we mean a family $I$ of subsets of the first infinite ordinal $\omega$ which satisfies (1) $\emptyset \in I$, $\omega \notin I$, (2) if $B \in I$ and $A \subseteq B$ then $A \in I$, and (3) if $A, B \in I$ then $A \cup B \in I$. Every ideal on $\omega$ can be considered as a subspace of the Cantor space $2^\omega$. When we say that an ideal is $F_\sigma$, Borel, analytic, etc, we mean it is with respect to the product topology of the Cantor space.

A **submeasure** on a set $X$ is a real-valued function $\varphi$ whose domain is a family of subsets of $X$ and satisfies $\varphi(\emptyset) = 0$ and $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$. A submeasure $\varphi$ is **lower semicontinuous** (lsc) if for any set $A$ in $\text{dom}(\varphi)$, any $F$ finite subset of $A$, $F \in \text{dom}(\varphi)$ and $\varphi(A)$ is the supremum of $\varphi(F)$ over all finite subsets $F$ of $A$.

Note that if $\text{dom}(\varphi) = [X]<^{\aleph_0}$ then there is a unique lsc submeasure $\varphi$ whose domain is $\mathcal{P}(X)$ and $\varphi \upharpoonright [X]<^{\aleph_0} = \varphi$. There are two ideals naturally associated with any lsc submeasure $\varphi$ on $\omega$:

$$\text{Fin}(\varphi) = \{ A \subseteq \omega : \varphi(A) < \infty \},$$

and

$$\text{Exh}(\varphi) = \{ A \subseteq \omega : \lim_{n \to \infty} \varphi(A \setminus n) = 0 \}.$$
K. Mazur in [3] proved that for every $F_\sigma$-ideal $I$, $I = \text{Fin}(\varphi)$ for some lsc submeasure $\varphi$.

An ideal $I$ on $\omega$ is a $P$-ideal if for every countable subfamily $\{I_n : n < \omega\}$ of $I$, there is $I \in I$ such that $|I_n \setminus I| < \infty$ for all $n < \omega$. S. Solecki [4] proved that for each analytic $P$-ideal $I$ on $\omega$, $I = \text{Exh}(\varphi)$ for some lsc submeasure $\varphi$. In particular, all the analytic $P$-ideals are $F_{\sigma\delta}$. We remark that, in Mazur’s (respectively, Solecki’s) proof, the construction of such a lsc submeasure was done by extending an integer-valued (resp. rational-valued) submeasure on $[\omega]^{<\aleph_0}$.

Set theoretic notation we use is standard and follows [2]. In particular, a natural number is identified with the set of all smaller natural numbers.

1. Universal submeasures

We construct two submeasures on $[\omega]^{<\aleph_0}$. The first, $\rho$, integer-valued, and the other, $\rho'$, rational-valued as Fraïssé limits. We present a detailed construction of $\rho$, while $\rho'$ can be constructed by a simple modifications to the construction of $\rho$.

**Theorem 1.1.** There is an integer-valued submeasure $\rho$ (respectively, rational-valued submeasure $\rho'$) on $[\omega]^{<\aleph_0}$ such that:

For every $a \in [\omega]^{<\aleph_0}$, every $z \notin a$ and every integer-valued (resp. rational-valued) submeasure $\varphi$ on $\mathcal{P}(a \cup \{z\})$, if $\varphi \upharpoonright a = \rho \upharpoonright a$ (resp $\varphi \upharpoonright a = \rho' \upharpoonright a$), then there is $l \in \omega$ such that $\text{id}_{a \cup \{l\}}$ is an isomorphism from $\langle a \cup \{l\}, \rho \upharpoonright a \cup \{l\} \rangle$ (resp. $\langle a \cup \{l\}, \rho' \upharpoonright a \cup \{l\} \rangle$) onto $\langle a \cup \{z\}, \varphi \rangle$.

**Proof.** Let $\langle \{s_n, \varphi_n\} : n \in \omega \rangle$ be an enumeration of the family of all pairs $\langle s, \varphi \rangle$, where $s \in \omega \setminus \{0\}$ and $\varphi$ is an integer-valued submeasure on $\mathcal{P}(s)$. We can assume that this enumeration satisfies the following conditions for all $n$ and $m$:

1. if $\max\{s_n, \varphi_n(s_n)\} < \max\{s_m, \varphi_m(s_m)\}$ then $n < m$, and
2. if $\max\{s_n, \varphi_n(s_n)\} = \max\{s_m, \varphi_m(s_m)\}$ and $s_n < s_m$ then $n < m$.

Recursively, we define:

- an increasing sequence $\langle M_n : n < \omega \rangle$ of natural numbers, and
- an $\subseteq$-increasing sequence $\langle \rho_n : n < \omega \rangle$ of submeasures on each respective $\mathcal{P}(M_n)$;

satisfying that for every $n < \omega$, every $a \subseteq M_n$ and every $j \leq n$, if $\langle a, \rho_n \upharpoonright a \rangle \cong \langle s_j \setminus \{s_j - 1\}, \varphi_j \upharpoonright (s_j \setminus \{s_j - 1\}) \rangle$ then there is $k < M_{n+1}$ such that $\langle a \cup \{k\}, \rho_{n+1} \upharpoonright a \cup \{k\} \rangle \cong \langle s_j, \varphi_j \rangle$. 

Define \( M_0 = 0, \rho_0(\emptyset) = 0 \); and for every \( n \), let \( \{ (a, m, f) : l < p_n \} \) an enumeration of the finite set of 3-tuples \( (a, m, f) \) so that \( a \subseteq M_n, m \leq n \) and \( f \) is an isomorphism from \( (s_{m-1}, \varphi_m \upharpoonright s_{m-1}) \) onto \( (a, \rho_n \upharpoonright a) \). Now we define \( M_{n+1} = M_n + p_n \) and \( \rho_{n+1} = \bigcup_{l=0}^{p_n} \rho'_n \) where \( \rho'_n \) is defined on \( \mathcal{P}(M_n + l + 1) \) as follows:

(a) If \( l = 0 \), extend \( f_0 \) to an isomorphism \( f'_0 \) from \( (s_m, \varphi_m) \) onto \( a_0 \cup \{ M_n \} \) and define \( \rho'_n(b) = \max\{ \varphi_m(f'_0^{-1}[b]), \rho_n(b \setminus a_0) \} \) for all \( b \subseteq M_n + 1 \).

(b) If \( 0 < l < p_n \), extend \( f_l \) to an isomorphism \( f'_l \) from \( (s_m, \varphi_m) \) onto \( a_l \cup \{ M_n + l \} \) and define \( \rho'_n(b) = \max\{ \varphi_m(f'_l^{-1}[b]), \rho_n^{-1}(b \setminus a_l) \} \) for all \( b \subseteq M_n + l \).

Let us check that \( \rho = \bigcup_n \rho_n \) works. Let \( a \) be a finite subset of \( \omega \), and suppose \( z \notin a \) and \( \varphi \) a submeasure on \( \mathcal{P}(a \cup \{ z \}) \) so that \( \rho \upharpoonright a = \varphi \upharpoonright a \). Let \( m \) be so that \( (a, \rho \upharpoonright a) \cong (s_m, \varphi_m) \), witnessed by a function \( h \). Clearly \( h' = h \cup \{(z, s_m)\} \) induces a submeasure \( \psi \) on \( s_m \), which makes \( h' \) an isomorphism. By (2), there is \( k > m \) so that \( \langle s_{m+1}, \psi \rangle = \langle s_k, \varphi_k \rangle \). Take \( N = \max(a \cup \{ k \}) + 1 \). Then \( a \subseteq M_N \) and consequently, there is \( l < p_N \) such that \( id_a \cup \{ (l, z) \} \) is an isomorphism from \( \langle a \cup \{ l \}, \rho \upharpoonright a \cup \{ l \} \rangle \) onto \( \langle a \cup \{ z \}, \varphi \rangle \).

An easy modification to the construction of \( \rho \) enables us to construct \( \rho' \): In conditions (1) and (2) for the ordering on submeasures, replace \( \varphi_n(s_n) \) for \( \max\{ j : (\exists a \subseteq s_n)(\varphi_n(a) = q_j) \} \), where \( \{ q_j : j \in \omega \} \) is a fixed enumeration of the non-negative rational numbers with \( q_0 = 0 \). This modification works because again, \( \langle s_n - 1, \rho_n \upharpoonright s_n - 1 \rangle = \langle s_k, \rho_k \rangle \) for some \( k < n \).

**Theorem 1.2.** There is a lsc submeasure \( \overline{\rho} \) on \( \mathcal{P}(\omega) \) such that for all lsc submeasures \( \varphi \), if \( \varphi(a) \in \mathbb{N} \) for all \( a \in [\omega]^{<\aleph_0} \) then there is \( X \subseteq \omega \) such that \( \langle \omega, \varphi \rangle \cong \langle X, \overline{\rho} \upharpoonright X \rangle \).

Analogously, there is a lsc submeasure \( \overline{\rho}' \) on \( \omega \) such that for all lsc submeasures \( \varphi \), if \( \varphi(a) \in \mathbb{Q} \) for all \( a \in [\omega]^{<\aleph_0} \) then exists \( X \subseteq \omega \) such that \( \langle \omega, \varphi \rangle \cong \langle X, \overline{\rho}' \upharpoonright X \rangle \).

**Proof.** Consider \( \overline{\rho} \) and \( \overline{\rho}' \) as the unique lower semicontinuous extensions to \( \mathcal{P}(\omega) \) of the submeasures \( \rho \) and \( \rho' \) from the previous lemma.

**Remark 1.3.** From the proof of Lemma 1.1, it is easy to see that the class \( \mathcal{K} \) (respectively, \( \mathcal{K}' \)) of all the integer (resp., rational)-valued submeasures on finite sets is a Frāssé class \([1]\), i.e., satisfies:

1. **hereditarity:** If \( \varphi \) is a submeasure on \( a \) and \( b \subseteq a \) then \( \varphi \upharpoonright b \) is a submeasure on \( b \),
(2) **joint embedding property:** If \( \varphi, \psi \) are submeasures on finite sets \( a \) and \( b \), then there is a submeasure \( \chi \) on a set \( c \) and embeddings \( f \) and \( g \) from \( \langle a, \varphi \rangle \) and \( \langle b, \psi \rangle \) into \( \langle c, \chi \rangle \).

(3) **amalgamation property:** If \( f : a \to c \) and \( g : a \to d \) are embeddings of \( \langle a, \varphi \rangle \) in \( \langle c, \chi \rangle \) and \( \langle d, \rho \rangle \) respectively, then there is a submeasure \( \psi \) on a set \( b \) and embeddings \( f' \) and \( g' \) from \( c \) and \( d \) to \( b \), respectively, so that \( f' \circ f = g' \circ g \), and

(4) \( K \) contains, up to isomorphism, only countably many submeasures and contains submeasures of arbitrarily large finite cardinalities.

In particular, \( \rho \) and \( \rho' \) are Fraïssé structures, i.e. they are countable, **locally finite** (finitely generated substructures are finite) and **ultrahomogeneous:** If \( f \) is an isomorphism from \( \langle a, \rho \upharpoonright a \rangle \) onto \( \langle b, \rho \upharpoonright b \rangle \) (resp, replacing \( \rho \) with \( \rho' \)), then \( f \) is extendable to an automorphism of \( \langle \omega, \rho \rangle \) (resp, \( \rho' \)). Moreover, \( \rho \) (resp, \( \rho' \)) is the Fraïssé limit of \( K \) (resp, \( K' \)), i.e., each submeasure in \( K \) (resp, \( K' \)) is embedded in \( \rho \) (resp, \( \rho' \)). Fraïssé limits are unique up to isomorphims, and satisfy the following Ramsey property (see [1]):

**Theorem 1.4.** For all \( A \subseteq \omega \), either \( \langle A, \rho \upharpoonright A \rangle \cong \langle \omega, \rho \rangle \) or \( \langle \omega \setminus A, \rho \upharpoonright \omega \setminus A \rangle \cong \langle \omega, \rho \rangle \). \( \square \)

### 2. Universal ideals

Let \( \mathcal{M} \) be a class of ideals on \( \omega \). We say that an ideal \( l \in \mathcal{M} \) is **universal** for \( \mathcal{M} \) if for every ideal \( J \in \mathcal{M} \) there is an \( l \)-positive set \( X \) such that \( J \cong l \upharpoonright X \). We say that \( l \in \mathfrak{M} \) is Fraïssé-universal for \( \mathcal{M} \) if, moreover, for every \( A \subseteq \omega \), either \( l \upharpoonright A \cong l \) or \( l \upharpoonright (\omega \setminus A) \cong l \).

An immediate consequence of the last theorem is that there are Fraïssé-universal ideals for the class of \( F_{\sigma} \)-ideals and the class of analytic P-ideals.

**Theorem 2.1.**

1. There is a Fraïssé-universal \( F_{\sigma} \)-ideal.
2. There is a Fraïssé-universal analytic P-ideal.

**Proof.** Let \( J \) be an \( F_{\sigma} \)-ideal. By Mazur’s theorem, there is a lsc submeasure \( \varphi \) such that \( J = \text{Fin}(\varphi) \). From Mazur’s proof, we can assume \( \varphi \upharpoonright [\omega]^{<\aleph_0} \) only takes integer values. Then there is \( X \in \text{Fin}(\rho)^+ \) so that \( \varphi \upharpoonright [\omega]^{<\aleph_0} \cong \rho \upharpoonright [X]^{<\aleph_0} \). Finally, the unique lower semicontinuous extension of \( \rho \upharpoonright [X]^{<\aleph_0} \) is \( \rho \upharpoonright \mathcal{P}(X) \) and is isomorphic to \( \varphi \). Hence, \( J \cong \text{Fin}(\rho \upharpoonright [X]^{<\aleph_0}) \). The proof of the second part is analogous. \( \square \)

We pose the following general question:
Question 2.2. For which families $\mathcal{M}$ of definable ideals on $\omega$ is there a (Fraïssé)-universal ideal $I$? In particular:

1) Is there a (Fraïssé)-universal $F_{\sigma\delta}$-ideal?
2) Is there a (Fraïssé)-universal analytic ideal?

Let us remark that there is no universal ideal $I$ for the class of all Borel ideals: If a Borel ideal $I$ is say $\Sigma^0_\alpha$ then all restrictions are at most $\Sigma^0_\alpha$. As there are Borel ideals of arbitrarily high Borel complexity the claim follows.

Our last question is motivated by [1]:

Question 2.3. Is the automorphism group of $\langle \omega, \rho \rangle$ (resp. $\langle \omega, \rho' \rangle$) extremely amenable?

REFERENCES


