Preservation theorems for Namba forcing

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Abstract

We study preservation properties of Namba forcing on κ. It turns out that Namba forcing is very sensitive to the properties of the ground model. We prove that if $\mathcal{I}$ is an ideal with a Borel base on ω^ω and $\kappa > \omega_1$ is a regular cardinal less than the uniformity number or bigger than the covering number of $\mathcal{I}$, then the κ-Namba forcing preserves covering of $\mathcal{I}$ (i.e. $\bigcup(V \cap \mathcal{I}) = \omega^\omega$). This result also holds for $\kappa = \omega_1$ in case Club Guessing holds or if $\mathfrak{d} = \omega_1$. On the other hand, this fails in case $add(I) = cof(\mathcal{I}) = \kappa$. We answer a question of Hrušák, Simon and Zindulka regarding partition properties on trees.

1 Introduction

Namba forcing was introduced in [12] in order to show that one can change the cofinality of ω_2 to ω while preserving ω_1, it also adds a new countable sequence of ordinals, yet it may not add new reals. In this paper, we will prove that Namba forcing may behave very differently in distinct models of set theory, for example it is consistent that Namba forcing adds Cohen reals, while it is also consistent that it has the Sacks property.

Let $\mathcal{I} \in V$ be an ideal in μ^ω with a Borel base. The main results of the paper are best described using the notion of a quasigeneric sequence: If W is a model of ZFC extending V, we say $r \in \mu^\omega \cap W$ is an *$\mathcal{I}$-quasigeneric sequence (over V)* if $r \notin B^W$ for every Borel set $B \in \mathcal{I} \cap V$. (By B^W we denote the reinterpretation of the Borel set B in W). For example, if $\mathcal{M}$ denotes the ideal of meager sets and $\mathcal{N}$ the ideal of null sets on ω^ω, then the $\mathcal{M}$-quasigeneric reals are the Cohen reals and the $\mathcal{N}$-quasigeneric are the random reals. One of the main aims of this paper is to study when does $\mathbb{NB}(\kappa)$ add quasigeneric sequence for certain σ-ideals. We will prove (a more general version of) the following result:

Theorem 1 *Let $\kappa > \omega_1$ be a regular cardinal, $\mu < \kappa$ and $\mathcal{I}$ a σ-ideal in μ^ω with a Borel base.*

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1. *If $\kappa <$ non$(\mathcal{I})$ then $\mathbb{NB}(\kappa)$ does not add $\mathcal{I}$-quasigeneric sequences.*
2. *If cov$(\mathcal{I}) < \kappa$ then $\mathbb{NB}(\kappa)$ does not add $\mathcal{I}$-quasigeneric sequences.*
3. *If add$(\mathcal{I}) =$ cof$(\mathcal{I}) = \kappa$ then $\mathbb{NB}(\kappa)$ adds $\mathcal{I}$-quasigeneric sequences.*

This result is true for $\kappa = \omega_1$ in case there is a Club Guessing sequence or if the dominating number is equal to the first uncountable cardinal. We will discuss in more detail the case $\kappa = \omega_1$, we do not know if the proposition is still true in its complete generality (still we will prove it is true for some specific σ-ideals). In [7] Simon, Hrušák and Zindulka asked if $\mathfrak{b}$ is the first regular, uncountable cardinal κ such that $\mathbb{NB}(\kappa)$ adds an unbounded real[1]. We will answer positively their question with the tools we developed. In [6] the relationship between Namba forcing and weak partition properties will be further studied.

Our notation is mostly standard. If X is a set, by $\wp(X)$ we denote the power set of X. An ideal $\mathcal{I} \subseteq \wp(X)$ on X is a collection of subsets of X closed under taking subsets and unions, for convenience, all our ideals will be proper (i.e. $X \notin \mathcal{I}$). A σ-ideal is an ideal closed under countable unions. If X is a topological space, we say $\mathcal{I}$ has a Borel base if every element of $\mathcal{I}$ is contained in a Borel set in $\mathcal{I}$. In this paper, the expression "for almost all" means for all except finitely many. The definition of the cardinal invariants used in this paper may be consulted in [3]. For more on Namba forcing, the reader may consult [5], [10] and [9].

2 Basic properties of Namba forcing and absoluteness results

Let κ be a cardinal, a tree $T \subseteq \kappa^{<\omega}$ is called a *κ-Namba tree* (or just Namba tree if the cardinal κ is clear by context) if there is $s \in T$ (called the *stem of* T) such that every $t \in T$ is comparable with s; furthermore if $t \sqsubset s$ then t has just one immediate successor and if $s \sqsubseteq t$ then t has κ many immediate successors. By $\mathbb{NB}(\kappa)$ we will denote the set of all κ-Namba trees ordered by inclusion; in this way, $\mathbb{NB}(\omega)$ is the Laver forcing. A generic filter for $\mathbb{NB}(\kappa)$ may be coded as a sequence which we will denote by $\mathfrak{n}_{gen} : \omega \longrightarrow \kappa$. It is easy to see that $\mathbb{NB}(\kappa)$ forces κ to have countable cofinality. Given S and T two κ-Namba trees, $S \leq_0 T$ will mean that $S \leq T$ and both S and T have the same stem. By $[T]$ we denote the set of branches of T and if $s \in T$ then we define T_s as the set of all $t \in T$ such that either $t \sqsubseteq s$ or $s \sqsubseteq t$ and $suc_T(s) = \{\alpha \in \kappa \mid s^\frown\alpha \in T\}$. By $B(T)$ we denote the set of nodes of T that extend the stem. By *stem*(T) we

[1]In [7] there is no mention of Namba forcing, rather about a weak partition property of trees, but it is easy to see that their question is equivalent to the one we stated.

denote the stem of T and $\mathbb{NB}_0(\kappa)$ will denote the set of all κ-Namba trees with empty stem.

The following ideals will be very useful to establish the preservation properties of Namba forcing on κ. For every function $F : \kappa^{<\omega} \longrightarrow [\kappa]^{<\kappa}$ we define $C(F) = \{f \in \kappa^\omega \mid \exists^\infty n\,(f(n) \in F(f \upharpoonright n))\}$ as well as the set $C^0(F) = \{f \in \kappa^\omega \mid \exists n\,(f(n) \in F(f \upharpoonright n))\}$. The *$\kappa$-Namba ideal* $\mathcal{L}_\kappa$ is the ideal in κ^ω generated by $\left\{C(F) \mid F : \kappa^{<\omega} \longrightarrow [\kappa]^{<\kappa}\right\}$ and let $\mathcal{L}^0_\kappa$ be the ideal generated by $\left\{C^0(F) \mid F : \kappa^{<\omega} \longrightarrow \kappa\right\}$. Note that if κ is a regular cardinal, it is enough to consider functions of the form $F : \kappa \longrightarrow \kappa$. In this way, $\mathcal{L}_\omega$ is the usual Laver ideal in ω^ω. (see [13] page 44).

Given $A \subseteq \kappa^\omega$ consider the following game $\mathcal{R}(A)$:

I	X_0		X_1		X_2		X_3	$\cdots$
II		α_0, i_0		α_1, i_1		α_2, i_2		$\cdots$

Where the $X_n \in [\kappa]^{<\kappa}$, $\alpha_n \in \kappa$ and $i_n \in 2$ for all $n \in \omega$. Then player II wins if and only if the following conditions hold:

1. $\langle \alpha_n \rangle_{n\in\omega} \in A$.
2. There is $n \in \omega$ such that $i_n = 1$.
3. If $i_n = 1$ and $m \geq n$ then $\alpha_m \notin X_m$.

Note that if $i_n = 1$ and $m \geq n$ then i_m is irrelevant, so we may ignore it. We now have the following result, which is only a slight generalization of [13] example 2.1.13:

Proposition 2 *Let $A \subseteq \kappa^\omega$ then the following holds:*

1. *Player I has a winning strategy in $\mathcal{R}(A)$ if and only if $A \in \mathcal{L}_\kappa$.*
2. *Player II has a winning strategy in $\mathcal{R}(A)$ if and only if there is $T \in \mathbb{NB}(\kappa)$ such that $[T] \subseteq A$.*

Proof. It is easy to see that if $A \in \mathcal{L}_\kappa$ then player I has a winning strategy. Let σ be a winning strategy for player I. Note that for every $t \in \kappa^{<\omega}$ there are at most $|t|$ possible ways in which player II reached t and player I was following σ. Define $F : \kappa^{<\omega} \longrightarrow [\kappa]^{<\kappa}$ to be the union of all this possibilities. It is then easy to see that $A \subseteq C(F)$.

It is easy to see that if there is $T \in \mathbb{NB}(\kappa)$ such that $[T] \subseteq A$ then player II has a winning strategy. Let σ be a winning strategy for player II. We can then find $n \in \omega$ and $t \in \kappa^n$ such that i_n is the first such that $i_n = 1$ and player II reached t during this partial play. It is now easy to see that there is $T \in \mathbb{NB}(\kappa)$ with stem t such that $[T] \subseteq A$. ■

By Borel determinacy we can then conclude the following result which will be used several times.

Corollary 3 *Let κ be a cardinal and $B \subseteq \kappa^{\omega}$ a Borel set. Then exactly one of the following possibilities holds:*

1. *There is $T \in \mathbb{NB}(\kappa)$ such that $[T] \subseteq B$.*
2. *$B \in \mathcal{L}_{\kappa}$.*

In particular, $\mathbb{NB}(\kappa)$ is forcing equivalent to Borel $(\kappa^{\omega})/\mathcal{L}_{\kappa}$.

For $A \subseteq \kappa^{\omega}$ we consider another game $\mathcal{H}(A)$:

I	X_0		X_1		X_2		X_3	$\cdots$	
II		α_0		α_1		α_2		$\cdots$	$\langle \alpha_n \rangle_{n \in \omega} \in A$

Where $X_n \in [\kappa]^{<\kappa}$ and $\alpha_n \in \kappa \setminus X_n$ for all $n \in \omega$. Player II wins if the game if $\langle \alpha_n \rangle_{n \in \omega} \in A$. The following lemma is easy and is left to the reader:

Lemma 4 *Let $A \subseteq \kappa^{\omega}$ then the following holds:*

1. *Player I has a winning strategy in $\mathcal{H}(A)$ if and only if $A \in \mathcal{L}^0_{\kappa}$.*
2. *Player II has a winning strategy in $\mathcal{H}(A)$ if and only if there is $T \in \mathbb{NB}_0(\kappa)$ such that $[T] \subseteq A$.*

By applying Borel Determinacy (see [8]) we conclude the following:

Corollary 5 *Let $A \subseteq \kappa^{\omega}$ be a Borel set, then exactly one of the following holds:*

1. *$A \in \mathcal{L}^0_{\kappa}$.*
2. *There is $T \in \mathbb{NB}_0(\kappa)$ such that $[T] \subseteq A$.*

We can also conclude the following lemma that will be used later:

Corollary 6 *Let κ be a cardinal, $\mu < \mathsf{cof}(\kappa)$ and let $\{A_{\alpha} \mid \alpha \in \mu\}$ be a family of Borel sets of κ^{ω} such that $\kappa^{\omega} = \bigcup\limits_{\alpha<\mu} A_{\alpha}$. Then there is $T \in \mathbb{NB}_0(\kappa)$ and $\alpha < \mu$ such that $[T] \subseteq A_{\alpha}$.*

Let $T \in \mathbb{NB}(\kappa)$ and $D \subseteq \mathbb{NB}(\kappa)$ be an open dense set below T. We define a rank function $rk_D : B(T) \longrightarrow OR \cup \{\infty\}$ as follows:

1. $rk_D(s) = 0$ if there is $S \leq_0 T$ such that $S \in D$.
2. $rk_D(s) \leq \alpha$ if $|\{\xi \in suc_T(s) \mid rk_D(s^\frown\xi) < \alpha\}| = \kappa$.
3. $rk_D(s) = \alpha$ if $rk_D(s) \leq \alpha$ and there is no $\beta < \alpha$ such that $rk_D(s) \leq \beta$.
4. In case there is no α such that $rk_D(s) \leq \alpha$ then $rk_D(s) = \infty$.

However, we will see that the last possibility can never happen. Note that if there was an $s \in B(T)$ such that $rk(s) = \infty$ then the set of all $\xi \in suc_T(s)$ such that $rk(s^\frown\xi) \neq \infty$ must have size less than κ.

Lemma 7 *Let $T \in \mathbb{NB}(\kappa)$ and $D \subseteq \mathbb{NB}(\kappa)$ be an open dense set below T. Then there is $S \leq_0 T$ such that the following holds:*

1. *$rk_D(s) \neq \infty$ for every $s \in B(T)$.*
2. *If X is the set of all $s \in B(S)$ such that s is minimal with $rk_D(s) = 0$ then X is a front in S (i.e. X is an antichain and for every $y \in [T]$ there is $n \in \omega$ such that $y \restriction n \in X$).*
3. *If $s \in X$ then $S_s \in D$.*

Proof. In case there was an $s \in T$ for which $rk_D(s) = \infty$, we could recursively construct a κ-Namba tree $S \leq T_s$ such that $rk_D(t) = \infty$ for all $t \in S$. However, we may then find $S' \leq S$ such that $S' \in D$. However, if t is the stem of S' then $rk_D(t) = 0$, which is a contradiction.

We can then recursively build a tree $S \leq_0 T$ such that rk_D is decreasing and if $s \in X$ (where X is the set of all s such that s is minimal with $rk_D(s) = 0$) then $S_s \in D$. Note that X must be a front in S since rk_D is decreasing. ■

We can then conclude that Namba forcing has the continuous reading of names:

Proposition 8 (Continous reading of names) *Let κ, μ be two cardinals, $T \in \mathbb{NB}(\kappa)$ and $\dot{y}$ such that $T \Vdash$ "$\dot{y} \in \mu^\omega$" then there is $S \leq T$ and a continuous function $F : [S] \longrightarrow \mu^\omega$ such that $S \Vdash$ "$F(\mathfrak{n}_{gen}) = \dot{y}$".*

We will now prove a version of the pure decision property for κ-Namba forcing:

Lemma 9 *Let κ be a cardinal, $T \in \mathbb{NB}(\kappa)$, $\mu <$ cof(κ) and $\dot{a}$ such that $T \Vdash$ "$\dot{a} \in \mu$", then there is $S \leq_0 T$ such that S decides $\dot{a}$.*

Proof. Let $D = \{S \leq T \mid \exists \alpha \, (S \Vdash \text{“}\dot{a} = \alpha\text{”})\}$ which clearly is an open dense set below T. Let $S \leq_0 T$ and X as in the previous lemma. For every $\alpha < \mu$ let $A_\alpha = \bigcup \{[S_t] \mid t \in X \wedge (S_t \Vdash \text{“}\dot{a} = \alpha\text{”})\}$. Clearly each A_α is a relative open set in $[S]$ and $[S] = \bigcup\limits_{\alpha<\mu} A_\alpha$ since X is a front in S. Since $\mu < \mathsf{cof}(\kappa)$ there is α and $S' \leq_0 S$ such that $[S'] \subseteq A_\alpha$ and then $S' \Vdash \text{“}\dot{a} = \alpha\text{”}$. ■

We now fix some notation that will be used in the rest of the paper. Given $F : T \longrightarrow \mu$ define the function $\overline{F} : [T] \longrightarrow \mu^\omega$ such that if $x \in \kappa^\omega$ and $n \in \omega$ then $\overline{F}(x) \upharpoonright n = F(x)$. A function $H : [T] \longrightarrow \mu^\omega$ is called *Lipschitz* if there is a function $F : T \longrightarrow \mu$ such that $H = \overline{F}$. Clearly every Lipschitz function is continuous. If $G : \kappa^\omega \longrightarrow \mu^\omega$ is a continuous function, define $G^* : \kappa^{<\omega} \longrightarrow \mu^{<\omega}$ where $G^*(s) = (\bigcup \{t \mid G[\langle s\rangle] \subseteq \langle t\rangle\}) \upharpoonright |s|$.

Corollary 10 (Lipschitz reading of names) *Let $\mu < \mathsf{cof}(\kappa)$, $T \in \mathbb{NB}(\kappa)$ and $\dot{y}$ a $\mathbb{NB}(\kappa)$-name such that $T \Vdash \text{“}\dot{y} \in \mu^\omega\text{”}$. Then there is $S \leq_0 T$ such that:*

1. *If $s \in S$ then S_s decides $\dot{y} \upharpoonright (|s| + 1)$.*
2. *There is $F : S \longrightarrow \mu$ such that $S \Vdash \text{“}\overline{F}(\mathfrak{n}_{gen}) = \dot{y}\text{”}$.*

The following absoluteness result will be useful in later sections:

Proposition 11 *Let κ be a cardinal, $M \subseteq V$ be a model of (a large portion of)* ZFC *such that $\kappa \in M$, $\kappa \subseteq M$ and $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$. If $B, C \in M$ are Borel sets of κ^ω and $M \models C \subseteq B$ then $V \models C \subseteq B$. In particular, the membership of Borel sets to $\mathcal{L}_\kappa$ is absolute between M and V.*

Proof. Let $f \in C$ and since $[\kappa]^\omega \cap M$ is cofinal in $[\kappa]^\omega$ there is $A \in [\kappa]^\omega \cap M$ such that $f \in A^\omega$. Let $B_1 = B \cap A^\omega$ and $C_1 = C \cap A^\omega$, note that $B_1, C_1 \in M$ and they are both Borel sets of a Polish space. Since $M \models C_1 \subseteq B_1$ then by Shoenfield's absoluteness we conclude that $C_1 \subseteq B_1$, hence $f \in B_1 \subseteq B$.

If $B \in M$ is a Borel set, then $M \models B \notin \mathcal{L}_\kappa$ if and only if there is $T \in \mathbb{NB}(\kappa)$ such that $M \models [T] \subseteq B$ and $M \models B \in \mathcal{L}_\kappa$ if and only if there is $F \in M$ such that $M \models B \subseteq C_F$. Wit this remarks we conclude the absoluteness of the membership of Borel sets to $\mathcal{L}_\kappa$. ■

In particular, we can conclude the following result:

Corollary 12 *Let $M \subseteq V$ be a model of (a large portion of)* ZFC *such that $\omega_1 \in M$ and $\omega_1 \subseteq M$. If $B \in M$ is a Borel set of ω_1^ω then $M \models B \in \mathcal{L}_{\omega_1}$ if and only if $V \models B \in \mathcal{L}_{\omega_1}$.*

The cardinal invariant $\mathsf{non}(\mathcal{L}_\kappa)$ will play a key role in the following sections. We will prove that $\mathsf{non}(\mathcal{L}_\kappa) = \kappa$ for every regular cardinal bigger than ω_1. Given an uncountable regular cardinal κ, by E^κ_ω we denote the set of all ordinals smaller than κ with cofinality ω. $\mathsf{CG}_\omega(\kappa)$ is the statement that there is a sequence $\overline{C} = \langle C_\alpha \mid \alpha \in E^\kappa_\omega \rangle$ where $C_\alpha \subseteq \alpha$ is a cofinal set of order type ω such that for every club $D \subseteq \kappa$ there is α for which $C_\alpha \subseteq D$. We call such $\overline{C}$ a *club guessing sequence.*

We will show that the existence of a Club Guessing sequence at κ implies that the uniformity of $\mathcal{L}_\kappa$ is precisely κ.

Proposition 13 *Let $\kappa > \omega$ be a regular cardinal. Then the principle $\mathsf{CG}_\omega(\kappa)$ implies $\mathsf{non}(\mathcal{L}_\kappa) = \kappa$.*

Proof. Let $\overline{C} = \{C_\alpha \mid \alpha \in E^\kappa_\omega\}$ be a club guessing sequence. Enumerate each $C_\alpha = \{\alpha_n \mid n \in \omega\}$ in an increasing way, we may further assume $0 \notin C_\alpha$ for every $\alpha \in LIM(\omega_1)$. We now define $f_\alpha : \omega \longrightarrow \kappa$ where $f_\alpha(n) = \alpha_n$, we will show that $X = \{f_\alpha \mid \alpha \in E^\kappa_\omega\} \notin \mathcal{L}_\kappa$.

Let $F : \kappa^{<\omega} \longrightarrow \kappa$, we must show that X is not contained in $C(F)$. Let $D \subseteq \kappa$ be a club such that if $\alpha \in D$ and $s \in \alpha^{<\omega}$ then $F(s) < \alpha$. Since $\overline{C}$ is a club guessing sequence, then there is $\alpha \in D$ such that $C_\alpha \subseteq D$. It is then easy to see that $f_\alpha \notin C(F)$. ■

It is a remarkable result of Shelah that $\mathsf{CG}_\omega(\kappa)$ holds for every regular cardinal bigger that ω_1:

Theorem 14 (Shelah, see [1]) *If κ is regular and $\omega_1 < \kappa$ then $\mathsf{CG}_\omega(\kappa)$ is true.*

It is well known that $\mathsf{CG}_\omega(\omega_1)$ may consistently fail, for example, this is the situation in the presence of the Proper Forcing Axiom. Moreover, we will later prove that the inequality $\omega_1 < \mathsf{non}(\mathcal{L}_{\omega_1})$ is consistent.

We say $T \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n)$ is a tree if whenever $(s,t) \in T$ and $n < |s|$ then $(s \restriction n, t \restriction n) \in T$. The set of branches of T will be defined as $[T] = \{(x,y) \mid \forall n\,((x \restriction n, y \restriction n) \in T)\}$ and its projection $p[T] = \{x \mid \exists y\,((x,y) \in [T])\}$. A set $A \subseteq \kappa^\omega$ is called *κ-analytic* if there is a tree $T \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n)$ such that $A = p[T]$. The following is well known,

Lemma 15 *Let κ be a cardinal, W a model of ZFC extending V and $T, S \subseteq \bigcup\limits_{n\in\omega} \kappa^n \times \kappa^n$ trees. Then $p[T] \cap p[S] = \emptyset$ if and only if $W \models p[T] \cap p[S] = \emptyset$.*

Proof. Given trees T and S, define $Z \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n \times \kappa^n)$ as the set of all (s,t,l) such that $(s,t) \in T$ and $(s,l) \in S$. Clearly Z is a tree and note that

$p[T] \cap p[S] \neq \emptyset$ if and only if Z is not well founded. Since being well founded is absolute between ZFC models, we conclude that $p[T] \cap p[S] = \emptyset$ if and only if $W \models p[T] \cap p[S] = \emptyset$. $\blacksquare$

We will need the following lemma, which is just the generalization of the fact that every Borel set (in ω^ω) is also analytic.

Lemma 16 *Every Borel set in κ^ω is κ-analytic. Moreover, for every Borel set $B \subseteq \kappa^\omega$ there is a tree $T_B \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n)$ for which $B = p[T_B]$, $p[T_B] \cap p\left[T_{\kappa^\omega \setminus B}\right] = \emptyset$ and if W is model of ZFC extending V then $W \models p[T_B] \cup p\left[T_{\kappa^\omega \setminus B}\right] = \kappa^\omega$.*

Proof. We prove it by induction on the complexity of B. If B is a closed set, let $S = \{s \in \kappa^{<\omega} \mid \langle s \rangle \cap B \neq \emptyset\}$, it is easy to see that S is a tree and $B = [S]$. Let $T_B = \{(s, s) \mid s \in S\}$ then clearly $B = p[T_B]$. Let $W = \{s_\alpha \mid \alpha \in \kappa\} \subseteq \kappa^{<\omega}$ such that $\kappa^\omega \setminus B = \bigcup\limits_{\alpha\in\kappa} \langle s_\alpha \rangle$. Define $T_{\kappa^\omega \setminus B} \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n)$ as the set of all (s, t) such that $s = t = \emptyset$ or (s, t) satisfy the following properties:

1. There is α such that t is the constant α sequence of length $|t|$.
2. either $s \subseteq s_\alpha$ or $s_\alpha \subseteq s$.

It is then easy to see that $\kappa^\omega \setminus B = p\left[T_{\kappa^\omega \setminus B}\right]$ and $W \models p[T_B] \cup p\left[T_{\kappa^\omega \setminus B}\right] = \kappa^\omega$. Now, assume $B = \bigcup\limits_{n\in\omega} B_n$ and each B_n satisfy the conclusion of the lemma. Let $T_B \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n)$ consist of all (s, t) such that $s = t = \emptyset$ or there is $n \in \omega$ and t' such that $t = n^\frown t'$ and $(s \restriction (|s| - 1), t') \in T_{B_n}$. It is easy to see that $B = p[T_B]$.

Fix $\mathcal{P} = \{P_n \mid n \in \omega\}$ a partition of ω into infinite sets. Given $t = \langle \alpha_0, ...\alpha_m \rangle \in \kappa^{<\omega}$ and $n \in \omega$ we define $t^n \in \kappa^{<\omega}$ as follows, if $P_n \cap dom(t) = \{k_0, ..., k_r\}$ (and $k_i < k_j$ whenever $i < j$) then $t_n = \langle \alpha_{k_0}, ..., \alpha_{k_r} \rangle$. Let $A = \kappa^\omega \setminus B$ and $A_n = \kappa^\omega \setminus A_n$ for each $n \in \omega$. We now define $T_A \subseteq \bigcup\limits_{n\in\omega} (\kappa^n \times \kappa^n)$ as the set of all (s, t) such that $(s \restriction dom(t^n), t^n) \in T_{A_n}$ for every $n \in \omega$. It is easy to see that $A = p[T_A]$ and $W \models p[T_B] \cup p[T_A] = \kappa^\omega$. $\blacksquare$

One could think that if T and S are two trees such that $p[S] \subseteq p[T]$ and W is a model of ZFC extending V then $W \models p[S] \subseteq p[T]$. However this is not the case, we will provide an example of this fact. For every $\alpha < \mathfrak{c}$ let $c_\alpha \in \mathfrak{c}^\omega$ denote the constant function α and let $F : \mathfrak{c} \longrightarrow \mathfrak{c}^\omega$ be a bijection. Define $T = \{(F(\alpha) \restriction n, c_\alpha \restriction n) \mid n \in \omega, \alpha < \mathfrak{c}\}$ and $S = \mathfrak{c}^{<\omega}$, clearly $p[S] = p[T]$. However, if W is any model extending V then $W \models p[T] = \mathfrak{c}^\omega \cap V$ (this is because T is a discrete set) while $W \models p[S] = \mathfrak{c}^\omega$. In this way, if $\mathfrak{c}^\omega \cap V \neq \mathfrak{c}^\omega \cap W$ then $W \models p[S] \nsubseteq p[T]$. Later we will show that this can not occur if W is a

$\mathbb{NB}(\kappa)$-forcing extension of V and T and S are trees on $\bigcup_{n\in\omega} \mu^n \times \mu^n$ where $\mu < \mathsf{cof}(\kappa)$.

It is also not true that if $\kappa^\omega = p[T] \cup p[S]$ and $V \subseteq W$ then $W \models \kappa^\omega = p[T] \cup p[S]$ (for instance, let T be the tree as in the previous example and S an empty tree). We will say that a tree $T \subseteq \bigcup_{n\in\omega} \lambda^n \times \lambda^n$ is a *huge tree in κ* if there is a tree $S \subseteq \bigcup_{n\in\omega} \lambda^n \times \lambda^n$ such that $\kappa^\omega = p[T] \cup p[S]$, $p[T] \cap p[S] = \emptyset$ and this relations holds in any other model extending V. In this way, if $B \subseteq \kappa^\omega$ is a Borel set then T_B is a huge tree in κ. The following is a simple consequence of Shoenfield's absoluteness:

Lemma 17 *If $T \subseteq \bigcup_{n\in\omega} \omega^n \times \omega^n$ is a tree such that $p[T]$ is Borel then T is a huge tree in ω.*

The following result follows from the definitions and the previous results:

Lemma 18 *Let κ, μ be two cardinals and W a model of ZFC extending V.*

1. *If T is a huge tree in κ and $S \subseteq \bigcup_{n\in\omega} \mu^n \times \mu^n$ is a tree such that $p[S] \subseteq p[T]$ then $W \models p[S] \subseteq p[T]$.*
2. *If T and S are huge trees in κ such that $p[T] = p[S]$ then $W \models p[T] = p[S]$.*
3. *If $B \subseteq \kappa^\omega$ is a Borel set then there is a tree T such that $B = p[T]$ and if S is another tree such that $p[S] \subseteq B$ then $W \models p[S] \subseteq p[T]$.*

The following lemma will be used in the next chapter:

Lemma 19 *Let $F : \kappa^\omega \longrightarrow \mu^\omega$ be a continuous function and $T \subseteq \bigcup_{n\in\omega} \mu^n \times \mu^n$ be a tree such that $p[T]$ is Borel. Then the following holds:*

1. *There is a tree $T_F \subseteq \bigcup_{n\in\omega} \lambda^n \times \lambda^n$ such that $p[T_F] = F^{-1}(p[T])$ (where λ is the maximum of κ and μ) and this holds in any model extending V.*
2. *If T is a huge tree for μ then T_F is a huge tree for κ.*
3. *If T is a huge tree for μ then there is $S \subseteq \bigcup_{n\in\omega} \kappa^n \times \kappa^n$ a huge tree for κ such that $p[S] = F^{-1}(p[T])$ and this holds in any model extending V.*

Proof. Define T_F as the set of all $(s,t) \in \bigcup_{n\in\omega} \lambda^n \times \lambda^n$ such that $s \in \kappa^{<\omega}$, $t \in \mu^{<\omega}$ and $(F^*(s), t \restriction |F^*(s)|) \in T$. It is easy to see that T_F has the desired properties. Now assume that T is a huge tree for μ, it is easy to see that if Z is a tree that witness the hugeness of T then Z_F witness the hugeness of T_F. Finally, since $F^{-1}(p[T])$ is a Borel set, there is $S \subseteq \bigcup_{n\in\omega} \kappa^n \times \kappa^n$ a huge tree for κ such that $p[S] = F^{-1}(p[T])$. Since both S and T_F are both huge trees, their projections are the same in any model extending V. ■

If $B \subseteq \lambda^\omega$ is a Borel set and W is a model of ZFC extending V, then *the reinterpretation of B in W* is defined as $B^W = p[S]$ where $S \in V$ is any huge tree such that $p[S] = B$.

Proposition 20 *Let κ, μ be cardinal such that $\mu < \mathsf{cof}(\kappa)$ and A be any set. If $B \subseteq A^\omega \times \mu^\omega$ is a Borel set such that if $x \in A^\omega$ then $B_x = \{y \in \mu^\omega \mid (x,y) \in B\} \neq \emptyset$, then $\mathbb{NB}(\kappa) \Vdash$ "$\forall x \in A^\omega (B_x \neq \emptyset)$".*

Proof. Let $\dot{x}$ be a $\mathbb{NB}(\kappa)$-name for an element of A^ω. We can then find $T \in \mathbb{NB}(\kappa)$ and a continous function $F : [T] \longrightarrow A^\omega$ such that $T \Vdash$ "$F(\dot{\mathfrak{n}}_{gen}) = \dot{x}$". For simplicity, we will assume T has empty stem. Consider the following game:

I	X_0		X_1		X_2		X_3	$\cdots$
II		α_0, β_0		α_1, β_1		α_2, β_2		$\cdots$

Where $X_n \in [\kappa]^{<\kappa}$, $\beta_n \in \kappa$ and $\alpha_n \in \mu$. Player II wins the game if the following holds:

1. $\beta_n \notin X_n$ for every $n \in \omega$.
2. $b = \langle \beta_n \rangle_{n\in\omega} \in [T]$.
3. $(F(b), a) \in B$ where $a = \langle \alpha_n \rangle_{n\in\omega} \in \mu^\omega$.

We will show that Player I does not have a winning strategy. To prove this claim, assume Player I does have a winning strategy. Then, she has a winning strategy that ignores the α_n since $\mu < \mathsf{cof}(\kappa)$. Let σ be a winning strategy for Player I that ignores the α_n and let $b = \langle \beta_n \rangle_{n\in\omega} \in [T]$ such that each $\beta_n \notin X_n$ (where X_n was played according to σ). Since $B_{F(b)} \neq \emptyset$ there is $a = \langle \alpha_n \rangle_{n\in\omega}$ such that $(F(b), a) \in B$. But then Player II wins the game by playing $\langle \alpha_n, \beta_n \rangle_{n\in\omega}$ which is a contradiction.

By Borel Determinacy, we conclude that Player II has a winning strategy. In this way, we can build $T' \leq_0 T$ and $H : T' \longrightarrow \mu$ such that if $b \in [T']$ then $\left(F(b), \overline{H}(b)\right) \in B$. If $\mathfrak{n} \in [T']$ is a generic sequence then $V[\mathfrak{n}] \models (\dot{x}[\mathfrak{n}], \overline{H}(\mathfrak{n})) = (F(\mathfrak{n}), \overline{H}(\mathfrak{n})) \in B$. ■

With the previous result we can conclude the following:

Corollary 21 *Let κ and μ be cardinals such that $\mu < \mathsf{cof}(\kappa)$ and $S_1, S_2 \subseteq \mu^{<\omega}$ are two trees. If $f_1 : [S_1] \longrightarrow \mu^\omega$ and $f_2 : [S_2] \longrightarrow \mu^\omega$ are two continuous functions such that $im(f_1) \subseteq im(f_2)$ then $\mathbb{NB}(\kappa) \Vdash$ "$im(f_1) \subseteq im(f_2)$".*

Proof. Define $B = \{(x, y) \mid x \notin [S_1] \vee (f_1(x) = f_2(y))\}$ then $B \subseteq \mu^\omega \times \mu^\omega$ is a Borel set such that $B_x \neq \emptyset$ for every $x \in \mu^\omega$. The result follows by the previous proposition. ■

In the following section, we will need the following proposition, which generalizes lemma 3 of [14].

Proposition 22 *Let $\mu < \mathsf{cof}(\kappa)$ and $\mathcal{I}$ be a family of Borel subsets of μ^ω such that no countable subcollection of $\mathcal{I}$ covers μ^ω. Then $\mathbb{NB}(\kappa)$ forces that no countable subcollection of $\mathcal{I}$ covers μ^ω.*

Proof. Let $T \in \mathbb{NB}(\kappa)$ and for each $n \in \omega$ let $\dot{S}_n$ a $\mathbb{NB}(\kappa)$-name for a Borel set such that T forces that $\dot{S}_n \in \mathcal{I}$. We must find $T' \leq T$ such that $T' \Vdash$ "$\bigcup \dot{S}_n \neq \mu^\omega$". For every $n \in \omega$ let $\dot{Z}_n$ be the name of a continous surjective function in V such that $T \Vdash$ "$\dot{Z}_n : \mu^\omega \longrightarrow \mu^\omega \setminus \dot{S}_n$". We may assume that there is a sequence $\langle F_n \rangle_{n\in\omega}$ with the following properties:

1. Each F_n is a front of T (i.e. $F_n \subseteq T$ is an antichain and every branch of T extends an element of F_n).
2. Every element of F_{n+1} properly extends an element of F_n.
3. If $t \in F_n$ then there is a continous function Z^t such that $T_t \Vdash$ "$\dot{Z}_n = Z^t$".

For simplicity we assume that T has empty stem. Consider the following game:

I	X_0		X_1		X_2		X_3	$\cdots$
II		β_0		β_1		β_2		$\cdots$

Where $X_n \in [\kappa]^{<\kappa}$, $\beta_n \in \kappa$ and let $b = \langle \beta_n \rangle_{n\in\omega}$. Furthermore, through the game, Player II is required to build sequences (one element at a time) $L^n = \{s_i^n \mid i \in \omega\} \subseteq \mu^{<\omega}$ (she is allowed to wait any number of finite steps before playing an s_i^n). Player II wins the game if the following condition holds:

1. $\beta_n \notin X_n$ for every $n \in \omega$.
2. $b \in [T]$.
3. $|s_i^n| = i$ for every $n, i \in \omega$.
4. $s_i^n \subseteq s_{i+1}^n$ for every $n, i \in \omega$.

5. If $t_1 = b \restriction m_1 \in F_{n_1}$ and $t_2 = b \restriction m_2 \in F_{n_2}$ then $Z^{t_1}\left(\bigcup s_i^{n_1}\right) = Z^{t_2}\left(\bigcup s_i^{n_2}\right)$.

We claim that Player I does not have a winning strategy. Assume Player I has a winning strategy, since $\mu < \mathsf{cof}(\kappa)$ it is easy to see that she has a winning strategy σ that ignores the L^n. Let M be a countable elementary submodel (of a big enough $\mathsf{H}(\theta)$) such that $T, \{(\dot{S}_n, \dot{Z}_n) \mid n \in \omega\}, \sigma \in M$. Since M is countable then, there is $x \notin \bigcup \mathcal{I} \cap M$. Let $b = \langle \beta_n \rangle_{n \in \omega} \in [T]$ be any sequence such that $b \in M$ and $\beta_n \notin X_n$ for every $n \in \omega$. For every $n \in \omega$ let $t_n \in F_n$ such that $t_n \subseteq b$. Since $t_n \in M$ then $Z^{t_n} \in M$ so we conclude that $x \in \bigcap\limits_{n \in \omega} im\left(Z^{t_n}\right)$. For every $n \in \omega$, let $s^n \in \kappa^\omega$ such that $Z^{t_n}(s^n) = x$. Then if Player II plays b and $L^n = \{s^n \restriction i \mid i \in \omega\}$ (which is possible since σ ignores the L^n) she will win the game, which is a contradiction.

By Borel Determinacy, we conclude that Player II has a winning strategy. We can then build a tree $T' \leq T$ such that for every $b \in [T]$ there are sequences $L^n = \{s_i^n \mid i \in \omega\}$ such that $s_i^n \subseteq s_i^{n+1}$ for every $n, i \in \omega$ and if $t_1 = b \restriction m_1 \in F_{n_1}$ and $t_2 = b \restriction m_2 \in F_{n_2}$ then $Z^{t_1}\left(\bigcup s_i^{n_1}\right) = Z^{t_2}\left(\bigcup s_i^{n_2}\right)$. We then can conclude that $T' \Vdash$ "$\bigcup p[\dot{S}_n] \neq \mu^\omega$". $\blacksquare$

3 $\mathbb{NB}(\kappa)$ for $\kappa > \omega_1$

Let $\mathcal{I} \in V$ be an ideal on μ^ω with a Borel base. If W is a model of ZFC extending V then $r \in \mu^\omega \cap W$ is called an *$\mathcal{I}$-quasigeneric sequence (over V)* if $W \models r \notin p[T]$ for every tree $T \subseteq \bigcup\limits_{n \in \omega} \mu^n \times \mu^n$ such that $T \in V$ and $p[T]$ is a Borel set in $\mathcal{I}$ (note it is enough to consider only huge trees). We say that a forcing notion $\mathbb{P}$ *preserves covering of* $\mathcal{I}$ if $\mathbb{P}$ does not add $\mathcal{I}$-quasigeneric sequences. In this paper, we are interested in determining when does $\mathbb{NB}(\kappa)$ preserve the covering of certain ideals. We will need the following definition:

Definition 23 *Let X, Y be two topological spaces, $\mathcal{I}_X$ and ideal on X and $\mathcal{I}_Y$ an ideal on Y.*

1. *We say a continuous function $F : X \longrightarrow Y$ is a* continuous Katětov-morphism *from $(X, \mathcal{I}_X)$ to $(Y, \mathcal{I}_Y)$ if $F^{-1}(A) \in \mathcal{I}_X$ for every $A \in \mathcal{I}_Y$.*

2. *We say $\mathcal{I}_Y$ is* continuously Katětov below *$\mathcal{I}_X$ (which we will denote by $\mathcal{I}_Y \leq_{\mathsf{CK}} \mathcal{I}_X$) if there is a continuous Katětov-morphism from $(X, \mathcal{I}_X)$ to $(Y, \mathcal{I}_Y)$.*

The reason we are interested in the continuos Katětov order is the following proposition:

Proposition 24 *Let $\mathcal{I}$ be an ideal on κ^ω and $\mathcal{J}$ an ideal on μ^ω such that $\mathcal{J} \leq_{\mathsf{CK}} \mathcal{I}$. If a forcing $\mathbb{P}$ adds an $\mathcal{I}$-quasigeneric sequence, then $\mathbb{P}$ adds an $\mathcal{J}$-quasigeneric sequence.*

Proof. Let $F : \kappa^\omega \longrightarrow \mu^\omega$ be a continuous Katětov-morphism from $(\kappa^\omega, \mathcal{I})$ to $(\mu^\omega, \mathcal{J})$. Let $G \subseteq \mathbb{P}$ be a generic filter and $r \in \kappa^\omega$ an $\mathcal{I}$-quasigeneric sequence. We will prove that $F(r)$ is an $\mathcal{J}$-quasigeneric sequence. Let $T \subseteq \bigcup\limits_{n\in\omega} \mu^n \times \mu^n$ a huge tree on μ such that $p[T] \in \mathcal{J}$ and $T \in V$. We then know that there is a huge tree $T_F \subseteq \bigcup\limits_{n\in\omega} \kappa^n \times \kappa^n$ such that $T_F \in V$ and $V[G] \models p[T_F] = F^{-1}(p[T])$. Since F is a Katětov-morphism then $r \notin p[T_F]$ so $F(r) \notin p[T]$. ■

The following result will be very useful:

Proposition 25 *Let X, Y be two topological spaces, $\mathcal{I}_X$ and ideal on X and $\mathcal{I}_Y$ an ideal on Y. If $\mathcal{I}_Y \leq_{\mathsf{CK}} \mathcal{I}_X$ then $\mathsf{cov}(\mathcal{I}_X) \leq \mathsf{cov}(\mathcal{I}_Y)$ and $\mathsf{non}(\mathcal{I}_Y) \leq \mathsf{non}(\mathcal{I}_X)$.*

Proof. Let $F : X \longrightarrow Y$ be a continuous Katětov-morphism from $(X, \mathcal{I}_X)$ to $(Y, \mathcal{I}_Y)$. We will first show that $\mathsf{cov}(\mathcal{I}_X) \leq \mathsf{cov}(\mathcal{I}_Y)$. Let $\{A_\alpha \mid \alpha \in \mathsf{cov}(\mathcal{I}_Y)\} \subseteq \mathcal{I}_Y$ such that $Y = \bigcup\{A_\alpha \mid \alpha \in \mathsf{cov}(\mathcal{I}_Y)\}$. Then $X = \bigcup\{F(A_\alpha)^{-1} \mid \alpha \in \mathsf{cov}(\mathcal{I}_Y)\}$ and $F(A_\alpha)^{-1} \in \mathcal{I}_X$ for every $\alpha \in \mathsf{cov}(\mathcal{I}_Y)$ (since F is a Katětov-morphism) so $\mathsf{cov}(\mathcal{I}_X) \leq \mathsf{cov}(\mathcal{I}_Y)$.

We will now prove that $\mathsf{non}(\mathcal{I}_Y) \leq \mathsf{non}(\mathcal{I}_X)$. Let $B \subseteq X$ such that $B \notin \mathcal{I}_X$, it is easy to see that $F[B] \notin \mathcal{I}_Y$ so $\mathsf{non}(\mathcal{I}_Y) \leq \mathsf{non}(\mathcal{I}_X)$. ■

Regarding the ideals related to Namba forcing, we have the following:

Proposition 26 *If κ is a cardinal then $\mathcal{L}_{\mathsf{cof}(\kappa)} \leq_{\mathsf{CK}} \mathcal{L}_\kappa$.*

Proof. Let $\{P_\alpha \mid \alpha \in \mathsf{cof}(\kappa)\}$ be a partition of κ such that each P_α has size less than κ. Let $h : \kappa \longrightarrow \mathsf{cof}(\kappa)$ such that $\beta \in P_{h(\beta)}$ for every $\beta \in \kappa$. We now define $F : \kappa^\omega \longrightarrow \mathsf{cof}(\kappa)^\omega$ where if $x \in \kappa^\omega$ and $n \in \omega$ then $F(x)(n) = h(x(n))$. Clearly F is a continuous function, we must only show that it is a Katětov-morphism. Let $G : \mathsf{cof}(\kappa)^{<\omega} \longrightarrow \mathsf{cof}(\kappa)$, we must prove that $F^{-1}(C_G) \in \mathcal{L}_\kappa$. Assume this is false, since $F^{-1}(C_G)$ is a Borel set then there is $T \in \mathbb{NB}(\kappa)$ such that $[T] \subseteq F^{-1}(C_G)$. In this way, $F[T] \subseteq C_G$ but it is easy to see that $F[T] \in \mathbb{NB}(\mathsf{cof}(\kappa))$ but this is a contradiction since no $\mathsf{cof}(\kappa)$-Namba tree can be an element of $\mathcal{L}_{\mathsf{cof}(\kappa)}$. ■

We can now characterize when $\mathbb{NB}(\kappa)$ adds a $\mathcal{I}$-quasigeneric sequences.

Proposition 27 *Let κ, μ be two cardinals and $\mathcal{I}$ an ideal on μ^ω with a Borel base. Then the following are equivalent:*

1. *$\mathbb{NB}(\kappa)$ adds an $\mathcal{I}$-quasigeneric sequence.*
2. *$\mathcal{I} \leq_{\mathsf{CK}} \mathcal{L}_\kappa$.*

Proof. Clearly $\mathbb{NB}(\kappa)$ adds an $\mathcal{L}_\kappa$-quasigeneric sequence so if $\mathcal{I} \leq_{\mathsf{CK}} \mathcal{L}_\kappa$ then $\mathbb{NB}(\kappa)$ adds an $\mathcal{I}$-quasigeneric sequence. Now assume that $\mathbb{NB}(\kappa)$ adds an $\dot{r}$ which is forced to be an $\mathcal{I}$-quasigeneric sequence. We can then find $T \in \mathbb{NB}(\kappa)$ and a continuous function $F : [T] \longrightarrow \mu^\omega$ such that $T \Vdash$ "$F(\mathfrak{n}_{gen}) = \dot{r}$". Let $G : \kappa^\omega \longrightarrow [T]$ be an homeomorphism, it is easy to see that $FG : \kappa^\omega \longrightarrow \mu^\omega$ is a continuous Katětov-morphism from $(\kappa^\omega, \mathcal{L}_\kappa)$ to $(\mu^\omega, \mathcal{I})$. ■

We can then conclude the following:

Corollary 28 *Let κ, μ be two cardinals and $\mathcal{I}$ an ideal on μ^ω with a Borel base. If $\mathbb{NB}(\kappa)$ preserves covering of $\mathcal{I}$ then $\mathbb{NB}(\mathsf{cof}(\kappa))$ preserves covering of $\mathcal{I}$.*

However, $\mathbb{NB}(\mathsf{cof}(\kappa))$ may not embed in $\mathbb{NB}(\kappa)$, we will provide (consistently) an example of this fact at the end of this section. Note that $\mathsf{cof}(\kappa) \leq \mathsf{add}(\mathcal{L}_\kappa)$, $\kappa \leq \mathsf{non}(\mathcal{L}_\kappa)$ and if κ has uncountable cofinality then $\mathsf{cov}(\mathcal{L}_\kappa) \leq \mathsf{cof}(\kappa)$. Hence if κ has uncountable cofinality then $\mathsf{cof}(\kappa) = \mathsf{add}(\mathcal{L}_\kappa) = \mathsf{cov}(\mathcal{L}_\kappa)$. By Proposition 22 and the previous equalities we may conclude the following:

Proposition 29 *Let κ, μ be two cardinals and $\mathcal{I}$ an ideal on μ^ω with a Borel base. Then the following holds:*

1. *If $\mathsf{non}(\mathcal{L}_\kappa) < \mathsf{non}(\mathcal{I})$ then $\mathbb{NB}(\kappa)$ preserves covering of $\mathcal{I}$.*
2. *If κ has uncountable cofinality and $\mathsf{cov}(\mathcal{I}) < \mathsf{cof}(\kappa)$ then $\mathbb{NB}(\kappa)$ preserves covering of $\mathcal{I}$.*

As an application of the previous result, we can conclude the following:

Corollary 30 *If $\mu < \mathrm{cof}(\kappa)$ is a cardinal of uncountable cofinality, then $\mathbb{NB}(\kappa) \Vdash$* "$\mathrm{cof}(\mu) > \omega$".

Proof. For every $\alpha \in \mu$ let $B(\alpha) = \{f \in \mu^\omega \mid im(f) \subseteq \alpha\}$ define $\mathcal{I}$ as the ideal generated by $\{B(\alpha) \mid \alpha < \mu\}$. Note that $\mathcal{I}$ is a σ-ideal on μ^ω since μ has uncountable cofinality and $cov(\mathcal{I}) \leq \mu < cof(\kappa)$. We can then conclude that $\mathbb{NB}(\kappa)$ does not add $\mathcal{I}$-quasigeneric sequences, in particular, it does not add a countable cofinal set to μ. ■

We will now show that the dominating number is an upper bound for $\mathsf{non}(\mathcal{L}_{\omega_1})$. Call $Part$ the set of all interval partitions (partitions in finite sets) of ω. We may define an order in $Part$ as follows, given $\mathcal{P}, \mathcal{Q} \in Part$ we say $\mathcal{P} \leq \mathcal{Q}$ if for all $Q \in \mathcal{Q}$ there is $P \in \mathcal{P}$ such that $P \subseteq Q$. In [3] it is proved that the smallest size of a dominating family of interval partitions is precisely $\mathfrak{d}$.

Proposition 31 $\mathsf{non}(\mathcal{L}_{\omega_1}) \leq \mathfrak{d}$.

Proof. Let $\mathcal{P} = \{\mathcal{P}_\gamma \mid \gamma \in \mathfrak{d}\}$ be a dominating family of interval partitions where $\mathcal{P}_\gamma = \{[P_\gamma(n), P_\gamma(n+1)) \mid n \in \omega\}$. For every limit ordinal $\alpha < \omega_1$, choose $C_\alpha = \langle \alpha_n \rangle_{n\in\omega}$ an increasing sequence cofinal in α. For every $\alpha < \omega_1$ and $\gamma < \mathfrak{d}$ we define $g^\gamma_\alpha : \omega \longrightarrow \omega_1$ given by $g^\gamma_\alpha(n) = \alpha_{P_\gamma(n+1)}$. We claim that $X = \{g^\gamma_\alpha \mid \alpha \in LIM(\omega_1) \wedge \gamma \in \mathfrak{d}\}$ is not an element of $\mathcal{L}$.

Let $F : \omega_1^{<\omega} \longrightarrow \omega_1$ and as before, let $D \subseteq \omega_1$ be a club such that if $\alpha \in D$ and $s \in \alpha^{<\omega}$ then $F(s) < \alpha$. Choose any $\alpha \in D$ which is also a limit point of D. Now we define an interval partition $\mathcal{Q} = \{[Q(n), Q(n+1)) \mid n \in \omega\}$ such that $[\alpha_{Q(n)}, \alpha_{Q(n+1)}) \cap D \neq \emptyset$ for every $n \in \omega$. Since $\mathcal{P}$ is a dominating family of interval partitions, then there is $\gamma < \mathfrak{d}$ such that $\mathcal{Q} \leq \mathcal{P}_\gamma$. It is then easy to see that $g^\gamma_\alpha \notin C(F)$. ■

We can then conclude the following:

Corollary 32 *Let κ be a regular cardinal, μ a cardinal and $\mathcal{I}$ an ideal in μ^ω with a Borel base. Then the following holds:*

1. *If $\kappa > \omega_1$ and $\kappa <$ $\mathsf{non}(\mathcal{I})$ then $\mathbb{NB}(\kappa)$ preserves covering of $\mathcal{I}$.*
2. *If $\mathfrak{d} = \omega_1$ or $\mathsf{CG}_\omega(\omega_1)$ holds and $\omega_1 <$ $\mathsf{non}(\mathcal{I})$ then $\mathbb{NB}(\omega_1)$ preserves covering of $\mathcal{I}$.*

The case $\mathbb{NB}(\omega_1)$ will be further studied in the next section. We will now recall some σ-ideals for which we will apply this result.

1. Let ctble be the ideal of all countable subsets of 2^ω. It is easy to see that non(ctble)$= \omega_1$ and cov(ctble)$= \mathfrak{c}$. The ctble-quasigeneric reals are the new reals.
2. For any $f \in \omega^\omega$ let $K_f = \{g \mid g \leq^* f\}$ and define $\mathcal{K}_\sigma$ as the ideal in ω^ω generated by $\{K_f \mid f \in \omega^\omega\}$. It is well known that $\mathcal{K}_\sigma$ is the σ-ideal generated by all compact subsets of ω^ω, $\mathsf{non}(\mathcal{K}_\sigma) = \mathfrak{b}$ and $\mathsf{cov}(\mathcal{K}_\sigma) = \mathfrak{d}$. The $\mathcal{K}_\sigma$-quasigeneric reals are the unbounded reals.
3. For any $f \in \omega^\omega$ let $L_f = \{g \mid f \nleq^* g\}$ and define $\mathcal{L}_\sigma$ the ideal generated by $\{L_f \mid f \in \omega^\omega\}$. Note that $\mathsf{non}(\mathcal{L}_\sigma) = \mathfrak{d}$ and $\mathsf{cov}(\mathcal{L}_\sigma) = \mathfrak{b}$. The $\mathcal{L}_\sigma$-quasigeneric reals are the dominating reals.
4. For every $A \in [\omega]^\omega$ define $M(A) = \{f \in 2^\omega \mid (A \cap f^{-1}(1) =^* \emptyset) \vee (A \setminus f^{-1}(1) =^* \emptyset)\}$. Let $\mathcal{I}_\mathcal{S}$ be the ideal generated by $\{M(A) \mid A \in [\omega]^\omega\}$. In this case, $\mathsf{non}(\mathcal{I}_\mathcal{S}) = \mathfrak{s}$ and $\mathsf{cov}(\mathcal{I}_\mathcal{S}) = \mathfrak{r}$. The $\mathcal{I}_\mathcal{S}$-quasigeneric reals are the splitting reals.
5. For every $f \in \omega^\omega$ define $H(f) = \{g \in \omega^\omega \mid |f \cap g| = \omega\}$ and let $\mathcal{E}$ be the ideal generated by $\{H(f) \mid f \in \omega^\omega\}$. In this case $\mathsf{non}(\mathcal{E}) = \mathsf{cov}(\mathcal{M})$ and

$\mathsf{cov}(\mathcal{E}) = \mathsf{non}(\mathcal{M})$. The $\mathcal{E}$-quasigeneric reals are the eventually different reals. We say a forcing notion $\mathbb{P}$ *destroys category* if there is $p \in \mathbb{P}$ such that $p \Vdash$ "$\omega^{\omega} \cap V \in \mathcal{M}$". It is a well known fact that a partial order $\mathbb{P}$ does not destroy category if and only if $\mathbb{P}$ does not add an eventually different real (under any condition).

6. We say $S : \omega \longrightarrow [\omega]^{<\omega}$ is a *slalom* if $|S(n)| \leq 2^n$ for every $n \in \omega$ and by $\mathcal{SL}$ we will denote the set of all slaloms. If $f \in \omega^{\omega}$ and $S \in \mathcal{SL}$ define $f \sqsubseteq^* S$ if $f(n) \in S(n)$ holds for almost all $n \in \omega$. Given $S \in \mathcal{SL}$ let $C(S) = \{f \in \omega^{\omega} \mid f \sqsubseteq^* S\}$ and define $\mathcal{I}_{\mathcal{SL}}$ as the ideal generated by $\{C(S) \mid S \in \mathcal{SL}\}$. Then $\mathsf{non}(\mathcal{I}_{\mathcal{SL}}) = \mathsf{add}(\mathcal{N})$ and $\mathsf{cov}(\mathcal{I}_{\mathcal{SL}}) = \mathsf{cof}(\mathcal{N})$. Note that a forcing $\mathbb{P}$ has the Sacks property if and only if $\mathbb{P}$ does not add $\mathcal{I}_{\mathcal{SL}}$-quasigeneric reals.

7. Let $\mathcal{J}_{\omega_1}$ be the ideal in ω_1^{ω} generated by $\{\alpha^{\omega} \mid \alpha \in \omega_1\}$. Then $\mathsf{cov}(\mathcal{J}_{\omega_1}) = \mathsf{non}(\mathcal{J}_{\omega_1}) = \omega_1$.and W has $\mathcal{J}_{\omega_1}$-quasigeneric sets over V if and only if ω_1^V is countable in W.

Applying the above result to the ideals we defined before, we conclude the following:

Theorem 33 (see [12] for 1 and 2) *Let κ be a regular cardinal such that either $\kappa > \omega_1$ or $\kappa = \omega_1$ and $\mathfrak{d} = \omega_1$ or $\mathsf{CG}_{\omega}(\omega_1)$ is true. Then the following holds:*

1. *If $\kappa > \omega_1$ then $\mathbb{NB}(\kappa)$ does not collapse ω_1;*
2. *If $\mathfrak{c} < \kappa$ then $\mathbb{NB}(\kappa)$ does not add new reals;*
3. *If $\mathsf{cov}(\mathcal{M}) < \kappa$ or $\kappa < \mathsf{non}(\mathcal{M})$ then $\mathbb{NB}(\kappa)$ does not add Cohen reals;*
4. *If $\mathsf{cov}(\mathcal{N}) < \kappa$ or $\kappa < \mathsf{non}(\mathcal{N})$ then $\mathbb{NB}(\kappa)$ does not add random reals;*
5. *If $\mathfrak{d} < \kappa$ or $\kappa < \mathfrak{b}$ then $\mathbb{NB}(\kappa)$ does not add unbounded reals;*
6. *$\mathbb{NB}(\kappa)$ adds a dominating real if and only if $\mathfrak{b} = \mathfrak{d} = \kappa$;*
7. *If $\mathfrak{r} < \kappa$ or $\kappa < \mathfrak{s}$ then $\mathbb{NB}(\kappa)$ does not add splitting reals;*
8. *If $\mathsf{non}(\mathcal{M}) < \kappa$ or $\kappa < \mathsf{cov}(\mathcal{M})$ then $\mathbb{NB}(\kappa)$ preserves category;*
9. *If $\mathsf{add}(\mathcal{N}) < \kappa$ or $\kappa < \mathsf{cof}(\mathcal{N})$ then $\mathbb{NB}(\kappa)$ has the Sacks property.*

With respect to adding new reals, we have the following result:

Proposition 34 *If $\kappa \leq \mathfrak{c}$ then $\mathbb{NB}(\kappa)$ adds a new real. Hence, if κ is a regular cardinal then $\mathbb{NB}(\kappa)$ adds a new real if and only if $\mathfrak{c} < \kappa$.*

Proof. We will prove that if $\kappa \leq \mathfrak{c}$ then $\mathsf{ctble} \leq_{\mathsf{CK}} \mathcal{L}_{\kappa}$. Recursively build a tree $R = \{p_s \mid s \in \kappa^{<\omega}\}$ such that for every $s, t \in \kappa^{<\omega}$ the following holds:

1. $p_s \subseteq 2^{<\omega}$ is a Sacks tree (i.e. $[p_s]$ is a perfect compact set).
2. If $s \in \kappa^n$ then the stem of p_s has size at least n.
3. If $t \subseteq s$ then $p_s \subseteq p_t$.
4. If $\alpha, \beta \in \kappa$ and $\alpha \neq \beta$ then $[p_{s^\frown\alpha}] \cap [p_{s^\frown\beta}] = \emptyset$.

This construction is easy to do since 2^{ω} can be partition into $\mathfrak{c}$ disjoint compact sets (recall that $2^{\omega\times\omega}$ is homeomorphic to 2^{ω}). We now define $F : \kappa^{\omega} \longrightarrow 2^{\omega}$ such that if $x \in \kappa^{\omega}$ then $\bigcap\limits_{n\in\omega} [p_{x\restriction n}] = \{F(x)\}$. It is easy to see that F is a continuous Katětov-morphism from $(\kappa^{\omega}, \mathcal{L}_{\kappa})$ to $(2^{\omega}, \mathsf{ctble})$. ■

We will now give a condition under which $\mathbb{NB}(\kappa)$ does not preserve the covering of an ideal:

Proposition 35 *Let κ be a cardinal, $\mu < \mathsf{cof}(\kappa)$ and $\mathcal{I}$ a σ-ideal generated by Borel sets in μ^{ω}. If $\mathsf{add}(\mathcal{I}) = \mathsf{cof}(\mathcal{I}) = \kappa$ then $\mathbb{NB}(\kappa)$ does not preserve the covering of $\mathcal{I}$.*

Proof. It is easy to see that if $\mathsf{add}(\mathcal{I}) = \mathsf{cof}(\mathcal{I}) = \kappa$ then there is a cofinal set $S = \{B_{\alpha} \mid \alpha \in \kappa\} \subseteq \mathcal{I}$ such that if $\alpha < \beta$ then $B_{\alpha} \subseteq B_{\beta}$. Let $\mathfrak{n} : \omega \longrightarrow \kappa$ be a generic sequence and in $V[\mathfrak{n}]$ define $S' = \{B_{\mathfrak{n}(n)} \mid n \in \omega\} \subseteq \mathcal{I}$, since $\mathcal{I}$ still generates a σ-ideal then there is $x \notin \bigcup S'$. By genericity, S' is a cofinal set, so x is an $\mathcal{I}$-quasigeneric sequence. ■

We will now provide an example where $\mathbb{NB}(\mathsf{cof}(\kappa))$ does not embed in $\mathbb{NB}(\kappa)$. Recall that a forcing notion $\mathbb{P}$ has *minimal real degree of constructibility* if for every generic filter $G \subseteq \mathbb{P}$ if $x \in V[G] \cap 2^{\omega}$ then either $x \in V$ or $G \in V[x]$. The following result was proved by Miller:

Proposition 36 ([11]) *If κ has uncountable cofinality and $\mathsf{cof}\left([\kappa]^{<\kappa}\right) < \mathfrak{p}$ then $\mathbb{NB}(\kappa)$ has minimal real degree of constructibility.*

With the result of Miller we can build our example:

Proposition 37 *The following statement is consistent with* ZFC : *There is a cardinal κ of uncountable cofinality such that $\mathbb{NB}(\kappa)$ does not add generic sequences for $\mathbb{NB}(\mathsf{cof}(\kappa))$.*

Proof. We start with a model of GCH and $\mathsf{CG}(\omega_1)$. Let $\mathbb{P}$ be a ccc forcing notion that forces $\aleph_{\omega_1+1} < \mathfrak{p}$. Note that since $\mathbb{P}$ is ccc, then $\mathsf{cof}([\aleph_{\omega_1}]^{<\aleph_{\omega_1}}) = \aleph_{\omega_1+1} < \mathfrak{p}$ holds in $V[G]$. In this way, both $\mathbb{NB}(\omega_1)$ and $\mathbb{NB}(\aleph_{\omega_1})$ both have minimal real degree of constructibility and both add reals so if $\mathbb{NB}(\aleph_{\omega_1})$ would add a generic sequences for $\mathbb{NB}(\omega_1)$ then we would have that $\mathbb{NB}(\aleph_{\omega_1})$ and $\mathbb{NB}(\omega_1)$ are in fact equivalent as forcing notions. However, this is a contradiction since $\mathbb{NB}(\omega_1)$ can not add an $\mathcal{L}_{\aleph_{\omega_1}}$-quasigeneric sequence. ■

4 Preservation results of $\mathbb{NB}(\omega_1)$

The case $\kappa = \omega_1$ is particularly interesting. We do not know the answer of the following question:

Problem 38 *If $\mathcal{I}$ is an ideal generated by Borel sets in ω^ω and $\omega_1 < \mathsf{non}(\mathcal{I})$, is it true that $\mathbb{NB}(\omega_1)$ preserves covering of $\mathcal{I}$?*

Nevertheless, we were able to answer the question positively for some specific ideals. Of course, the main issue is that while $\mathsf{CG}_\omega(\kappa)$ is true for every regular $\kappa > \omega_1$, $\mathsf{CG}_\omega(\omega_1)$ may consistently fail. We will now prove that the inequality $\mathsf{non}(\mathcal{L}_{\omega_1}) > \omega_1$ is consistent and we will use Baumgartner's forcing for adding a club with finite conditions. Let $\mathbb{BA}$ be set of all finite functions $p \subseteq \omega_1 \times \omega_1$ with the property that there is a function enumerating a club $g : \omega_1 \longrightarrow \omega_1$ such that $p \subseteq g$ and $im(g)$ consists only of indecomposable ordinals. We order $\mathbb{BA}$ by inclusion, it is well known that $\mathbb{BA}$ is a proper forcing and adds a club, whose name we will denote by $\dot{D}_{gen}$. Given a club $D \subseteq \omega_1$, define a function $F_D : \omega_1^{<\omega} \longrightarrow \omega_1$ given by $F_D(s) = min\{\gamma \in D \mid im(s) \subseteq \gamma\}$. Recall that if $F : \omega_1^{<\omega} \longrightarrow \omega_1$ we defined $C(F) = \{f \in \omega_1^\omega \mid \exists^\infty n (f(n) \in F(f \restriction n))\}$. Note that if $f \in \omega_1^\omega$ then the following holds:

1. If $f[\omega]$ has a maximum then $f \in C(F_D)$.
2. If $\bigcup f[\omega]$ is not a limit point of D then $f \in C(F_D)$.

Lemma 39 *If $f \in \omega_1^\omega$ then $E_f = \{p \in \mathbb{BA} \mid p \Vdash$ "$f \in C(F_{\dot{D}_{gen}})$"$\}$ is a dense set.*

Proof. Let $p \in \mathbb{BA}$, we may assume $f[\omega]$ has no maximum and p forces that $\gamma = \bigcup f[\omega]$ is a limit point of $\dot{D}_{gen}$ (in particular γ must be an indecomposable ordinal) so there must be a limit ordinal $\beta < \omega_1$ such that $p(\beta) = \gamma$, Let $q \leq p$ and $n \in \omega$, we must prove that there is $q_1 \leq q$ and $m > n$ such that $q_1 \Vdash$ "$f(m) < F_{\dot{D}_{gen}}(f \restriction m)$". Let $g : \omega_1 \longrightarrow \omega_1$ be a function enumerating a club such that $q \subseteq g$ and $im(g)$ consists only of indecomposable ordinals. Let $\beta_0 = max(\beta \cap dom(q))$ note we may assume that $f(0), ... f(n) < q(\beta_0)$ (if this is not the case we just extend q in order to obtain this condition). Let m be the smallest natural number for which $q(\beta_0) < f(m)$. Since q forces that γ is a limit point of $\dot{D}_{gen}$, there must be $\beta_0 < \beta_1 < \beta$ such that $f(m), f(m+1) < g(\beta_1)$. We then define q_1 and g_1 as follows:

1. $q_1 = q \cup \{(\beta_0 + 1, g(\beta_1))\}$.

2. $g \restriction (\beta_0 + 1)$, $g \restriction [\beta, \omega_1) \subseteq g_1$.

3. $g_1 (\beta_0 + 1) = g (\beta_1)$.

4. If $\xi \in (\beta_0 + 1, \beta)$ then $g_1 (\xi) = g (\beta_1 + \xi)$.

Note that q_1 is a condition of $\mathbb{BA}$ (as witness by g_1) extending q and $q_1 \Vdash$ "$f (m + 1) < F_{\dot{D}_{gen}} (f \restriction m + 1)$". ■

Since $\mathbb{BA}$ is a proper forcing, we conclude the following:

Proposition 40 *The Proper Forcing Axiom implies* $\mathsf{non}(\mathcal{L}_{\omega_1}) > \omega_1$.

The following lemma is easy and it is left to the reader,

Lemma 41 *If W is a ccc forcing extension of V and $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in W$ then $(\omega_1^{\omega} \cap V) \setminus C (F) \neq \emptyset$.*

We can then prove the following:

Proposition 42 *If $\omega_1 < \mathsf{cov}(\mathcal{M})$ then $\mathbb{NB} (\omega_1)$ does not destroy category.*

Proof. We need to prove that for every continuous function $H : \omega_1^{\omega} \longrightarrow \omega^{\omega}$ there is $h \in \omega^{\omega}$ such that the preimage of $\{f \in \omega^{\omega} \mid |f \cap h| = \omega\}$ is not in $\mathcal{L}_{\omega_1}$. Let M be an elementary submodel (of some $H (\theta)$ for some big enough θ) such that $H \in M$, $\omega_1 \subseteq M$ and $|M| = \omega_1$. Since $\omega_1 < \mathsf{cov}(\mathcal{M})$, there is $c : \omega \longrightarrow \omega$ which is Cohen over M. Let $B = \{f \in \omega^{\omega} \mid |f \cap c| = \omega\}$, clearly B is a Borel set and $B \in M [c]$. Let $A = H^{-1} (B)$ we now claim $M [c] \models A \notin \mathcal{L}_{\omega_1}$. We argue in $M [c]$, let $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in M [c]$. Since $M [c]$ is a ccc extension of M, then there is $g \in (\omega_1^{\omega} \cap M) \setminus C (F)$. In this way, $H (g) \in M$ and since c is Cohen over M, then $H (g) \cap c \neq \emptyset$ so $g \in A \setminus C (F)$. In this way, $M [c] \models A \notin \mathcal{L}_{\omega_1}$ and then $A \notin \mathcal{L}_{\omega_1}$. ■

By $\mathbb{H}$ we denote the *Hechler forcing*, an element of $\mathbb{H}$ is of the form (s, f) where $s \in \omega^{<\omega}$ and $f \in \omega^{\omega}$. The order relation is given by $(s, f) \leq (z, g)$ if $z \subseteq s$, $g \leq f$ and if $i \in dom (s) \setminus dom (z)$ then $s (i) \geq g (i)$. If $\mathcal{F} \subseteq \omega^{\omega}$ we denote by $\mathbb{H} (\mathcal{F})$ as the suborder of $\mathbb{H}$ of all $(s, f) \in \mathbb{H}$ such that $f \in \mathcal{F}$. By $f <_n g$ we will mean that $f (m) < g (m)$ for every $m \geq n$.

Lemma 43 *Assume M is an elementary submodel and $f \in V$ is an increasing dominating real for M. Then there is a partial order $\mathbb{Q} \in V$ that adds a $g : \omega \longrightarrow \omega$ such that $g \leq f$ and g is Hechler over M.*

Proof. Let $\mathbb{Q}$ be the suborder of $\mathbb{H}$ to be the set of all pairs $(s,h) \in \mathbb{H} \cap M$ such that $s \leq f$ and $h \leq_{|s|} f$. Clearly $\mathbb{Q}$ adds a function $g : \omega \longrightarrow \omega$ and $g \leq f$. We will show that g is Hechler over M (note that $\mathbb{Q}$ is not in M). It is enough to show that if $D \in M$ and $D \subseteq \mathbb{H}$ is open dense then $D \cap \mathbb{Q}$ is dense for $\mathbb{Q}$.

Pick any $(s,h) \in \mathbb{Q}$ with $|s| = n_0$, for every $i > n_0$ let $s^i = s^\frown h \upharpoonright [n_0, i]$. Note that $\left(s^i, h\right) \in \mathbb{Q}$ and it extends (s,h). Inside M, we recursively construct two sequences $\{(s_i, h_i) \mid i \in \omega\} \subseteq \mathbb{H}$ and $\{n_i \mid i \in \omega\} \subseteq \omega$ such that for every $i \in \omega$ the following holds:

1. $(s_0, h_0) = (s, h)$,
2. $(s_{i+1}, h_{i+1}) \in D$,
3. $|s_i| = n_i$,
4. $(s_{i+1}, h_{i+1}) \leq (s^{n_i}, h)$.
5. $h_i \leq_{n_{i+1}} h_{i+1}$.

We then define $l = s^\frown (s_1 + h_1) \upharpoonright [n_0, n_1)^\frown (s_2 + h_1 + h_2) \upharpoonright [n_2, n_3)^\frown ...$ and note that $l \in M$, therefore, there is $i \in \omega$ such that $l <_{n_i} f$. This entails that $(s_{i+1}, h_i) \in \mathbb{Q}$. ■

We are now ready to prove the following,

Proposition 44 *If $\omega_1 < \mathfrak{b}$ then $\mathbb{NB}(\omega_1)$ does not add unbounded reals.*

Proof. We need to prove that $\mathcal{L}_{\mathsf{CK}}$ is not continuously Katětov below $\mathcal{K}_\sigma$. Let $H : \omega_1^\omega \longrightarrow \omega^\omega$ be a continuous function and M an elementary submodel of size ω_1 such that $\omega_1 \subseteq M$ and $H \in M$. Since $\omega_1 < \mathfrak{b}$ there is an increasing function $f \in \omega^\omega$ that is a dominating real over M. Let $g : \omega \longrightarrow \omega$ be given by the previous lemma, so $M[g]$ is a Hechler extension of M. Let $B = \{h \in \omega^\omega \mid h \leq^* g\}$ and $A = H^{-1}(B)$ which is a Borel set in $M[g]$, we will prove that $M[g] \models A \notin \mathcal{L}_{\omega_1}$. Let $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in M[g]$ and since $M[g]$ is a ccc extension of M then $(\omega_1^\omega \cap M) \setminus C(F) \neq \emptyset$. Let $x \in (\omega_1^\omega \cap M) \setminus C(F)$ then $H(x) \in M$ and since g is Hechler over M we conclude that $H(x) \leq^* g$ so $x \in A \setminus C(F)$. Since $M[g] \models A \notin \mathcal{L}_{\omega_1}$ by absoluteness, we conclude that $V[g] \models A \notin \mathcal{L}_{\omega_1}$.

Let $A_0 = \{x \in \omega_1^\omega \mid H(x) \leq^* f\}$ and since $g \leq f$ we know that $V[g] \models A_0 \notin \mathcal{L}_{\omega_1}$ and then by absoluteness, $V \models A_0 \notin \mathcal{L}_{\omega_1}$. ■

We can then conclude the following result, which answers a question of [7]:

Corollary 45 *$\mathfrak{b}$ is the first uncountable regular cardinal κ such that $\mathbb{NB}(\kappa)$ adds an unbounded real.*

We will now prove the following:

Lemma 46 *Assume M is an elementary submodel and $c \in V$ is an increasing unbounded real for M. Then there is a partial order $\mathbb{Q} \in V$ that adds a function $d : \omega \longrightarrow \omega$ such that $d \leq c$ and d is Cohen over M.*

Proof. Let $\mathbb{Q}$ be the suborder of $\omega^{<\omega}$ given by $\mathbb{Q} = \{s \in \omega^{<\omega} | s \leq c\}$, clearly $\mathbb{Q}$ adds a function $d : \omega \longrightarrow \omega$ and $d \leq c$. We will show that d is Cohen over M and it is enough to show that if $D \in M$ and $D \subseteq \omega^{<\omega}$ is open dense then $D \cap \mathbb{Q}$ is dense for $\mathbb{Q}$.

Let $s \in \mathbb{Q}$ with $|s| = n_0$ and for every $i > n_0$ define $s^i = s^\frown \overline{0} \restriction [n_0, i]$ where $\overline{0}$ is the constant 0 function. Note that $s^i \in \mathbb{Q}$ and it extends s. Inside M, we recursively construct two sequences $\{s_i \mid i \in \omega\} \subseteq \mathbb{C}$ and $\{n_i \mid i \in \omega\} \subseteq \omega$ such that for every $i \in \omega$ the following holds:

1. $s_0 = s$,
2. $s_{i+1} \in D$,
3. $|s_i| = n_i$,
4. $n_i < n_{i+1}$,
5. $s_{i+1} \leq s^{n_i}$,

We now define $l : \omega \longrightarrow \omega$ where $l(i)$ is the biggest value in the range of s_{i+1} and note that $l \in M$, therefore, there is $i \in \omega$ such that $l(i) < c(i)$ and since $i \leq n_i$ then there is a condition $s_i \in D$ extending s. ■

Using the same method as before, we can prove the following:

Proposition 47 *If $\omega_1 < \mathfrak{d}$ then $\mathbb{NB}(\omega_1)$ does not add dominating reals. In this way, $\mathbb{NB}(\omega_1)$ adds a dominating real if and only if $\mathfrak{d} = \omega_1$.*

Proof. We need to prove that for every continuous function $H : \omega_1^\omega \longrightarrow \omega^\omega$ there is $f \in \omega^\omega$ such that the preimage of $\{h \in \omega^\omega \mid f \nleq^* h\}$ is not in $\mathcal{L}_{\omega_1}$. Let M be an elementary submodel of size ω_1 such that $\omega_1 \subseteq M$ and $H \in M$. Since $\omega_1 < \mathfrak{d}$ there is a function $f \in \omega^\omega$ that is unbounded over M. Let $g : \omega \longrightarrow \omega$ be given by the previous lemma, so $M[g]$ is a Cohen extension of M. Let $B = \{h \in \omega^\omega \mid f \nleq^* h\}$ and $A = H^{-1}(B)$ which is a Borel set in $M[g]$, we will prove that $M[g] \models A \notin \mathcal{L}$. Let $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in M[g]$ and since $M[g]$ is a ccc extension of M then $(\omega_1^\omega \cap M) \setminus C(F) \neq \emptyset$. Let $h \in (\omega_1^\omega \cap M) \setminus C(F)$ then clearly $H(h) \in M$ and since g is Cohen over M then $g \nleq^* H(h)$ so $h \in A \setminus C(F)$. The last part of the argument is similar to the previous one. ■

Given $n \in \omega$ the n-Amoeba forcing $\mathbb{A}_n$ is define as the set of all open subsets of 2^ω with Lebesgue measure less than $\frac{1}{n}$. If $U_1, U_2 \in \mathbb{A}$ then $U_1 \leq U_2$ if $U_1 \subseteq U_2$. It can be proved that $\mathbb{A}_n$ and $\mathbb{A}_m$ are forcing equivalent for every $n, m \in \omega$ (see [2] lemma 3.1.11). In this way, forcing with $\mathbb{A}_2$ adds a null set containing every ground model null set. It is well known that $\mathbb{A}_2$ is ccc and Judah and Repický proved that the Martin number of $\mathbb{A}_2$ is $\mathsf{add}(\mathcal{N})$ (see [2] theorem 3.4.17).

Proposition 48 *If $\omega_1 < \mathsf{add}(\mathcal{N})$ then $\mathbb{NB}(\omega_1)$ has the Sacks property.*

Proof. We need to prove that for every continuous function $F : \omega_1^\omega \longrightarrow \omega^\omega$ there is a slalom S such that the preimage of $\{f \in \omega^\omega \mid f \sqsubseteq^* S\}$ is not in $\mathcal{L}_{\omega_1}$. Let M be an elementary submodel of size ω_1 such that $F \in M$ and $\omega_1 \subseteq M$. Since $\omega_1 < \mathsf{add}(\mathcal{N})$ then there is a filter $G \subseteq \mathbb{A}_2$ that is $(M, \mathbb{A}_2)$-generic. In this way, in $M[G]$ there is a null set containing every null set from M so then there is $S \in \mathcal{SL}$ such that $f \sqsubseteq^* S$ for every $f \in M$. Let $B = \{f \in \omega^\omega \mid f \sqsubseteq^* S\}$ and $A = F^{-1}(B)$ which is a Borel set. We claim that $M[G] \models A \notin \mathcal{L}_{\omega_1}$, let $H : \omega_1 \longrightarrow \omega_1 \in M$ and since $M[G]$ is a ccc extension of M, then there is $x \in M \cap (\omega_1^\omega \setminus C_H)$. But then $F(x) \in M$ so $F(x) \sqsubseteq^* S$ hence $x \in A$ which implies that A is not contained in C_H so $M[G] \models A \notin \mathcal{L}_{\omega_1}$ and then $A \notin \mathcal{L}_{\omega_1}$ by absoluteness. ■

We remark that neither $\mathbb{NB}(\mathsf{add}(\mathcal{N}))$ nor does $\mathbb{NB}(\mathsf{add}(\mathcal{N}))$ have the Sacks property. This will be proved in [6]. We will consider briefly a forcing very similar to $\mathbb{NB}(\kappa)$ (see [4]):

Definition 49 *Let κ be a regular cardinal.*

1. *We say a tree $T \subseteq \kappa^{<\omega}$ is a* κ-Bukovský tree *if the following conditions hold:*
   (a) *If $s \in T$ then either $|suc_T(s)| = 1$ or $|suc_T(s)| = \kappa$.*
   (b) *For every $s \in T$ there is $t \in T$ extending s such that $|suc_T(t)| = \kappa$.*
2. *By $\mathbb{M}(\kappa)$ we denote the set of all κ-Bukovský tree ordered by inclusion.*

In this way, $\mathbb{M}(\omega)$ is the usual Miller forcing. For every $F : \kappa^{<\omega} \longrightarrow \kappa$ define $D_F = \{x \in \kappa^\omega \mid \forall^\infty n \in \omega\, (x(n) < F(x \upharpoonright n))\}$ and let $\mathcal{K}(\kappa)$ be the ideal generated by $\{D_F \mid F : \kappa^{<\omega} \longrightarrow \kappa\}$. It is easy to see that if $\kappa > \omega$ is a regular cardinal then $\mathcal{K}(\kappa)$ is a σ-ideal. For a set $B \subseteq \kappa^\omega$ consider the following game $\mathcal{G}(B)$:

I	s_0		s_1		s_2		s_3	$\cdots$	$\bigcup s_n \in B$
II		α_0		α_1		α_2		$\cdots$	

Such that $s_n \in \kappa^{<\omega}$, $s_n \subseteq s_{n+1}$, $\alpha_n \in \kappa$ and $\alpha_n < s_{n+1}(|s_n|)$ for every $n \in \omega$. Player I wins the game if $\bigcup s_n \in B$. The following proposition is easy and left to the reader:

Proposition 50 *Let $\kappa > \omega$ be a regular cardinal and $B \subseteq \kappa^{\omega}$.*

1. *Player I has a winning strategy in $\mathcal{G}(B)$ if and only if there is $T \in \mathbb{M}(\kappa)$ such that $[T] \subseteq B$.*
2. *Player II has a winning strategy in $\mathcal{G}(B)$ if and only if $B \in \mathcal{K}(\kappa)$.*
3. *Every Borel set of κ^{ω} either contains the branches of a κ-Bukovský tree or belongs to $\mathcal{K}(\kappa)$.*
4. *$\mathbb{M}(\kappa)$ is forcing equivalent to Borel $(\kappa^{\omega}) \diagup \mathcal{K}(\kappa)$.*

We have the following result:

Proposition 51 *Let κ, μ cardinals such that $\kappa > \omega$ is a regular cardinal. Let $\mathcal{I} \subseteq \mu^{\omega}$ be an ideal with a Borel base. If $\mathbb{NB}(\kappa)$ preserves covering of $\mathcal{I}$ then $\mathbb{M}(\kappa)$ preserves covering of $\mathcal{I}$.*

Proof. It is easy to see that $\mathbb{M}(\kappa)$ has continous reading of names. By a similar argument to the one of $\mathbb{NB}(\kappa)$, it can be proved that $\mathbb{M}(\kappa)$ preserves covering of $\mathcal{I}$ if and only if $\mathcal{I} \nleq_{CK} \mathcal{K}(\kappa)$. Since $\mathcal{K}(\kappa) \leq_{CK} \mathcal{L}(\kappa)$ the result follows. ■

5 Open questions

Here is a list of some questions that we were unable to answer.

Problem 52 *If $\mathcal{I}$ is a σ-ideal with Borel base and $\omega_1 < \mathsf{non}(\mathcal{I})$, then does $\mathbb{NB}(\omega_1)$ preserve the covering of $\mathcal{I}$? (probably the most interesting cases are for the ideals of meager and null sets).*

Let $Col(\omega, \omega_1)$ be the usual forcing for collapsing ω_1 to ω with finite conditions. It is clear that $Col(\omega, \omega_1)$ adds Cohen reals, so if $\omega_1 < \mathfrak{b}$ then $Col(\omega, \omega_1)$ does not embed in $\mathbb{NB}(\omega_1)$. This motivates the following question,

Problem 53 *When does $Col(\omega, \omega_1)$ regularly embed in $\mathbb{NB}(\omega_1)$?*

We studied when Namba forcing does not add $\mathcal{I}$-quasigeneric sequences, one could take this line of research a step further,

Problem 54 *When does $\mathbb{NB}(\kappa)$ add Sacks, Laver, Mathias or Miller reals?*

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