Cofinalities of Borel ideals*

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Abstract

We study the possible values of the cofinality invariant for various Borel ideals on the natural numbers. We introduce the notions of a fragmented and gradually fragmented F_σ ideal and prove a dichotomy for fragmented ideals. We show that every gradually fragmented ideal has cofinality consistently strictly smaller than the cardinal invariant $\mathfrak{b}$ and produce a model where there are uncountably many pairwise distinct cofinalities of gradually fragmented ideals.

1 Introduction

This paper concerns the possibilities for the cofinalities of Borel ideals on ω. Here, an *ideal* is a subset of $\mathcal{P}(\omega)$ closed under subsets and unions; in order to avoid trivialities, we will always assume that the ideal contains all finite sets and is not generated by a countable collection of sets. The space $\mathcal{P}(\omega)$ is equipped with the usual Polish topology, and therefore it makes sense to speak about descriptive set theoretic complexity of ideals on ω. Finally, the *cofinality* of an ideal $\mathcal{I}$, $\mathtt{cof}(\mathcal{I})$, is the least cardinality of a collection $A \subset \mathcal{I}$

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such that every set in the ideal has a superset in the collection A; thus our ideals will always have uncountable cofinality. The cofinality of an ideal is a cardinal number less or equal to the continuum. The comparison of these numbers with traditional cardinal invariants and with each other in various models of set theory carries information about the structure of the underlying ideals. A survey of known results will generate several natural questions and hypotheses, of which we address two.

Question 1.1. What are the possible cofinalities of Borel ideals?

Only four possible uncountable values of standard ideals were known: $\mathfrak{d} = \mathtt{cof}(\mathrm{Fin} \times \mathrm{Fin})$, $\mathtt{cof}(\mathtt{meager}) = \mathtt{cof}(\mathrm{nwd}(\mathbb{Q}))$, $\mathtt{cof}(\mathtt{null}) = \mathtt{cof}(Z)$ and $\mathfrak{c} = \mathtt{cof}(ED)$, where $\mathrm{nwd}(\mathbb{Q})$ is the ideal of nowhere dense subsets of the rationals, Z is the ideal of sets of natural numbers of asymptotic density 0 and ED is the ideal on the square $\omega \times \omega$ generated by vertical sections and graphs of functions. A possible conjecture that these are the only values fails badly, we will produce many F_σ ideals such that the inequalities between their cofinalities can be manipulated arbitrarily in various generic extensions.

Question 1.2. What is the smallest cofinality of a Borel ideal?

It is not difficult to argue that every F_σ ideal has cofinality larger or equal to $\mathtt{cov}(\mathtt{meager})$, and a result of Louveau and Velickovic [3] shows that every non-F_σ Borel ideal has cofinality at least $\mathfrak{d}$. In view of known examples, the natural conjecture was that $\mathfrak{d}$ is, in fact, the smallest possible cofinality of a Borel ideal. We will show that there are F_σ ideals whose cofinality is consistently less than $\mathfrak{b}$, and therefore even this conjecture fails.

The notation of this paper is standard and follows [1]. For a tree $T \subset (\omega \times \omega)^{<\omega}$, the symbol $[T]$ stands for its set of cofinal branches as a subset of $\omega^\omega \times \omega^\omega$, and $p[T]$ is the projection of this set into the first coordinate.

2 The smallest possible cofinality

Regarding the lower bound on the cofinality of a Borel ideal, we first record two known facts.

Fact 2.1. *[3]* $\mathtt{cof}(\mathcal{I}) \geq \mathfrak{d}$ *for every Borel non-F_σ ideal.*

Fact 2.2. *(folklore)* $\mathtt{cof}(\mathcal{I}) \geq \mathtt{cov}(\mathtt{meager})$ *for every F_σ ideal.*

Recall that $\mathtt{cov}(\mathtt{meager}) \leq \mathfrak{d}$ holds in ZFC. Since we do not know an example of an F_σ ideal with cofinality equal to $\mathtt{cov}(\mathtt{meager})$, it seems natural to conjecture that, in fact, $\mathfrak{d}$ is the smallest possible value for the cofinality of any Borel ideal. However, we will identify a whole array of F_σ ideals whose cofinality is equal to $\aleph_1$ in the Laver model. Since in that model, $\aleph_2 = \mathfrak{b} = \mathfrak{d}$, this refutes the conjecture. In view of the results of this paper, it is difficult to replace it with any other reasonable conjecture.

Most ideals discussed in this paper are F_σ, and are in fact of a quite special form that sets them apart from the analytic P-ideals.

Definition 2.3. An ideal $\mathcal{I}$ on ω is *fragmented* if there is a partition of $\omega = \bigcup_j a_j$ into finite sets and submeasures φ_j on each of them such that

$$\mathcal{I} = \{b \subset \omega : \exists k \ \forall j \ \varphi_j(a_j \cap b) < k\}.$$

The ideal $\mathcal{I}$ represented as in the previous sentence is *gradually fragmented* if for every k there is an m such that for all l, for all but finitely many j and for any B subset of $P(a_j)$, if $|B| = l$ and $\varphi_j(b) < k$ (for each $b \in B$), then $\varphi_j(\bigcup B) < m$.

Note that every fragmented ideal is F_σ. The ideal of sets of *polynomial growth* $\mathcal{P} = \{A \subseteq \omega : (\exists k \in \omega)(\forall n \in \omega) \ |A \cap 2^n| \leq n^k\}$ introduced in [3] is a typical example of a gradually fragmented ideal. Many ideals which in retrospect are gradually fragmented were also considered by K. Mazur in [4].

Next we show that any proper forcing notion having the *Laver property* [1] preserves cofinalities of gradually fragmented ideals. As a corollary we get the following:

Theorem 2.4. *In the iterated Laver model,* $\mathtt{cof}(\mathcal{I}) = \aleph_1 < \mathfrak{b} = \mathfrak{c} = \aleph_2$ *for every gradually fragmented ideal* $\mathcal{I}$.

Recall that a forcing notion has the *Laver property* if for every function $f \in \omega^\omega$ in the extension which is dominated by a ground model function, there is a ground model function $g : \omega \to [\omega]^{<\aleph_0}$ such that for every $i \in \omega$, $|g(i)| \leq i + 1$ and $f(i) \in g(i)$. As the terminology suggests, the Laver forcing as well as its countable support iterations have the Laver property (see [1]).

Proposition 2.5. *Let* $\mathbb{P}$ *be a proper forcing notion having the Laver property and let* $\mathcal{I}$ *be a gradually fragmented ideal. Then in the* $\mathbb{P}$*-extension,* $\mathcal{I} \cap V$ *is cofinal in* $\mathcal{I}$.

Proof. Let $\mathcal{I}$ be an ideal gradually fragmented via $\langle a_j : j \in \omega \rangle$ and $\varphi = \sup_j \varphi_j$. Let $\dot{a}$ be an $\mathbb{P}$-name and $p \in \mathbb{P}$ a condition such that $p \Vdash \dot{a} \in \mathcal{I}$. Find $p' \leq p$ and $k \in \omega$ such that $p' \Vdash \varphi(a) < k$. Use the gradual fragmentation to find a number $m \in \omega$ as well as numbers $0 = l_0 < l_1 < l_2 < \ldots$ so that for every $i \in \omega$ and for every l, if $l_i \leq l < l_{i+1}$ and $B \subset P(a_l)$ is a collection of size $\leq i + 1$ consisting of sets of submeasure $< k$, then $\varphi_l(\bigcup B) < m$. Use the Laver property of $\mathbb{P}$ to find a function $g : \omega \to [\omega]^{<\omega}$ (in V) and a condition $q \leq p'$ such that for all $i \in \omega$, the value $g(i)$ is a collection of size $\leq i + 1$ consisting of subsets of $\bigcup_{l_i \leq l < l_{i+1}} a_l$, each set in $g(i)$ has submeasure $< k$, and $q \Vdash \forall i \in \omega \ \dot{a} \cap \bigcup_{l_i \leq l < l_{i+1}} a_l \in g(i)$. Let $b = \bigcup_i \bigcup g(i)$. The properties of the sequence $l_i : i \in \omega$ imply that $\varphi(b) < m$, so $b \in \mathcal{I}$ and clearly $q \Vdash \dot{a} \subset b$. $\square$

The previous result should be contrasted with the provably high cofinality of fragmented ideals which are not gradually fragmented. Recall (e.g. [3]) that a subset P of an ideal $\mathcal{I}$ is *strongly unbounded* if P contains no infinite bounded subset, i.e the union of every infinite subset of P is $\mathcal{I}$-positive. Clearly, every ideal $\mathcal{I}$ which contains a strongly unbounded subset of size $\mathfrak{c}$ has $\mathtt{cof}(\mathcal{I}) = \mathfrak{c}$.

Theorem 2.6. *If* $\mathcal{I}$ *is a fragmented ideal then either*

1. *$\mathcal{I}$ is gradually fragmented, or*

2. *$\mathcal{I}$ contains a perfect strongly unbounded subset.*

Proof. Let $\mathcal{I}$ be fragmented (via $\langle a_j : j \in \omega\rangle$ and $\varphi = \sup_j \varphi_j$) which is not gradually fragmented. If $k \in \omega$ is where graduality fails, then there is an infinite set $C \subset \omega$, a sequence $\langle B_j : j \in C\rangle$ (with $B_j \subset \mathcal{P}(a_j)$ and $\varphi(b) < k$ for all $b \in B_j$) and a partition $\{C_m : m \in \omega\}$ of C into infinite sets, such that for each $m \in \omega$ there is an $l_m \in \omega$ such that:

$$j \in C_m \Rightarrow |B_j| = l_m \text{ and } \varphi(\bigcup B_j) > m.$$

For $j \in C_m$ write $B_j = \{K_i^j : i < l_m\}$. Now, for each $m \in \omega$, let $\{C_m^n : n \in \omega\}$ be a partition of C_m into infinite sets, and set:

$$C^n = \bigcup_{m\in\omega} C_m^n, \quad X_n = \bigcup\{a_j : j \in C^n\} \quad \text{and} \quad X = \bigcup_{n\in\omega} X_n$$

We will use the following simple fact:

Claim 2.7. *For all $N \in \omega$, there is a sequence of functions $\langle f_n : n \in \omega\rangle$ from ω to N such that:*

$$(\forall A \in [\omega]^N)(\exists M \in \omega)([0, N) \subseteq \{f_n(M) : n \in A\})$$

Proof. Fix $N \in \omega$, for each $t \in N^{<\omega}$ define A_t an infinite subset of ω by recursion on the length of t as follows: Let $A_\emptyset = \omega$, if A_t has been defined for all $t \in N^n$, let $\{A_{t^\frown\langle j\rangle} : j < N\}$ be a partition of A_t into infinite sets. Let $f_0 : \omega \to N$ be the function such that $f_0 \restriction A_{\langle j\rangle} = j$ (for each $j \in N$). Define $f_{n+1} : \omega \to N$ by: $f_{n+1} \restriction A_{t^\frown\langle j\rangle} = j$ (for each $t \in N^n$ and $j \in N$). The sequence $\langle f_n : n \in \omega\rangle$ has the desired property: If $A = \{n_0, \ldots, n_{N-1}\} \in [\omega]^N$ is such that $n_i < n_j$, let $t \in N^{n_{N-1}+1}$ such that $t(n_i) = i$, then for $M \in A_t$ and for $i < N$, $f_{n_i}(M) = i$. $\square$

Apply the claim to each C_m^n and $N = l_m$, in order to obtain a sequence of functions $\langle f_p^{\langle n,m\rangle} : p \in \omega\rangle$ from C_m^n to l_m. Then, define a sequence of functions $\langle f_p : p \in \omega\rangle$ from C to ω by:

$$f_p = \bigcup_{n,m\in\omega} f_p^{\langle n,m\rangle}$$

and a sequence $\langle J_p : p \in \omega\rangle$, of subsets of ω:

$$J_p = \bigcup_{j\in C} K^j_{f_p(j)}$$

Clearly $\varphi(J_p) < k$, as each $K^j_{f_p(j)} \subset a_j$ is of φ-mass less than k. For $n, p \in \omega$, let $J_p^n = X_n \cap J_p$.

Claim 2.8. *For each $n, m \in \omega$ and $A \in [\omega]^{l_m}$,*

$$\varphi(\bigcup_{p\in A} J_p^n) > m$$

In particular, for each n, the sequence $\langle J_p^n : p \in \omega\rangle$ is strongly unbounded.

Fix $n, m \in \omega$ and $A \in [\omega]^{l_m}$, by the choice of the sequence $\langle f_p^{n,m} : p \in \omega\rangle$ (Claim 2.7), there is $M \in C_m^n$ such that $[0, l_m) = \{f_p^{\langle n,m\rangle}(M) : p \in A\}$. So

$$\bigcup B_M = \bigcup_{p\in A} K^M_{f_p(M)} \subset \bigcup_{p\in A} J_p^n$$

and $\varphi(\bigcup B_M) > m$.

We now define the perfect strongly unbounded subset of $\mathcal{I}$: Let $\mathcal{A} \subset \omega^\omega$ be a perfect family of eventually-different functions of cardinality $\mathfrak{c}$. Define $G : \mathcal{A} \to \mathcal{I}$ by

$$G(g) = \bigcup_{n\in\omega} (X_n \cap J_{g(n)})$$

It is clear that $\varphi(G(g)) < k$ and that G is a 1-1 well defined function.

Claim 2.9. *The set $G''\mathcal{A}$ is strongly unbounded.*

Let $\langle G_r = G(g_r) : r \in \omega\rangle$ be an infinite subset of $G''\mathcal{A}$. First, observe that, since $\mathcal{A}$ is an eventually-different family of functions, for each $m \in \omega$ there is $L \in \omega$ such that for each $n \geq L$, the set $\{g_r(n) : r \leq l_m\}$ has cardinality l_m. Now, set $m \in \omega$, $n \geq L$ and $A = \{g_r(n) : r \leq l_m\}$. By Lemma 2.7, there is $M \in C_m^n$ such that

$$\varphi(\bigcup_{a\in A} K^M_{f_a(M)}) > m$$

However $\bigcup B_M = \bigcup_{a\in A} K^M_{f_a(M)} \subset \bigcup_{r\in\omega} G_r$. Hence $\varphi(\bigcup_{r\in\omega} G_r) = \infty$.

□

While the cofinality of gradually fragmented ideals is consistently small, it is also true that their cofinality is consistently quite large in comparison to traditional cardinal invariants.

There is a natural forcing associated to every Borel ideal $\mathcal{I}$, which adds a new element of $\mathcal{I}$ not contained in any ground model set in $\mathcal{I}$.

Definition 2.10. Let $\mathcal{I}$ be a Borel ideal. Let J be the σ-ideal on $\mathcal{I}$ generated by the family $\{\mathcal{P}(a) : a \in \mathcal{I}\}$. Denote by $\mathbb{P}_\mathcal{I}$ the forcing $Borel(\mathcal{I})/J$.

The forcing $\mathbb{P}_\mathcal{I}$ falls naturally into the scope of [6]. Formally, one should define J as the σ-ideal on $\mathcal{P}(\omega)$ generated by singletons and the sets in the family $\{\mathcal{P}(a) : a \in \mathcal{I}\}$, hence dealing with the quotient $P_J = Borel(\mathcal{P}(\omega))/J$. The Borel ideal $\mathcal{I}$ is then itself a condition in P_J (recall that $\mathcal{I}$ is not countably generated) and $\mathbb{P}_\mathcal{I}$ is just a restriction of P_J below $\mathcal{I}$. General theorems of [6, Section 4.1] and simple genericity arguments give:

Proposition 2.11. *Let $\mathcal{I}$ be a Borel ideal and let $\mathbb{P}_\mathcal{I}$ be the corresponding forcing. Then:*

1. *$\mathbb{P}_\mathcal{I}$ is proper.*
2. *$\mathbb{P}_\mathcal{I}$ preserves* $\mathtt{non}(\mathtt{meager})$.
3. *$\mathbb{P}_\mathcal{I}$ preserves* $\mathtt{cof}(\mathtt{meager})$ *and preserves P-points, provided that $\mathcal{I}$ is F_σ.*
4. *$\mathbb{P}_\mathcal{I}$ adds an unbounded element of $\mathcal{I}$.*

Proof. Items 1 and 2 follow directly from the fact that the ideal J is σ-generated by compact sets [6, Theorem 4.1.2], item 4 is a straightforward genericity argument (here we use the restriction to $\mathcal{I}$). To see item 3, one only needs to realize that if $\mathcal{I}$ is an F_σ ideal on ω, then the σ-ideal J is, in fact, σ-generated by a σ-compact collection of compact sets. By [6, Theorem 4.1.8] $\mathbb{P}_\mathcal{I}$ is ω^ω-bounding (does not add unbounded reals) which together with (2) implies that $\mathtt{cof}(\mathtt{meager})$ is preserved. The fact that $\mathbb{P}_\mathcal{I}$ preserves P-points is proved yet not stated in [6, Theorem 4.1.8]. □

As a corollary one gets the following:

Theorem 2.12. *It is consistent that* $\mathtt{cof}(\mathtt{meager}) = \aleph_1 < \mathtt{cof}(\mathcal{I}) = \mathfrak{c} = \aleph_2$ *for all uncountably generated F_σ ideals $\mathcal{I}$ at once.*

Proof. To construct the model witnessing the statement of the theorem, start with a model of CH and use a suitable bookkeeping tool to set up a countable support iteration of forcings of the form $\mathbb{P}_\mathcal{I}$ defined above, as $\mathcal{I}$ varies over all possible F_σ ideals in the extension. Suitable iteration theorems show that the iteration is proper, bounding, preserves Baire category (and also preserves P-points). Thus, in the resulting model the desired statement holds.

□

Another property of the forcing $\mathbb{P}_\mathcal{I}$ used heavily in the next section is the *continuoius reading of names*: For every $\mathcal{J}$-positive Borel subset B of $\mathcal{I}$ and a Borel function $f : B \to 2^\omega$ there is a $\mathcal{J}$-positive Borel subset C of B such that f restricted to C is continuous (see [6, Theorem 4.1.2]).

3 Nonclassification of possible cofinalities

This section aims to produce many F_σ ideals whose cofinality invariants can take quite independent values in various generic extensions. These will be gradually fragmented ideals with an additional weak boundedness property.

Definition 3.1. Let $g \in {}^\omega\omega$ be defined by: $g(0) = 2$, and for $k > 0$

$$g(k+1) = g(k)^{g(k)^{g(k)}}.$$

Let $m_0 = 0$, $m_k = g(k)^k$; $n_k = \Sigma_{i\leq k} m_i$, and $a_k = [n_{k-1}, n_k)$ (so $|a_k| = m_k = g(k)^k$) and define φ_k (with support a_k) by $\varphi_k(b) = \log_{g(k)}(|b|)$ (for $k > 0$ and $b \neq \emptyset$, otherwise $\varphi_0(b) = 0 = \varphi_k(\emptyset)$).

Finally, for each infinite set $u \subset \omega$ define $\mathcal{I}_u$ to be the ideal on the countable set $\bigcup_{i\in u} a_i$ given by

$$\mathcal{I}_u = \{b : \varphi(b) < \infty\},$$

where $\varphi(b) = \sup_{i\in u} \varphi_i(b \cap a_i)$.

Some basic properties of this fragmentation, that will be used later, are summarized in the following lemma:

Lemma 3.2. *Let a_k and φ_k be defined as above and let $r_k = \Pi_{i\leq k}|a_k|$. Then:*

1. *$\varphi_k(a_k) = k$.*
2. *Let $\hat{k} > k$ and $b \subseteq a_{\hat{k}}$ such that $|b| \leq r_k$ then $\varphi_{\hat{k}}(b) < (r_k 2^{\hat{k}})^{-1}$.*
3. *Let $\hat{k} > k$, $c, d \subset a_{\hat{k}}$ such that $|c| = |d| - 1$ then $\varphi_{\hat{k}}(d) < \varphi_{\hat{k}}(c) + (r_k 2^{\hat{k}})^{-1}$.*
4. *The* very slow fragmentation *property: $(\forall r, \epsilon > 0)(\exists i)(\forall j > i)(\forall b, c \subset a_j)$ if $\varphi_j(b), \varphi_j(c) < r$, then $\varphi_j(b \cup c) < r + \epsilon$.*

Proof. $\varphi_k(a_k) = \log_{g(k)}(g(k)^k) = k$. For $\hat{k} > k$:

$$\varphi_{\hat{k}}(b) = \log_{g(\hat{k})} |b| \leq \log_{g(\hat{k})} r_k < \log_{g(\hat{k})}(g(k)^{k^2}) = \frac{k^2}{h(\hat{k})} < \frac{1}{r_k 2^{\hat{k}}}$$

where $h(\hat{k})$ is such taht $g(\hat{k}) = g(k)^{h(\hat{k})}$. Also, for $c, d \subset a_{\hat{k}}$,

$$\varphi_{\hat{k}}(d) \leq \log_{g(\hat{k})} 2|c| = \log_{g(\hat{k})} 2 + \varphi_{\hat{k}}(c) < \frac{1}{r_k 2^{\hat{k}}} + \varphi_{\hat{k}}(c)$$

Finally, take i_0 large enough such that $\log_{g(i_0)} 2 < \epsilon$ $\square$

By taking k large enough so that $(r_k 2^{k+1})^{-1} < \epsilon$, we get the following.

Corollary 3.3. *Let $r, \epsilon > 0$. There is $k = k_\epsilon$ such that for any family B of sets b with $\varphi(b) < r$ and $|\{b \cap a_i : b \in B\}| < \Pi_{l<i}|a_l|$, if $j > k$, then $\varphi_j((\bigcup B) \cap a_j) < r + \epsilon$.*

We will show that whenever $u, v \subset \omega$ are almost disjoint infinite sets then the inequalities $\mathtt{cof}(\mathcal{I}_u) > \mathtt{cof}(\mathcal{I}_v)$ and $\mathtt{cof}(\mathcal{I}_v) > \mathtt{cof}(\mathcal{I}_u)$ are both consistent, and this effect can be reached in both iteration-type and product-type extensions. The product method even leads to the consistency of the cofinalities of many of these ideals being mutually distinct at the same time (A somewhat similar result has been proved in [2]).

Theorem 3.4. *It is relatively consistent with* ZFC *that there are uncountably many distinct cofinalities of ideals of the form $\mathcal{I}_u$.*

The basic forcing $\mathbb{P}_{\mathcal{I}}$ to achieve this has already been introduced in proposition 2.11 as the forcing $P_J = Borel(\mathcal{P}(\omega))/J$, where J is the σ-ideal on $\mathcal{P}(\omega)$ generated by singletons and the sets in the family $\{\mathcal{P}(a) : a \in \mathcal{I}\}$ restricted to $\mathcal{I}$ (considered as a condition of P_J). Here we will strengthen the initial condition and give a different presentation of the forcing for the case of the fragmented ideals $\mathcal{I}_u$.

Let $u \subset \omega$ be an infinite set. Set $T = \bigcup_j \Pi_{i \in j \cap u} a_i$ and let $\mathcal{J}_u$ be the ideal on $\Pi_{i \in u} a_i = [T]$ generated by all products $\Pi_{i \in u} b_i$ of sets b_i whose φ_i-masses are uniformly bounded by some real number. This is equivalent to generating the ideal by sets $A \subset \Pi_{i \in u} a_i$ such that $\bigcup_{f \in A} \operatorname{rng}(f) \in \mathcal{I}_u$. So the quotient forcing $P_{\mathcal{J}_u}$ of Borel $\mathcal{J}_u$-positive subsets of $[T]$ ordered by inclusion is a proper, bounding forcing preserving Baire category and adding an unbounded element of $\mathcal{J}_u$ ([6, Section 4.1] and proposition 2.11) .

Identifying functions in the product with their ranges, it is quite clear that in fact $P_{\mathcal{J}_u}$ is equivalent to the forcing $\mathbb{P}_{\mathcal{I}_u}$ below the set of all selectors on the sets $a_i : i \in u$.

We will give a combinatorial form of the quotient forcing $P_{\mathcal{J}_u}$. Say that a tree $S \subset T$ is a *large tree* if for every real number r, every node of T can be extended to a splitnode s at some level $i \in u$ such that the φ_i-mass of the set of immediate successors of the splitnode is at least r. As in [6, Claim 4.1.9] the following lemma holds.

Lemma 3.5. *Every analytic $\mathcal{J}_u$-positive set contains all branches of a large tree.*

Thus, the poset of large trees ordered by inclusion is naturally densely embedded in $P_{\mathcal{J}_u}$ by the embedding $S \mapsto [S]$.

Proof. Suppose that $A \subset \Pi_{i \in u} a_i$ is an analytic $\mathcal{J}_u$-positive set, a projection of some tree $S \subset (\omega \times \omega)^{<\omega}$. Thinning out the tree S if necessary we may assume that for every node $t \in S$, $p[S \restriction t] \notin \mathcal{J}_u$. By recursion on $n \in \omega$ build finite trees U_n as well as functions f_n so that

- $0 = U_0$ and U_{n+1} is an end-extension of U_n. The tree $U = \bigcup_n U_n$ will be the sought large tree;
- $f_0 \subset f_1 \subset \dots$ are functions such that $\operatorname{dom}(f_n) \subset U_n$ is a set including all endnodes of U_n, and $f_n(t) \in S$ is a pair of finite sequences of which t is the first, for every $t \in \operatorname{dom}(f_n)$. Thus, for every point $x \in [U]$, the union $\bigcup_n f_n(x \restriction n)$ witnesses the fact that $x \in A$ and therefore $[U] \subset A$;
- for every endnode $t \in U_n$ there is an extension $s \in U_{n+1}$ such that, writing $i = \min(u \setminus \operatorname{dom}(s))$, $\varphi_i\{j \in a_i : s^\frown\langle i, j\rangle \in U_{n+1}\} > n$. This guarantees the largeness of the tree U.

The recursion is straightforward: Suppose that U_n, f_n have been constructed, fix an endnode $t \in U_n$ and construct the part of U_{n+1} and f_{n+1} above t in the following way. There must be a finite sequence s extending t such that writing

$i = \min(u \setminus \mathrm{dom}(s))$, $\varphi_i\{j \in a_i : \exists x \in p[S \restriction f_n(t)]\ s^\frown\langle i, j\rangle \subset x\} > n$. For if such a sequence s did not exist, the subadditivity requirements on φ_i would imply that for every $i \in u \setminus \mathrm{dom}(t)$, $\varphi_i\{j \in a_i : \exists x \in p[S \restriction f_n(t)]\ x(i) = j\} < n + 1$ and therefore the set $p[S \restriction f_n(t)]$ would be in the ideal $\mathcal{I}_u$. Pick such a finite sequence s, write $i = \min(u \setminus \mathrm{dom}(s))$, for every number $j \in a_i$ such that $\exists x \in p[S \restriction f_n(t)]\ s^\frown\langle i, j\rangle \subset x$ put the sequence $s^\frown\langle i, j\rangle$ into U_{n+1} and pick a node $f_{n+1}(s^\frown\langle i, j\rangle)$ in the tree $S \restriction f_n(t)$ whose first coordinate is this sequence, and proceed to another endnode of U_n. □

Corollary 3.6. *For every analytic $\mathcal{J}_u$-positive set A there is an infinite set $u' \subset u$, sets $b_i, i \in u'$ and a continuous function $G : \Pi_{i\in u'} b_i \to A$ such that*

1. *$b_i \subset a_i$ and the numbers $\varphi_i(b_i), i \in u'$ tend to infinity;*
2. *for every collection of nonempty sets $c_i \subset b_i, i \in u'$ such that the numbers $\varphi_i(c_i), i \in u'$ tend to infinity, the image $\mathrm{rng}(G \restriction \Pi_{i\in u'} c_i)$ is $\mathcal{J}_u$-positive.*

Proof. Find a large tree T such that $[T] \subset A$ and thin it out if necessary to find an infinite set u' such that every level of T contains at most one splitnode, for every number $i \in u'$ there is a splitnode at level i with a set $b_i \subset a_i$ of immediate successors, and the numbers $\varphi_i(b_i), i \in u'$ tend to infinity. The function G is then defined in such a way that $G(x)$ is the unique path y through the tree T such that whenever $i \in u'$ is such that $x \restriction i$ is a splitnode of T then $x(i) = y(i)$. It is easy to verify the required properties of the function G.

□

We will show that if $v \subset \omega$ is an infinite set with finite intersection with u, then both countable support iterations and countable support products of quotient forcing $P_{\mathcal{J}_u}$ preserve the cofinality of $\mathcal{I}_v$.

Lemma 3.7. *In the $P_{\mathcal{J}_u}$ extension, every set in $\mathcal{I}_v$ can be covered by a ground model set in $\mathcal{I}_v$.*

Proof. In order to be able to generalize the method of proof to the product and iteration cases, we will use several auxiliary claims about products of finite sets.

Claim 3.8. *Let $j \in \omega$ and let $w \subset \omega$ be a finite set with $\min(w) > j$. Suppose that $b_i, i \in w$ are subsets of $a_i, i \in w$ and $f : \Pi_{i\in w} b_i \to \Pi_{i\in j} |a_i|$ is a function. Then there exist sets $c_i, i \in w$, subsets of b_i such that $\varphi_i(c_i) > \varphi_i(b_i) - 2^{-i}$ and $f \restriction \Pi_{i\in w} c_i$ is constant.*

The claim is proved by induction on the size of the set w: For $b \subseteq a_{i_0}$, $j < i_0$ and $f : b \to \Pi_{i\in j} |a_i|$, let $r_j = \Pi_{i\in j} |a_i|$. If $|b| = r_j q + s$ $(0 \le s < r_j)$, there is a $c \subseteq b$ such that $|c| = q$ and f is constant on c. By lemma 3.2: $\varphi_{i_0}(b) < (2^{i_0} r_j)^{-1} + \varphi_{i_0}(c) < 2^{-i_0} + \varphi_{i_0}(c)$.

For $w = \{i_1, \ldots, i_{m+1}\}$, $j < i_1$, $b_i \subseteq a_i$ $(i \in w)$ and $f : \Pi_{i\in w} b_i \to r_j$, set $k = j_{m+1}$, $r = r_{j_m}$ and $b = b_k$. Let $\{t_1, \ldots, t_l\}$ be an enumeration of $\Pi_{i\in u\setminus\{k\}} b_i$, and let $T_1 = \{t_1^\frown\langle x\rangle : x \in b\}$. As before (since $|T_1| = |b|$), there is $d_1 \subseteq b$ such that $f \restriction \{t_1^\frown\langle x\rangle : x \in d_1\}$ is constant, and (since $k > j_m$) $\varphi_k(b) < (r2^k)^{-1} +$

$\varphi_k(d_1)$. Recursively define $T_j = \{t_j^\frown\langle x\rangle : x \in d_{j-1}\}$ and $d_j \subseteq d_{j-1}$ such that $f \restriction \{t_j^\frown\langle x\rangle : x \in d_j\}$ is constant, and $\varphi_k(d_{j-1}) < (r2^k)^{-1} + \varphi_k(d_j)$ for $1 < j \leq l$. Finally, let $d = d_l$, so $\varphi_k(b) < l(r2^k)^{-1} + \varphi_k(d_1) < 2^{-k} + \varphi_k(d)$ (as $l < r$). Now, let $x_0 = \min d$ and let $g : \Pi_{i\in w\setminus\{k\}} b_i \to r_j$ be defined by: $g(t) = f(t^\frown\langle x_0\rangle)$. By the inductive hypothesis, there are $c_i \subseteq b_i$ $(i \in w \setminus \{k\})$ such that $g \restriction \Pi_{i\in w\setminus\{k\}} c_i$ is constant, say, of constant value $\alpha \in r_j$ and $\varphi_i(b_i) < 2^{-i} + \varphi_i(c_i)$. Set $c_k = d$, then it is clear that $f \restriction \Pi_{i\in w} c_i$ is constant (with constant value α as well), and that the sequence $c_i, i \in w$ is as desired.

Claim 3.9. *Suppose that $b_i, i \in u$ are finite sets such that their submeasures $\varphi_i(b_i), i \in u$ tend to infinity. Suppose $r \in \mathbb{R}$ is a real number and $F : \Pi_{i\in u} b_i \to \mathcal{I}_v$ is a continuous function whose range consists of sets of φ-mass $< r$. Let $\epsilon > 0$ be a real number. Then there are sets $c_i \subset b_i$ such that the numbers $\varphi_i(c_i), i \in u$ still tend to infinity and such that the values of the function $F \restriction \Pi_{i\in u} c_i$ can be all enclosed in a single set of submeasure $< r + \epsilon$.*

Fix the continuous function F. Then for every $j \in v$ there is $k_j \in \omega$ such that the value $F(x) \cap a_j$ depends only on $x \restriction (u \cap k_j)$ whenever $x \in \Pi_{i\in u} b_i$. Apply the previous claim repeatedly at each number $j \in v$ (with $w = u \cap (j, k_j)$) to obtain sets $c_i \subset b_i$ for $i \in u$ such that their submeasures still tend to infinity and $F(x) \cap a_j$ depends only on $x \restriction (u \cap j + 1)$ whenever $x \in \Pi_{i\in u} c_i$. There is a number m such that for every $j \in v \setminus m$, $j \notin u$ the union of a family of $\Pi_{i\in j} |a_i|$ subsets of a_j each of φ_j-mass $< r$ has φ_j-mass $< r + \epsilon$. By thinning out finitely many sets c_i to singletons if necessary we may arrange that the set $F(x) \cap \bigcup_{j\in v\cap m} a_j$ is the same for all $x \in \Pi_{i\in u} c_i$. It follows that $\bigcup F''\Pi_{i\in u} c_i$ has mass $< r + \epsilon$ as required.

To conclude the proof, suppose that $B \in P_{\mathcal{J}_u}$ is a condition forcing $\dot{a} \in \mathcal{I}_v$ is a set of φ-mass $< r$. Find a large tree T and a continuous function f such that $[T] \subset B$, $f : [T] \to \mathcal{I}_v$ and $B \Vdash \dot{f}(\dot{x}_{gen}) = \dot{a}$; by thinning out the tree T we may assume that the range of the function f consists only of sets of mass $< r$. Now, for every real number $\epsilon > 0$ the conjunction of Claim 3.9 and Corollary 3.6 yields a subtree $S \subset T$ such that $\varphi(\bigcup \mathrm{rng} f \restriction [S]) < r + \epsilon$. The lemma follows.

□

Lemma 3.10. *In the extension obtained by the countable support product of $P_{\mathcal{J}_u}$ forcing, every set in $\mathcal{I}_v$ can be covered by a ground model set in $\mathcal{I}_v$.*

Proof. The countable support product of definable forcings that are proper, bounding, and preserve Baire category is treated in [6, Theorem 5.2.6]. In particular, for the countable support product of length κ of $P_{\mathcal{J}_u}$ forcing, whenever $\dot{a}$ is a name for a set in $\mathcal{I}_v$, there is a countable set $b \subset \kappa$ and large trees $T_i : i \in b$ and a continuous function $f : \Pi_i[T_i] \to \mathcal{I}_v$ such that $\langle T_i : i \in b\rangle \Vdash \dot{a} = \dot{f}(\vec{x}_{gen} \restriction b)$. Thinning out the trees T_i we may assume that the range of f consists of sets of mass $< r$ for some fixed real number r, and there are no two splitnodes at the same level. The conjunction of Claim 3.9 and Corollary 3.6 yields trees $S_i \subset T_i$ such that $\varphi(\bigcup \mathrm{rng}(f \restriction \Pi_i[S_i])) < r + \epsilon$ for any positive real $\epsilon > 0$ given beforehand. The lemma follows.

$\square$

We will need a slight strengthening of the lemma. Let K be an arbitrary set, let $u_k, k \in K$ be infinite subsets of ω, and consider the countable support product $P = \prod_k P_{u_k}$.

Lemma 3.11. *Let $u \subset \omega$ be an infinite set. P forces that every set in I_u is covered by a set in I_u that belongs to the model given by $\prod\{P_{u_k} : u_k \cap u$ is infinite$\}$.*

Proof. We begin by proving the following strengthening of Claim 3.9:

Claim 3.12. *Let $b_i \subset a_i$ be sets such that the numbers $\{\varphi_i(a_i) : i \in \omega\}$ tend to infinity. Let r be a real number, and let $f : \prod_i b_i \to \prod_i \mathcal{P}(a_i)$ be a continuous function such that for every $v \in \prod_i b_i$ and every $i \in \omega$, $\varphi_i(f(v)(i)) < r$. Then for every $\epsilon > 0$ there are sets $c_i \subset b_i$ such that the numbers $\{\varphi_i(c_i) : i \in \omega\}$ tend to infinity and a continuous function $g : \prod_i c_i \to \prod_i \mathcal{P}(a_i)$ such that for every $v \in \prod_i c_i$ and every $i \in \omega$,*

1. *$\varphi_i(g(v)(i)) < r + \epsilon$,*
2. *$f(v)(i) \subset g(v)(i)$, and*
3. *$g(v)(i)$ depends only on $v(i)$.*

Proof. Let $k = k_\epsilon$ be given by Corollary 3.3. By induction on $j \in \omega$ build sets $b_i^j, i \in \omega \setminus k+1$ so that:

- $b_i^0 = b_i$, $b_i^{j+1} \subset b_i^j$, and if $j < i$ then $b_i^j = b_i^{j+1}$;
- $\varphi_i(b_i^{j+1}) > \phi_i(b_i^j) - 1/i$;
- for every j, for $v \in \prod_i b_i^j$, the value $f(v)(j-1)$ depends only on $v \restriction j$.

If this succeeds, let $c_i = \{\min b_i\}$ (for $i \leq k$) and $c_i = b_i^i$ for $i > k$. Define $g : \prod_i c_i \to \prod_i \mathcal{P}(a_i)$ by $g(v)(i) = \bigcup\{f(w)(i) : w \in \prod_j c_j, w(i) = v(i)\}$. By Corollary 3.3 and the third item above, this function will satisfy the demands of the claim.

The induction itself is easy. Suppose that the sets b_i^j have been found for some $j \in \omega$. Let $m_0 \in \omega$ be a number such that $f(v)(j)$ depends only on $v \restriction m_0$ for $v \in \prod_i b_i^j$. Such a number has to exist since the product is compact and the function f is continuous on it. Now, by downwards induction on $j \leq m \leq m_0$ construct sets $b_m^{j+1} \subset b_m^j$, $\phi_m(b_m^{j+1}) > \phi_m(b_m^j) - 1/m$ so that the value of $f(v)(j)$ depends only on $v \restriction m$ for each $v \in \prod_{i \leq k} c_i \times \prod_{i<m} b_i^j \times \prod_{m \leq i \leq m_0} b_i^{j+1} \times \prod_{i>m_0} b_i^j$. This downwards induction is easily performed by the subadditivity properties of the submeasures given by Lema 3.2. In the end, let $b_i^{j+1} = b_i^j$ for all $i < j$ and $i > m_0$.

$\square$

Let us set up some useful standard notation for the product. A condition in the product is a function p with a countable domain $\mathrm{dom}(p) \subset K$ such that for each $k \in \mathrm{dom}(p)$ the value $p(k)$ is a tree in the poset P_{u_k}. The set $[p]$ is defined as the subset of $2^{\omega\,\mathrm{dom}(p)}$ consisting of those sequences $\vec{x}$ such that for every $k \in \mathrm{dom}(p)$, $\vec{x}$ is a branch through the tree $p(k)$. A splitnode of p is a splitnode of one of the trees in $\mathrm{rng}(p)$. The generic object for the product is identified with the sequence $\vec{x}_{gen} : K \to 2^\omega$ such that for every condition p in the generic filter, $\vec{x}_{gen} \restriction \mathrm{dom}(p) \in [p]$.

Let $p \in P$ and let τ be a P-name for a set in I_u. The usual countable support product fusion arguments yield a condition $q \leq p$, a real number r and a continuous function $f : [q] \to \prod_i \mathcal{P}(a_i)$ such that $q \Vdash \forall i \in u\ \tau \cap a_i \subset \dot{f}(\vec{x}_{gen} \restriction \mathrm{dom}(q))(i)$, and for every $\vec{x} \in [q]$ and every $i \in \omega$, $\phi_i(f(\vec{x})(i)) < r$. Thinning out the trees on the condition q if necessary, we may assume that any two distinct splitnodes are at distinct levels.

For every $i \in \omega$, write t_i for the unique splitnode in the condition q at level i if it exists, and let b_i be the set of all of its immediate successors. Let $v \subset \omega$ be the set of all natural numbers i for which t_i exists. Thinning out the condition q if necessary, we may assume that the numbers $\phi_i(b_i), i \in v$ tend to infinity, and if t_i is a splitnode of q at a coordinate $j \in \mathrm{dom}(q)$ such that the set $u_j \cap u$ is finite, then actually $i \notin u$. There is a natural map $\pi : \prod_{i \in v} b_i \to [q]$ sending every point $z \in \Pi_{i \in v} b_i$ to the unique $x \in [q]$ such that if t_i is a splitnode which is an initial segment of $\vec{x}(k)$ for some $k \in K$ then $t_i^\frown z(i) \subset \vec{x}(k)$. Consider the function $\hat{f} = f \circ \pi$. The Claim 3.12 shows that there are sets $c_i \subset b_i$ for $i \in v$ and a function $\hat{g} : \prod_{i \in v} c_i \to I_u$ such that the numbers $\phi_i(c_i), i \in v$ tend to infinity, for every $i \in \omega$, $\hat{g}(v)(i) \supset \hat{f}(v)(i)$, $\phi_i(\hat{g}(v)(i)) < r + 1$, and the value $\hat{g}(v)(i)$ depends only on $v(i)$ (if $i \notin v$ then this value is constant).

Consider the condition $q' \leq q$ obtained from q by thinning out all branchings of the splitnodes of q from b_i to c_i. Clearly, this is a condition in the product P with the same domain as q. Consider the name σ for a set in the ideal I_u defined by the following: If $i \in u$ is a number such that t_i is defined, and for the unique $k \in K$ such that t_i is a splitnode of the tree $q(k)$ it is the case that $t_i \subset \vec{x}_{gen}(k)$, then $\sigma \cap a_i = \vec{x}_{gen}(j)|t_i|$; if $i \in u$ does not satisfy these conditions, then $\sigma \cap a_i = \bigcap\{\hat{g}(v)(j) : v \in \prod_j c_j\}$. A review of definitions shows that σ is a name in the product $\prod\{P_{u_k} : k \in \mathrm{dom}(q), u_k \cap u \text{ is infinite}\}$, and $q' \Vdash \tau \subset \sigma$ as desired.

□

Proof. (of theorem 3.4) Let V be a model of CH and let $u_\alpha, \alpha < \omega_1$ be an almost disjoint family of infinite subsets of ω. Let $\mathbb{P}_\alpha$ be a countable support product of $\omega_{\alpha+1}$ copies of the forcing $P_{\mathcal{J}_{u_\alpha}}$ and let $\mathbb{P}$ be a countable support product of the $\mathbb{P}_\alpha$, $\alpha < \omega_1$. Then:

1. $\mathbb{P}$ is proper and ω_2-c.c., hence it does not collapse cardinals.
2. Each $\mathbb{P}_\alpha$ forces $\mathtt{cof}(\mathcal{I}_{u_\alpha}) \geq \omega_{\alpha+1}$, and $\mathbb{P}_\alpha$ forces $\mathtt{cof}(\mathcal{I}_{u_\alpha}) = \omega_{\alpha+1}$ assuming the ground model is a model of CH.

3. $\mathbb{P}$ forces $\texttt{cof}(\mathcal{I}_{u_\alpha}) = \omega_{\alpha+1}$ for every $\alpha < \omega_1$.

As V is a model of CH and each forcing in the product has size $\mathfrak{c}$ the ω_2-c.c. follows from a standard Δ-system argument. The properness of $\mathbb{P}$ easily follows from lemma 3.4 and a standard Sacks-type fusion argument.

By a simple genericity argument all of the generic reals added by $\mathbb{P}_\alpha$ are mutually independent elements of $\mathcal{I}$ each unbounded over the rest. If the ground-model is a model of CH, then $\mathbb{P}_\alpha$ forces $\mathfrak{c} = \omega_{\alpha+1}$.

To see (3) first note that $\mathbb{P}$ forces $\texttt{cof}(\mathcal{I}_{u_\alpha}) \geq \omega_{\alpha+1}$ by (2). On the other hand, the fact that $\mathbb{P}$ forces $\texttt{cof}(\mathcal{I}_{u_\alpha}) \leq \omega_{\alpha+1}$ follows directly from Lemma 3.11. $\square$

As mentioned before, also countable support iteration can be used to separate the cofinalities of the ideals $\mathcal{I}_u$.

Theorem 3.13. *Let u, v be infinite almost disjoint subsets of ω, let $\mathbb{P}$ be a countable support iteration of length ω_2 of the forcing $P_{\mathcal{J}_u}$, and let G be $\mathbb{P}$-generic over a model of CH. Then $V[G] \models \texttt{cof}(\mathcal{I}_v) < \texttt{cof}(\mathcal{I}_u)$*

The theorem follows directly from properness, genericity and the following lemma.

Lemma 3.14. *In the extension obtained by the countable support iteration of $P_{\mathcal{J}_u}$ forcing, every set in $\mathcal{I}_v$ can be covered by a ground model set in $\mathcal{I}_v$.*

Proof. This is a consequence of the first preservation theorem [1, Theorem 6.1.13]. Say that a forcing P *strongly preserves* the ideal $\mathcal{I}_v$, if every set $a \in \mathcal{I}_v$ in the extension can be covered by a ground model set of an arbitrary close φ-mass. This is exactly the case for the forcing $P_{\mathcal{J}_u}$ by the proof of Lemma 3.7. We will argue that the strong preservation of the ideal $\mathcal{I}_v$ falls into the scheme of the first preservation theorem, proving the lemma. The following easy claim, which follows directly from the very slow fragmentation property of the submeasure, is the starting point.

Claim 3.15. *Suppose that $b_n : n \in \omega$ are sets in $\mathcal{I}_v$ and $r > 0$ is a real number such that $\varphi(b_n) < r$ holds for every number n. Suppose $\epsilon > 0$. Then there is an infinite set $b \subset \omega$ such that $\varphi(\bigcup_{n \in b} b_n) < r + \epsilon$.*

Fix positive rationals $r, \epsilon > 0$. Let X be the space of all sequences $x = \langle r_x, x(0), x(1), \ldots \rangle$ where $r_x \in \mathbb{Q}$ is a positive rational smaller than r and $(\forall i \in \omega)(x(i) \subset a_{j_i} \wedge \varphi_i(x(i)) \leq r_x)$, where j_i is the i-th element of the set v. Let Y be the set of all sequences $y = \langle y(0), y(1), \ldots \rangle$ such that $(\forall i \in \omega)(y(i) \subset a_{j_i} \wedge \varphi_i(y(i)) \leq r + \epsilon)$. Let $\sqsubseteq_n$ be the relation on $X \times Y$ defined by: $x \sqsubseteq_n y$ if for every $i > n$, $x(i) \subset y(i)$. Let $\sqsubseteq = \bigcup_n \sqsubseteq_n$. It is not difficult to verify that these relations fall into the framework of [1, Definition 6.1.6]. In particular, if $\langle x_n : n \in \omega \rangle$ are countably many elements of the space X and $\langle i_n : n \in \omega \rangle$ is an increasing sequence of numbers such that for every $j > i_n$ and sets $B, C \subset a_j$, if $\varphi_j(B), \varphi_j(C) < r + \epsilon - \epsilon/(n+1)$, then $\varphi_j(B \cup C) < r + \epsilon - \epsilon/(n+2)$, then the sequence $y = \langle \bigcup_{n: i_n < i} x_n(i) : i \in \omega \rangle \in Y$ $\sqsubseteq$-dominates all the points in $\{x_n : n \in \omega\}$.

We will show that if the forcing P strongly preserves $\mathcal{I}_v$, then it preserves $\sqsubseteq$ in the sense of [1, Definition 6.1.10]. The preservation theorem [1, Theorem 6.1.13] then completes the proof of the lemma.

Suppose that M is a countable elementary submodel of a large structure, $\langle \dot{x}_l : l \in k \rangle$ are finitely many names for elements of the space X in the model M, suppose $\langle p_n : n \in \omega \rangle$ is a decreasing collection of conditions in the model M such that p_n decides $\dot{x}_l \restriction n$, yielding sequences $\langle \bar{x}_l : l \in k \rangle$ in $X \cap M$, and suppose that $y \in Y$ is a point such that $\forall x \in X \cap M$, $x \sqsubseteq y$. We must find a condition $q \leq p_0$ such that

- q is M-master for P;
- $q \Vdash \forall x \in X \cap M[G]$ $x \sqsubseteq y$; and
- for all $l \in k$ for all $n \in \omega$ $q \Vdash \bar{x}_l \sqsubseteq_n y \to \dot{x}_l \sqsubseteq_n y$.

To find the condition q, first work in the model M. Fix a rational $r' < r$ greater than all the numbers $r_{\bar{x}_l}$. By assumption, for each n there is a condition p'_n and a set $b_{ln} \in \mathcal{I}_v$ such that $(\forall i \geq n)(\varphi_i(b_{ln} \cap a_{j_i}) < r')$, $p'_n \Vdash \dot{x}_l(i) \subset (b_{ln} \cap a_{j_i})$ and for $i \in n$, $a_{ln} \cap a_{j_i} = \bar{x}_l(i)$.

Use Claim 3.15 to find an infinite set $d \subset \omega$ such that for each l less than or equal to k, $\forall i \varphi_i(\bigcup_{n \in d} b_{ln} \cap a_{j_i}) < r$. Set $b = \bigcup_{n \in d, l \in k} b_{ln}$, then by the very slow fragmentation, there is $i_0 \in \omega$ such that $(\forall i > i_0)$ $\varphi_i(b \cap a_{j_i}) < r$. Since y $\sqsubseteq$-dominates all elements of $X \cap M$, there must be $j > i_0$ such that for every $i > j$, $b \cap a_{j_i} \subset y(i)$. Let $n > j$ be a number in the set d, and use the properness of the forcing P to find a master condition $q \leq p'_n$. The last item holds by the choice of the condition p'_n: for $l \leq k$,

$$p'_n \Vdash \dot{x}_l(i) \subset (b_{ln} \cap a_{j_i}) \subset (b \cap a_{j_i}) \subset y(i)$$

for all $i > n$. The first item holds by the choice of the condition q. The second item is an immediate consequence of the first, and the fact that the forcing P strongly preserves the ideal $\mathcal{I}_v$: If $\dot{x}$ is a name in M for an element of $X \cap M[G]$, such that $\varphi(\dot{x}) < r$, by assumption, there is $a \in M$ such that $\varphi(a) < r$ and $\dot{x} \subset a$. But y bounds M, therefore $\dot{x} \sqsubseteq y$.

□

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