Topology of Mrówka-Isbell spaces

Fernando Hernández-Hernández

Michael Hrušák

Facultad de Ciencias Físico Matemáticas, UMSNH, Morelia, Michoacán, México.
E-mail address: fhernandez@fismat.umich.mx

Centro de Ciencias Matemáticas, UNAM, Campus Morelia, Apartado Postal 61-3 (Xangari), Morelia, Michoacán, México, C.P. 58089
E-mail address: michael@matmor.unam.mx

2010 *Mathematics Subject Classification.* Primary: 54D80; Secondary: 03E35

Key words and phrases. almost disjoint families, Mrówka family, pseudocompactness, hyperspace, Luzin gap

The authors gratefully acknowledge support from CONACyT grants 169078 and 177758 as well as a PAPIIT grant IN108014.

Abstract. The purpose of this chapter is to present Mrówka-Isbell spaces associated to almost disjoint families as versatile topological tools with special emphasis on their relationship with pseudocompactness.

Contents

CHAPTER 1

Topology of Mrówka-Isbell spaces

1. Introduction to Mrówka-Isbell spaces

An infinite family $\mathscr{A}$ of infinite subsets of the natural numbers, ω, is *almost disjoint (AD)* if the intersection of any two distinct elements of $\mathscr{A}$ is finite. It is *maximal almost disjoint (MAD)* if given an infinite $X \subseteq \omega$ there is an $A \in \mathscr{A}$ such that $|A \cap X| = \omega$, in other words, if the family $\mathscr{A}$ is not included in any larger almost disjoint family.

Given an almost disjoint family $\mathscr{A}$, the Ψ-space or the *Mrówka-Isbell space* associated to $\mathscr{A}$ (denoted by $\Psi(\mathscr{A})$) has $\omega \cup \mathscr{A}$ as the underlying set, the points of ω being isolated, while the basic open neighborhoods of $A \in \mathscr{A}$ are of the form

$$\{A\} \cup (A \setminus F),$$

where F ranges over all finite subsets of ω.

There are few immediate properties of the Mrówka-Isbell spaces: they are Hausdorff spaces due to the assumption that the families are almost disjoint, since we are using subsets of a countable set and finite subsets of them, it follows that $\Psi(\mathscr{A})$ is a separable, first countable space. Another easy observation is that each point has a neighborhood base consisting of compact open subsets and hence $\Psi(\mathscr{A})$ is a locally compact zero dimentional space. In fact, any subspace of $\Psi(\mathscr{A})$ is a locally compact subspace as one can easily verify. Kannan y Rajagopalan [**55**] noted that separable spaces with all of their subspaces locally compact are, in fact, exactly the Mrówka-Isbell spaces (incuding degenerate situations). In the space $\Psi(\mathscr{A})$ the subspace $\mathscr{A}$ is infinite closed discrete and hence the Mrówka-Isbell spaces are not countably compact whenever the corresponding AD family $\mathscr{A}$ is infinite, and they are metrizable if and only if $\mathscr{A}$ is countable.

The first documented space of Mrówka-Isbell type was described by Alexandroff and Urysohn in [**2**, 1, Chapter V, 1.3] in 1925. It was later rediscovered independently by Mrówka in [**70**], and Gillman and Jerrison in [**41**] (there attributed to Isbell, hence the name), where the term Ψ-space probably appeared first, in order to give an example of a pseudocompact space that is not countably compact. Indeed, $\Psi(\mathscr{A})$ is pseudocompact if and only if $\mathscr{A}$ is a MAD family as we shall show next. If $\mathscr{A}$ is not maximal, there is an infinite set $J \subseteq \omega$ almost disjoint from each element of $\mathscr{A}$. This set J is closed discrete and locally finite subset of $\Psi(\mathscr{A})$, therefore any (unbounded) function from J to $\mathbb{R}$ can be continuously extended to $\Psi(\mathscr{A})$. This shows that $\Psi(\mathscr{A})$ is not pseudocompact. On the other hand, if $f : \Psi(\mathscr{A}) \to \mathbb{R}^+$ is continuous and unbounded, then any infinite set $\{k_n : n \in \omega\} \subseteq \omega$ such that $|f(k_{n+1})| > \max\{|f(k_i)| : i \leq n\}$ is almost disjoint from every element of the family.

Over the years Ψ-spaces associated with almost disjoint families have evolved into extremely useful and versiatile tools. Their applications range from the study of Fréchet and sequential spaces, compactifications, continuous selections, spaces of continuous functions both endowed with the topology of pointwise convergence, and as Banach spaces or Banach algebras, to connections with the normal Moore space problem.

The purpose of this chapter is to provide a selective survey on Mrówka-Isbell spaces, with the intention of providing the reader with a good sample of results and techniques to allow him/her to appreciate the depth, breadth, beauty and flexibility of their applications.

We conclude this section by mentioning a simple application of Mrówka-Isbell spaces over maximal almost disjoint families used by Arhangel'skii and Buzyakova in [**5**], as presented at https://dantopology.wordpress.com/tag/maximal-almost-disjoint-family/ by Dan Ma. Recall that a space X is said to have *G_δ-diagonal* if the diagonal $\Delta = \{(x,x) : x \in X\} \subseteq X \times X$ is G_δ in $X \times X$. Of course, any metrizable space has G_δ-diagonal. A countably compact space with G_δ-diagonal must be metrizable, by a theorem of Chaber [**22**].

A space (X,τ) is said to be *submetrizable* if there is a topology τ^* weaker than τ such that (X,τ^*) is metrizable. Every submetrizable space has a G_δ-diagonal since the diagonal Δ is a G_δ-set in the metric square $X \times X$, and hence also in the square in the original topology. We shall see that the property of having a G_δ-diagonal is strictly weaker than that of being submetrizable.

Given a MAD family $\mathscr{A}$, we already know that $\Psi(\mathscr{A})$ is a pseudocompact space, and it is easy to observe that it has G_δ-diagonal. The set $\mathscr{A}$ is an uncountable closed discrete subset of the space, thus $\Psi(\mathscr{A})$ is not Lindelöf but is separable, hence not metrizable. However, every pseudocompact submetrizable space is metrizable. To see this it suffices to show that any closed set in the original topology τ is also closed in the weaker topology τ^*. If C is closed in the original topology, then

$$C = \bigcap \{\mathrm{cl}_\tau(U) : U \text{ is open and } C \subseteq U\}.$$

This is so because the space is regular. Now, note that the sets $\mathrm{cl}_\tau(U)$ are pseudocompact with respect to the topology τ, hence they are also pseudocompact with respect to τ^* as this is a weaker topology. However, this is a metric topology, hence the sets $\mathrm{cl}_\tau(U)$ are compact with respect to τ^* and it follows that C is also closed with respect to τ^*.

2. Basic combinatorics of almost disjoint families

There are almost disjoint families of size continuum. There are two standard ways to construct them, one (used in the above mentioned paper by Alexandroff and Urysohn) is to fix for each real number $r \in \mathbb{R}$, an infinite sequence of rationals $S_r \subseteq \mathbb{Q}$ convergent to r. It is obvious that the family $\{S_r : r \in \mathbb{R}\}$ is almost disjoint and identifying $\mathbb{Q}$ with ω one gets an almost disjoint family of subsets of ω of size $\mathfrak{c}$.

Another way to construct an AD family of size $\mathfrak{c}$ is to identify ω with the binary tree $2^{<\omega}$. The key observation is that two different branches through $2^{<\omega}$ are almost disjoint subsets of nodes, thus given an $X \subseteq 2^\omega$, the family

$$\mathscr{A}_X = \{B_x : x \in X\},$$

where $B_x = \{x \restriction n : n \in \omega\}$, for $x \in X$, is an almost disjoint family of size $|X|$. This approach has the advantage that the almost disjoint family $\mathscr{A}_X$ may reflect in some way the topological properties of X, see e.g. Proposition 3.3.

A standard application of Zorn's Lemma gives a simple yet crucial fact that *every almost disjoint family can be extended to a maximal one.*

Another standard fact is that there are no countably infinite MAD families. This follows by an applications of Cantor's diagonalization method. Indeed, if $\mathscr{A} = \{A_n : n \in \omega\}$ is an almost disjoint family then one can choose $x_0 \in A_0$ and $x_{n+1} \in A_{n+1} \setminus (\bigcup_{i \leq n} A_i \cup \{x_i : i \leq n\})$, for $n \in \omega$. Of course, this is possible since A_{n+1} is assumed to be infinite and it shares only finitely many elements with each of the A_i, for $i \leq n$. Therefore the set $\{x_n : n \in \omega\}$ is almost disjoint from each of the sets in $\mathscr{A}$. Thus, the size of a MAD family ranges between $\aleph_1$ and $\mathfrak{c}$, the minimal size of a MAD family is denoted by

$$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A} \text{ is a MAD family}\}.$$

It is well known that there are models of set theory in which all possibilities of the inequalities $\aleph_1 \leq \mathfrak{a} \leq \mathfrak{c}$ hold. The interested reader can consult [**10, 23**] or [**18**] for information relevant to this affirmation and many other facts about the cardinal $\mathfrak{a}$ and other combinatorial characteristics of the continuum.

Recall that $\mathscr{I} \subsetneq \mathscr{P}(X)$ is an ideal on a set X if it is non-empty, closed under taking subsets and finite unions of its elements. Given any almost disjoint family $\mathscr{A}$, the ideal generated by $\mathscr{A}$ is

$$\mathscr{I}_{\mathscr{A}} = \left\{ I \subseteq \omega : \ (\exists H \in [\mathscr{A}]^{<\omega})(|I \setminus \bigcup H| < \omega) \right\}.$$

The collection of $\mathscr{I}_{\mathscr{A}}$-positive sets, that is, the subsets of ω that are not elements of $\mathscr{I}_{\mathscr{A}}$ is denoted by $\mathscr{I}_{\mathscr{A}}^+$.

A very useful combinatorial property of almost disjoint families is the following due to J. Dočkálková.

LEMMA 2.1. [**8**] *If $\mathscr{A}$ is an almost disjoint family and $\{X_n : n \in \omega\} \subseteq \mathscr{I}_{\mathscr{A}}^+$ with $X_{n+1} \subseteq X_n$, for each $n \in \omega$, then there is $X \in \mathscr{I}_{\mathscr{A}}^+$ which is almost contained in each of the sets X_n, for every $n \in \omega$.*

PROOF. Assume that $\mathscr{A}$ is MAD for if not $\mathscr{A}$ can be extended to a MAD family in such a way that each of the X_n's remain positive. Recursively, pick for each $n \in \omega$ a set $A_n \in \mathscr{A} \setminus \{A_m : m < n\}$ such that $X_n \cap A_n$ is infinite. Choose, for $n \in \omega$, an infinite pseudointersection $Y_n \subseteq A_n$ of $\{X_n : n \in \omega\}$. Finally set $X = \bigcup Y_n$ and observe X is the set we were looking for. $\square$

A useful equivalent of Lemma 2.1 is that for every partition of ω, $\{Y_n : n \in \omega\}$, either there is $n \in \omega$ such that $Y_n \in \mathscr{I}_{\mathscr{A}}^+$ or there is $I \in \mathscr{I}_{\mathscr{A}}^+$ such that $I \cap Y_n$ is finite for all $n \in \omega$. Another basic property is:

LEMMA 2.2. [**61**] *Let $\mathscr{A}$ be an almost disjoint family. Then any colouring $\varphi : [\omega]^2 \to 2$ is constant on the pairs of an $\mathscr{I}_{\mathscr{A}}$-positive set; that is, there is an $\mathscr{I}_{\mathscr{A}}$-positive homogeneous set for the colouring.*

PROOF. We first proof a preliminary fact: For every partition of ω into finite pieces $\{F_n : n \in \omega\}$ there is $I \in \mathscr{I}_{\mathscr{A}}^+$ such that $|I \cap F_n| \leq 1$ for all $n \in \omega$.

Choose an infinite $\{A_n : n \in \omega\} \subseteq \mathscr{A}$, and then construct $\{a_n : n \in \omega\}$ such that $a_n \in F_n$ and $(\forall k \in \omega)(\exists^\infty n \in \omega)(a_n \in A_k)$. Then the set $\{a_n : n \in \omega\}$ hits in an infinite set A_n, for all $n \in \omega$, hence it is in $\mathscr{I}_{\mathscr{A}}^+$.

Now we claim that if $\{Y_n : n \in \omega\}$ is a partition of ω such that $Y_n \notin \mathscr{I}^+_{\mathscr{A}}$ for all $n \in \omega$, then there is $Z \in \mathscr{I}^+_{\mathscr{A}}$ such that $|Z \cap Y_n| \leq 1$ for all $n \in \omega$. Indeed, by the remark before this lemma there is $X \in \mathscr{I}^+_{\mathscr{A}}$ which meets every Y_n in a finite set. Let $\{y_n : n \in \omega\}$ be an enumeration of $\omega \setminus X$. Consider now the partition given by $F_n = (X \cap Y_n) \cup \{y_n\}$ for $n \in \omega$. The preliminary fact then implies that there is $Z \in \mathscr{I}^+_{\mathscr{A}}$ such that $|Z \cap F_n| = 1$. This set Z works.

We are now ready for the proof. Observe that given an infinite $G \subset \omega$, either $G \in \mathscr{I}^+_{\mathscr{A}}$ or $\omega \setminus G \in \mathscr{I}^+_{\mathscr{A}}$, otherwise $\omega \in \mathscr{I}_{\mathscr{A}}$. For each $n \in \omega$, define

$$X_n = \{k > n : \varphi(\{k, n\}) = 1\}.$$

Since $\omega = \{n \in \omega : X_n \in \mathscr{I}^+_{\mathscr{A}}\} \cup \{n \in \omega : \omega \setminus X_n \in \mathscr{I}^+_{\mathscr{A}}\}$, without loss of generality, assume $Y = \{n \in \omega : X_n \in \mathscr{I}^+_{\mathscr{A}}\} \in \mathscr{I}^+_{\mathscr{A}}$. Set, for $n \in \omega$, $X_n^{'} = \bigcap\{X_i : i \in Y \cap n+1\} \in \mathscr{I}^+_{\mathscr{A}}$. By Lemma 2.1, there is $W \in \mathscr{I}^+_{\mathscr{A}}$ which is almost contained in each X'_n. It is easy to construct an increasing sequence $\{k_n : n \in \omega\}$ such that $W \setminus k_{n+1} \subseteq X'_{k_n}$, for all $n \in \omega$. Then either $\bigcup_{n\in\omega}[k_{2n}, k_{2n+1}) \in \mathscr{I}^+_{\mathscr{A}}$ or $\bigcup_{n\in\omega}[k_{2n+1}, k_{2n+2}) \in \mathscr{I}^+_{\mathscr{A}}$. Assume the latter and take $X = Y \cap W \cap \bigcup_{n\in\omega}[k_{2n+1}, k_{2n+2}) \in \mathscr{I}^+_{\mathscr{A}}$. By the preliminary fact, we may assume that X contains at most one point from each interval. For every $n \in \omega$ we have $X \setminus k_{2n} = X \setminus k_{2n+1} \subseteq X'_{k_{2n}}$ and if $a, b \in X$ are such that $k_{2m+1} \leq a < k_{2m+2} < k_{2n+1} \leq b < k_{2n+2}$, then $b \in X \setminus k_{2n} = X \setminus k_{2n+1} \subseteq X'_{k_{2n}} = \bigcap\{X_i : i \in Y \cap (k_{2n}+1\}$, in particular $b \in X_a$ and therefore $\varphi(\{a, b\}) = 1$. $\square$

3. Mrówka-Isbell spaces as Moore spaces

Almost disjoint families and the corresponding Ψ-spaces played their role in the solution to the normal Moore space problem [**82**]. Recall that a space X is a *Moore space* if it is a developable regular space, that is, there is a sequence of open covers of the space such that for any closed set C and any point $p \in X \setminus C$ there exists a cover in the collection such that every neighborhood of p in that cover is disjoint from C. Note that every Ψ-space is a Moore space. Tall in his thesis [**82**] gave a solution to the normal Moore space problem for separable spaces as follows

Theorem 3.1. [**82**] *The following are equivalent*

(1) *There is a separable normal non-metrizable Moore space,*
(2) *There is an uncountable normal Ψ-space,*
(3) *There is an uncountable Q-set.* [1]

Proof. A consequence of Bing's Theorem [**33**, p. 329] (that a normal Moore space is metrizable if and only if it is collectionwise normal) is that separable normal but non-metrizable Moore space has an uncountable closed discrete subset. So, if X is a separable normal non-metrizable Moore space, $N \subseteq X$ is a countable dense set and $A \subseteq X \setminus N$ is an uncountable closed discrete set, then taking sequences from N which converge to the points of A one gets an uncountable almost disjoint family $\mathscr{A}$. Normality of X implies that of $\Psi(\mathscr{A})$. This shows that (1) implies (2) and we just mentioned that (2) implies (1) as Ψ-spaces are Moore spaces.

The proof that (2) and (3) are equivalent is the Proposition 3.3 below. $\square$

[1] Recall that a separable metrizable space X is called a *Q-set* if every subset of X is G_δ in X.

Note that, in particular, (as observed previously by F. B. Jones [**33**, p. 60]) assuming the Continuum Hypothesis every separable normal Moore space is metrizable. $\Psi(\mathscr{A})$ is not a normal space whenever $\mathscr{A}$ is a MAD family. If $\mathscr{A}$ is countable, the space $\Psi(\mathscr{A})$ is normal, in fact metrizable. The following proposition is standard and easy to prove; it characterizes when the space $\Psi(\mathscr{A})$ is normal.

Proposition 3.2. *$\Psi(\mathscr{A})$ is a normal space if and only if for every $\mathscr{B} \subseteq \mathscr{A}$ there is a $J \subseteq \omega$ such that*

$$\mathscr{B} = \{A \in \mathscr{A} : A \subseteq^* J\} \quad \textit{and } \mathscr{A} \setminus \mathscr{B} = \{A \in \mathscr{A} : A \cap J =^* \emptyset\}.$$

The set J in the conclusion of the proposition is called *partitioner* for $\mathscr{B}$ and $\mathscr{A} \setminus \mathscr{B}$. Notice that if $\mathscr{B} = \{A \in \mathscr{A} : A \subseteq^* J\}$, then

$$\mathscr{B} = \bigcup_{n\in\omega} \bigcap_{m\in\omega} \{A \in \mathscr{A} : m \in A \setminus n \Rightarrow m \in J\};$$

that is, $\mathscr{B}$ is an F_σ-set of $\mathscr{A}$ as subspace of $\mathscr{P}(\omega)$. Therefore, if $\Psi(\mathscr{A})$ is a normal space then $\mathscr{A}$ is a Q-set (as subspace of $\mathscr{P}(\omega)$).

Silver [**82**] proved that it is consistent that a Q-set exists, thus the existence of a non-metrizable separable normal Moore space is consistent with ZFC.

On the other hand, recall that to any $X \subseteq 2^\omega$ corresponds an almost disjoint family $\mathscr{A}_X$ consisting of the branches B_x, for $x \in X$. We have also the following.

Proposition 3.3. [**82**] *Let $X \subseteq 2^\omega$ and $\mathscr{A}_X$ be the almost disjoint family corresponding to X. Then X is a Q-set if and only if $\Psi(\mathscr{A}_X)$ is a normal space.*

Proof. Assume X is a Q-set, $\mathscr{B} \subseteq \mathscr{A}$, and let $B = \{x \in X : A_x \in \mathscr{B}\}$. Since X is a Q-set there are closed subsets F_n and G_n of X such that $B = \bigcup_{n\in\omega} F_n$ and $X \setminus B = \bigcup_{n\in\omega} G_n$. Define $J_o = \widehat{F}_0$, $K_0 = \widehat{G}_0 \setminus \widehat{F}_0$ and $J_n = \widehat{F}_n \setminus \bigcup_{i<n} \widehat{G}_i$ as well as $K_n = \widehat{G}_n \setminus \bigcup_{i\le n} \widehat{F}_n$ for $n > 0$. Put $J = \bigcup_{n\in\omega} J_n$ and observe that $J \cap K_m =^* \emptyset$ for every $m \in \omega$. If $A_x \in \mathscr{B}$, then there is some $n \in \omega$ such that $x \in F_n$. Moreover, since each G_i is closed in X and $G_i \cap B = \emptyset$, for $i < n$, there is some $k \in \omega$ such that $[x \restriction k] \cap \bigcup_{i<n} G_i = \emptyset$. This implies that $A_x \subseteq^* J_n \subseteq J$. Similarly if $x \in X \setminus B$ there are $k, m \in \omega$ such that $x \in G_m$ and $[x \restriction k] \cap \bigcup_{i\le n} F_i = \emptyset$; this implies $A_x \cap J =^* \emptyset$. By Proposition 3.2 this suffices to show that $\Psi(\mathscr{A})$ is normal. The other direction follows from the comments following Proposition 3.2. □

To finish this section we are going back to the problem of determining the normality of the space $\Psi(\mathscr{A})$. A direct application of Jones Lemma shows that $\Psi(\mathscr{A})$ to be normal it is necessary that $|\mathscr{A}| < \mathfrak{c}$. With the aid of Martin's Axiom one can characterize the normality of $\Psi(\mathscr{A})$. This result appeared in [**45**].

An example of a non-normal Ψ-space was given by Luzin:

Theorem 3.4. [**59**] *There is an uncountable almost disjoint family such that no two uncountable subfamilies can be separated.*

Proof. Recursively choose A_α in the following way. Start by taking a partition of ω into infinite sets $\{A_n : n \in \omega\}$, then if A_β have been chosen for $\beta < \alpha$, enumerate them as $\{B_n : n \in \omega\}$ and choose $a_n \subseteq B_n \setminus \bigcup_{k<n} B_k$ of size n and let $A_\alpha = \bigcup_{n\in\omega} a_n$.

The family just constructed has the following property:

$$(\forall \alpha < \omega_1)(\forall n \in \omega)(\{\beta < \alpha : A_\beta \cap A_\alpha \subseteq n\} \text{ is finite}).$$

Families like this are called *Luzin gaps*.

If $\mathscr{B}, \mathscr{C} \subseteq \mathscr{A}$ are uncountable, then if they can be separated there is $m \in \omega$ and uncountable subfamilies $\mathscr{B}' \subseteq \mathscr{B}$ and $\mathscr{C}' \subseteq \mathscr{C}$ such that $\bigcup \mathscr{B}' \cap \bigcup \mathscr{C}' \subseteq m$, and as these families are both uncountable there is $A_\alpha \in \mathscr{B}'$ such that $A_\beta \in \mathscr{C}'$ for infinitely many $\beta < \alpha$; nevertheless $A_\alpha \cap A_\beta \nsubseteq m$ for some of those $\beta < \alpha$, which is a contradiction. $\square$

The notion of an n-Luzin gap is a weakening of a notion recently introduced in [**85, 34**] which is in turn weaker than the familiar notion of a Luzin gap defined above. Let $n \in \omega$ and $\mathscr{B}_i = \left\{B^i_\alpha \mid \alpha \in \omega_1\right\}$ be disjoint subfamilies of an AD family $\mathscr{A}$ for $i < n$. We call $\langle \mathscr{B}_i | \ i < n\rangle$ an *n-Luzin gap* if there is $m \in \omega$ such that

(1) $B^i_\alpha \cap B^j_\alpha \subseteq m$ for all $i \neq j$, $\alpha < \omega_1$, and
(2) $\bigcup_{i \neq j}(B^i_\alpha \cap B^j_\beta) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.

We say that $\mathscr{A}$ contains an n-Luzin gap if there is an n-Luzin gap $\langle \mathscr{B}_i | i < n\rangle$ where each $\mathscr{B}_i$ is a subfamily of $\mathscr{A}$. For more on Luzin gaps and related properties see [**1, 73, 50**].

THEOREM 3.5. *Assume* MA. *Let $\mathscr{A}$ be an AD family. Then $\Psi(\mathscr{A})$ is normal if and only if $|\mathscr{A}| < \mathfrak{c}$ and $\mathscr{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

4. Extensions of continuous functions

For a normal Ψ-space $\Psi(\mathscr{A})$ any function $f : \mathscr{A} \to \mathbb{R}$ has a *full* continuous extension $\widehat{f} : \Psi(\mathscr{A}) \to \mathbb{R}$, a fact that charactarizes normality of $\Psi(\mathscr{A})$, so in particular any function defined on a countable $\mathscr{A}$ has a full extension. On the other hand:

PROPOSITION 4.1. [**60**] *There is a Ψ-space of size $\aleph_1$ such that every function $f : \mathscr{A} \to \mathbb{R}$ with at least two different uncountable fibers has no full extension.*

PROOF. Follows easily from Theorem 3.4. $\square$

In [**60**], Malykhin and Tamariz-Mascarúa introduced the notion of an *essential extension* $\widehat{f} : N \cup \mathscr{A} \to \mathbb{R}$ of a given function $f : \mathscr{A} \to \mathbb{R}$ as one that is continuous, and such that $\mathscr{A} \subseteq \mathrm{cl}_{\Psi(\mathscr{A})}(N)$. The main results concerning this notion can be summarized as follows:

THEOREM 4.2 (see [**60**] and [**3**]). (1) *Every Ψ-space of size $\mathfrak{c}$ admits a function $f : \mathscr{A} \to 2$ with no essential extension.*
(2) *(Assuming $2^\omega < 2^{\omega_1}$) There is a function $f : \mathscr{A} \to 2$ without an essential extension for any uncountable AD family*
(3) *There is a function $f : \mathscr{A} \to 2$ without an essential extension for any MAD family.*
(4) *(Assuming* MA*) Every function $f : \mathscr{A} \to \mathbb{R}$ defined on an AD family of size less than $\mathfrak{c}$ has an essential extension.*

PROOF. (1) and (2) follow directly from the observation that if $\mathscr{A}$ is an AD family such that $2^{|\mathscr{A}|} > \mathfrak{c}$ (in particular if $|\mathscr{A}| = \mathfrak{c}$), then there is a function $f : \mathscr{A} \to 2$ which has no essential extension: To see this note that there are $2^{|\mathscr{A}|}$-many functions on $\mathscr{A}$, while there are only $\mathfrak{c}$-many possible essential extensions, as these are uniquely determined by their value on an infinite set N.

For (3), assume that $\mathscr{B} = \{A_n : n \in \omega\}$ is a countable subset of a MAD family $\mathscr{A}$, and $f : \mathscr{A} \to 2$ is such that, say, $\mathscr{B} = f^{-1}(1)$, and assume that $\widehat{f} : N \cup (\mathscr{A}) \to \mathbb{R}$ is an essential extension of f. Let $C = f^{-1}((\frac{1}{2}, \infty))$, and note that by continuity $A \subseteq^* C$ for every $A \in \mathscr{B}$, while $A \cap C$ is finite for every $A \in \mathscr{A} \setminus \mathscr{B}$. In particular, $C \in \mathscr{I}^+_{\mathscr{A}}$.

Now, recursively pick $k_n \in C \cap A_n \setminus \bigcup_{m<n} A_m$, and let $D = \{k_n : n \in \omega\}$. Note that D is almost disjoint from all elements of $\mathscr{A} \setminus \mathscr{B}$ being a subset of C, and by the construction, it is also almost disjoint from all elements of $\mathscr{A}$, having only finite intersection with each of the A_n's, contradicting maximality of $\mathscr{A}$.

For the proof of (4) consult [**60**]. $\square$

Note that, in particular, it is consistent with ZFC (follows from MA) that there is an AD family $\mathscr{A}$ which admits essential but not necessarily full extensions for functions $f : \mathscr{A} \to \mathbb{R}$. Similar issues were delt with by Kulesza and Levy in [**57**] where they constructed assuming MA a MAD family $\mathscr{A}$ such that every countable subset of $\mathscr{A}$ is C^*-embedded in $\Psi(\mathscr{A})$ but no infinite subset of $\mathscr{A}$ is 2-embedded in $\Psi(\mathscr{A})$, i.e. for every countable $\mathscr{B} \subseteq \mathscr{A}$ there is a function $f : \mathscr{B} \to 2$ which does not have a continuous extension $\widehat{f} : \Psi(\mathscr{A}) \to 2$.

Fibers of continuous real-valued functions on Ψ-spaces were studied by Vaughan, and Payne in [**86**].

The study of groups of homeomorphisms of Ψ-spaces was initiated by García-Ferreira in [**37**]. This is, of course, equivalent to asking under which conditions a permutation $\pi : \omega \to \omega$ extends to a homeomorphism $\widehat{\pi} : \Psi(\mathscr{A}) \to \Psi(\mathscr{A})$. The simplest possible case being that the almost disjoint family $\mathscr{A}$ is invariant under π, i.e., $\pi[A] \in \mathscr{A}$ for all $A \in \mathscr{A}$. Following [**37**] we associate to each AD family $\mathscr{A}$ the subgroup $Inv(\mathscr{A})$ of $Sym(\omega)$ which consists of those permutations π which are invariant under $\mathscr{A}$, and $Inv^*(\mathscr{A}) = \{f \in Sym(\omega) : (\forall A \in \mathscr{A})(\exists A' \in \mathscr{A})(A' =^* f[A])\}$. We consider $Sym(\omega)$ as a topological group with the subspace topology of the product ω^ω. Garcia-Ferreira in [**37**] showed that:

Theorem 4.3. [**37**] *Let G be a countable subgroup of $Sym(\omega)$. Then there is a MAD family $\mathscr{A}$ such that $G \subseteq Inv(\mathscr{A})$.*

On the other hand:

Proposition 4.4. [**4**] *There is a MAD family $\mathscr{A}$ such that $Inv(\mathscr{A}) = \{Id\}$.*

Proof. Before embarking on the proof let us make two trivial observations:

(1) If $\mathscr{A}$ and $\mathscr{B}$ are MAD families such that, for any $A \in \mathscr{A}$ there is a $B \in \mathscr{B}$ so that $A =^* B$, then $Inv^*(\mathscr{A}) = Inv^*(\mathscr{B})$.
(2) Let $\mathscr{A}$ be a MAD family, $g \in Inv^*(\mathscr{A})$, and $\mathscr{B} \subset \mathscr{A}$ with $|\mathscr{B}| < |\mathscr{A}|$. There are $X, Y \in \mathscr{A} \setminus \mathscr{B}$ such that $Y =^* g[X]$.

In order to construct the MAD family $\mathscr{A}$, let $\mathscr{C}$ be any MAD family of cardinality $\mathfrak{c}$, and let $\{f_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the set $Inv^*(\mathscr{C}) \setminus \{Id\}$. Applying the above observation recursively, one can chose $\{B^i_\beta : i < 2, \beta < \mathfrak{c}\} \subseteq \mathscr{C}$ so that, for every $\alpha < \mathfrak{c}$,

(1) $\{B^0_\alpha, B^1_\alpha\} \cap \{B^i_\beta : i < 2, \beta < \alpha\} = \emptyset$, and
(2) $B^1_\alpha =^* f_\alpha[B^0_\alpha]$.

For each $\alpha < \mathfrak{c}$, then pick $n_\alpha, m_\alpha \in \omega$ so that$n_\alpha \neq m_\alpha$ and $f_\alpha(m_\alpha) = n_\alpha$, let $A^0_\alpha = B^0_\alpha \cup \{m_\alpha\}$ and $A^1_\alpha = B^1_\alpha \setminus \{n_\alpha\}$, and define

$$\mathscr{A} = (\mathscr{C} \setminus \{B^i_\alpha : i \in 2, \alpha < \kappa\}) \cup \{A^i_\alpha : i \in 2, \alpha < \kappa\}.$$

$\mathscr{A}$ is then a MAD family, and $Inv^*(\mathscr{A}) = Inv^*(\mathscr{C})$.

Aiming for a contradiction, assume that $f \in Inv(\mathscr{A}) \setminus \{Id\}$. Then there is an $\alpha < \mathfrak{c}$ such that $f = f_\alpha$. By the construction,

$$f_\alpha[A^0_\alpha] = f_\alpha[B^0_\alpha \cup \{m_\alpha\}] =^* B^1_\alpha =^* A^1_\alpha$$

but also $f_\alpha[A^0_\alpha] \neq A^1_\alpha$, which contradicts the fact that $f \in Inv(\mathscr{A})$. $\square$

Another result of [**4**] shows that $Inv(\mathscr{A})$ can also be dense in $Sym(\omega)$. It seems to be an interesting problem to characterize those subgroups of $Sym(\omega)$ which are of the form $Inv(\mathscr{A})$, $Inv^*(\mathscr{A})$, and $\{h \restriction \omega : h \in Hom(\Psi(\mathscr{A}))\}$.

5. Fréchet and sequential spaces

There is an extremely close relationship between almost disjoint families, and Fréchet spaces. The topology of any such space is uniquely determined by taking for each point a maximal almost disjoint family of sequences converging to this point. It is therefore no surprise that Mrówka-Isbell spaces play central role in this part of topology.

Given an almost disjoint family $\mathscr{A}$, the *Franklin compactum* $\mathscr{F}(\mathscr{A}) = \Psi(\mathscr{A}) \cup \{\infty\}$ is the one point compactification of the Ψ-space $\Psi(\mathscr{A})$. It is easy to observe that $\mathscr{F}(\mathscr{A})$ is a Fréchet space if and only if for every $X \notin \mathscr{I}_{\mathscr{A}}$, the restriction of $\mathscr{A}$ to X, that is $\{A \cap X : A \in \mathscr{A}\}$, is not MAD on X. Moreover, if $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1$ is is an almost disjoint family with $\mathscr{A}_0 \cap \mathscr{A}_1 = \emptyset$, then the product $\mathscr{F}(\mathscr{A}_0) \times \mathscr{F}(\mathscr{A}_1)$ is not Fréchet if $\mathscr{A}$ is somewhere MAD, which means that the restriction of $\mathscr{A}$ is MAD on some element of $\mathscr{I}^+_{\mathscr{A}}$. Note that if $\mathscr{A}_0 \cup \mathscr{A}_1 \restriction X$ is maximal, then $\langle \infty, \infty \rangle$ is in the closure of $\{\langle n, n \rangle : n \in X\}$ but no subsequence converging to $\langle \infty, \infty \rangle$.

THEOREM 5.1. [**77**] *There is a MAD family $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1$ such that $\mathscr{F}(\mathscr{A}_i)$ is a Fréchet space for $i \in 2$, yet $\mathscr{F}(\mathscr{A}_0) \times \mathscr{F}(\mathscr{A}_1)$ is not.*

PROOF. By the comments before the statement of this theorem, all we need is to show some MAD family $\mathscr{A}$ which can be written as a disjoint union $\mathscr{A}_0 \cup \mathscr{A}_1$ of nowhere MAD families.

Proceed towards a contradiction assuming that for each MAD family $\mathscr{A}$ on a countably infinite set and for each partition $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1$ there is $i \in 2$ and a set $X_i \in \mathscr{I}^+_{\mathscr{A}}$ such that the restriction of $\mathscr{A}_i$ to X_i is MAD. We shall write below $\mathscr{I}(\mathscr{A})$ instead of $\mathscr{I}_{\mathscr{A}}$ if there are subindexes adorning $\mathscr{A}$.

Let $\mathscr{A}$ be a MAD family of size continuum enumerated as $\{A_f : f \in 2^\omega\}$. Put $\mathscr{A}_{n,i} = \{A_f : f(n) = i\}$ for $n \in \omega$ and $i \in 2$. Of course, for each $n \in \omega$, $\mathscr{A} = \mathscr{A}_{n,0} \cup \mathscr{A}_{n,1}$ and $\mathscr{A}_{n,0} \cap \mathscr{A}_{n,1} = \emptyset$.

Use induction on n to get a decreasing sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ and a sequence $\langle i_n : n \in \omega \rangle \in 2^\omega$ such that $X_n \in \mathscr{I}^+(\mathscr{A}_{n,i_n})$ and the restriction of $\mathscr{A}_{n,i_n}$ is MAD on X_n. By Lemma 2.1, there is $Y \in \mathscr{I}^+_{\mathscr{A}}$ which is almost contained in every X_n, $n \in \omega$. Since $Y \notin \mathscr{I}_{\mathscr{A}}$, $|Y \cap A_g| = \aleph_0$ for infinitely many $g \in 2^\omega$. Pick $g \in 2^\omega \setminus \{\langle i_n : n \in \omega \rangle\}$ and fix n such that $g(n) \neq i_n$.

Since $Y \setminus X_n$ is finite and $Y \cap A_g$ is infinite, it follows that $X_n \cap A_g$ is also infinite. Now $A_g \notin \mathscr{A}_{n,i_n}$, hence $A_g \cap A$ is finite for each $A \in \mathscr{A}_{n,i_n}$ and $X_n \cap A_g$ is infinite, yet the restriction of $\mathscr{A}_{n,i_n}$ to X_n is MAD, a contradiction. □

The proof of the theorem can be strengthened to show that given any MAD family $\mathscr{A}$ there is an $\mathscr{I}_{\mathscr{A}}$-positive set X such that $\mathscr{A}$ can be partitioned into two nowhere maximal families. It was unclear for a long time whether the restriction in the result was necessary. Dow in [**27**] showed that it is. He constructed a consistent example of a MAD family $\mathscr{A}$ of size ω_2 such that for any subfamily $\mathscr{B} \subseteq \mathscr{A}$ of size ω_2 there is an $\mathscr{I}_{\mathscr{A}}$-positive set X such that $\mathscr{B} \restriction X$ is maximal.

Recall that a subset A of a topological space X is *sequentially closed* if every convergent sequence of points in A has its limit point in A. A space X is *sequential* if every sequentially closed subset of X is closed. Given a subset A of X the *sequential closure* of A is defined as

$$seqcl(A) = \{x \in X : (\exists (a_n)_{n\in\omega} \subset A)(a_n \to x)\}.$$

Iterating the procedure one defines $seqcl^0(A) = A$,

$$seqcl^{\alpha+1}(A) = seqcl(seqcl^{\alpha}(A))$$

for $\alpha < \omega_1$, and $seqcl^{\alpha}(A) = \bigcup_{\beta<\alpha} seqcl^{\beta}(A)$ in case $\alpha \le \omega_1$ is a limit ordinal. The *sequential order* of a topological space X, denoted by $so(X)$, is the minimal $\alpha \le \omega_1$ such that for every $A \subset X$ the set $seqcl^{\alpha}(A)$ is sequentially closed.

The one point compactification of the Mrówka-Isbell space of a MAD family is a compact sequential space of sequential order 2. It is unknown if there is a compact sequential space of sequential order bigger than 2 in ZFC alone. In 1974, Bashkirov [**11**] proved that, under CH, there are compact sequential spaces of any sequential order. We prove Bashkirov's result next and take this opportunity to extend the concept of almost disjointness and show its usefulness. The proof presented here follows closely the argument given in [**39**]. If $\mathscr{I}$ is an ideal over ω and $A, B \subseteq \omega$ we say that they are *$\mathscr{I}$-almost disjoint* whenever $A \cap B \in \mathscr{I}$. We will use the following two facts.

Lemma 5.2. *Suppose that there is a family $\{X_\alpha : \alpha < \omega_1\}$ of compact scattered sequential spaces such that each X_α has $so(X_\alpha) < \omega_1$. If the family $\{so(X_\alpha) : \alpha < \omega_1\}$ is cofinal in ω_1, then the one point compactification of $\bigoplus_{\alpha<\omega_1} X_\alpha$ is a compact scattered space of sequential order ω_1.*

Lemma 5.3 (Folklore). *Let X be a compact scattered space of countable scattered height. Then X is sequential.*

Thus, to define the space of sequential order ω_1 it is enough to define, for each ordinal $\eta < \omega_1$, a compact sequential scattered space of sequential order $\eta + 1$. The key of our construction is the following.

Definition 5.4. Let η be an infinite countable ordinal. A family of subsets of ω will be called *η-layered* if $\mathscr{A} = \bigcup_{\xi\le\eta} L_\xi(\mathscr{A})$, where

(1) $L_0(\mathscr{A}) = \{\{n\} : n \in \omega\}$, $L_\xi(\mathscr{A})$ is a countable family of proper infinite subsets of ω for $0 < \xi < \eta$ and $L_\eta(\mathscr{A}) = \{\omega\}$,
(2) for each $\xi < \eta$ and $A, B \in L_\xi(\mathscr{A})$, $A \cap B \in \mathscr{I}_\xi$, where $\mathscr{I}_\xi$ denotes the ideal generated by the family $\bigcup_{\gamma<\xi} L_\gamma(\mathscr{A})$,

(3) for every $\xi < \zeta < \eta$, $A \in L_\xi(\mathscr{A})$ and $B \in L_\zeta(\mathscr{A})$, either $A \setminus B \in \mathscr{I}_\xi$ or $A \cap B \in \mathscr{I}_\xi$.

Given $A \in \mathscr{A}$, we say that is on the *level* ξ in $\mathscr{A}$ and write $L(A) = \xi$ if $A \in L_\xi(\mathscr{A})$. Say that an η-layered family $\mathscr{A} \subseteq \mathscr{P}(\omega)$ is a *canonical η-layered* family if given $A, B \in \mathscr{A}$, either $A \subseteq B$ or $A \cap B = \emptyset$. We also write $A \subset_{\mathscr{I}} B$ to mean that $A \setminus B$ belongs to the ideal $\mathscr{I}$ and we say that A is contained in B mod $\mathscr{I}$. We also consider the notion of equivalence of η-layered families. Layered families $\mathscr{A}$ and $\mathscr{B}$ are *equivalent* if:

- they generate the same Boolean subalgebra of $\mathscr{P}(\omega)$,
- $\mathscr{I}_\xi(\mathscr{A}) = \mathscr{I}_\xi(\mathscr{B}) = \mathscr{I}_\xi$ for every $\xi < \eta$, and
- there is a bijective function $\varphi : \mathscr{A} \to \mathscr{B}$ which preserves layers such that $A_\xi \triangle \varphi(A_\xi) \in \mathscr{I}_\xi$ for every $\xi < \eta$.

Note that, in particular, $A^m_\beta \triangle \varphi(A^m_\beta)$ belongs to the Boolean algebra generated by the layered families.

Lemma 5.5. *Each countable η-layered family $\mathscr{A}$ is equivalent to a canonical η-layered family $\mathscr{B}$.*

Given an η-layered family $\mathscr{B}$, call an infinite $Y \subseteq \mathscr{B}$ *slim* if there is a $\xi \leq \eta$ such that $\langle L(y) : y \in Y \rangle$ increasingly converges to ξ, if ξ is limit, or $Y \subseteq L_\zeta(\mathscr{A})$, if $\xi = \zeta + 1$, and there is a $C \in \mathscr{B}$ such that

- $(\forall^\infty y \in Y)(y \subseteq_{\mathscr{I}_{L(y)}} C)$,
- $\{y \in Y : y \subseteq_{\mathscr{I}_{L(y)}} D\}$ is finite for every $D \in \mathscr{A}$ such that $L(D) < L(C)$.

Note that such a $C \in \mathscr{A}$ is uniquely determined by Y.

Theorem 5.6. [**11**] *The Continuum Hypothesis,* CH*, implies that for each $\eta < \omega_1$ there is a compact scattered space of sequential order $\eta + 1$.*

Proof. Let $\eta < \omega_1$. Suppose that there is an η-layered family $\mathscr{A}$ such that for every slim $Y \in [\mathscr{A}]^\omega$ there is a $C \in \mathscr{A}$ such that

(1) $(\forall^\infty y \in Y)(y \subseteq_{\mathscr{I}_{L(y)}} C)$,
(2) $|\{y \in Y : y \subseteq_{\mathscr{I}_{L(y)}} D\}| < \omega$ for every $D \in \mathscr{A}$ with $L(D) < L(C)$, and
(3) $L(C) = \sup\{L(y) + 1 : y \in Y\}$.

Let $\mathbb{B}_{\mathscr{A}} \subseteq \mathscr{P}(\omega)$ be the Boolean algebra generated by $\mathscr{A}$. Observe that every $A \in \mathscr{A}$ produces an ultrafilter x_A on $\mathbb{B}_{\mathscr{A}}$ defined by

$$x_A = \{B \in \mathbb{B}_{\mathscr{A}} : A \subseteq_{\mathscr{I}_{L(A)}} B\}.$$

Let $X = St(\mathbb{B}_{\mathscr{A}}) = \{x_A : A \in \mathscr{A}\}$ be the Stone space of $\mathbb{B}_{\mathscr{A}}$. A slim subset Y of the η-layered family $\mathscr{A}$, corresponds to a convergent sequence in X and, if $C \in \mathscr{A}$ is the witness to Y being slim in $\mathscr{A}$, then the sequence $\{x_y : y \in Y\}$ converges to x_C. Also observe that the scattered levels, $X^{(\gamma)}$, of the space X correspond to the levels of the η-layered family $\mathscr{A}$. Thus X is a compact scattered space of height $\eta + 1$ and, by Lemma 5.3, X is also sequential.

To prove that X is of sequential order $\eta + 1$, consider the level 0 of our space. By the properties of the η-layered family $\mathscr{A}$ asserted above, if $\{y_n : n \in \omega\}$ is contained in $\bigcup_{\xi<\gamma} L_\xi(\mathscr{A})$, then every slim subset of $\{y_n : n \in \omega\}$ is witnessed by some element of $\bigcup_{\xi<\gamma+1} L_\xi(\mathscr{A})$. That is, every convergent subsequence of a sequence contained in $X^{(\leq\gamma)}$, the first γ levels of X, has a its limit in $X^{(\leq\gamma+1)}$.

This shows that $seqcl^{\alpha}(X^{(0)}) \subseteq \bigcup_{\beta\leq\alpha} X^{(\beta)}$ for all $\alpha < \eta$. So, we are left to show that the family $\mathscr{A}$ can be constructed if CH is assumed.

Fix an enumeration $\{Y_\alpha : \alpha < \omega_1\}$ of all countable infinite subsets of $\mathscr{P}(\omega)$ in such a way that each of them appears cofinally many times. Let $\mathscr{A}_0$ be an arbitrary canonical countable η-layered family, suppose $\mathscr{A}_\alpha$ have been constructed, for some $\alpha < \omega_1$. Applying Lemma 5.5, assume that $\mathscr{A}_\alpha$ is canonical. Either if Y_α is not a slim subset of $\mathscr{A}_\alpha$ or if Y_α is slim witnessing by $C_{Y_\alpha} \in \mathscr{A}_\alpha$ and by $\xi \leq \eta$, according to the definition of slim, and $L(C_{Y_\alpha}) = \xi$, then $\mathscr{A}_{\alpha+1} = \mathscr{A}_\alpha$. Otherwise, we shall construct a set $D(C_{Y_\alpha}, \xi)$.

To define $D(C_{Y_\alpha}, \xi)$ enumerate as $\{C_n : n \in \omega\}$ the family

$$\{A \in \mathscr{A}_\alpha : L(A) < \xi \ \&\ A \subset C_{Y_\alpha}\}$$

and define $k_\alpha(0) = 0$ and

$$k_\alpha(n+1) = \min\{k \in \omega : k > n + k_\alpha(n) \ \&\ (\exists y \in Y_\alpha)(y \subseteq C_k)\}.$$

Let

$$D(C_{Y_\alpha}, \xi) = \bigcup_{n\in\omega} \Big(\bigcup\{C_i : i < k_\alpha(n+1) \ \&\ L(C_i) < \xi\} \setminus \bigcup\{C_j : j < k_\alpha(n)\} \Big).$$

Then let $L_\xi(\mathscr{A}_{\alpha+1}) = L_\xi(\mathscr{A}_\alpha) \cup \{D(C_{Y_\alpha}, \xi)\}$. Clearly $\mathscr{A}_{\alpha+1}$ is a η-layered family. For $\alpha \leq \omega_1$, a limit ordinal, if $\mathscr{A}_\beta$ is defined for all $\beta < \alpha$, define $\mathscr{A}_\alpha = \bigcup_{\beta<\alpha} \mathscr{A}_\beta$.

It only remains to show that the η-layered family $\mathscr{A}_{\omega_1}$ has the required properties. Let Y be a slim subset of $\mathscr{A}_{\omega_1}$, choose α such that $Y = Y_\alpha \subseteq \mathscr{A}_\alpha$. After applying Lemma 5.5 we may assume that $\mathscr{A}_\alpha$ is a canonical η-layered family. Let $C_Y \in \mathscr{A}_\alpha$ and $\xi < \eta$ be the witness of Y being slim with respect to $\mathscr{A}_\alpha$. Then either $L(C_Y) = \xi$ and we are done or each element $A \in \mathscr{A}_\beta$ below C_Y contains only finitely many elements of Y, the definition of $D(C_Y, \xi)$ ensure that it contains infinitely many elements of Y in case $L(C_Y) > \xi$. This concludes the proof. □

It should be noted here that what we constructed was a special compactification of a Mrówka-Isbell space. In [**26**], Dow extended Bashkirov's result by constructing a compact space of sequential order 4 assuming $\mathfrak{b} = \mathfrak{c}$. He pointed out that his method cannot be generalized to get larger sequential order, and in [**28**] he proved that, under PFA, the sequential order of any compact sequential scattered space for which the sequential order and the Cantor-Bendixson rank coincide cannot be greater than ω. In the process he showed that under the same hypothesis, every MAD family contains a Luzin gap. The original question (due to Arhangel'skiĭ and Franklin [**6**]) whether there is, in ZFC, a compact sequential space of sequential order larger than 3 remains open.

6. Compactifications of Ψ-spaces

A very interesting construction of a MAD family was given by Mrówka in [**71**] where he presented an almost compact Ψ-space. Recall that a Tychonoff space is *almost compact* if its Čech-Stone compactification coincides with its one-point compactification. A MAD family $\mathscr{A}$, such that $|\beta\Psi(\mathscr{A}) \setminus \Psi(\mathscr{A})| = 1$, is called a *Mrówka family*. As we shall see later, it is one of the most useful constructions of AD families for applications in various branches of topology.

Theorem 6.1. [**71**] *There is a Mrówka MAD family $\mathscr{A}$.*

Proof. We shall construct a Mrówka family on $2^{<\omega}$. For each $f \in 2^\omega$, let $A_f = \{f \restriction n : n \in \omega\}$ and set $\mathscr{B}_0$ be a MAD family containing $\{A_f : f \in 2^\omega\}$. Use some $X \subseteq 2^\omega$ to enumerate $\mathscr{B}_0 \setminus \{A_f : f \in 2^\omega\}$ as $\{B_g : g \in X\}$. Modify $\mathscr{B}_0$ as

$$\mathscr{B}_1 = \{A_f : f \in 2^\omega \setminus X\} \cup \{A_g \cup B_g : g \in X\}.$$

Recall that $P \subseteq 2^{<\omega}$ a *partitioner* of $\mathscr{B}_1$ if for every $B \in \mathscr{B}_1$ either $P \cap B$ is finite or $B \setminus P$ is also finite. Observe that for a non-trivial[2] partitioner P the set

$$\{f \in 2^\omega : A_f \subseteq^* P\} = \{f \in 2^\omega : (\exists n \in \omega)(\forall m \geq n)(f \restriction m \in P)\}$$

is an F_σ set and hence of size $\mathfrak{c}$ since it is not countable as $\mathscr{B}_1 \restriction P$ is a MAD family. Therefore every non-trivial partitioner of $\mathscr{B}_1$ almost contains $\mathfrak{c}$ elements of $\mathscr{B}_1$.

Let $\{P_\alpha : \alpha < \kappa\}$ be all the non-trivial partitioner of $\mathscr{B}_1$, for some $\kappa \leq \mathfrak{c}$. Define a sequence $\langle A_\alpha, B_\alpha : \alpha < \kappa\rangle$ in such a way that $A_\alpha \subseteq^* P_\alpha$ and $B_\alpha \cap P_\alpha =^* \emptyset$; it is easy because there are plenty of elements from $\mathscr{B}_1$ to be chosen due to the last paragraph's claim.

Modify now the family $\mathscr{B}_1$ to the family

$$\mathscr{B}_2 = \{A_\alpha \cup B_\alpha : \alpha < \kappa\} \cup (\mathscr{B}_1 \setminus \{A_\alpha, B_\alpha : \alpha < \kappa\}).$$

Note that $\mathscr{B}_2$ has no non-trivial partitioner. Now enumerate $\mathbb{R}^\omega$ as $\{f_\alpha : \alpha < \mathfrak{c}\}$ and recursively choose elements A_α, B_α of $\mathscr{B}_2$ so that in case f_α extends to a non-trivial function $\overline{f_\alpha} : \Psi(\mathscr{B}_2) \to \mathbb{R}$, then $\overline{f_\alpha}(A_\alpha) \neq \overline{f_\alpha}(B_\alpha)$. Then set

$$\mathscr{A} = \{A_\alpha \cup B_\alpha : \alpha < \mathfrak{c}\} \cup (\mathscr{B}_2 \setminus \{A_\alpha, B_\alpha : \alpha < \mathfrak{c}\}).$$

It is easy to verify that $\mathscr{A}$ is a MAD family. Assume towards a contradiction that $\mathscr{A}$ is not a Mrówka family. As a first step, $\beta(\Psi(\mathscr{A})) \setminus \Psi(\mathscr{A})$ is connected for if C is a non-trivial clopen subset then there are open subsets U and V of $\Psi(\mathscr{A})$ which separate C from $\beta(\Psi(\mathscr{A})) \setminus C$. Then $U \cup V$ covers $\beta(\Psi(\mathscr{A})) \setminus \Psi(\mathscr{A})$ and hence $F = \mathscr{A} \cap \beta(\Psi(\mathscr{A})) \setminus (U \cup V)$ must be finite. Thus $P = (U \cap 2^{<\omega}) \setminus \bigcup F$ would be a non-trivial partitioner of $\mathscr{A}$, which is impossible.

On the other hand, it is also a zero dimensional space since otherwise there would be a surjective continuous function from $\beta(\Psi(\mathscr{A})) \setminus \Psi(\mathscr{A})$ to $[0,1]$, and as $\beta(\Psi(\mathscr{A})) \setminus \Psi(\mathscr{A})$ is closed in $\beta(\Psi(\mathscr{A}))$ that implies the existence of a surjective continuous function $\overline{f} : \Psi(\mathscr{A}) \to [0,1]$, but the restriction $\overline{f} \restriction \omega$ was enumerated as f_α and by the last modification of the family $\mathscr{B}_2$, the function $\overline{f}$ cannot be continuous at $A_\alpha \cup B_\alpha \in \mathscr{A}$. $\square$

This novel method for constructing special MAD family has appeared in several sources, see for example [**12, 13, 47, 46**]. Terasawa (a student of Mrówka) extended Mrówka's result and method by showing:

Theorem 6.2. [**84**] *For every compact metric space X without isolated points there is a MAD family $\mathscr{A}$ such that $\beta(\Psi(\mathscr{A})) \setminus \Psi(\mathscr{A})$ is homeomorphic to X.*

Terasawa credited Mrówka with the result that there is a MAD family $\mathscr{A}$ such that $\beta(\Psi(\mathscr{A}))\setminus\Psi(\mathscr{A})$ is homeommorphic to ω_1+1. Dow and Vaughan [**31**] improved this result of Mrówka. Recall that the *tower number* $\mathfrak{t}$ is the minimum size of a tower in $[\omega]^\omega$, that is, the minimum size of a family $\mathscr{T} \subseteq [\omega]^\omega$ which is well-ordered by $\supseteq^*$ and has no infinite pseudointersection.

[2]A partitioner P is *trivial* if $P \in \mathscr{I}_{\mathscr{A}}$ or $\omega \setminus P \in \mathscr{I}_{\mathscr{A}}$.

THEOREM 6.3. [**31**] *For every ordinal $\gamma < \mathfrak{t}^+$ there is a MAD family $\mathscr{A}$ such that the Čech-Stone remainder of $\Psi(\mathscr{A})$ is homeomorphic to $\gamma + 1$ with the order topology.*

Dow and Vaughan pointed out that this theorem is the best possible in ZFC since it is consistent that $\mathfrak{t}^+ + 1$ is not the Čech-Stone remainder of any Ψ-space.

As noted by Kulesza and Levy in [**57**] the methods of Baumgartner and Weese [**14**] show that assuming CH every continuous image of $\beta\omega$ is homeomorphic to the remainder of a Ψ-space. On the other hand, Dow [**25**] has shown that this fails in general, as in the Cohen model every remainder of a Ψ-space has size at most $\mathfrak{c}$. The following question of Dow seems to be still open:

QUESTION 6.4. [**24**] Is every compact space of weight ω_1 homeomorphic to the remainder of a Ψ-space?

7. Spaces of continuous functions on Ψ-spaces

Given spaces X and Y, the space of continuous functions from X to Y with the pointwise convergence topology is denoted by $C_p(X, Y)$, with $C_p(X, \mathbb{R})$ written simply as $C_p(X)$. One of the major problems in the area deals with the Lindelöf property. It is well known that $C_p(X)$ is rarely Lindelöf. Buzyakova [**20**] showed that for a class of Ψ-like spaces $C_p(X)$ is Lindelöf: Given an ordinal α she considered the space X of successor ordinals and ordinals of countable cofinality below α. This result led Dow and Simon [**29**], and Hrušák, Szeptycki and Tamariz-Mascarúa [**52**], independently, to study spaces of continuous functions over $\Psi(\mathscr{A})$.

THEOREM 7.1. [**29**] *The space $C_p(\Psi(\mathscr{A}))$ is not Lindelöf for any MAD family $\mathscr{A}$.*

PROOF. Given a MAD family $\mathscr{A}$, and $A \in \mathscr{A}$, let

$$U_A = \{f \in C_p(\Psi(\mathscr{A})) : f(A) \neq 0\},$$

and given $k < m \in \omega$ let

$$U_{k,m} = \left\{f \in C_p(\Psi(\mathscr{A})) : f(m) < {}^1\!/_{k+1} \ \& \ (\forall n \in [k, m))(f(n) < {}^1\!/_2)\right\}.$$

We claim that $\mathcal{U} = \{U_A : A \in \mathscr{A}\} \cup \{U_{k,m} : k < m \in \omega\}$ is an open cover of $C_p(\Psi(\mathscr{A}))$ without a countable subcover.

Obviously all sets in $\mathcal{U}$ are open. To se that $\mathscr{U}$ is cover note that by maximality of $\mathscr{A}$

$$f \in C_p(\Psi(\mathscr{A})) \setminus \bigcup_{A \in \mathscr{A}} U_A \text{ if and only if } \lim_{n\to\infty} f(n) = 0,$$

in which case there are $k < m \in \omega$ such that $f \in U_{k,m}$.

To see that $\mathcal{U}$ does not have a countable subcover, consider countable $\mathcal{V} \subseteq \mathcal{U}$. As $\mathscr{A}$ is uncountable, there is an $A \in \mathscr{A}$ such that $U_A \notin \mathcal{V}$. Define $g : \Psi(\mathscr{A}) \to \mathbb{R}$ by putting $g(x) = 1$ if $x \in A \cup \{A\}$, $g(B) = 0$ if $B \in \mathscr{A} \setminus \{A\}$ and $g(n) = \frac{1}{|A \cap n|+1}$ for $n \in \omega \setminus A$. It should be obvious that $g \in C_p(\Psi(\mathscr{A}))$, and also that $g \notin U_B$ $B \in \mathscr{A} \setminus \{A\}$. To see that g is not covered by $\mathcal{V}$ it therefore suffices to see that $g \notin U_{k,m}$ for any $k < m \in \omega$. For this there are two cases: if $A \cap [k, m] \neq \emptyset$ then either there is an $n \in [k, m)$ such that $g(n) = 1$ or $g(m) = 1$, hence $g \notin U_{k,m}$. If $A \cap [k, m] = \emptyset$ then $A \cap n \subseteq k$ and consequently $g(m) = \frac{1}{|A \cap n|+1} \geq \frac{1}{1+k}$, hence $g \notin U_{k,m}$. $\square$

The situation changes if one restricts to the subspace of two-valued continuous functions, i.e. the space $C_p(\Psi(\mathscr{A}), 2)$.

THEOREM 7.2. (1) [**29**] $\mathfrak{b} > \omega_1$ *implies that the space* $C_p(\Psi(\mathscr{A}), 2)$ *is not Lindelöf for any MAD family* $\mathscr{A}$.

(2) (see [**29**] and [**52**]) *It is consistent that, there is a Mrówka MAD family* $\mathscr{A}_0$ *such that* $C_p(\Psi(\mathscr{A}_0), 2)$ *is Lindelöf.*

(3) *There is a Mrówka MAD family* $\mathscr{A}_1$ *such that* $C_p(\Psi(\mathscr{A}_1), 2)$ *is not Lindelöf.*

The second and third clause nicely illustrates the complexity of the Lindelöf property on function spaces, consistently there are two spaces which are virtually identical, both are Ψ-spaces with unique compactifications, yet one has the space of continuous functions Lindelöf while the other does not.

We restrict our attention to MAD families with no non-trivial partitioners only (in particular, Mrówka MAD families are such), because they have very simple spaces of continuous functions, and we can nicely characterize when $C_p(\Psi(\mathscr{A}, 2))$ is Lindelöf. Note that for a Mrówka MAD family $\mathscr{A}$

$$C_p(\Psi(\mathscr{A}, 2)) = \bigcup_{n\in\omega, i\in 2} \sigma^i_n(\mathscr{A}),$$

where

$$\sigma^i_n(\mathscr{A}) = \{f \in C_p(\Psi(\mathscr{A}, 2)) : |f^{-1}(i) \cap \mathscr{A}| \leq n\}.$$

for every $n \in \omega$ and $i \in 2$ is a closed subspace of $C_p(\Psi(\mathscr{A}, 2))$.

We say that an AD family $\mathscr{A}$ is *concentrated on* $[\omega]^{<\omega}$ if for every open set $U \subseteq \mathscr{P}(\omega)$ containing $[\omega]^{<\omega}$ there is a countable $\mathscr{B} \subseteq \mathscr{A}$ such that $\bigcup H \in U$ for all $H \in [\mathscr{A} \setminus \mathscr{B}] < \omega$. Here we consider $\mathscr{P}(\omega)$ as a compact metric space endowed with the product topology of 2^ω via characteristic functions. Consistent examples of MAD families with similar combinatorial properties were constructed by Brendle and Piper in [**19**] and Miller in [**66**].

PROPOSITION 7.3. *Let* $\mathscr{A}$ *be a Mrówka MAD family. Then* $C_p(\Psi(\mathscr{A}, 2))$ *is Lindelöf if and only if* $\mathscr{A}$ *is* concentrated on $[\omega]^{<\omega}$.

PROOF. To start with, note that $C_p(\Psi(\mathscr{A}, 2))$ is Lindelöf if and only if $\sigma^i_n(\mathscr{A})$ is Lindelöf for every $n \in \omega$ and $i \in 2$. Since $\sigma^0_n(\mathscr{A})$ and $\sigma^1_n(\mathscr{A})$ are naturally homeomorphic, we can only consider one of them, say $\sigma^1_n(\mathscr{A})$ which we denote from now on as $\sigma_n(\mathscr{A})$.

Assume first that $\mathscr{A}$ is concentrated on $[\omega]^{<\omega}$. We shall show by induction on n that all $\sigma_n(\mathscr{A})$ are Lindelöf. To begin with note that $\sigma_0(\mathcal{A})$ consists only of characteristic functions of finite subsets of ω, hence is countable, therefore Lindelöf.

For the inductive step assume that $\sigma_{n-1}(\mathscr{A})$ is Lindelöf, and let $\mathcal{U}$ be an open cover of $\sigma_n(\mathscr{A})$ by basic open sets in $C_p(\Psi(\mathscr{A}, 2))$. By inductive hypothesis there is a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ which covers $\sigma_{n-1}(\mathscr{A})$. Now for each $x \in [\mathscr{A}]^n$ let

$$F_x = \{f \in \sigma_n(\mathscr{A}) : f^{-1}(1) \cap \mathscr{A} = x\}.$$

As each F_x is homeomorphic to a subset of 2^ω, it is covered by a countable subfamily $\mathcal{U}_x$ of $\mathcal{U}$. Thus it suffices to prove that

Claim. The set $D = \{x \in [\mathscr{A}]^n : F_x$ is not covered by $\mathcal{V}\}$ is countable.

If the set D is uncountable, it contains an uncountable Δ-system $\{x_\alpha : \alpha < \omega_1\}$ with root r and for each α a function $f_\alpha \in F_{x_\alpha} \setminus \bigcup \mathcal{V}$. By possibly going to a subset of the Δ-system we may assume that there is a finite set $a \subseteq \omega$ such that each $f_\alpha \restriction \omega$ is the characteristic function of $a \triangle \bigcup x_\alpha$. As $|r| \leq n-1$, F_r is covered by $\mathcal{V}$. Let

$$W = \{f \restriction \omega : (\exists V \in \mathcal{V})(V \cap F_r \neq \emptyset \ \& \ f \in V)\},$$

and let

$$W_r = \left\{ Z \triangle \left(a \triangle \bigcup r \right) : \chi_Z \in W \right\}.$$

W_r is then an open set in $\mathscr{P}(\omega)$ containing $[\omega]^{<\omega}$. It is obviously open, to see that it covers $[\omega]^{<\omega}$ note that given a finite $b \subseteq \omega$, the function $g \in C_p(\Psi(\mathscr{A}, 2))$ defined by $g(x) = 1$ if and only if $x \in b \triangle (a \triangle \bigcup r) \cup r$ is an element of F_r, and as $\mathcal{V}$ covers F_r, there is a $V \in \mathcal{V}$ such that $g \in V$. That is $g \restriction \omega = \chi_{b \triangle (a \triangle \bigcup r)} \in W_r$ and hence $b = (b \triangle (a \triangle \bigcup r)) \triangle (a \triangle \bigcup r) \in W_r$.

As the family $\mathscr{A}$ is concentrated on $[\omega]^{<\omega}$, there is an $\alpha < \omega_1$ such that $\bigcup (x_\beta \setminus r) \in W_r$ and hence $f_\beta \restriction \omega = \chi_{a \triangle \bigcup x_\beta} \in W$ for all $\beta > \alpha$. If $\beta > \alpha$ is large enough so that the supports of all $v \in \mathcal{V}$ are contained in β, we get that f_β is covered by $\mathcal{V}$ which is a contradiction.

For the reverse, assume that $\mathscr{A}$ is not concentrated on $[\omega]^{<\omega}$ as witnessed by an open set U. Then, we can recursively choose disjoint finite $x_\alpha \subseteq \mathscr{A}$, $\alpha < \omega_1$, such that $\bigcup X_\alpha \notin U$. Define for each $\alpha < \omega_1$ an $f_\alpha \in C_p(\Psi(\mathscr{A}, 2))$ by

$$f_\alpha(x) = 1 \text{ if and only if } x \in A \cap \{A\} \text{ for some } A \in x_\alpha.$$

Note that, as the family $\{x_\alpha : \alpha < \omega_1\}$ is pairwise disjoint, any accumulation point of it is in $\sigma_0(\mathscr{A})$. On the other hand, $[\omega]^{<\omega} \subseteq U$ which implies that for each $f \in \sigma_0(\mathscr{A})$ there is an $m \in \omega$ and $s_f : m \to 2$ such that

(1) $f \restriction m = s_f$, and
(2) $\{A \in \mathscr{P}(\omega) : A \cap m = s_f^{-1}(1)\} \subseteq U$.

Now, note that for f, the set $\{g \in C_p(\Psi(\mathscr{A}, 2)) : s_f \subseteq g\}$ defines an open neighborhood of f in $C_p(\Psi(\mathscr{A}, 2))$ disjoint from the set $\{f_\alpha : \alpha < \omega_1\}$, which is therefore closed and descrete, hence $C_p(\Psi(\mathscr{A}, 2)$ is not Lindelöf. $\square$

Using this proposition we can now prove Theorem 7.2.

Proof. (1) Assume $\mathfrak{b} > \omega_1$, and let $\mathscr{A}$ be any MAD family. We shall see that $C_p(\Psi(\mathscr{A}, 2))$ is not Lindelöf. To that end choose $\{A_\alpha : \alpha < \omega_1\}$ distinct elements of $\mathscr{A}$. As $\mathfrak{b} > \omega_1$ there is an increasing function $g : \omega \to \omega$ such that $A_\alpha \cap [g(n), g(n+1)) \neq \emptyset$ for every $\alpha < \omega_1$ and all but finitely many $n \in \omega$.[3] Given $A \in \mathscr{A}$ let

$$U_A = \{f \in C_p(\Psi(\mathscr{A}), 2) : f(A) = 1\}.$$

Now, let $J = \min\{j : (\forall \beta < \omega_1)(\exists \alpha > \beta)(\forall n \geq j)(A_\alpha \cap [g(n), g(n+1)) \neq \emptyset)\}$. Given a finite set $F \subseteq \omega$, let $m(F) = J+1$ if $F \subseteq J+1$, otherwise, let $m(F) = \max(F)+1$ (note that in either case $m(F) \geq J$), and define

$$V_F = \{f \in C_p(\Psi(\mathscr{A}), 2) : (\forall n < g(m(F)))(f(n) = 1 \text{ if and only if } n \in F)\}.$$

It is easy to see that $\mathcal{U} = \{U_A : A \in \mathscr{A}\} \cup \{V_F : F \in [\omega]^{<\omega}\}$ is an open cover of $C_p(\Psi(\mathscr{A}), 2)$, as for every $f \in C_p(\Psi(\mathscr{A}), 2)$ either there is an $A \in \mathscr{A}$ such that $f(A) = 1$, and hence $f \in U_A$, or, by maximality of $\mathscr{A}$, the set $F = f^{-1}(1)$ is a finite subset of ω, and then $f \in V_F$.

[3]To see this let f be a function dominating all increasing enumarations e_α of the sets A_α and let $g(0) = f(0)$ and $g(n+1) = f(g(n)+1)$.

Now, if $\mathcal{V}$ is a countable subset of $\mathcal{U}$, then there is $\alpha < \omega_1$ such that

- $A_\alpha \cap [g(n), g(n+1)) \neq \emptyset$ for all $n \geq J$, and
- $U_{A_\alpha} \notin \mathcal{V}$.

The function $h \in C_p(\Psi(\mathscr{A}), 2)$ defined by $h(x) = 1$ if and only if $x \in A_\alpha \cup \{A_\alpha\}$ is then not covered by $\mathcal{V}$, as it does not belong to any of the $U_B \in \mathcal{V}$, and also not to any V_F, as $h(n) = 1$ for some $n \in [g(m(F) - 1), g(m(F)))$.

Part (2), was proved in [**29**] using $\diamondsuit$, and in [**52**] using CH. Here we choose to prove that the existence of a Lindelöf $C_p(\Psi(\mathscr{A}), 2)$ is also consistent with the negation of CH. Also, the usual construction of a Mrówka family produces one of size $\mathfrak{c}$. Here we show, that a Mrówka family can also have size strictly less than $\mathfrak{c}$. This is the only explicit forcing argument we decided to put in the text.

Claim. There is a Mrówka MAD family concentrated on $[\omega]^{<\omega}$ of size ω_1 in any model obtained by adding uncountably many Cohen reals.

To see this we recall the standard construction of a MAD family of size ω_1 added by the forcing $\mathbb{P} = Fn(\omega_1, \omega)$ for adding ω_1-many Cohen reals (see [**78**]). The forcing generically adds a function $f = \bigcup G : \omega_1 \to \omega$, where G is a filter generic for $\mathbb{P}$. Fix for each infinite $\alpha < \omega_1$ a bijection $e_\alpha : \omega \to \alpha$ and let:

$$A_n = \{i \in \omega : f(i) = n\}$$

and then recursively, for infinite $\alpha < \omega_1$, let first

$$B_0^\alpha = A_{e_\alpha(0)} \ \& \ B_n^\alpha = A_{e_\alpha(n)} \setminus \bigcup_{m<n} A_{e_\alpha(m)}, \text{ for } n > 0.$$

and then let

$$A_\alpha = \{n \in \omega : \ \exists k \ n \in B_k^\alpha \ \& \ n < f(\alpha \cdot \omega + k\}.$$

We claim that the family $\mathscr{A} = \{A_\alpha : \alpha < \omega_1\}$ is the family we are looking for. First note that by genericity each of the sets A_α is infinite, and it is then clear from the definition that the family $\mathscr{A}$ is AD. The arguments that it is maximal, Mrówka, and concentrated on $[\omega]^{<\omega}$ are all very similar.

To see that $\mathscr{A}$ is maximal, let $X \in V[G]$ be an infinite subset of ω. Then there is a an infinite α such that $X \in V[G_\alpha] = V[f \restriction \alpha \cdot \omega]$. Now if $A_\beta \cap X$ is finite for every $\beta < \alpha$ then, by genericity $A_\alpha \cap X$ is infinite, as the set

$$D_n = \{p \in \mathbb{P}_\alpha : p \vdash \text{“}(\exists k > n)(k \in X \cap \dot{A}_\alpha)\text{”}\}$$

is dense in $\mathbb{P}_\alpha = Fn([\alpha \cdot \omega, \alpha \cdot \omega + \omega), \omega)$ for every $n \in \omega$.

To see that $\mathscr{A}$ is concentrated on $[\omega]^{<\omega}$, let $U \in V[G]$ be an open set containing $[\omega]^{<\omega}$. Again, there is a an infinite α such that $U \in V[G_\alpha] = V[f \restriction \alpha \cdot \omega]$. It suffices to see that for any finite $F \subseteq \omega_1 \setminus \alpha$ the set

$$E_F = \{p \in Fn(\omega_1 \setminus \alpha \cdot \omega, \omega) : p \Vdash \text{“} \bigcap_{\gamma \in F} \dot{A}_\gamma \in U\text{”}\}$$

is dense in $Fn(\omega_1 \setminus \alpha \cdot \omega, , \omega)$. To see this let $p \in Fn(\omega_1 \setminus \alpha \cdot \omega, \omega)$ and let

$$a = \{n \in \omega : p \Vdash \text{“}n \in \bigcap_{\gamma \in F} \dot{A}_\gamma\text{”}\}.$$

The set a is finite hence there is an $m \in \omega$ and $s : m \to 2$ such that

(1) $a = s^{-1}(1)$
(2) $\{X \in \mathscr{P}(\omega) : X \cap m = a\} \subseteq U$.

One can then extend p to a condition $q \in Fn(\omega_1 \setminus \alpha \cdot \omega, , \omega)$ such that

$$q \vdash \text{“} \bigcap_{\gamma \in F} \dot{A}_\gamma \cap m = a\text{”}$$

and hence

$$q \vdash \text{“} \bigcap_{\gamma \in F} \dot{A}_\gamma \in U\text{”},$$

which completes the proof of density of E_F.

Tha fact that $\mathscr{A}$ is Mrówka is proved analogously: A candidate for a continuous function $F : \Psi(\mathscr{A}) \to \mathbb{R}$ is trapped at some stage $\alpha < \omega_1$, meaning that $f = F \restriction \omega \in V[G_\alpha]$, and assuming that f is not constant outside of a set in $\mathscr{I}_{\mathscr{A}}$, genericity argument shows that f can not be continuously extended to A_α.

For (3) , let $\mathscr{B}_0 \subseteq [\omega]^\omega$ be a perfect (in particular, closed in $\mathscr{P}(\omega)$ and of size $\mathfrak{c}$) AD family. Extend $\mathscr{B}_0$ to a MAD family $\mathscr{B}_1$. Then run the construction of a Mrówka family $\mathscr{A}_1$ just as in Theorem 6.1 with the extra hypothesis that in the glueing process $\mathfrak{c}$-many elements of $\mathscr{B}_0$ are not used, which is easy to do. That is $\mathscr{A}_1$ is a Mrówka MAD family such that $|\mathscr{B}_0 \cap \mathscr{A}_1| = \mathfrak{c}$. Now, note that $U = \mathscr{P}(\omega) \setminus \mathscr{B}_0$ is an open set in $\mathscr{P}(\omega)$ containing $[\omega]^{<\omega}$ which does not contain uncountably many elements of $\mathscr{A}_1$ (all elements of $\mathscr{B}_0 \cap \mathscr{A}_1$), hence is not concentrated on $[\omega]^{<\omega}$ and consequently $C_p(\Psi(\mathscr{A}_1), 2)$ is not Lindelöf. □

An extension of these methods to properties stronger than Lindelöf have been studied by Bernal-Santos in [**15, 16**], and Bernal-Santos and Tamariz-Mascarúa in [**17**]. Spaces of continuous functions on Ψ-spaces were also studied by Just, Sipacheva and Szeptycki in [**54**], where a non-normal Ψ-space of countable extent is consistently constructed.

Banach spaces of continuous functions over Franklin compacta (or equivalently over Ψ-spaces for maximal AD families), equipped with the supremum norm, are frequently used in functional analysis (see e.g. [**35, 36, 56, 40, 62**]). For much the same reason as above (the simple structure of the space of continuous functions) the research focuses mostly on Mrówka MAD families. The main observation is

Proposition 7.4. [**40**] *Let $\mathscr{A}$ be a Mrówka MAD family. Then the corresponding Banach space (or even C^*-algebra) of continuous functions satisfies the following short exact sequence:*

$$0 \to c_0 \to c_0(\Psi(\mathscr{A})) \to c_0(2^\omega) \to 0.$$

Proof. Let $\mathscr{J}$ be the ideal on $c_0(\Psi(\mathscr{A}))$ consisting of all the continuous functions which are constant 0 on $\mathscr{A}$, i.e. the set of continuous extensions of functions in c_0. Now, if $f \in c_0(\Psi(\mathscr{A}))$, then $f \restriction \mathscr{A} \in c_0(\mathscr{A})$, while (1) any function $g \in c_0(\mathscr{A})$ can be obtained as such a restriction, and (2) all continuous extensions of $f \restriction \mathscr{A}$ to $\Psi(\mathscr{A})$ are equivalent to F modulo $\mathscr{J}$. □

In the same paper Koszmider and Ghasemi show that there is a non-commutative variation on a Mrówka MAD family:

Theorem 7.5. [**40**] *There is a C^*-algebra $\mathscr{B}$ which satisfies the following short exact sequence:*

$$0 \to \mathscr{K}(\ell_2) \to \mathscr{B} \to \mathscr{K}(\ell_2(2^\omega)) \to 0.$$

Here $\mathscr{K}(X)$ denotes the the algebra/ideal of compact operators on the corresponding Hilbert space.

8. Pseudompactness of hyperspaces and products

Given a space X, 2^X denotes the collection of all nonempty closed subsets of X endowed with the *Vietoris topology*, τ_V which has as a subbase all subsets of 2^X of the forms

$$U^- = \{A \in 2^X : A \cap U \neq \emptyset\}$$

and

$$V^+ = \{A \in 2^X : A \subseteq V\},$$

where U and V are open subsets of X. One of the fundamental problems in the theory of hyperspaces is to decide how a topological property of X can be transfered to 2^X and vice versa. For instance, the famous Vietoris-Michael theorem [**64**] asserts that a space X is compact if and only if 2^X is compact. It is natural to wonder what kind of results one can have for other type of compactness properties such as countable compactness or pseudocompactness. Since any finite product of X is embedded as a closed subset of 2^X, if 2^X is countable compact then X^n is for all $n \in \omega$. Nevertheless, neither countable compactness nor pseudocompactness is (finitely) productive in the realm of Tychonoff spaces as Novák [**72**] and Terasaka [**83**] showed independently. Ginsburg realized that it was not a simple task to study the relationship between the countable compactness or pseudocompactness (feebly compactness) of the hyperspace 2^X and that of the finite powers of X. However, he was able to show, [**42**], the following: [4]

(1) If all powers of a space X are countably compact, then its hyperspace 2^X is countably compact;
(2) If the hyperspace 2^X of a space X is countably compact, then all finite powers of X are countably compact.

and asked: *Is there any relation between the pseudocompactness (countable compactness) of X^ω and that of 2^X?* Cao, Nogura and Tomita studied Ginsburg's question and showed:

THEOREM 8.1. [**21**]

(1) *Let X be a regular homogeneous space. If 2^X is countably compact, then X^ω is countably compact.*
(2) *Let X be a Tychonoff homogeneous space. If 2^X is feebly compact, then X^κ is pseudocompact for any cardinal κ.*

Once again Mrówka-Isbell spaces were useful here too. The first related result is the following.

PROPOSITION 8.2. [**48**] *$\Psi(\mathscr{A})^\omega$ is pseudocompact for every MAD family $\mathscr{A}$.*

PROOF. Since $\mathscr{A}$ is MAD, every infinite subset of ω has an accumulation point in $\Psi(\mathscr{A})$. One can mimic the proof that shows that countable product of sequentially compact spaces is sequentially compact, to obtain that any infinite subset of ω^ω has an accumulation point in $\Psi(\mathscr{A})^\omega$. This implies that any continuous function $f : \Psi(\mathscr{A})^\omega \to \mathbb{R}$ must be bounded; otherwise ω^ω would contain an infinite set closed discrete in $\Psi(\mathscr{A})^\omega$. $\square$

[4] It is known that 2^X is completely regular if and only if it is normal if and only if it is compact; hence we use the feebly compactness concept in the realm of the hyperspaces.

Ginsburg's question in the context of Mrówka-Isbell spaces had also some surprises. While it is consistent that for every MAD family $\mathscr{A}$ the hyperspace $2^{\Psi(\mathscr{A})}$ is feebly compact, it is also consitent that there is a MAD family for which it is not.

Recall that $\mathscr{D} \subseteq [\omega]^\omega$ is *dense* if for every $B \in [\omega]^\omega$ there is $D \in \mathscr{D}$ such that $D \subseteq^* B$. The *distributivity number* $\mathfrak{h}$ of $[\omega]^\omega$ is defined as the minimal size of a collection of dense downward closed subsets of $[\omega]^\omega$ whose intersection is empty. The following is the *base tree theorem* of Balcar, Pelant and Simon.

THEOREM 8.3. [**9**] *There is a family $\mathscr{T} \subseteq [\omega]^\omega$ such that*

(1) *$\mathscr{T}$ is a tree (ordered by $\supseteq^*$) of height $\mathfrak{h}$.*
(2) *Each level of $\mathscr{T}$ is a maximal antichain in $[\omega]^\omega$ (a MAD family).*
(3) *Each $D \in \mathscr{T}$ has $\mathfrak{c}$-many immediate successors.*
(4) *$\mathscr{T}$ is a dense subset of $[\omega]^\omega$.*

We are going to use the theorem to prove the existence of a MAD family $\mathscr{A}$ for which $2^\Psi(\mathscr{A})$ is not pseudocompact if we assume that the distributivity number is less than $\mathfrak{c}$, which is a well known consistent fact.

THEOREM 8.4. [**48**] *If $\mathfrak{h} < \mathfrak{c}$, then there is a MAD family $\mathscr{A}$ such that $2^{\Psi(\mathscr{A})}$ is not feebly compact.*

PROOF. Fix a base tree $\mathscr{T}$ of height $\mathfrak{h}$ as in Theorem 8.3. For $A \subseteq 2^{<\omega}$, let $\pi_A = \{n \in \omega : A \cap 2^n \neq \emptyset\}$. Use Zorn's Lemma to get a MAD family $\mathscr{A} \subseteq [2^{<\omega}]^\omega$ be such that

(1) $\mathscr{A}$ is a MAD family (of subsets of $2^{<\omega}$),
(2) every $A \in \mathscr{A}$ is either a chain or an antichain in $2^{<\omega}$,
(3) $\pi_A \in \mathscr{T}$ for all $A \in \mathscr{A}$,
(4) $A, B \in \mathscr{A}$ and $A \neq B$ implies $\pi_A \neq \pi_B$.

Let $Y = \{F_m : m \in \omega\}$, where $F_m = 2^m$, the set of all binary sequences of length m. We will show that Y has no accumulation point in $2^{\Psi(\mathscr{A})}$. From this, $2^{\Psi(\mathscr{A})}$ will have a infinite family of discrete open subsets and hence it will not be pseudocompact. Notice that an accumulation point F of Y, if there is any, must be contained in $\mathscr{A}$, for if $s \in F \cap 2^{<\omega}$, then $\{s\}^-$ is a neighborhood of F for which $|U \cap Y| \leq 1$. To see that there are no accumulation points, let $F \subseteq \mathscr{A}$.

If $|F| < \mathfrak{c}$, there is an $f \in 2^\omega$ such that $B_f = \{f \restriction n : n \in \omega\}$ has finite intersection with all members A of F. Then

$$U = (\Psi(\mathscr{A}) \setminus \mathrm{cl}_{\Psi(\mathscr{A})}(B_f))^+$$

is a neighborhood of F, which contains no F_n.

If on the contrary, $|F| = \mathfrak{c} > \mathfrak{h}$, by (3) and (4), the set $\{\pi_A : A \in F\} \subseteq \mathscr{T}$ is not a branch of the base tree $\mathscr{T}$. So, there are $A, B \in F$ such that $\pi_A \cap \pi_B \subseteq k$ for some $k \in \omega$. Then $W = (A \backslash k)^- \cap (B \backslash k)^-$ is neighborhood of F, yet $W \cap Y = \emptyset$. So, F is not an accumulation point of Y. □

In the rest of this section we prove that it is consistent that for every MAD family $\mathscr{A}$ the hyperspace $2^{\Psi(\mathscr{A})}$ is feebly compact

LEMMA 8.5. *Let X have a dense set D of isolated points. Then the following are equivalent:*

(1) *X is pseudocompact.*

(2) *D is relatively countably compact*[5] *in X.*

Let Fin denote the set of all non-empty finite subsets of ω. The following lemma is easy to prove.

Lemma 8.6. *If X is a topological space such that ω is the dense set of isolated points of X, then* Fin *is a dense set of isolated points in 2^X.*

Recall also that a family $\mathscr{F} \subseteq [\omega]^\omega$ is *centered* if the intersection of any finite subset of $\mathscr{F}$ is infinite. The *pseudo-intersection number* $\mathfrak{p}$ is defined as the minimal size of a centered family $\mathscr{F} \subseteq [\omega]^\omega$ without an infinite pseudo-intersection, i.e. the minimal size of a centered family $\mathscr{F} \subseteq [\omega]^\omega$ such that for every $A \in [\omega]^\omega$ there is an $F \in \mathscr{F}$ such that $A \setminus F$ is infinite.

Given a one-to-one sequence $Y = \langle F_n : n \in \omega \rangle \subseteq \text{Fin}$ and $A \subseteq \omega$, let

- $I_A = \{n \in \omega : A \cap F_n \neq \emptyset\}$,
- $M_A = \{n \in \omega : F_n \subseteq A\}$.

We also define for $F \in 2^{\Psi(\mathscr{A})}$

- $\mathscr{F}_F = \left\{I_{A\setminus k} : A \in F \cap \mathscr{A},\ k \in \omega\right\} \cup \{I_{\{n\}} : n \in F \cap \omega\}$.

Lemma 8.7. *Given a one-to-one sequence $Y = \langle F_n : n \in \omega \rangle \subseteq$ Fin. Then a closed set $F \subseteq \Psi(\mathscr{A})$ is an accumulation point of Y in the hyperspace $2^{\Psi(\mathscr{A})}$ if and only if the family $\mathscr{F}_F \cup \{M_P\}$ is centered, for every $P \subseteq \omega$ such that $F \cap \omega \subseteq P$ and $(\forall A \in F \cap \mathscr{A})\,(A \subseteq^* P)$.*

Proof. Assume that F is an accumulation point of Y and $P \subseteq \omega$ is as in the statement of the lemma. To see that $\mathscr{F}_F \cup \{M_P\}$ is centered consider the open set $V = P \cup (F \cap \mathscr{A}) \subseteq \Psi(\mathscr{A})$ which is a neighborhood of F. Let

$$\mathscr{Q} = \left\{I_{A_0\setminus k_0}, \dots, I_{A_m\setminus k_m}\right\} \cup \left\{I_{\{a_0\}}, \dots, I_{\{a_l\}}\right\} \cup \{M_P\} \subseteq \mathscr{F}_F \cup \{M_P\},$$

where $A_i \in F \cap \mathscr{A},\ k_i \in \omega$ for all $i \leq m$ and $a_i \in F \cap \omega$ for all $i \leq l$. Then

$$U = \langle V; \{A_0\} \cup A_0 \setminus k_0, \dots, \{A_m\} \cup A_m \setminus k_m, \{a_0\}, \dots, \{a_l\}\rangle$$

is a neighborhood of F in $2^{\Psi(\mathscr{A})}$ and therefore $Y \cap U$ is infinite. There is an $I \in [\omega]^\omega$ such that $(A_j \setminus k_j) \cap F_i \neq \emptyset$ and $\{a_k\} \cap F_i \neq \emptyset$ for all $i \in I$ and all $j \leq m,\ k \leq l$. This means that $\mathscr{F}_F \cup \{M_P\}$ is centered.

On the other hand, if $\mathscr{F}_F \cup \{M_P\}$ is centered and consider

$$U = \langle V; \{A_0\} \cup A_0 \setminus k_0, \dots, \{A_m\} \cup A_m \setminus k_m, \{a_0\}, \dots, \{a_l\}\rangle.$$

a neighborhood of F. Since $F \cap \omega \subseteq V \cap \omega$, $A \subseteq^* V \cap \omega$ for all $A \in F \cap \mathscr{A}$ and

$$\bigcap \left\{I_{A_i\setminus k_i} : i \leq m\right\} \cap \bigcap \left\{I_{\{a_i\}} : i \leq l\right\} \cap \{M_{V\cap\omega}\}$$

is infinite. Hence so is $U \cap Y$ which shows that F is an accumulation point of Y. $\square$

Theorem 8.8. *$\mathfrak{p} = \mathfrak{c}$ implies that the hyperspace $2^{\Psi(\mathscr{A})}$ is feebly compact for every MAD family $\mathscr{A}$.*

Proof. By Lemma 8.6 and Lemma 8.5, it suffices to show that if

$$Y = \langle F_n : n \in \omega \rangle \subseteq \text{Fin}$$

is a one-to-one sequence then it has an accumulation point in $2^{\Psi(\mathscr{A})}$.

[5] A subset A of topological space X is relatively countably compact in X if every $E \in [A]^\omega$ has an accumulation point in X.

Let $\{P_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$, where each element is listed $\mathfrak{c}$-many times and $P_0 = \omega$. Construct a family $\{E_\alpha : \alpha < \mathfrak{c}\}$ with the following properties, for every $\alpha < \mathfrak{c}$:

(1) $E_\alpha \subseteq \Psi(\mathscr{A})$,
(2) $|E_\alpha| \leq |\alpha| + \omega$,
(3) $\alpha \leq \beta$ implies $E_\alpha \subseteq E_\beta$,
(4) $\mathscr{F}_\alpha = \{I_{A\setminus k} : A \in E_\alpha \cap \mathscr{A},\ k \in \omega\} \cup \{I_{\{n\}} : n \in E_\alpha \cap \omega\}$ is centered, and
(5) one of the following occurs:
  (a) $(E_\alpha \cap \omega) \setminus P_\alpha \neq \emptyset$,
  (b) there is an $A \in E_\alpha \cap \mathscr{A}$ such that $A \not\subseteq^* P_\alpha$, or
  (c) $\mathscr{F}_\alpha \cup \{M_{P_\alpha}\}$ is centered.

Since $\mathscr{A}$ is a MAD family, there is $A \in \mathscr{A}$ such that $A \cap \bigcup Y$ is infinite. Let $E_0 = \{A\}$. As $P_0 = \omega$, (1) - (5) hold.

Assume that $0 < \alpha < \mathfrak{c}$ and that E_β has been constructed for all $\beta < \alpha$. Then $\mathscr{F} = \bigcup_{\beta<\alpha} \mathscr{F}_\beta$ is centered. If $\mathscr{F} \cup \{M_{P_\alpha}\}$ is also centered, then $E_\alpha = \bigcup_{\beta<\alpha} E_\beta$ works. If $\mathscr{F} \cup \{M_{P_\alpha}\}$ is not centered, then as $\mathscr{F}$ is centered and $|\mathscr{F}| < \mathfrak{p}$, there is a $J \in [\omega]^\omega$ almost contained in all elements of $\mathscr{F}$ such that $J \cap M_{P_\alpha} = \emptyset$.

Now either $\{n \in \omega : m \in F_n\}$ is finite for all $m \in \omega \setminus P_\alpha$ or there is an $m \in \omega \setminus P_\alpha$ such that $\{n \in J : m \in F_n\}$ is infinite. If the latter occurs, then let $E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{m\}$. All clauses but (4) are evidently true. To see that the fourth clause also holds for E_α, take

$$\mathscr{G} = \{I_{A_0\setminus k_0}, \ldots, I_{A_s\setminus k_s}, I_{\{a_0\}}, \ldots, I_{\{a_t\}}, I_{\{m\}}\} \subseteq \mathscr{F} \cup \{I_{\{m\}}\}.$$

Since $\{n \in J : m \in F_n\} = I_{\{m\}} \cap J$ is infinite and $J \subseteq^* F$ for all $F \in \mathscr{F}$, then

$$\{n \in J : m \in F_n\} \subseteq^* \bigcap_{i\leq s} I_{A_i\setminus k_i} \cap \bigcap_{i\leq t} I_{\{a_i\}} \cap I_{\{m\}},$$

thus $\bigcap \mathscr{G}$ is infinite and therefore $\mathscr{F}_\alpha$ is centered.

If $\{n \in \omega : m \in F_n\}$ is finite for all $m \in \omega \setminus P_\alpha$, it follows that $\bigcup_{n\in\omega} F_n \setminus P_\alpha$ is infinite and hence there is $A \in \mathscr{A}$ such that $A \cap \left(\bigcup_{n\in\omega} F_n\right) \setminus P_\alpha$ is infinite. In this case let $E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{A\}$. To check clause (4), take

$$\mathscr{G} = \{I_{A_0\setminus k_0}, \ldots, I_{A_s\setminus k_s}, I_{\{a_0\}}, \ldots, I_{\{a_t\}}, I_{A\setminus k}\} \subseteq \mathscr{F}.$$

As the set $\bigcup_{n\in\omega} F_n \setminus P_\alpha$ is infinite, so is the set $\{n \in J : A \cap F_n \neq \emptyset\}$. Moreover,

$$\{n \in J : A \cap F_n \neq \emptyset\} \subseteq J \subset^* F$$

for all $F \in \mathscr{F}$. So, $\{n \in J : A \cap F_n \neq \emptyset\} \subseteq \bigcap \mathscr{G}$ and therefore $\mathscr{F}_\alpha$ is centered.

Let E be the closure of $\bigcup_{\alpha<\mathfrak{c}} E_\alpha \subseteq \Psi(\mathscr{A})$. We shall show that E is an accumulation point of Y in $2^{\Psi(\mathscr{A})}$. By Lemma 8.7, it suffices to show that for every $P \subseteq \omega$ one of the following holds:

(1) $(E \cap \omega) \setminus P \neq \emptyset$,
(2) there is $A \in E \cap \mathscr{A}$ such that $|A \setminus P| = \aleph_0$, or
(3) $\mathscr{F}_E \cup \{M_{P_\alpha}\}$ is centered.

Before doing that, we first show that $\mathscr{F}_E$ is centered. Clearly $\mathscr{F} = \bigcup_{\alpha<\mathfrak{c}} \mathscr{F}_\alpha$ is centered and $E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha \subseteq \mathscr{A}$, since all elements of ω are isolated. We also have that, $(A \setminus k) \cap \bigcup_{\alpha<\mathfrak{c}} E_\alpha \neq \emptyset$ for each $A \in E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ and each $k \in \omega$. Observe that if $m \in (A \setminus k) \cap \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ we have that $I_{\{m\}} \subseteq I_{A\setminus k}$. This implies that for all $F \in \mathscr{F}_E$ there is an $G \in \mathscr{F}$ such that $G \subseteq F$. As $\mathscr{F}$ is centered, so is $\mathscr{F}_E$.

Finally, consider $P \subseteq \omega$. If $\mathscr{F}_E \cup \{M_P\}$ is not centered, there are $A_0, \ldots, A_n \in E \cap \mathscr{A}$, $k_0, \ldots, k_n \in \omega$ and $m_0, \ldots, m_k \in E \cap \omega$ such that

$$\bigcap_{i \leq n} I_{A_i \setminus k_i} \cap \bigcap_{i \leq k} I_{\{m_i\}} \cap M_P$$

is finite. For each $i \leq n$ such that $A_i \in E \setminus \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ there is $\alpha_i < \mathfrak{c}$ and there is $m_i \in E_{\alpha_i}$ such that $I_{\{m_i\}} \subseteq I_{A_i \setminus k_i}$, as we saw before. Choose $\beta < \mathfrak{c}$ such that $A_j \in E_\beta$ whenever $A_j \in \bigcup_{\alpha<\mathfrak{c}} E_\alpha$ or $m_j \in \bigcup_{\alpha<\mathfrak{c}} E_\alpha$, and $j \leq n$,. Let $\alpha < \mathfrak{c}$ be such that $P = P_\alpha$ and $\alpha > \beta$. Now it is easy to see that $\mathscr{F}_\alpha \cup \{M_P\}$ is not centered either. Therefore, $(E_\alpha \cap \omega) \setminus P \neq \emptyset$ or there is $A \in E_\alpha \subseteq E$ such that $|A \setminus P| = \aleph_0$. □

Inspired by this theorem the authors answered Ginsburg's question by constructing a subspace X of $\beta\omega$ such that X^ω is pseudocompact yet 2^X is not feebly compact in ZFC. One question left open in [**48**] asks:

Question 8.9. Is there, in ZFC, a MAD family $\mathscr{A}$ such that $2^{\Psi(\mathscr{A})}$ is pseudocompact?

9. Ψ-spaces and selections

Given a set $\mathscr{F} \subseteq 2^X$, a function $\varphi : \mathscr{F} \to X$ is a *selection* on $\mathscr{F}$ if $\varphi(F) \in F$ for all $F \in \mathscr{F}$. A selection on $[X]^2$ is called a *weak selection*, and when $\mathscr{F} = 2^X$, φ is referred simply as a selection. A selection φ on $\mathcal{F}$ is a *continuous selection* if it is continuous with respect to the Vietoris topology inherited from 2^X to $\mathscr{F}$.

The study of continuous selections was initiated by Ernest Michael in his 1951 paper [**64**]. There he showed, in particular, that a sufficient condition for a space X to admit a continuous weak selection is that it admits a weaker topology generated by a linear order, i.e. is *weakly orderable*. The question whether the converse holds, implicit in Michael's paper, was asked explicitly by van Mill and Wattel in [**65**] and became known as the *van Mill-Wattel selection problem.* Michael showed that every connected compact space which admits a continuous weak selection is orderable, van Mill and Wattel showed that connectedness was not necessary.

In [**53**] the authors started the study of continuous (weak) selections on Mrówka-Isbell spaces. Among other things they showed the following

Theorem 9.1. [**53**] *The space $\Psi(\mathscr{A})$ does not have a continuous weak selection for any MAD family $\mathscr{A}$.*

Proof. Aiming towards a contradiction assume that f is a continuous weak selection on $\Psi(\mathscr{A})$ for a MAD family $\mathscr{A}$ and let $c(\{m,n\}) = 0$ if $f(\{m,n\}) = \min\{m,n\}$, and $c(\{m,n\}) = 1$ otherwise. By Lemma 2.2 there is an $\mathscr{I}(\mathscr{A})$-positive set X homogeneous for c. As $\mathscr{A}$ is maximal, there are A_0, A_1 distinct elements of $\mathscr{A}$ which both intersect X on an infinite set. Now, assume that $f(A_0, A_1) = A_0$. By continuity of F this means that $f(n,m) = n$ for all but finitely many $n \in A_0$, $m \in A_1$, i.e for all n, m larger than some number N. Now, let $N < n_0 < m < n_1$ be such that $n_0, n_1 \in A_0 \cap X$ and $m \in A_1 \cap X$. Then $c(\{n_0, m\}) \neq c(\{n_1, m\})$ which contradicts homogeneity of X. □

A stronger result, an extension of the van Mill-Wattel result, was obtained by García-Ferreira and Sanchis in [**38**] building on results of Artico, Marconi, Pelant, Rotter and Tkachenko [**7**].

THEOREM 9.2. [**38**] *A pseudocompact space admits a continuous weak selection if and only if it is weakly-orderable.*

PROOF. The right to left implication follows from the above mentioned result of Michael. For the left to right direction it suffices to show that *any continuous weak selection on a pseudocompact space X can be extended to a continuous weak selection on βX*, as by the van Mill-Wattel theorem βX is then orderable, hence, X suborderable, and, in particular, weakly orderable.

Now, a continuous weak selection f on a psedo-compact space X can be treated as a continuous $f : X \times X \to X$ such that (1) $f(x,y) \in \{x,y\}$, and (2) $f(x,y) = f(y,x)$. It is immediate from the density of the space X in βX that any continuous extension of f to βX is itself a weak selection on βX. Hence to prove the theorem one only needs to note that $X \times X$ is pseudocompact, and Glicksberg's Theorem [**43**] provides the required extension (see **??**).

To that end, let X be a pseudocompact space X addmitting a continuous weak selection f.

Claim. Given a sequence $\{V_n : n \in \omega\}$ of non-empty pairwise disjoint open subsets of X there is a strictly increasing sequence $\{n_i : i \in \omega\}$ of integers, a sequence $\{W_i : i \in \omega\}$ of non-empty open sets in X and a a point $x \in X$ such that

(1) $W_i \subseteq V_{n_i}$ for every $i \in \omega$, and
(2) the sequence $\{W_i : i \in \omega\}$ *converges* to x, i.e. every neighborhood of X contains all but finitely many elements of $\{W_i : i \in \omega\}$.

We shall refer to the conclusion as saying that the sequence $\{V_n : n \in \omega\}$ has a *convergent shrinking subsequence.* Note that once the claim is proved it easily follows that $X \times X$ is pseudocompact, as the property that every sequence of parwise disjoint non-empty open sets has a convergent shrinking subsequence is easily seen to be finitely productive and stronger than pseudocompactness.

We shall prove the claim by contradiction. Note first that in a pseudocompact space a sequence of open sets converges if and only if it has a unique accumulation point. Assuming that the sequence $\{V_n : n \in \omega\}$ does not have a convergent shrinking subsequence, one can recursively construct non-empty open sets P_n, Q_n such that for alla $n \in \omega$

(1) $P_n \cap Q_n = \emptyset$,
(2) $f[P_n, Q_n] \subseteq P_n$
(3) $P_{n+1} \cup Q_{n+1} \subseteq Q_n$, and
(4) both $A_n = \{i \in \omega : P_n \cap V_i \neq \emptyset\}$ and $B_n = \{i \in \omega : Q_n \cap V_i \neq \emptyset\}$ are infinite.

Having done this we pick an increasing sequence $\{n_i : i \in \omega\}$ so that $n_i \in A_i$ and let $W_i = P_i \cap V_{n_i}$ for each $i \in \omega$. Note that then $f[W_i, W_j] \subseteq W_i$ for any $i < j \in \omega$. By pseudocompactness, $\{W_i : i \in \omega\}$ has an accumulation point x. We claim that x is the unique accumulation point, i.e. the sequence $\{W_i : i \in \omega\}$ converges to x. If not, i.e. if there is an accumulation point y different from x then there are disjoint non-empty open sets U and V each containing one of x and y such that $f[V, U] \subseteq V$. Pick $i < j \in \omega$ such that $W_i \cap V \neq \emptyset$ and $W_j \cap U \neq \emptyset$, and choose $x_i \in W_i \cap U$ and $x_j \in W_i \cap V$. Then, on the one hand $f(x_i, x_j) = x_i$ as $f[W_i, W_j] \subseteq W_i$, but on the other hand $f(x_i, x_j) = x_j$ as $f[V, U] \subseteq V$, which is a contradiction. $\square$

The van Mill-Wattel selection problem was eventually solved using Mrówka-Isbell spaces:

THEOREM 9.3. [**49**] *There is an almost disjoint family $\mathscr{A}$ such that the space $\Psi(\mathscr{A})$ admits a continuous weak selection but it is not weakly orderable.*

PROOF. We shall prove the theorem by a sequence of lemmata starting with the following simple fact[6]:

LEMMA 9.4. *Let φ be a weak selection on ω and let $\mathscr{A}$ be an almost disjoint family. Then φ extends (uniquely) to a continuous weak selection on $\Psi(\mathscr{A})$ if and only if*

(1) *$A||^*_\varphi B$ for all $A \neq B \in \mathscr{A}$ and*
(2) *$\{n\}||^*_\varphi A$ for all $n \in \omega$ and $A \in \mathscr{A}$.*

Our plan for constructing the space is to first find a suitable weak selection on ω and then to carefully construct an AD family to which the selection extends. This selection can be viewed as an "oriented" version of Rado's Random graph.

LEMMA 9.5. *There is a weak selection $\varphi : [\omega]^2 \to \omega$ such that for every disjoint $F, G \in [\omega]^{<\omega}$, there is an $n \in \omega \setminus (F \cup G)$ such that $F \rightrightarrows_\varphi \{n\} \rightrightarrows_\varphi G$.*

PROOF. Let $\mathscr{I} = \{I_n : n \in \omega\} \subseteq [\omega]^\omega$ be an independent family[7] such that $n \in I_m$ if and only if $m \notin I_n$, for every $n, m \in \omega$.[8] Let $\varphi : [\omega]^2 \to \omega$ be defined by $\varphi(\{n, m\}) = n$ if and only if $n \in I_m$.

Now, if $F, G \in [\omega]^{<\omega}$ are disjoint then $F \rightrightarrows_\varphi \{k\} \rightrightarrows_\varphi G$ for any $k \in (\bigcap_{n\in F} I_n) \cap (\bigcap_{m\in G} (\omega \setminus I_m))$. $\square$

The selection φ will be referred to as the *universal weak selection*. We denote by

$$\mathscr{R} = \{A \subseteq \omega : \varphi \restriction [A]^2 \approx \varphi\}$$

[6]For the rest of this section we shall fix the following notation concerning weak selections. Given sets X and Y, and $\psi : [X]^2 \to X$ and $\varphi : [Y]^2 \to Y$ weak selections, we will say that ψ and φ are *isomorphic*, $\psi \approx \varphi$, if there is a bijection $\rho : X \to Y$ such that $\psi(\{a, b\}) = \varphi(\{\rho(a), \rho(b)\})$ for every $a, b \in X$. We will also say that ψ is embedded in φ if $\psi \approx \varphi \restriction [A]^2$ for some $A \subseteq X$. Let φ be a weak selection on a set X and let $x, y \in X$. We will denote by $x \to_\varphi y$ the condition $\varphi(x, y) = y$. If $A, B \subseteq X$, we will say that B *dominates* A *with respect to* φ, denoted by $A \rightrightarrows_\varphi B$, if for every $a \in A$ and $b \in B$, $a \to_\varphi b$. We will also say that A and B are *aligned* with respect to φ and denote by $A||_\varphi B$, if $A \rightrightarrows_\varphi B$ or $B \rightrightarrows_\varphi A$. Given $A, B \in [\omega]^\omega$ and ψ a weak selection on ω, we will say that B *almost dominates* A *with respect to* ψ (or simply that B *almost dominates* A if ψ is clear from the context) and denote by $A \rightrightarrows^*_\psi B$, if there is a $k \in \omega$ such that $A \setminus k \rightrightarrows_\psi B \setminus k$. We will also say that A and B are *almost aligned* with respect to ψ, denoted by $A||^*_\psi B$, if $A \rightrightarrows^*_\psi B$ or $B \rightrightarrows^*_\psi A$. If $n \in \omega$ then we will say that A *is almost dominated by* $\{n\}$, which will be denoted by $A \rightrightarrows^*_\psi \{n\}$, whenever $A \setminus k \rightrightarrows_\psi \{n\}$ for some $k \in \omega$. In a similar way, we define $\{n\} \rightrightarrows^*_\psi A$ and $\{n\}||^*_\psi A$. When the selection is clear from the context, we suppress the use of the subscript. Given a weak selection φ, a triple $\{a, b, c\}$ is called a *3-cycle* if either $a \to b \to c \to a$ or $c \to b \to a \to c$.

[7]Recall that a family $\mathscr{I} \subseteq [\omega]^\omega$ is *independent* if $\bigcap \mathscr{F} \setminus \bigcup \mathscr{F}'$ is infinite for every $\mathscr{F}, \mathscr{F}'$ finite disjoint subsets of $\mathscr{I}$.

[8]To obtain such an independent family start with an arbitrary independent family $\mathscr{J} = \{J_n : n \in \omega\} \subseteq [\omega]^\omega$ and recursively define a family $\mathscr{I} = \{I_n : n \in \omega\}$ as follows:

- $I_0 = J_0$;
- $I_{n+1} = (J_{n+1} \setminus \{k \leq n : n+1 \in I_k\}) \cup \{k \leq n : n+1 \notin I_k\}$.

For every $n \in \omega$, the set $I_n \in \mathscr{I}$ is obtained by finite changes of J_n, guaranteeing that $\mathscr{I}$ is also an independent family such that, $n \in I_m$ if and only if $m \notin I_n$, for every $n, m \in \omega$.

the set of copies of φ in itself. The selection has the following basic properties (the corresponding properties are known to hold for the Random graph):

(a) Every weak selection ψ on ω can be embedded in φ.
(b) Given any partition $\{P_0, P_1\}$ of ω, there is an $i \in 2$ such that $P_i \in \mathscr{R}$.
(c) If $F, G \in [\omega]^{<\omega}$ are disjoint, then the set

$$\{k \in \omega \setminus (F \cup G) : F \rightrightarrows_\varphi \{k\} \rightrightarrows_\varphi G\} \in \mathscr{R}$$

(a) is easily proved by the back-and-forth argument.

For (b), suppose the contrary and let $\{P_0, P_1\}$ be a partition of ω such that neither P_0 nor P_1 is in $\mathscr{R}$. Then we can find $F_i, G_i \in [P_i]^{<\omega}$ disjoint such that every $n \in P_i$ does not dominate F_i or is not dominated by G_i, so there is an $m \in \omega$ so that $F_0 \cup F_1 \rightrightarrows \{m\} \rightrightarrows G_0 \cup G_1$. However $m \in P_0$ or $m \in P_1$, which in either case is a contradiction.

For (c), suppose that for a couple F, G of finite disjoint subsets of ω, the set $A = \{k \in \omega \setminus (F \cup G) : F \rightrightarrows \{k\} \rightrightarrows G\} \notin \mathscr{R}$. It follows by (c) that $\omega \setminus A$ is in $\mathscr{R}$ and so there is an $n \in \omega \setminus A$ that dominates F and is dominated by G, but this n must also be in A, which is a contradiction.

Let $\leq$ be a linear order on a set X and let $Y \subseteq X$ be infinite. We will say that a set Y is *monotone*, if either there is a downward closed set $S \subseteq X$ such that $Y \subseteq S$ and for every $s \in S$, $Y \cap (\leftarrow, s)_\leq$ is finite, or there is an upward closed set $T \subseteq X$ such that $Y \subseteq T$ and for every $t \in T$, $Y \cap (t, \rightarrow)_\leq$ is finite.

LEMMA 9.6. *Let φ be the universal selection and let $\preccurlyeq$ be a linear order on ω. If $X \subseteq \omega$ belongs to $\mathscr{R}$, then there are $X_0, X_1 \in [X]^\omega$ such that*

(1) $X_0 \cap X_1 = \emptyset$,
(2) $X_0 \rightrightarrows X_1$,
(3) $X_0 \cup X_1$ *is monotone.*

PROOF. If $X \cap (\leftarrow, 0)_\preccurlyeq \in \mathscr{R}$, then define $M_0 = X \cap (\leftarrow, 0)_\preccurlyeq$ and let $M_0 = X \cap [0, \rightarrow)_\preccurlyeq$ otherwise. As $X \in \mathscr{R}$, in either case it occurs that $M_0 \in \mathscr{R}$ by (c) above. Choose $a_0, b_0, c_0 \in M_0$ distinct so that $\{a_0, b_0, c_0\}$ is a 3-cycle in M_0. Choose now $x_0, y_0 \in \{a_0, b_0, c_0\}$ such that $x_0 \prec y_0$ and $x_0 \to y_0$ and define the set $D_1 = \{n \in M_0 : x_0 \to n \to y_0\} \setminus \{x_0, y_0\}$ which, by (c) above, is in $\mathscr{R}$. As before, let $M_1 = D_1 \cap (\leftarrow, 1)_\preccurlyeq$ if $D_1 \cap (\leftarrow, 1)_\preccurlyeq \in \mathscr{R}$ and let $M_1 = D_1 \cap [1, \rightarrow)_\preccurlyeq$ in the other case. Choose $a_1, b_1, c_1 \in M_1$ so that $\{a_1, b_1, c_1\}$ is a 3-cycle en M_1 and pick $x_1, y_1 \in \{a_1, b_1, c_1\}$ such that $x_1 \to y_1$ and $y_1 \prec x_1$. Notice that $\{x_0, x_1\} \rightrightarrows \{y_0, y_1\}$.

Following this procedure, we can form recursively $\{M_n : n \in \omega\} \subseteq \mathscr{R}$ and disjoint subsets $W_0 = \{x_n : n \in \omega\}$, $W_1 = \{y_n : n \in \omega\} \in [X]^\omega$ such that for every $n \in \omega$, $M_{n+1} \subseteq M_n$, $\{x_0, x_1, ..., x_n\} \rightrightarrows \{y_0, y_1, ..., y_n\}$, $x_n \prec y_n$ whenever n is even and $y_n \prec x_n$ if n is odd. Moreover, the set $S = \{n \in \omega : M_n \subseteq (n, \rightarrow)_\preccurlyeq\}$, if infinite, is $\preccurlyeq$-downward closed and the set $T = \{n \in \omega : M_n \subseteq (\leftarrow, n)_\preccurlyeq\}$ is $\preccurlyeq$-upward closed, if it is infinite. Notice also that for every $k \in S$, $(W_0 \cup W_1) \cap (\leftarrow, k)_\preccurlyeq$ is finite, as well as for every $k \in T$, $(W_0 \cup W_1) \cap (k, \rightarrow)_\preccurlyeq$ is finite.

To conclude the proof, notice that either $W_0 \cap S$ and $W_1 \cap S$ are infinite or $W_0 \cap T$ and $W_1 \cap T$ are. To see this, suppose e.g., that $W_0 \cap S$ is finite. As $S \cup T = \omega$, there is some $k \in \omega$ such that for all $n \geq k$, $x_n \in T$. Whenever $m \geq k$ is even, then $x_m \prec y_m$ and T is $\preccurlyeq$-upward closed, so $y_m \in T$, too. If both sets $W_0 \cap S, W_1 \cap S$ are infinite, define $X_0 = W_0 \cap S$ and $X_1 = W_1 \cap S$, if not, let $X_0 = W_0 \cap T$ and

$X_1 = W_1 \cap T$. The recursion guarantees that whenever $k \geq n$, then $x_k, y_k \in M_n$, consequently, the set $X_0 \cup X_1$ is monotone. $\square$

The following is the first approximation to the AD family we need:

LEMMA 9.7. *There is an AD family $\mathscr{A} \subseteq [\omega]^\omega$ such that:*
(1) $|\mathscr{A}| = \mathfrak{c}$,
(2) $\mathscr{A} \subseteq \mathscr{R}$ *and*
(3) $A||^*B$ *for every* $A \neq B \in \mathscr{A}$.

PROOF. Consider the complete binary tree $2^{<\omega}$ and for every $f \in 2^\omega$, consider the branch determined by f, $A_f = \{f \restriction n : n \in \omega\}$. For $f, g \in 2^{<\omega}$, we will write $f \perp g$ if there is an $n \in \omega$ so that $f(n) \neq g(n)$ and $f \not\perp g$ whenever either $f \subseteq g$ or $g \subseteq f$. Define the weak selection ψ on $2^{<\omega}$ by $\psi(\{f, g\}) = g$ if and only if either $f \not\perp g$ and $\varphi(\{|f|, |g|\}) = |g|$ or $f \perp g$ and $f(f \vartriangle g) = 0$, where $f \vartriangle g = \min\{k \in \omega : f(k) \neq g(k)\}$.

By the universality of φ, we can suppose, without loss of generality, that ψ is embedded in φ. It is easy to see that $A_f \in \mathscr{R}$ for every $f \in 2^\omega$. Moreover, it holds that $(A_f \setminus f \vartriangle g) \rightrightarrows (A_g \setminus f \vartriangle g)$ if $f(f \vartriangle g) = 0$ and $(A_g \setminus f \vartriangle g) \rightrightarrows (A_f \setminus f \vartriangle g)$ otherwise, which implies that $A_f||^*A_g$. Therefore $\mathscr{A} = \{A_f : f \in 2^\omega\}$ is the required family. $\square$

Next we will show how to refine $\mathscr{A}$ to "kill" all potential linear orders on ω. To that end enumerate $\mathscr{A}$ as $\{A_\alpha : \alpha < \mathfrak{c}\}$ and all linear orders on ω as $\{\leq_\alpha : \alpha < \mathfrak{c}\}$.

LEMMA 9.8. *For every $\alpha < \mathfrak{c}$, there are $X_0^\alpha, X_1^\alpha \in [A_\alpha]^\omega$ such that*
(1) $X_0^\alpha \cap X_1^\alpha =^* \emptyset$,
(2) $X_0^\alpha||^*X_1^\alpha$,
(3) *for every* $n \in \omega$ *and* $i \in 2$, $X_i^\alpha||^*\{n\}$ *and*
(4) $X_0^\alpha \cup X_1^\alpha$ *is* $\leq_\alpha$*-monotone.*

PROOF. Fix $\alpha < \mathfrak{c}$. By Lemma 9.7, $A_\alpha \in \mathscr{R}$ and by Proposition 9.6 we can find $X_0, X_1 \in [A_\alpha]^\omega$ such that $X_0 \rightrightarrows X_1$ and $X_0 \cup X_1$ is $\leq_\alpha$-monotone. Since for every $x \in X_0$, either $x \to 0$ or $0 \to x$, there is an infinite $C_0 \subseteq X_0$ such that $C_0||\{0\}$. Proceeding recursively, construct a family $\mathscr{C} = \{C_n : n \in \omega\}$ of infinite subsets of X_0 in such a way that for every $n \in \omega$, $C_{n+1} \subseteq C_n$ and $C_n||\{n\}$. Let X_0^α be a pseudointersection of $\mathscr{C}$, i.e. $X_0^\alpha \in [X_0]^\omega$ is such that $C_n \setminus X_0^\alpha$ is finite for every $n \in \omega$. Analogously, construct a family $\mathscr{E} = \{E_n : n \in \omega\}$ of infinite subsets of X_1 such that $E_{n+1} \subseteq E_n$ and $E_n||\{n\}$ for every $n \in \omega$. Therefore, if X_1^α is a pseudointersection of $\mathscr{E}$, then X_0^α, X_1^α satisfy (a), (b), (c) by the construction and (d) follows by the fact that both sets are infinite subsets of X_0 and X_1, which satisfy 9.6, (3). $\square$

We are ready now to prove the main result of this section. Let $\mathscr{B} = \{X_0^\alpha, X_1^\alpha : \alpha < \mathfrak{c}\}$, where X_i^α is as in the Lemma 9.8 for $i \in 2$, and consider $X = \Psi(\mathscr{B})$, the Mrówka-Isbell space associated to $\mathscr{B}$.

By Lemmata 9.7 and 9.8, φ satisfies the conditions of Lemma 9.4, hence there is a (unique) continuous weak selection $\overline{\varphi}$ on $\Psi(\mathscr{B})$ extending the universal weak selection φ.

To conclude the proof, it is enough to verify that X is not weakly orderable. Aiming towards a contradiction, suppose that there exists a linear order $\sqsubseteq$ on X

whose induced topology is coarser than the topology on X. Let $\alpha < \mathfrak{c}$ be such that $\sqsubseteq\restriction [\omega]^2 =\leq_\alpha$ and suppose, without loss of generality, that for the points $X_0^\alpha, X_1^\alpha \in \Psi(\mathscr{B})$ the inequality $X_0^\alpha \sqsubseteq X_1^\alpha$ holds. By Lemma 9.8, the infinite set $X_0^\alpha \cup X_1^\alpha$ is $\leq_\alpha$-monotone. Assume that $S \subseteq \omega$ is a witness: The set S is downward closed, contains $X_0^\alpha \cup X_1^\alpha$ and for every $s \in S$, $(\leftarrow, s)_{\leq_\alpha} \cap (X_0^\alpha \cup X_1^\alpha)$ is finite. If there is an $s \in S$, with $X_0^\alpha \sqsubseteq s$, then $(\leftarrow, s)_{\sqsubseteq}$ is an $\sqsubseteq$-open interval containing the point X_0^α, which meets the set X_0^α in finitely many points. However this contradicts the assumption that the $\sqsubset$-order topology on X is coarser that the original one. However, if $S \subseteq (\leftarrow, X_0^\alpha)_{\sqsubseteq}$, then the interval $(X_0^\alpha, \rightarrow)_{\sqsubseteq}$ contains the point X_1^α and is disjoint from the set X_1^α, which leads to the same contradiction. The case when $X_0^\alpha \cup X_1^\alpha$ is contained in an upward directed set T is treated analogously.

We have proved that the topology determined by the order $\sqsubseteq$ cannot be coarser than that of X and therefore X is not weakly orderable. $\square$

The authors of [**49**] showed that the space $\Psi(\mathscr{A})$ constructed in the previous theorem is not weakly orderable yet it admits a continuous selection for all compact sets, and in particular, for all finite sets. In fact, they showed that any separable space admitting a continuous weak selection admits a continuous weak selection on n-tuples for any finite n. It is an open problem due to Gutev and Nogura [**44**] whether the existence of a continuous weak selection on a space X gaurantees the existence of a continuous selection for triples on X.

Next theorem shows that even for separable spaces the existence of a continuous selections for triples does not guarantee the existence of a continuous selection for pairs.

THEOREM 9.9. [**49**] *There is an almost disjoint family $\mathscr{A}$ such that the space $\Psi(\mathscr{A})$ admits a continuous weak selection for triples but no continuous weak selection.*

PROOF. Identify ω with $2^{<\omega}$. For every $f \in 2^\omega$ let $A_f = \{f \restriction n : n \in \omega\}$ be the branch determined by f and let $\mathscr{A} = \{A_f : f \in 2^\omega\}$. Enumerate the AD family $\mathscr{A}$ by $\{A_\alpha : \alpha < \mathfrak{c}\}$. Enumerate also the set of all weak selections on $2^{<\omega}$ by $\{f_\alpha : \alpha < \mathfrak{c}\}$

For every $\alpha < \mathfrak{c}$ define $g_\alpha : [A_\alpha]^2 \to 2$ as follows:

$$g_\alpha(\{f \restriction m, f \restriction n\}) = \begin{cases} 0 & \text{if } f_\alpha(f \restriction m, f \restriction n) = f \restriction \min\{m, n\} \\ 1 & \text{if } f_\alpha(f \restriction m, f \restriction n) = f \restriction \max\{m, n\} \end{cases},$$

where $f \in 2^\omega$ and $A_\alpha = A_f$.

By Ramsey's Theorem, there is a g_α-homogeneous set $B_\alpha \in [A_\alpha]^\omega$ so that $g_\alpha''[B_\alpha]^2 = \{i\}$ for some $i \in 2$. Let $\{B_\alpha^0, B_\alpha^1\}$ be a partition of B_α such that $|B_\alpha^i| = \omega$ for $i \in 2$ and consider the AD family $\mathscr{B} = \{B_\alpha^0, B_\alpha^1 : \alpha < \mathfrak{c}\}$. Let $X = \Psi(\mathscr{B})$, the Mrówka-Isbell space associated to $\mathscr{B}$.

We define a relation $\leq$ on X in the following way:

$$x \leq y \text{ if and only if } \begin{cases} x = y \text{ or} \\ x, y \in 2^{<\omega} \text{ and } x \subseteq y \text{ or} \\ x = f \restriction n \in 2^{<\omega} \text{ and } y = B_f^i \text{ for some } i \in 2 \end{cases}$$

It is clear that $\leq$ is reflexive, antisymmetric and transitive.

If $x \nleq y$ and $y \nleq x$, we will write $x \perp y$. Now, for every pair of elements $x, y \in X$ with $x \perp y$, we can associate an element $\Delta_{x,y}$ of $\omega \cup \{\omega\}$ as follows:

$$\Delta_{x,y} = \begin{cases} \min\{n : x(n) \neq y(n)\} \text{ if } x, y \in 2^{<\omega} \\ \min\{n : x(n) \neq f(n)\} \text{ if } x \in 2^{<\omega} \text{ and } y = B_f^i \text{ for some } i \in 2 \\ \min\{n : f(n) \neq g(n)\} \text{ if } x = B_f^i,\ y = B_g^j \text{ with } i, j \in 2 \text{ and } f \neq g \\ \omega \text{ if } \{x, y\} = \{B_f^0, B_f^1\} \text{ for some } f \in S \end{cases}$$

Notice that if $x \perp y$ and $y \leq z$ then $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.

We define the function $\rho : [X]^3 \to X$ by $\rho(\{x, y, z\}) = x$, if either $x \leq y$ and $x \leq z$ or $x \perp y$, $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.

Let us first prove that ρ is well defined. Let $F = \{x, y, z\} \in [X]^3$. Notice that F has at most one element comparable with all its elements. In this case, the function is well defined by construction. So we can suppose that $x \perp y$ and $x \perp z$. If $y \leq z$ then $\Delta_{x,y} = \Delta_{x,z}$ and, since y and z are comparable, then $\rho(\{x, y, z\}) = x$. In the same way, if $x \perp z$ and $z \leq y$ then $\rho(\{x, y, z\}) = x$. Therefore, we can suppose that $x \perp y$, $x \perp z$ and $y \perp z$. If $\Delta_{x,y} = \Delta_{x,z}$ then $\Delta_{y,z} > \Delta_{x,y}$ and so $\rho(\{x, y, z\}) = x$. Otherwise, if $\Delta_{x,y} < \Delta_{x,z}$ then $\Delta_{y,z} = \Delta_{x,y}$ and then $\rho(\{x, y, z\}) = y$. Finally, if $\Delta_{x,y} > \Delta_{x,z}$ then $\Delta_{y,z} = \Delta_{x,z}$ and $\rho(\{x, y, z\}) = z$.

To prove that ρ is continuous, let $\{x, y, z\} \in [X]^3$ $\rho(\{x, y, z\}) = x$.

Case 1: $x \leq y$ and $x \leq z$.

Since $x \in 2^{<\omega}$, there are $f \in S$ and $n \in \omega$ such that $x = f \upharpoonright n$. If $y = f \upharpoonright m$ for some $m > n$, then let $U_y = \{f \upharpoonright m\}$. Otherwise, if $y = B_f^i$ for some $i \in 2$, let $U_y = \{y\} \cup (B_f^i \setminus \{f \upharpoonright k : k \leq n\})$. In a similar way, we can consider a neighborhood U_z for z. It is not difficult to verify that $\mathscr{U} = \langle \{x\}, U_y, U_z \rangle$ is a neighborhood of $\{x, y, z\}$ with $\rho[\mathscr{U}] = \{x\}$.

Case 2: $x \perp y$, $x \perp z$ and $\Delta_{x,y} = \Delta_{x,z}$.

Let us suppose first that $x \in 2^{<\omega}$ and let $U_x = \{x\}$. Let $U_y = \{y\}$ if $y \in 2^{<\omega}$ and $U_y = \{y\} \cup (B_g^j \setminus \{g \upharpoonright k : k \leq \Delta_{x,y}\})$ if $y = B_g^j$ for $g \in S$ and $j \in 2$. Define U_z in the same form. Finally, consider the neighborhood $\mathscr{U} = \langle U_x, U_y, U_z \rangle$ of $\{x, y, z\}$. Notice that for every $y_0 \in U_y$ and $z_0 \in U_z$, $x \perp y_0$, $x \perp z_0$ and $\Delta_{x,y_0} = \Delta_{x,z_0} = \Delta_{x,y}$. Therefore, $\rho[\mathscr{U}] = \{x\}$.

On the other hand, let us suppose that $x = B_f^i$ for some $f \in S$ and $i \in 2$ and let U be a neighborhood of x. We can find $n \in \omega$ so that $\{x\} \cup (B_f^i \setminus \{f \upharpoonright k : k \leq n\}) \subseteq U$. Let $m = \max\{n, \Delta_{x,y}\}$ and let $U_x = \{x\} \cup (B_f^i \setminus \{f \upharpoonright k : k \leq m\})$. If $y \in 2^{<\omega}$, consider the neighborhood $U_y = \{y\}$. If otherwise, $y = B_g^j$ for some $g \in S$ and $j \in 2$, let $U_y = \{y\} \cup (B_g^j \setminus \{g \upharpoonright k : k \leq m\})$. In a similar way, we can find a neighborhood U_z for z. As before, if $\mathscr{U} = \langle U_x, U_y, U_z \rangle$, it is not hard to verify that $\rho[\mathscr{U}] \subseteq U_x \subseteq U$ and we can conclude with this that ρ is continuous on $\{x, y, z\}$.

Finally, to prove that the space X does not admit a continuous weak selection, let h be any weak selection on X. Then $h \upharpoonright 2^{<\omega} = f_\alpha$ for some $\alpha < \mathfrak{c}$. Let $f \in 2^\omega$ so that $A_\alpha = A_f$ and assume, without loss of generality, that $h(\{B_\alpha^0, B_\alpha^1\}) = B_\alpha^0$. Let $\mathscr{U}$ be a basic neighborhood of (B_α^0, B_α^1). We can find a $k \in \omega$ in such a way that $(B_\alpha^0 \setminus \{f \upharpoonright l : l < k\}) \cap (B_\alpha^1 \setminus \{f \upharpoonright l : l < k\}) = \emptyset$ and $\langle \{B_\alpha^0\} \cup (B_\alpha^0 \setminus \{f \upharpoonright l : l < k\}), \{B_\alpha^1\} \cup (B_\alpha^1 \setminus \{f \upharpoonright l : l < k\}) \rangle \subseteq \mathscr{U}$. If $f_\alpha(\{f \upharpoonright m, f \upharpoonright n\}) = f \upharpoonright \min\{m, n\}$ for every $f \upharpoonright n, f \upharpoonright m \in B_\alpha$, choose $n, m \in \omega$, with $n > m$ and such that $f \upharpoonright n \in$

$B^0_\alpha \setminus \{f \restriction l : l < k\}$ and $f \restriction m \in B^1_\alpha \setminus \{f \restriction l : l < k\}$. Then $(f \restriction n, f \restriction m) \in \mathscr{U}$ and $h(\{f \restriction n, f \restriction m\}) = f \restriction m \notin B^0_\alpha$. In the other case, if $g''_\alpha[B_\alpha]^2 = \{1\}$, choose $n, m \in \omega$, with $n < m$, such that $f \restriction n \in B^0_\alpha \setminus \{f \restriction l : l < k\}$ and $f \restriction m \in B^1_\alpha \setminus \{f \restriction l : l < k\}$. Then $(f \restriction n, f \restriction m) \in \mathscr{U}$ and $h(\{f \restriction n, f \restriction m\}) = m \notin B^1_\alpha$. We conclude that h is not continuous at (B^0_α, B^1_α). $\square$

10. Concluding remarks

This survey has a non-zero overlap but also a substential symmetric difference with a similar survey [**46**] written recently by the second author. As mentioned in the introduction the survey presented here was not meant to be exhaustive as such a task would require a book of its own. So, several interesting topics were left off or only touched upon only lightly.

One of these topics is the study of weak covering properties in Ψ-spaces. Matveev in [**63**] introduced the notion of property (a)[9] motivated by the fact that a space is absolutely countably compact if and only if it is countably compact and has property (a), hence the name. The study of property (a) in Ψ-spaces was undertaken by Szeptycki and Vaughan in [**79, 80**], where they showed, in particular, that $\Psi(\mathscr{A})$ has property (a) if and only if for every $f : \mathscr{A} \to \omega$ there is a set Y intersecting each $A \setminus f(A)$ in a finite non-empty set. The relationship of property (a) and other covering properties of Ψ-spaces with the parametrized $\diamondsuit$-principles introduced in [**67**] was studied by Morgan and da Silva in a series of articles [**69, 74, 75, 68, 76**].

There are natural variants of Ψ-spaces on uncountable cardinals, both for almost disjoint families of countable sets and for (strongly) almost disjoint families of uncountable sets. Some of these were studied by Szeptycki in [**81**], Hrušák, Raphael and Woods in [**51**], Dow and Vaughan in [**32**] and Vaughan and Payne in [**86**]. In particular, various variants on a Mrówka family were constructed.

Finally with one exception we have avoided forcing arguments in the text as they do not fall within the scope of this book.

[9] A space X is said to have *property (a)* provided for every open cover $\mathcal{U}$ of X and every dense subset $D \subseteq X$ there is a set F closed discrete in X contained in D such that $st(F,\mathcal{U}) = X$, where $st(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$

Bibliography

[1] U. Abraham, S. Shelah. Lusin sequences under CH and under Martin's axiom. Fund. Math. 169 (2001), no. 2, 97–103.

[2] P. Alexandroff, P. Urysohn. Mémoire sur les espaces topologiques compacts. Verh. Akad. Wetensch. Amsterdam 14 (1926) 1-96.

[3] M. Arciga-Alejandre. Extensiones continuas sobre espacios de Isbell-Mrówka, Master's Thesis, Universidad Michoacana de San Nicolás de Hidalgo, Mexico (2007).

[4] M. Arciga-Alejandre, M. Hrušák, C. Martínez-Ranero. Invariance properties of almost disjoint families. J. Symbolic Logic 78 (2013), no. 3, 989-999.

[5] A. V. Arhangelskii, R. Z. Buzyakova. The rank of the diagonal and submetrizability. Comment. Math. Univ. Carolin. 47 (2006), no. 4, 585–597.

[6] A. V. Arhangelskiĭ, S. P. Franklin. Ordinal invariants for topological spaces. Michigan Math. J. 15 (1968), 313-320.

[7] G. Artico, U. Marconi, J. Pelant, L. Rotter, M Tkachenko. Selections and suborderability. Fund. Math. 175 (2002), no. 1, 133.

[8] B. Balcar, J. Dočkálková, P. Simon. Almost disjoint families of countable sets. Finite and infinite sets, Vol. I, II (Eger, 1981), 59–88, Colloq. Math. Soc. János Bolyai, 37, North-Holland, Amsterdam, 1984.

[9] B. Balcar, J. Pelant, P. Simon. The space of ultrafilters on **N** covered by nowhere dense sets. Fund. Math. 110 (1) (1980) 11-24.

[10] T. Bartoszyński, H. Judah. *Set theory. On the structure of the real line.* A K Peters, Ltd., 1985.

[11] A.I. Bashkirov. The classification of quotient maps and sequential bicompacta. Soviet Math. Dokl. 15 (1974) 1104-1109.

[12] A. I. Bashkirov. On continuous maps of Isbell spaces and strong 0-dimensionality. Bull. Acad. Polon. Sci. Sr. Sci. Math. 27 (1979), no. 7-8, 605–611 (1980).

[13] A. I. Bashkirov. On Stone-Čech compactifications of Isbell spaces. Bull. Acad. Polon. Sci. Sr. Sci. Math. 27 (1979), no. 7-8, 613–619 (1980).

[14] J. E. Baumgartner, M. Weese. Partition algebras for almost-disjoint families. Trans. Amer. Math. Soc. 274 (1982), no. 2, 619–630.

[15] D. Bernal-Santos. The Rothberger property on $C_p(\Psi(\mathcal{A}), 2)$. Comment. Math. Univ. Carolin. 57 (2016), no. 1, 83–88.

[16] D. Bernal-Santos. The Rothberger property on $C_p(X, 2)$. Topology Appl. 196 (2015), part A, 106–119.

[17] D. Bernal-Santos, A. Tamariz-Mascarúa. The Menger property on $C_p(X, 2)$. Topology Appl. 183 (2015), 110–126.

[18] A. Blass. Combinatorial cardinal characteristics of the continuum. In *Handbook of set theory.* Vols. 1, 2, 3, pages 395-489. Springer, Dordrecht, 2010.

[19] J. Brendle, G. Piper. MAD families with strong combinatorial properties. Fund. Math. 193 (2007), no. 1, 7–21.

[20] R. Z. Buzyakova. In search for Lindelöf C_p's. Comment. Math. Univ. Carolin. 45 (2004) 145-151.

[21] J. Cao, T. Nogura, A. H. Tomita. Countable compactness of hyperspaces and Ginsburg's questions. Topology Appl. 144 (1-3) (2004) 133-145.

[22] J. Chaber. Conditions which imply compactness in countably compact spaces. Bull. Acad. Pol. Sci. Ser. Math., 24 (1976) 993-998.

[23] E. K. van Douwen. The integers and topology. Handbook of set-theoretic topology, 111–167, North-Holland, Amsterdam, 1984.

[24] A. Dow. Dow's questions. Open problems in topology, 5–11, North-Holland, Amsterdam, 1990.
[25] A. Dow. Large compact separable spaces may all contain $\beta\mathbb{N}$. Proc. Amer. Math. Soc. 109 (1990), no. 1, 275–279.
[26] A. Dow. Sequential order under $\mathsf{M}A$. Topology Appl. 146-147 (2005) 501-510.
[27] A. Dow. A non-partitionable mad family. Spring Topology and Dynamical Systems Conference. Topology Proc. 30 (2006), no. 1, 181–186.
[28] A. Dow. Sequential order under PFA. Canad. Math. Bull. 54(2) (2011) 270-276.
[29] A. Dow. P. Simon. Spaces of continuous functions over a Ψ-space. Top. Appl. 153 (2006) 2260-2271.
[30] A. Dow, J. E. Vaughan. Ordinal remainders of ψ-spaces. Topology Appl. 158 (2011), no. 14, 1852–1857.
[31] A. Dow, J. Vaughan. Ordinal remainders of classical ψ-spaces. Fund. Math. 217 (2012) 83-93.
[32] A. Dow, J. E. Vaughan. Mrówka maximal almost disjoint families for uncountable cardinals. Topology Appl. 157 (2010), no. 8, 1379–1394.
[33] R. Engelking. *General topology*, second ed. Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989. Translated from the Polish by the author.
[34] I. Farah. Luzin gaps. Trans. Amer. Math. Soc. 356(6) (2004) 2197-2239.
[35] J. Ferrer. On the Controlled Separable Projection Property for some C(K) spaces, Acta Math. Hung. 124 (2009), no. 12, 145-154.
[36] J. Ferrer, P. Koszmider and W. Kubiś. Almost disjoint families of countable sets and separable complementation properties. J. Math. Anal. Appl. 401 (2013), no. 2, 939–949.
[37] S. García-Ferreira. Continuous functions between Isbell-Mrówka spaces. Comment. Math. Univ. Carolin. 39 (1998), no. 1, 185-195.
[38] S. García-Ferreira, M. Sanchis. Weak selections and pseudocompactness. Proc. Amer. Math. Soc. 132 (2004) 1823-1825.
[39] M. A. Gaspar, F. Hernández-Hernández, M. Hrušák. Scattered spaces from weak diamonds. Preprint (2017).
[40] S. Ghasemi, P. Koszmider. An extension of compact operators by compact operators with no nontrivial multipliers. Preprint (2017)
[41] L. Gillman, M. Jerison. *Rings of Continuous Functions*. Van Nostrand, Princeton, NJ. 1960
[42] J. Ginsburg. Some results on the countable compactness and pseudocompactness of hyperspaces. Canad. J. Math. 27 (1975) 1392-1399.
[43] I. Glicksberg. Stone-Čech compactifications of products. Trans. Amer. Math. Soc. 90 (1959) 369-382.
[44] V. Gutev, T. Nogura. Selection problems for hyperspaces. In Open Problems in Topology II, E. Pearl, editor. Elsevier B. V. 2007.
[45] O. Guzmán, M. Hrušák. n-Luzin gaps. Top. appl. 160 (2013) 1364-1374.
[46] M. Hrušák. Almost disjoint families and topology. Recent progress in general topology. III, 601-638, Atlantis Press, Paris, 2014.
[47] M. Hrušák, S. García-Ferreira. Ordering MAD families a la Katětov. J. Symbolic Logic 68 (2003), no. 4, 1337–1353.
[48] M. Hrušák, F. Hernández-Hernández, I. Martínez-Ruiz. Pseudocompactness of hyperspaces. Topol. Appl. 154 (2007) 3048-3055.
[49] M. Hrušák, I. Martínez-Ruiz. Selections and weak orderability. Fund. Math. 203 (2009) 1-20.
[50] M. Hrušák, C. J. D. Morgan, S. G. da Silva. Luzin gaps are not countably paracompact. Questions Answers Gen. Topology 30 (2012), no. 1, 59–66.
[51] M. Hrušák, R. Raphael, R. G. Woods. On a class of pseudocompact spaces derived from ring epimorphisms. Topology Appl. 153 (2005), no. 4, 541–556.
[52] M. Hrušák, P. J. Szeptycki, Á. Tamariz-Mascarúa. Spaces of continuous functions defined on Mrówka spaces. Top. Appl. 148 (2005) 239-252.
[53] M. Hrušák, P. J. Szeptycki, A. H. Tomita. Selections on Ψ-spaces. Comment Math. Univ. Carolin. 42 (2001) 763-769.
[54] W. Just, O. V. Sipacheva, P. J. Szeptycki. Nonnormal spaces $C_p(X)$ with countable extent. Proc. Amer. Math. Soc. 124 (1996), no. 4, 1227–1235.
[55] V. Kannan, M. Rajagopalan. Hereditarily locally compact separable spaces. In *Categorical topology (Proc. Internat. Conf. Free Univ. Berlin)*. Lecture Notes in Math. Vol. 71 pages 185-195. Springer, Berlin 1979.

[56] P. Koszmider. On decompositions of Banach spaces of continuous functions on Mrówka's spaces. Proc. Amer. Math. Soc. 133 (2005), no. 7, 2137–2146.
[57] J. Kulesza, R. Levy. Separation in Ψ-spaces. Topology Appl. 42 (1991), no. 2, 101–107.
[58] K. Kunen. *Set theory: an introduction to independence proofs*, vol. 102, North-Holland Publishing Co., Amsterdam, 1980.
[59] N. N. Luzin. On subsets of the series of natural numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat. 11 (1947) 403–410.
[60] V. I. Malykhin, A. Tamariz-Mascarúa. Extensions of functions in Mrówka-Isbell spaces. Topology Appl. 81 (1997), no. 1, 85102.
[61] A. R. D. Mathias. Happy families. Annals of Mathematical Logic 12 (1977) 59-111.
[62] W. Marciszewski. A function space $C(K)$ not weakly homeomorphic to $C(K)\times C(K)$.. Studia Math. 88 (1988), no. 2, 129–137.
[63] M. V. Matveev. Some questions on property (a). Questions Answers Gen. Topology 15 (1997), no. 2, 103–111.
[64] E. Michael. Topologies on spaces of subsets. Trans. Amer. Math. Soc. 71 (1951) 152-182.
[65] J. van Mill, E. Wattel. Selections and orderability. Proc. Amer. Math. Soc. 83 (1981) 601-605.
[66] A. W. Miller. A MAD Q-set. Fund. Math. 178 (2003), no. 3, 271–281.
[67] J. T. Moore, M. Hrušák, M. Džamonja. Parametrized $\diamondsuit$ principles. Trans. Amer. Math. Soc. 356 (2004), no. 6, 2281–2306.
[68] C. J. G. Morgan, S. G. da Silva. Almost disjoint families and "never" cardinal invariants. Comment. Math. Univ. Carolin. 50 (2009), no. 3, 433–444.
[69] C. J. G. Morgan, S. G. da Silva. Selectively (a)-spaces from almost disjoint families are necessarily countable under a certain parametrized weak diamond principle. Houston J. Math. 42 (2016), no. 3, 1031–1046.
[70] S. Mrówka. On completely regular spaces. Fund. Math. 41 (1954) 105-106.
[71] S. Mrówka. Some set-theoretic constructions in topology. Fund. Math., 94, (1977) 83-92.
[72] J. Novák. On the Cartesian product of two compact spaces. Fund. Math. 40 (1953) 106-112.
[73] J. Roitman, L. Soukup. Luzin and anti-Luzin almost disjoint families. Fund. Math. 158 (1998), no. 1, 51–67.
[74] S. G. da Silva. On the extent of separable, locally compact, selectively (a)-spaces. Colloq. Math. 141 (2015), no. 2, 199–208.
[75] S. G. da Silva. (a)-spaces and selectively (a)-spaces from almost disjoint families. Acta Math. Hungar. 142 (2014), no. 2, 420–432.
[76] da Silva, Samuel Gomes. On the presence of countable paracompactness, normality and property (a) in spaces from almost disjoint families. Questions Answers Gen. Topology 25 (2007), no. 1, 1–18.
[77] P. Simon. A compact Fréchet space whose square is not Fréchet. Comment. Math. Univ. Carolin. 21 (1980) 749-753.
[78] J. Steprāns. Combinatorial consequences of adding Cohen reals. Set theory of the reals (Ramat Gan, 1991), 583–617, Israel Math. Conf. Proc., 6, Bar-Ilan Univ., Ramat Gan, 1993.
[79] P. J. Szeptycki. Soft almost disjoint families. Proc. Amer. Math. Soc. 130 (2002), no. 12, 3713–3717.
[80] P. J. Szeptycki. J. E. Vaughan. Almost disjoint families and property (a). Fund. Math. 158 (1998), no. 3, 229–240.
[81] P. Szeptycki. Countable metacompactness in Ψ-spaces. Proc. Amer. Math. Soc. 120 (1994), no. 4, 1241–1246.
[82] F. D. Tall. Set-Theoretic Consistency Results and Topological Theorems Concerning The Normal Moore Space Conjecture and Related Problems. Thesis (Ph.D.) The University of Wisconsin - Madison. 1969. 105 pp
[83] H. Terasaka. On Cartesian products of compact spaces. Osaka Math. J. 4 (1952) 11-15.
[84] J. Terasawa. Spaces $N \cup \mathscr{R}$ and their dimensions. Topology Appl. 11 (1980) 93-102.
[85] S. Todorčević. Analytic gaps. Fund. Math. 150(1) (1996) 55-66.
[86] J. E. Vaughan, C. Payne. Fibers of continuous real-valued functions on ψ-spaces. Topology Appl. 195 (2015), 256–264.

Index