COUNTABLE IRRESOLVABLE SPACES AND CARDINAL INVARIANTS

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ABSTRACT. Answering a question of M. Scheepers we show that that the cardinal invariant $\mathfrak{d}$ is a lower bound on $\mathfrak{irr}$ the minimal weight (equivalently, minimal π-weight) of a countable regular irresolvable space. We consider related cardinal invariants such as $\mathfrak{r}_{\mathsf{scat}}$ the reaping number of the quotient algebra $\mathcal{P}(\mathbb{Q})$ mod the ideal of scattered subsets of the rationals and prove that $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ implies that $\mathfrak{irr} = \omega_1$.

1. Introduction

All topological spaces considered are regular and *crowded*, i.e., have no isolated points. A topological space X is said to be *irresolvable* provided there are no disjoint dense subsets Y, $W \subseteq X$. Otherwise, X is *resolvable.*

It is easy to see that $\mathbb{Q}$ is a resolvable space. It follows, due to a well known theorem of W. Sierpiński, that every countable first countable crowded regular space is resolvable. So, if X is a countable regular irresolvable space, $w(X)$ should be uncountable. In fact, the same is true for countable regular spaces with countable π-weight.

M. Scheepers [6] defines the *irresolvability number* as follows:

$$\mathfrak{irr} = \min\{\pi w((\omega, \tau)) : \tau \subseteq \mathcal{P}(\omega) \textit{ is an irresolvable } T_3 \textit{ topology on } \omega\}$$

It is folklore knowledge that $\mathfrak{r} \leq \mathfrak{irr} \leq \mathfrak{i}$ (see [6, 3]), where $\mathfrak{r}$ denotes the *reaping number* (the minimal size of a *reaping (or unsplittable)* family, i.e. the minimal size of a family $\mathcal{R} \subseteq [\omega]^\omega$ such that for any $X \in [\omega]^\omega$ there is an $R \in \mathcal{R}$ such that $R \subseteq X$ or $R \cap X = \emptyset$), and $\mathfrak{i}$ is the minimal size of *maximal independent* family (see [5, 3]). In [6], M. Scheepers asks whether the equality $\mathfrak{r} = \mathfrak{irr}$ is provable in ZFC. We will show that the *dominating number* $\mathfrak{d}$ (see [5]) is also a lower bound for $\mathfrak{irr}$ hence, in particular, it is relatively consistent with ZFC to have $\mathfrak{r} < \mathfrak{irr}$. We also consider related cardinal invariants such as the reaping

Date: April 15, 2013.
2010 *Mathematics Subject Classification.* 03E17, 54G99, 54A25, 54A35.
Key words and phrases. Irresolvable space, reaping family.
Research was supported by CONACyT scholarship for Doctoral Students. The second listed author acknowledges support from PAPIIT grant IN102311 and CONACyT grant 177758.

number of the quotient algebra $\mathcal{P}(\mathbb{Q})$ mod the ideal of scattered subsets of the rationals and compare them to the cardinal invariants already mentioned.

2. The cardinal invariant $\mathfrak{irr}$ and other cardinals.

The following proposition tells us that we can replace the π-weight by *weight* in the definition of $\mathfrak{irr}$.

Proposition 2.1. The irresolvability number $\mathfrak{irr}$ is equal to the minimum weight of a countable irresolvable T_3 space.

Proof. It is clear that $\mathfrak{irr}$ is less or equal to the minimum weight of a countable irresolvable T_3 space.

Let τ be an irresolvable T_3 topology on ω of minimal π-weight and let $\mathcal{B}$ be a π-base witnessing the minimality of $\pi w((\omega, \tau))$. We claim that there is an $X \subseteq \omega$ and a topology τ' on X such that (X, τ') is irresolvable T_3 and has weight at most $\mathfrak{irr}$. Let $X = \bigcup \mathcal{B}$, and for each $U \in \mathcal{B}$, and each pair of distinct points $x, y \in U$, pick $W_x(U, x, y), W_y(U, x, y)$ disjoint clopen sets such that $x \in W_x(U, x, y)$, $y \in W_y(U, x, y)$, and $W_x(U, x, y)$, $W_y(U, x, y) \subseteq U$. Now, consider the following family of sets:

$$\mathcal{B}' = \{W_x(U, x, y), W_y(U, x, y) : U \in \mathcal{B}, x, y \in U, x \neq y\} \cup \\ \{X \setminus W_x(U, x, y), X \setminus W_y(U, x, y) : U \in \mathcal{B}, x, y \in U, x \neq y\}.$$

Finally, let τ' be the topology on X generated by $\mathcal{B}'$. Then (X, τ') is a countable irresolvable T_3 space of weight at most $\mathfrak{irr}$. □

Recall that a set $X \subseteq \mathbb{Q}$ is *scattered* if every non-empty $Y \subseteq X$ has an isolated point. The collection of all scattered subsets of $\mathbb{Q}$ forms a proper ideal which will be denoted by scat, and the family $\mathcal{P}(\mathbb{Q}) \setminus \mathsf{scat}$ of scat-positive sets will be denoted by scat^+.

Definition 2.1. A family $\mathcal{R} \subseteq \mathsf{scat}^+$ is called *scattered-reaping*, if for every $X \in \mathsf{scat}^+$, there is a $Y \in \mathcal{R}$ such that $Y \subseteq X$ or $X \cap Y = \emptyset$. The *scattered-reaping number*, which we denote by $\mathfrak{r}_{\mathsf{scat}}$, is defined as the minimum size of a scattered-reaping family

$$\mathfrak{r}_{\mathsf{scat}} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathsf{scat}^+ \\ (\forall X \in \mathsf{scat}^+)(\exists Y \in \mathcal{R})(Y \subseteq X \vee Y \cap X = \emptyset)\}.$$

It is easy to see that $\mathfrak{r}_{\mathsf{scat}}$ is equal to $\mathfrak{r}(\mathcal{P}(\mathbb{Q})/\mathsf{scat})$ - the reaping number of the Boolean algebra $\mathcal{P}(\mathbb{Q})/\mathsf{scat}$.

Proposition 2.2. $\mathfrak{r}_{\mathsf{scat}} \leq \mathfrak{irr}$.

Proof. Let τ be an irresolvable topology on ω, and let $\mathcal{M} \preccurlyeq H(\theta)$ be a countable elementary submodel with $\tau \in \mathcal{M}$, and $\mathcal{B}$ a π-basis for τ. Take $\mathcal{B}' = \tau \cap \mathcal{M}$. Due to Sierpiński's theorem, $\mathcal{B}'$ generates a topology

τ' which is homeomorphic to the topology of $\mathbb{Q}$, so we can assume that $\mathsf{scat} = \mathsf{scat}_{(\omega,\tau')}$. Note that every non-empty $U \in \tau$ is a scat-positive set. Let $X \in \mathsf{scat}^+$. Since (ω, τ) is irresolvable, one of X and $\omega \setminus X$ has non-empty τ-interior. If $int_\tau(X)$ is not empty, we get $U \in \mathcal{B}$ such that $U \subseteq X$. If $int_\tau(\omega \setminus X) \neq \emptyset$, we get basic open $U \in \mathcal{B}$ such that $U \cap X = \emptyset$. So, $\mathcal{B}$ is a scattered-reaping family. □

Lemma 2.1. For every $A \in \mathsf{scat}^+$, there is a crowded closed nowhere-dense set B such that $B \subseteq A$.

Proof. Let $A \in \mathsf{scat}^+$. Without loss of generality A is a crowded set. For each $n \in \omega$, let $\{B_{n,m} : m \in \omega\}$ be a local basis of clopen sets at n. Recursively, construct an increasing sequence $\{F_m : m \in \omega\}$ of finite subsets of A, and an increasing sequence of clopen sets $\{U_m : m \in \omega\}$ satisfying the following:

a) For all $n \in \omega$, there is m such that $n \in F_m$ or $n \in U_m$.
b) For all $m \in \omega$, $F_m \cap U_m = \emptyset$.
c) For all $m \in \omega$, for all $k \in F_m$ and all $i > m$, $B_{k,i} \cap F_i \setminus \{k\} \neq \emptyset$.

Suppose both sequences have been sucessfully constructed. Put $F = \bigcup_{n\in\omega} F_n$. The clause c) ensures that F is a crowded set, while a) and b) tell us that F is closed. Since every F_n is a subset of A, we have $F \subseteq A$. If F is not nowhere dense, replace it by any of its closed crowded nowhere dense subsets.

In order to carry out the construction, let $k_0 = \min(A)$ and $F_0 = \{k_0\}$. Pick a clopen set U_0 such that $\{i : i < k_0\} \subseteq U_0$ and $k_0 \notin U_0$. Now, suppose that F_m and U_m have been defined . Then $F_m \subseteq A \setminus U_m$ and $A \setminus U_m$ is crowded. For each $k \in F_m$, pick $n_k \in B_{k,m+1} \cap A \setminus U_m$, and let $F_{m+1} = F_m \cup \{n_k : k \in F_m\}$. Finally, let $j = \min(A \setminus (F_{m+1} \cup U_m))$, and pick a clopen set V such that $j \in V$ and $V \cap F_{m+1} = \emptyset$, and let $U_{m+1} = U_m \cup V$. Obviously a), b) and c) are satisfied. □

Proposition 2.3. $\mathfrak{d} \leq \mathfrak{r}_{\mathsf{scat}}$.

Proof. Let $\mathcal{F} \subseteq \mathsf{scat}^+$ be a collection of crowded sets of cardinality less than $\mathfrak{d}$. We will find a set $Y \in \mathsf{scat}^+$ such that for all $X \in \mathcal{F}$, both $X \cap Y$ and $X \setminus Y$ are in scat^+. By Lemma 2.1, we can assume that each $X \in \mathcal{F}$ is a crowded closed nowhere dense set. Also, we can assume that $\omega = \bigcup \mathcal{F}$. For each $n \in \omega$, let C_n be the set of all $X \in \mathcal{F}$ such that $n \in X$. Note that C_n has size less than $\mathfrak{d}$. Recursively, construct two sequences $\{A_n : n \in \omega\}$, $\{B_n : n \in \omega\}$ of subsets of ω such that:

i) $A_0 = B_0 = \emptyset$.
ii) For all n, $A_n \cap B_n = \emptyset$.
iii) For all n, $A_n, B_n \in \mathsf{scat}$.
iv) For all n, $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$.
v) For all n, $n \in \bar{A}_{n+1} \cap \bar{B}_{n+1}$.
vi) For all n and for all $X \in C_n$, $A_{n+1} \cap X \neq \emptyset$ and $B_{n+1} \cap X \neq \emptyset$.

It is clear from the construction that $A = \bigcup_{n\in\omega} A_n$ and $B = \bigcup_{n\in\omega} B_n$ are disjoint dense sets, and item vi) implies that for all $X \in \mathcal{F}$, $A \cap X$ and $A \setminus X \supseteq X \cap B$ are both infinite.
Suppose that both A_n, B_n have been defined. If $n \in \bar{A}_n \cap \bar{B}_n$ put $A_{n+1} = A_n$ and $B_{n+1} = B_n$. If $n \notin \bar{A}_n$, let $\{W_m : m \in \omega\}$ be a partition of $\omega \setminus (\bar{A}_n \cup \bar{B}_n \cup \{n\})$ into clopen sets. Note that none of them has n in its closure, so for all $X \in C_n$ and all $k \in \omega$, $X \nsubseteq W_k$. Moreover, for infinitely many $k \in \omega$, $W_k \cap X$ is infinite (otherwise X would be a scattered set with n as its unique limit point). For each $X \in C_n$, let $\tilde{X}(k)$ to be the minimum $i \geq k$ such that $X \cap W_i \neq \emptyset$. Then define the following function:

$$f_X(i) = \min(X \cap W_{\tilde{X}(i)}) + 1$$

Since $|C_n| < \mathfrak{d}$, there is an increasing function f which is not dominated by $\{f_X : X :\in C_n\}$. Having fixed such f let

$$A_{n+1} = A_n \cup \bigcup_{k\in\omega} W_k \cap f(k)$$

It is easily seen that A_{n+1} satisfies i) - vi). Now, consider $W'_k = W_k \setminus A_{n+1}$. Note that again, for all $X \in C_n$ there are infinitely many $k \in \omega$ such that $X \cap W'_k$ is infinite. Let $\tilde{X}'(k)$ be the minimum $i \geq k$ such that $X \cap W'_i \neq \emptyset$. Define a new family of functions $\{g_X : X \in C_n\}$ as follows:

$$g_X(i) = \min(X \cap W'_{\tilde{X}'(i)}) + 1$$

Again, fix an increasing function $g : \omega \to \omega$ which is not dominated by $\{g_X : X \in C_n\}$ and let

$$B_{n+1} = B_n \cup \bigcup_{k\in\omega} W'_k \cap g(k)$$

Then B_{n+1} satisfies i) to vi). □

Corolary 2.1. $\max\{\mathfrak{r}, \mathfrak{d}\} \leq \mathfrak{r}_{\mathsf{scat}} \leq \mathfrak{irr}$.

Let us turn our attention to the question of M. Scheepers. It is well known that in the *Miller model* (see [2, 4]) $\mathfrak{r} < \mathfrak{d}$. In particular, in this model $\mathfrak{r} < \mathfrak{r}_{\mathsf{scat}} = \mathfrak{irr}$ holds.

Corolary 2.2. It is relatively consistent with ZFC that $\mathfrak{r} < \mathfrak{irr}$.

3. A diamond for $\mathfrak{r}_{\mathsf{scat}}$

It is well known that for many non-Borel cardinal invariants there is a Borel cardinal invariant such that its associated $\diamondsuit$-principle implies the former to be equal to ω_1 (see [5]). Some examples of this phenomena are the cases of $\mathfrak{b}$ and $\mathfrak{a}$, $\mathfrak{r}$ and $\mathfrak{u}$, and $\mathfrak{r}_{\mathbb{Q}}$[1] and $\mathfrak{i}$ (see [1, 5]). This section

[1] $\mathfrak{r}_{\mathbb{Q}} = \mathfrak{r}(\mathcal{P}(\mathbb{Q})/\mathsf{nwd})$ is the reaping number of the Boolean algebra $\mathcal{P}(\mathbb{Q})/\mathsf{nwd}$, where nwd denotes the the ideal of nowhere dense subsets of the rationals

is devoted to proving that the relation between $\mathfrak{r}_{\mathsf{scat}}$ and $\mathfrak{irr}$ has the same flavor.

Definition 3.1. $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ is the following statement:

$\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ For every Borel function[2] $F : 2^{<\omega_1} \to \mathsf{scat}^+$ there is a $g : \omega_1 \to \mathsf{scat}^+$ such that for all $f \in 2^{\omega_1}$ the set $\{\alpha \in \omega_1 : g(\alpha) \subseteq F(f \restriction \alpha) \vee F(f \restriction \alpha) \cap g(\alpha) = \emptyset\}$ is stationary.

The function g given by $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ is called a $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$-guessing sequence for F.

Theorem 3.1. $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$ implies $\mathfrak{irr} = \omega_1$

Proof. By a suitable coding, we will assume that the domain of our F is the set of all ordered pairs $(A, \vec{I})$, where $\vec{I} = \langle I_\beta : \beta < \alpha \rangle \subseteq \mathcal{P}(\omega)$ is a sequence of lenght $\alpha \in \omega_1$, and A is a subset of ω. Define F as follows:

- If $\{I_\beta : \beta < \alpha\} \cup \{\omega \setminus I_\beta : \beta \in \alpha\}$ is not a subbasis for a topology homeomorphic to the usual topology on $\mathbb{Q}$, then $F(A, \vec{I}) = \mathbb{Q}$.
- If $\{I_\beta : \beta < \alpha\} \cup \{\omega \setminus I_\beta : \beta \in \alpha\}$ is a subbasis for a topology homeomorphic to the usual topology on $\mathbb{Q}$, and A is scattered relative to this topology, then $F(A, \vec{I}) = \mathbb{Q}$.
- If $\{I_\beta : \beta < \alpha\} \cup \{\omega \setminus I_\beta : \beta \in \alpha\}$ is a subbasis for a topology homeomorphic to the usual topology on $\mathbb{Q}$, and A is not scattered relative to this topology, pick $h_{\vec{I}} : \omega \to \mathbb{Q}$ a recursive homeomorphism, and define $F(A, \vec{I}) = h_{\vec{I}}[A]$.

Here the homeomorphism $h_{\vec{I}}$ depends (in a recursive, or Borel way) only on $\vec{I}$, in particular, it is the same homeomorphism for all pairs $(A, \vec{I})$ with the same second coordinate.

Now, let $g : \omega_1 \to \mathsf{scat}^+$ be a $\diamondsuit(\mathfrak{r}_{\mathsf{scat}})$-guessing sequence, and recursively define a family of subbases as follows:

(1) Let $\mathcal{B}_0 = \langle U_n : n \in \omega \rangle$ be a basis for the usual topology on $\mathbb{Q}$.
(2) Suppose we have defined $\mathcal{B}_\beta$ for all $\beta < \alpha$. If α is a limit ordinal, then make $\mathcal{B}_\alpha = \bigcup_{\beta<\alpha} \mathcal{B}_\beta$. For $\alpha = \beta + 1$, look at $g(\beta) \in \mathsf{scat}^+$. By lemma 2.1, there is a perfect nowhere-dense set B_β contained in $g(\beta)$. Also $\langle U_\gamma : \gamma < \alpha \rangle$ generates a topology homeomorphic to the usual topology on $\mathbb{Q}$, so in the definition of F, we make use of the recursive homeomorphism $h_{\mathcal{B}_\alpha}$. Let $U_\alpha = h^{-1}_{\mathcal{B}_\alpha}[B_\alpha]$. It is not hard to see that $\mathcal{B}_\alpha = \mathcal{B}_\beta \cup \{\mathcal{U}_\alpha\} \cup \{\omega \setminus \mathcal{U}_\alpha\}$ generates a topology homeomorphic to the rationals.

We claim that $\{U_\alpha : \alpha \in \omega_1\}$ generates an irresolvable T_3 topology τ_{ω_1} on ω. Since we are making each U_α clopen, then the topology we get is 0-dimensional. Let us see that it is irresolvable. That means,

[2]A function F from $2^{<\omega_1}$ to a metric space X is *Borel* if all of its restrictions to levels 2^α are Borel. Here we consider scat^+ as a subspace of $\mathcal{P}(\mathbb{Q})$ endowed with the product topology.

for every $A \subseteq \omega$ either A or $\omega \setminus A$ has non-empty interior in τ_{ω_1}. We only have to worry about of the sets $A \in \mathsf{scat}^+_{\tau_{\omega_1}}$ (if $A \in \mathsf{scat}_{\tau_{\omega_1}}$ then obviously $\omega \setminus A$ has non-empty interior in τ_{ω_1}). Pick one of such A. Then, in particular, $A \in \mathsf{scat}^+$. If g guesses $(A, \langle U_\alpha : \alpha \in \omega_1 \rangle)$ at γ, then $h_{\langle U_\alpha : \alpha < \gamma \rangle}[A] \supseteq g(\gamma)$ or $h_{\langle U_\alpha : \alpha < \gamma \rangle}[A] \cap g(\gamma) = \emptyset$. So, either A or $\omega \setminus A$ has non-empty interior in τ_{ω_1}. In the former case, we have $A \supseteq U_\gamma$, and in the later case $A \cap U_\gamma = \emptyset$, so it is not possible for A be both dense and codense. By Proposition 2.1 we are done. □

4. Related facts and questions

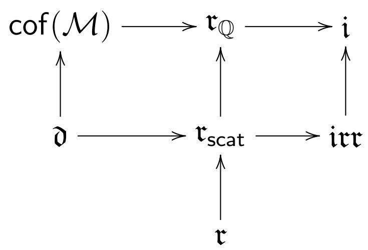
In [1], it is proved that $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{r}_{\mathbb{Q}} \leq \mathfrak{i}$.

Proposition 4.1. $\mathfrak{r}_{\mathsf{scat}} \leq \mathfrak{r}_{\mathbb{Q}}$.

Proof. Let $\{D_\alpha : \alpha \in \kappa\}$ be a *Dense*$(\mathbb{Q})$-reaping family, and $\mathcal{B}$ a basis for the usual topology on $\mathbb{Q}$. The following family witness a scattered-reaping family:

$$\mathcal{R}_S = \{A \cap U : A \in \mathcal{R} \wedge U \in \mathcal{B}\}$$ □

The following diagram summarizes some of the results related with those presented here:



Some inequalities are folklore knowledge. The inequalities $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{r}_{\mathbb{Q}} \leq \mathfrak{i}$ were proved in [1]. We have the following questions concerning some of the cardinal invariants in the above diagram:

(1) Is $\mathfrak{r}_{\mathsf{scat}} = \mathfrak{r}_{\mathbb{Q}}$?
(2) Is $\mathfrak{r}_{\mathsf{scat}} = \max\{\mathfrak{d}, \mathfrak{r}\}$?
(3) Is there a model where $\mathfrak{r}_{\mathsf{scat}} < \mathfrak{irr}$?
(4) Is $\mathfrak{irr} = \mathfrak{i}$?
(5) Is $\mathsf{cof}(\mathcal{M}) \leq \mathfrak{r}_{\mathsf{scat}}$?
(6) Are $\mathsf{cof}(\mathcal{M})$ and $\mathfrak{irr}$ provably comparable?
(7) Are $\mathfrak{r}_{\mathbb{Q}}$ and $\mathfrak{irr}$ provably comparable?

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