ON $\mathbb{R}$-EMBEDDABILITY OF ALMOST DISJOINT FAMILIES AND AKEMANN-DONER C*-ALGEBRAS

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Abstract. An almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ is said to be $\mathbb{R}$-embeddable if there is a function $f : \mathbb{N} \to \mathbb{R}$ such that the sets $f[A]$ are ranges of real sequences converging to distinct reals for distinct $A \in \mathcal{A}$. It is well known that almost disjoint families which have few separations, such as Luzin families, are not $\mathbb{R}$-embeddable. We study extraction principles related to $\mathbb{R}$-embeddability and separation properties of almost disjoint families of $\mathbb{N}$ as well as their limitations. An extraction principle whose consistency is our main result is:

- every almost disjoint family of size continuum contains an $\mathbb{R}$-embeddable subfamily of size continuum.

It is true in the Sacks model. The Cohen model serves to show that the above principle does not follow from the fact that every almost disjoint family of size continuum has two separated subfamilies of size continuum. We also construct in ZFC an almost disjoint family, where no two uncountable subfamilies can be separated but always a countable subfamily can be separated from any disjoint subfamily.

Using a refinement of the $\mathbb{R}$-embeddability property called a controlled $\mathbb{R}$-embedding property we obtain the following results concerning Akemann-Doner C*-algebras which are induced by uncountable almost disjoint families:

- In ZFC there are Akemann-Doner C*-algebras of density $\mathfrak{c}$ with no commutative subalgebras of density $\mathfrak{c}$,
- It is independent from ZFC whether there is an Akemann-Doner algebra of density $\mathfrak{c}$ with no nonseparable commutative subalgebra.

This completes an earlier result that there is in ZFC an Akemann-Doner algebra of density ω_1 with no nonseparable commutative subalgebra.

1. Introduction

A family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is *almost disjoint* if any two distinct elements of $\mathcal{A}$ have finite intersection. The earliest uncountable almost disjoint families considered by Sierpiński were defined as the ranges of sequences of rationals converging to distinct reals. Hence, we say that an almost disjoint family $\mathcal{A}$ is $\mathbb{R}$-*embeddable* if there is a function (called an *embedding*) $f : \mathbb{N} \to \mathbb{R}$ such that the sets $f[A]$ for $A \in \mathcal{A}$ are the ranges of sequences converging to distinct reals (see e.g.[14, 13]). Two families $\mathcal{B}, \mathcal{C}$ of subsets of $\mathbb{N}$ are *separated* if there is $X \subseteq \mathbb{N}$ such that:

(1) If $B \in \mathcal{B}$ then $B \setminus X$ is finite.
(2) If $C \in \mathcal{C}$ then $C \cap X$ is finite.

Considering disjoint neighbourhoods of two condensation points of the limits of converging sequences we see that $\mathbb{R}$-embeddable almost disjoint families contain many pairs of uncountable subfamilies which are separated. On the other hand it

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is an old and beautiful result of Luzin ([19]) that there is an almost disjoint family $\mathcal{A}$ of size ω_1 such that no two uncountable subfamilies of $\mathcal{A}$ can be separated. We will call such families *inseparable*. To highlight the relationship between inseparable and $\mathbb{R}$-embeddable families, recall a dichotomy of [14] where it is shown that assuming the proper forcing axiom (PFA) every almost disjoint family of size ω_1 is either $\mathbb{R}$-embeddable or contains an inseparable subfamily, while Dow [7] showed that under the same assumption every maximal almost disjoint family contains an inseparable subfamily.

An uncountable almost disjoint family $\mathcal{A}$ is called a *Q-family* if for every $\mathcal{B} \subseteq \mathcal{A}$ the families $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ are separated (sometimes called a *separated* family). One of the earliest applications of Martin's axiom (MA) was proving the consistency of the existence of Q-families (which is false under the continuum hypothesis (CH) by a counting argument). All Q-families are $\mathbb{R}$-embeddable and moreover they have a stronger uniformization type property: for every $\phi : \mathcal{A} \to \mathbb{R}$ there is $f : \mathbb{N} \to \mathbb{R}$ such that $f[A]$ is the range of a sequence converging to $\phi(A)$ (in other words $\lim_{n \in A}(f(n) - \phi(A)) = 0$) for each $A \in \mathcal{A}$ ([13, Propositions 2.1., 2.3]).

It is natural, and useful (see e.g., [2, Theorem 2.39]), to consider versions of the above notions which are more cardinal specific: Let κ be a cardinal, then

- an almost disjoint family $\mathcal{A}$ has the *κ-controlled $\mathbb{R}$-embedding property* if for every $\phi : \mathcal{A} \to \mathbb{R}$ there is $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ and $f : \mathbb{N} \to \mathbb{R}$ such that $f[B]$ is the range of a sequence converging to $\phi(B)$ for every $B \in \mathcal{B}$,
- an almost disjoint family $\mathcal{A}$ of size κ is *κ-inseparable* if no two subfamilies of $\mathcal{A}$ both of size κ can be separated,
- an almost disjoint family $\mathcal{A}$ is *κ-anti Lusin* if it has cardinality κ and for every subfamily $\mathcal{B} \subseteq A$ of cardinality κ there are two subfamilies $\mathcal{B}_0, \mathcal{B}_1 \subseteq \mathcal{B}$ of cardinality κ which can be separated ([26]).

This paper is a contribution to the study of extraction principles for almost disjoint families in the context of the above properties. Our main positive results concern the cardinality of the continuum $\mathfrak{c}$ and are:

- It is consistent that every almost disjoint family of size $\mathfrak{c}$ contains an $\mathbb{R}$-embeddable subfamily of size $\mathfrak{c}$ (Theorem 31).
- It is consistent that every almost disjoint family of size $\mathfrak{c}$ has the ω_1-controlled $\mathbb{R}$-embedding property (Theorem 41).
- The above extraction principles are not consequences of every almost disjoint family of size $\mathfrak{c}$ containing a $\mathfrak{c}$-anti Luzin subfamily (Theorems 14 and 17).

The first two extraction principles above are obtained in the iterated Sacks model. As a side product we also prove that in that model every partial function $f : X \to 2^{\mathbb{N}}$ for $X \subseteq 2^{\mathbb{N}}$ of cardinality $\mathfrak{c}$ is uniformly continuous on an uncountable $Y \subseteq X$ (Theorem 39). We do not know if the consistency of this property of functions can be concluded from known results like in [28] or [6] or the fact that under PFA every function is monotone on an uncountable set (see [3]).

The third result above is obtained in the Cohen model from a result of Dow and Hart (Theorem 14) stating that in that model every almost disjoint family is $\mathfrak{c}$-anti Luzin ([8, Proposition 2.6.] using Steprāns's characterization of $\mathcal{P}(\mathbb{N})/Fin$ in that model ([27]) and from the first of our negative results below:

- In the Cohen model there is an almost disjoint family of cardinality $\mathfrak{c}$ with no uncountable $\mathbb{R}$-embeddable subfamily (Theorem 17).
- In the Cohen model no uncountable almost disjoint family has ω_1-controlled $\mathbb{R}$-embedding property (Theorem 18).

We should recall here that by a result of A. Avilés, F. Cabello Sánchez, J. Castillo, M. González and Y. Moreno it is consitent (follows from MA) that $\mathfrak{c}$-inseparable families exist ([2, Lemma 2.36]) ($\mathfrak{c}$-inseparable families are called $\mathfrak{c}$-Lusin families in [8, 2]).

On the other hand, we also discover some ZFC limitations to other extraction principles:

- No almost disjoint family of size $\mathfrak{c}$ has the $\mathfrak{c}$-controlled embedding property (Theorem 6).
- There is in ZFC an inseparable family of cardinality ω_1 which has all possible separations (i.e., separating its countable parts from the rest of the family) (Corollary 11).

The second result is not only natural in the above context by showing that one cannot even consistently hope for extracting from every inseparable family an uncountable subfamily with even fewer separations (for example like Mrówka's family where one can only separate finite subfamilies from the rest of the family). It has also found a natural application in a construction of a thin-tall scattered operator algebra in [10]. Note that under the hypothesis of $\mathfrak{b} > \omega_1$ all inseparable families have the properties of our family from Corollary 11 (see [31, Theorem 3.3]).

Some of the above results concerning the $\mathbb{R}$-embeddability of almost disjoint families find immediate applications in the theory of C*-algebras. It was in the paper [1] of Akemann and Doner where certain C*-algebras were associated to an almost disjoint family $\mathcal{A}$ and a function $\phi : \mathcal{A} \to [0, 2\pi)$. We call these algebras Akemann-Doner algebras and denote them by $AD(\mathcal{A}, \phi)$. For the construction see Section 6 or the papers [1, 5]. These algebras, initially for $\mathcal{A}$ and ϕ constructed only under CH in [1], were the first examples providing negative answer to a question of Dixmier whether every nonseparable C*-algebra must contain a nonseparable commutative C*-subalgebra. Later S. Popa found in [25] a different and a ZFC example, the reduced group C*-algebra of an uncountable free group. However, the latter C*-algebra is very complicated (e.g. it has no nontrivial idempotents [24] etc.) while Akemann-Doner algebras are approximately finite dimensional in the sense of [9] that is, there is a directed family of finite-dimensional C*-subalgebras whose union is dense in the entire C*-algebra. In [5] it was noted that employing an inseparable family $\mathcal{A}$ one can obtain in ZFC a nonseparable Akemann-Doner algebra with no nonseparable commutative subalgebra. Such ZFC examples must be obtained from almost disjoint families $\mathcal{A}$ of cardinality ω_1. This is because we have, for example, the above mentioned result of Dow and Hart that it is consistent that every almost disjoint family of cardinality $\mathfrak{c}$ is $\mathfrak{c}$-anti-Lusin. The cardinality of the almost disjoint family $\mathcal{A}$ is the density of the C*-algebra $AD(\mathcal{A}, \phi)$, that is minimal cardinality of a norm-dense set. Some natural questions remained, for example, if one can have in ZFC an Akemann-Doner algebra of density $\mathfrak{c}$ with no nonseparable commutative subalgebra or another question if it is consistent that every Akemann-Doner algebra of density $\mathfrak{c}$ has a commutative C*-subalgebra of density $\mathfrak{c}$. Here we answer these question proving that:

- In ZFC there are Akemann-Doner C*-algebras of density $\mathfrak{c}$ with no commutative subalgebras of density $\mathfrak{c}$ (Theorem 44).
- It is independent from ZFC whether there is an Akemann-Doner algebra of density $\mathfrak{c}$ with no nonseparable commutative subalgebra (Theorem 45 and the result of [1]).

In fact, we also prove in Theorems 46 and 47 that the existence of nonseparable commutative C*-subalgebras in every Akemann-Doner algebra does not follow from the negation of CH.

The structure of the paper is as follows: in Section 2 we prove some preliminary ZFC results concerning $\mathbb{R}$-embeddability, Section 3 is devoted to the construction of an inseparable almost disjoint family where all countable parts can be separated from the remaining part of the family, Section 4 is devoted to the results mentioned above that hold in the Cohen model and Section 5 to the results that hold in the Sacks model. The last section 6 concerns the consequences of the previous results for the Akemann-Doner C*-algebras.

The set-theoretic terminology and notation is standard and can be found in [16]. The knowledge on C*-algebras required to follow Section 6 does not exceed a linear algebra course concerning 2×2 matrices. Any additional background can be found in [22].

All almost disjoint families are assumed to be infinite and consist of infinite sets. $A \subseteq^* B$ means that $B \setminus A$ is finite. We use $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ for nonnegative integers, reals and rationals respectively. When we view elements of $\mathbb{N}$ as von Neumann ordinals, i.e. subsets and/or elements of each other then we use ω for $\mathbb{N}$. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. If κ is a cardinal and X is a set, then $[X]^\kappa$ denotes the family of all subsets of X of cardinality κ. In particular $[A]^2$ is the set of all pairs $\{a, b\}$ of elements of A. Elements of A^n for $n \in \omega$ are n-tuples of A i.e., $t = (t(0), t(2), \ldots, t(n-1))$. We consider $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ with the inclusion as a tree, we also consider its subtrees T and then $[T]$ denotes the set of all branches of T. The terminology concerning the Cohen forcing $\mathbb{C}$ and the Sacks forcing $\mathbb{S}$ is recalled at the beginning of Sections 4 and 5 respectively.

2. Preliminaries

2.1. **$\mathbb{R}$-embedability of almost disjoint families.** Recall the definition of an $\mathbb{R}$-embeddable almost disjoint family from the introduction. A useful tool for describing properties of almost disjoint families are Ψ-spaces associated with them ([15]). The Ψ-space corresponding to an almost disjoint family $\mathcal{A} \subseteq \wp(\mathbb{N})$ whose points are identified with $\mathbb{N} \cup \mathcal{A}$ is denoted by $\Psi(\mathcal{A})$.

Lemma 1. *Suppose that $\mathcal{A}$ is an almost disjoint family. There is a 1-1 correspondence between continuous functions $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ and functions $f : \mathbb{N} \to \mathbb{R}$ for which $x_A = \lim_{n\in A} f(n)$ exists for each $A \in \mathcal{A}$. It is given by $f = \phi \restriction \mathbb{N}$. Then $x_A = \phi(A)$ for each $A \in \mathcal{A}$.*

Lemma 2. *Let $\mathcal{A} \subseteq \wp(\mathbb{N})$ be an almost disjoint family. Consider $\mathbb{N}^{<\omega} \cup \mathbb{N}^\omega$ with the topology where $\mathbb{N}^{<\omega}$ is discrete and the basic neighbourhoods of $x \in \mathbb{N}^\omega$ are of the form*

$$\{y \in \mathbb{N}^{<\omega} \cup \mathbb{N}^\omega \mid y(n) = x(n) \text{ for all } n \in F\},$$

for any finite $F \subseteq \omega$. The following conditions are equivalent (to the property of being $\mathbb{R}$-embeddable):

(1) *There is a continuous* $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ *such that* $\phi \restriction \mathcal{A}$ *is injective,*
(2) *There is a continuous* $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ *such that* $\phi \restriction \mathcal{A}$ *is injective and* $\phi[\mathcal{A}]$ *has dense complement in* $\mathbb{R}$*,*
(3) *There is a continuous* $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ *such that* $\phi \restriction \mathcal{A}$ *is injective and* $\phi[\mathcal{A}] \subseteq \mathbb{R} \setminus \mathbb{Q}$*,*
(4) *There is a continuous* $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ *such that* ϕ *is injective,* $\phi[\mathcal{A}] \subseteq \mathbb{R} \setminus \mathbb{Q}$ *and* $\phi[\mathbb{N}] \subseteq \mathbb{Q}$*,*
(5) *There is a continuous* $\phi : \Psi(\mathcal{A}) \to \mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega}$ *such that* ϕ *is injective,* $\phi[\mathcal{A}] \subseteq \mathbb{N}^{\omega}$ *and* $\phi[\mathbb{N}] \subseteq \mathbb{N}^{<\omega}$*,*
(6) *There is a continuous* $\phi : \Psi(\mathcal{A}) \to 2^{\omega}$ *such that* $\phi \restriction \mathcal{A}$ *is injective,*

Proof. (1) $\Rightarrow$ (2) We may assume that $\mathcal{A}$ is infinite. Let $U \subseteq \mathbb{R}$ be the union of all open intervals included in $\phi[\mathcal{A}]$. If it is empty, we are done. Otherwise let $E = \{e_n \mid n \in \mathbb{N}\} \subseteq U$ be countable and dense in U. A continuous $\phi' : \Psi(\mathcal{A}) \to \mathbb{R} \setminus E$ such that $\phi'[\Psi(\mathcal{A})] \subseteq \phi[\Psi(\mathcal{A})]$ will satisfy (2). Let $\{x^n_k \mid n, k \in \mathbb{N}\} \subseteq \phi[\mathcal{A}]$ be distinct where $x^n_0 = e_n$ for each $n \in \mathbb{N}$ and such that $|x^n_k - x^n_{k+1}| < 1/(n+k)$. We may choose such x^k_ns since e_n's are in the interior of $\phi[\mathcal{A}]$. Let $A^n_k \in \mathcal{A}$ be such that $\phi(A^n_k) = x^n_k$ for each $n, k \in \mathbb{N}$. Find finite $G^n_k \subseteq A^n_k$ so that $A^n_k \setminus G^n_k$'s are all pairwise disjoint and $|\phi(i) - x^n_k| < 1/(n+k)$ for each $i \in A^n_k \setminus G^n_k$ for each $n, k \in \mathbb{N}$.

Modify ϕ to obtain ϕ' in the following way: Put $\phi' \restriction A^n_k \setminus G^n_k$ to be constantly x^n_{k+1} for each $n, k \in \mathbb{N}$ and $\phi'(A^n_k) = x^n_{k+1}$ for each $n, k \in \mathbb{N}$ and put ϕ' to be equal to ϕ on the remaining points of $\Psi(\mathcal{A})$.

Injectivity of $\phi' \restriction \mathcal{A}$ and the inclusion $\phi'[\Psi(\mathcal{A})] \subseteq \phi[\Psi(\mathcal{A})] \setminus E$ are clear. So we are left with the continuity. ϕ' is clearly continuous at each A^n_k for $n, k \in \mathbb{N}$. Let $A \in \mathcal{A}$ be distinct than each A^n_k. Then each intersection $A \cap A^n_k$ is finite. As $|\phi'(i) - \phi(i)| < 2/(n+k)$ for $i \in A^n_k$ for each $n, k \in N$, it follows that $\lim_{i \in A} |\phi'(i) - \phi(i)| = 0$, that is

$$\lim_{i \in A} \phi'(i) = \lim_{i \in A} \phi(i) = \phi(A) = \phi'(A).$$

(2) $\Rightarrow$ (3) Choose dense countable $E \subseteq \mathbb{R} \setminus \phi[\mathcal{A}]$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that $\eta[E] = \mathbb{Q}$ and consider $\phi' = \eta \circ \phi$.

(3) $\Rightarrow$ (4) Take ϕ satisfying (3) and modify it on $\mathbb{N}$ to obtain ϕ' in such a way that $\phi'(n)$s are distinct rationals for all $n \in \mathbb{N}$ and $|\phi(n) - \phi'(n)| < 1/n$ for all $n \in \mathbb{N}$.

(4) $\Leftrightarrow$ (5) First we construct certain bijection $\rho : \mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega} \to \mathbb{R}$ such that $\rho[\mathbb{N}^{<\omega}] = \mathbb{Q}$ and $\rho[\mathbb{N}^{\omega}] = \mathbb{R} \setminus \mathbb{Q}$. First define a family $(I_s \mid s \in \mathbb{N}^{<\omega})$ of open intervals with rational end-points with the following properties:

(1) $I_\emptyset = \mathbb{R}$,
(2) $\bigcup\{\overline{I_{s \frown n}} \mid n \in \mathbb{N}\} = I_s$,
(3) each end-point of an interval I_s is an endpoint of another interval $I_{s'}$ for $|s| = |s'|$,
(4) the diameter of I_s is smaller than $1/|s|$ for $s \neq \emptyset$,
(5) for every $s \in \mathbb{N}^{<\omega}$ we have $I_{s \frown n} \cap I_{s \frown n'} = \emptyset$ for distinct $n, n' \in \mathbb{N}$,
(6) Every rational is used as an end-point of two (and necessarily only two adjacent, by the previous properties) of the intervals I_s for $s \in \mathbb{N}^{<\omega}$. 0 is end-point of two of I_ss for some $|s| = 1$.

First define ρ on $\mathbb{N}^{<\omega}$ by defining $\rho(s)$ by induction on $|s|$. Let $\rho(\emptyset) = 0$. If $|s| = 1$, then $\rho(s)$ is the right end-point of I_s if I_s consists of positive reals and $\rho(s)$ is the left end-point of I_s if I_s consists of negative reals. If $|s| > 1$, then $\rho(s)$ is the left end-point of I_s. For $x \in \mathbb{N}^\omega$ let $\rho(x)$ be the only point of $\bigcap_{n\in\omega} I_{x\restriction n}$.

Note that ρ is continuous and that $\rho^{-1}(x_n) \to \rho^{-1}(x)$ if x_n is a sequence of rationals converging to an irrational x. This proves (4) $\Leftrightarrow$ (5).

(5) $\Rightarrow$ (6) First note that there $\eta : \mathbb{N}^{<\omega} \cup \mathbb{N}^\omega \to \mathbb{N}^\omega$ which is continuous and the identity on $\mathbb{N}^\omega$. Namely send $s \in \mathbb{N}^{<\omega}$ to the sequence $s^\frown 0^\omega$. Now note that there is $\zeta : \mathbb{N}^\omega \to 2^\omega$ which is continuous. So use the composition of these functions to obtain (6) from (5).

(6) $\Rightarrow$ (1) is clear. □

Remark 3. *Using Lemma 1 we obtain versions of the conditions from Lemma 2 for functions from $\mathbb{N}$ into $\mathbb{R}$. In particular the definition of an $\mathbb{R}$-embeddable almost disjoint from the introduction which is a version of item (1) of Lemma 2 is equivalent to version in the literature, e.g. in* [13] *which are versions of item (4) of Lemma 2.*

The following is a simple condition that allows us to get $\mathbb{R}$-embeddability.

Lemma 4. *Let $T \subseteq 2^{<\omega}$ be a tree, $Z \subseteq [T]$ and $\mathcal{A} = \{A_r \mid r \in Z\}$ an almost disjoint family of subsets of $\mathbb{N}$. If there is a family $\{B_s \mid s \in T\} \subseteq [\mathbb{N}]^\omega$ with the following properties:*

(1) $B_t = \bigcup\{B_{t^\frown i} \mid t^\frown i \in T,\ i \in \{0,1\}\}$ for all $t \in T$,
(2) $B_s \cap B_t$ is finite whenever $s, t \in T$ are incompatible.
(3) $A_r \subseteq \bigcap_{n\in\omega} B_{r\restriction n}$ for every $r \in Z$.

Then, $\mathcal{A}$ is $\mathbb{R}$-embeddable.

Proof. Define $\phi : \Psi(\mathcal{A}) \longrightarrow 2^\omega$ by puting $\phi(A_r) = r$ for all $r \in Z$ and $\phi(n) = s^\frown 0^\omega$ if $n \in B_s$, $|s| \geq n$ and s is the first in the lexicographic order which satisfies the previous requirements. If there is no such $s \in 2^{<\omega}$, then put $\phi(n) = 0^\omega$. Clearly $\phi \restriction \mathcal{A}$ is injective, so we are left with the continuity to check (1) of Lemma 2.

By (3) if $k \in A_r$, then $k \in B_{r\restriction n}$ for every $n \in \omega$. Fix $n \in \omega$. So if we take

$$(*) \qquad k \in A_r \setminus \bigcup\{B_t \mid |t| = n,\ t \neq r|n\},$$

then the condition " $k \in B_s$ and $|s| \geq n$" implies $r \restriction n \subseteq s$ by (1). By (2) the set in (*) almost covers A_r, and so for almost all elements of $k \in A_r$ we have $r \restriction n \subseteq \phi(k)$. As $n \in \omega$ was arbitrary, it follows that $\lim_{k\in A_r} \phi(k) = r = \phi(A_r)$, as required for the continuity. □

Remark 5. *By transfinite induction one can construct a family of sequences $(q_n^\alpha)_{n\in\mathbb{N}}$ for $\alpha < \mathfrak{c}$ in such a way that no tree $T \subseteq 2^{<\omega}$ and no collection $\{B_t \mid t \in T\}$ satisfies the hypothesis of Lemma 4 for any family of $\wp(\mathbb{N})$ obtained through a bijection between $\mathbb{N}$ and $\mathbb{Q}$ from $\{\{q_n^\alpha\}_{n\in\mathbb{N}} \mid \alpha < \mathfrak{c}\}$. It follows that the condition from Lemma 4 is not equivalent to the $\mathbb{R}$-embeddability. This way one can also conclude that there are $\mathbb{R}$-embeddable almost disjoint families of subsets of $\mathbb{N}$ which are not equivalent to a family of branches of $2^{<\omega}$.*

2.2. **κ-controlled $\mathbb{R}$-embedding property.** Recall the definition of the κ-controlled $\mathbb{R}$-embedding property from the introduction.

Theorem 6. *No almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ has $\mathfrak{c}$-controlled $\mathbb{R}$-embedding property.*

Proof. Let $\mathcal{A}$ be an almost disjoint family of size $\mathfrak{c}$ consisting of infinite sets. Let $(M_\alpha)_{\alpha<\mathfrak{c}}$ be a well-ordered, continuous, increasing chain of sets satisfying

(1) $|M_\alpha| \leq \max(|\alpha|, \omega)$ for each $\alpha < \mathfrak{c}$,
(2) $\mathbb{R}^{\mathbb{N}}, \wp(\mathbb{N}) \subseteq \bigcup_{\alpha<\mathfrak{c}} M_\alpha$,
(3) If $A \in M_\alpha \cap \wp(\mathbb{N})$ and $f \in M_\alpha \cap \mathbb{R}^{\mathbb{N}}$ and $\lim_{n\in A} f(n)$ exists, then it belongs to $M_{\alpha+1}$.

It should be clear that one can construct such a sequence $(M_\alpha)_{\alpha<\mathfrak{c}}$. Define $\phi : \mathcal{A} \to [0, 1]$ so that $\phi(A) \in \mathbb{R} \setminus M_{\alpha(A)+1}$ for $A \in \mathcal{A}$, where

$$\alpha(A) = \min\{\alpha < \mathfrak{c} \mid A \in M_\alpha\}.$$

This can be arranged by (2) and by (1). Now suppose $\mathcal{A}' \subseteq \mathcal{A}$ has cardinality $\mathfrak{c}$ and $f : \mathbb{N} \to \mathbb{R}$. By (2) there is $\alpha_0 < \mathfrak{c}$ such that $f \in M_{\alpha_0}$. Take $A \in \mathcal{A}'$ such that $\alpha(A) \geq \alpha_0$ which exists by (1) as $\mathcal{A}'$ has cardinality $\mathfrak{c}$. Then $A \in M_{\alpha(A)} \cap \wp(\mathbb{N})$ and $f \in M_{\alpha(A)} \cap [0, 1]^{\mathbb{N}}$, so by (3), if $\lim_{n\in A} f(n)$ exists, then it belongs to $M_{\alpha(A)+1}$. But $\phi(A) \notin M_{\alpha(A)+1}$ by the definition of ϕ, so $\lim_{n\in A} f(n) \neq \phi(A)$. $\square$

However, it is quite possible to have almost disjoint families of cardinality κ with κ-controlled embedding property:

Proposition 7. [13, cf. 2.3.] *Let κ be a cardinal. Assume MA_κ. Then every subfamily $\mathcal{A}$ of cardinality κ of the Cantor family $\mathcal{C} = \{A_x \mid x \in 2^\omega\} \subseteq \wp(2^{<\omega})$, where $A_x = \{x \restriction n \mid n \in \omega\}$ for $x \in 2^\omega$, has the following strong version of the κ-controlled embedding property: For every function $\phi : \mathcal{A} \to [0, 1]$ there is a function $f : 2^{<\omega} \to [0, 1]$ such that for all $A \in \mathcal{A}$*

$$\lim_{s\in A} f(s) = \phi(F).$$

Proof. It is well known that under the above hypothesis all subsets of 2^ω of cardinality κ are Q-sets and that it implies that all subfamilies of the Cantor family of cardinality κ can be separated from the rest of the family, i.e. they are Q-families in our terminology from the introduction. It follows that $\Psi(\mathcal{A})$ is a normal topological space. As the nonisolated points of $\Psi(\mathcal{A})$ correspond to $\mathcal{A}$ and form a discrete closed subset of $\Psi(\mathcal{A})$ any function ϕ on them is continuous and extends by the Tietze extension theorem to a continuous $\widetilde{\phi} : \Psi(\mathcal{A}) \to [0, 1]$. So put $f = \widetilde{\phi} \restriction 2^{<\omega}$ and use Lemma 1 identifying $2^{<\omega}$ and $\mathbb{N}$. $\square$

3. A Luzin family with all possible separations in ZFC

The main striking property of a Luzin family is that it is inseparable. On the other hand, there is also an almost disjoint family $\mathcal{A}$ of size $\aleph_1$ such that every countable $\mathcal{B} \subseteq \mathcal{A}$ can be separated from $\mathcal{A} \setminus \mathcal{B}$ (see [23]). Here we construct an almost disjoint family which satisfies both properties simultaneously. As both of these properties are hereditary with respect to uncountable subfamilies this shows certain limitations to any further extraction principles.

To construct the almost disjoint family with the aboved-mentioned properties we need colorings of pairs of countable ordinals with properties first obtained by S. Todorcevic in [29] (cf. [30]). In fact, the concrete construction we choose, due to Velleman ([32]), is based on a family of finite subsets of ω_1. It was C. Morgan who connected these two ideas ([21]). For functions $c : [\omega_1]^2 \to \mathbb{N}$ we will abuse notation and denote $c(\{\alpha, \beta\})$ by $c(\alpha, \beta)$.

Theorem 8. *There is a sequence* $(g_\alpha \mid \alpha < \omega_1) \subseteq \{0,1,2\}^{\mathbb{N}}$ *and a coloring* $c : [\omega_1]^2 \to \mathbb{N}$ *satisfying the following:*

(1) *For all* $\beta < \alpha < \omega_1$ *for all* $k > c(\beta, \alpha)$ *we have* $\{g_\beta(k), g_\alpha(k)\} \neq \{1,2\}$,
(2) *For all* $\beta < \alpha < \omega_1$ *we have* $g_\beta(c(\beta,\alpha)) = 1$ *and* $g_\alpha(c(\beta,\alpha))\} = 2$,
(3) *For all* $\gamma < \beta < \alpha < \omega_1$ *if* $c(\gamma,\beta) > c(\alpha,\beta)$, *then* $c(\gamma,\beta) = c(\gamma,\alpha)$,
(4) *For all* $\alpha < \omega_1$ *and all* $m \in \mathbb{N}$ *the set* $\{\beta < \alpha \mid c(\beta,\alpha) < m\}$ *is finite.*
(5) *For all* $\alpha < \omega_1$ *the sets and* $g_\alpha^{-1}[\{1\}]$ *and* $g_\alpha^{-1}[\{2\}]$ *are infinite.*

Proof. We choose the approach from Section 5 of [17]. Thus our $c : [\omega_1]^2 \to \mathbb{N}$ is m of Definition 5.1. of [17], i.e., $c(\alpha,\beta)$ is the minimal rank of an element $X \in \mu$ such that $\alpha, \beta \in X$ where μ is an (ω, ω_1)-cardinal.

The functions g_α for $\alpha < \omega_1$ are defined as follows, for $n = 0$ we put $g_\alpha(0) = 0$ for any $\alpha < \omega_1$ and for any $n \in \mathbb{N}$ we put:

$$g_\alpha(n+1) = \begin{cases} 0 & \text{if } \exists X_1 * X_2 \in \mu\ rank(X_1) = rank(X_2) = n,\ \alpha \in X_1 \cap X_2, \\ 1 & \text{if } \exists X_1 * X_2 \in \mu\ rank(X_1) = rank(X_2) = n,\ \alpha \in X_1 \setminus X_2, \\ 2 & \text{if } \exists X_1 * X_2 \in \mu\ rank(X_1) = rank(X_2) = n,\ \alpha \in X_2 \setminus X_1. \end{cases}$$

Here $X_1 * X_2$ is as in the definition 1.1. (5) of [17]. First let us argue that the g_αs are well defined. By Definition 1.1. (6) and (7) of [17] each element $\alpha \in \omega_1$ is in an element of rank zero of (ω, ω_1)-cardinal μ. Now by Velleman's Density Lemma 2.3. of [17] it follows that α is in an element of rank n of μ for any $n \in \mathbb{N}$. By Definition 1.1. (5) of [17] each element X of μ of rank bigger than zero is of the form $X_1 * X_2$ which means in particular that $X = X_1 \cup X_2$ and $X_1 \cap X_2 < X_1 \setminus X_2 < X_2 \setminus X_1$. Now suppose that $\alpha \in X = X_1 * X_2$ and $\alpha \in Y = Y_1 * Y_2$ and the ranks of X_1, X_2, Y_1, Y_2 are elements of μ of fixed rank $n \in \mathbb{N}$. By Definition 1.1. (3) of [17] there is an order preserving $f_{Y,X} : X \to Y$, which by By Definition 1.1. (3) and (5) of [17] must satisfy $f[X_1] = Y_1$ and $f[X_2] = Y_2$ and moreover $f \restriction (X \cap (\alpha+1))$ is the identity on $X \cap (\alpha + 1)$ be the coherence lemma 2.1 of [17], so $f_{Y,X}(\alpha) = \alpha$ and $f[X_1 \cap X_2] = Y_1 \cap Y_2$, $f[X_1 \setminus X_2] = Y_1 \setminus Y_2$ and $f[X_2 \setminus X_1] = Y_2 \setminus Y_1$ and so the value of $g_\alpha(n+1)$ does not depend if we applied the definition of $g_\alpha(n+1)$ to $X_1 * X_2$ or $Y_1 * Y_2$ which completes the proof of the claim that the g_αs are well defined.

Now we will prove (1) and (2) for $\alpha < \beta < \omega_1$ such that $c(\alpha,\beta) > 0$. For (1) let $n + 1 = k > rank(X)$ such that $\alpha, \beta \in X \in \mu$. Let Y (which exists by the above-mentioned Density Lemma) be such that $X \subseteq Y \in \mu$ and $rank(Y) = k$. $Y = Y_1 * Y_2$. By Definition 1.1. (5) of [17] we have that $X \subseteq Y_1$ or $X \subseteq Y_2$, so $\{g_\beta(k), g_\alpha(k)\} \neq \{1,2\}$. (2) follows from the definition of c, i.e., from the minimality of the rank of $X \ni \alpha, \beta$, which is of the form $X_1 \cup X_2$ with $X_1 \setminus X_2 < X_2 \setminus X_1$ by By Definition 1.1. (5) of [17] and by the hypothesis that $c(\alpha,\beta) > 0$. Property (3) is Corollary 5.4 (2) of [17]. Property (4) is Proposition 5.3 (a) of [17].

To obtain property (5), recall from [17, Theorem 4.5] that $(g_\alpha^{-1}[\{1\}], g_\alpha^{-1}[\{2\}])_{\alpha<\omega_1}$ is a Hausdorff gap, so the sets must be infinite from some point on, so it is enough to remove possibly countably many $\alpha < \omega_1$ and renumerate the remaining ones.

So we are left with removing the hypothesis $c(\alpha,\beta) > 0$ from (1) and (2). Note that what we have proved so far is valid for α, β, γ from any subset of ω_1, in other words we can pass to an uncountable subset X of ω_1 and consider only g_αs for $\alpha \in X$ and then re-enumerate X as ω_1 in an increasing manner. So we need to argue that there is an uncountable $X \subseteq \omega_1$ such that $c(\alpha,\beta) > 0$ for every $\alpha < \beta$ and $\alpha, \beta \in X$. To obtain X apply the Dushnik-Miller theorem (Theorem 9.7 of

[16]) to a coloring $d : [\omega_1]^2 \to \{0,1\}$ given by $d(\alpha,\beta) = \min\{1, c(\alpha,\beta)\}$ knowing that all elements of rank zero must have fixed finite cardinality.

□

Theorem 9. *There are families $(X_\alpha, Y_\alpha, A_\alpha, B_\alpha \mid \alpha < \omega_1)$ of subsets of $\mathbb{N}$ such that*

(1) $X_\alpha = A_\alpha \cup B_\alpha$ is infinite, $A_\alpha \cap B_\alpha = \emptyset$ for all $\alpha < \omega_1$,
(2) $X_\beta \cap X_\alpha =^* \emptyset$ for all $\beta < \alpha < \omega_1$,
(3) $Y_\beta \subseteq^* Y_\alpha$ for all $\beta < \alpha < \omega_1$,
(4) $X_\beta \subseteq^* Y_\alpha$ for all $\beta < \alpha < \omega_1$,
(5) $X_\alpha \cap Y_\alpha = \emptyset$ for all $\alpha < \omega_1$,
(6) For every $\alpha < \omega_1$ and every $k \in \mathbb{N}$ for all but finitely many $\beta < \alpha$ there is $l > k$ such that

$$l \in A_\beta \cap B_\alpha.$$

Proof. Define all the sets as subsets of $[\{0,1,2\}^{<\omega}]^2$ instead of $\mathbb{N}$. For $\alpha < \omega_1$ put $X_\alpha = A_\alpha \cup B_\alpha$, where

$$A_\alpha = \{\{g_\alpha \restriction (n+1), s\} \mid s \in \{0,1,2\}^{n+1},\ g_\alpha(n) = 1, s(n) = 2\},\ n \in \mathbb{N}\}.$$

$$B_\alpha = \{\{g_\alpha \restriction (n+1), s\} \mid s \in \{0,1,2\}^{n+1},\ g_\alpha(n) = 2, s(n) = 1\},\ n \in \mathbb{N}\}.$$

So (1) is clear by Theorem 8 (5).

If $\beta < \alpha < \omega_1$ and $\{r,s\} \in X_\alpha \cap X_\beta$ and $g_\alpha \restriction (n+1) \neq g_\beta \restriction (n+1)$, then $\{r,s\} = \{g_\alpha \restriction (n+1), g_\beta \restriction (n+1)\}$ and $\{r(n), s(n)\} = \{1,2\}$ which means that $n \leq c(\alpha,\beta)$ by (1) and (2) of Theorem 8. So we obtain (2).

Note that if $\beta < \alpha < \omega_1$, then $\{g_\alpha \restriction c(\alpha,\beta), g_\beta \restriction c(\alpha,\beta)\} \in A_\beta \cap B_\alpha$ by (1) of Theorem 8, so we obtain (6).

For $\alpha < \omega_1$ define

$$Y_\alpha = \bigcup_{\beta<\alpha} \Big(X_\beta \setminus \bigcup_{i \leq c(\beta,\alpha)} [\{0,1,2\}^{i+1}]^2 \Big).$$

If follows that $X_\beta \subseteq Y_\alpha$ if $\beta < \alpha < \omega_1$, so we have (4). Also $Y_\alpha \cap X_\alpha = \emptyset$ holds because $X_\beta \cap X_\alpha \subseteq \bigcup_{i \leq c(\beta,\alpha)} [\{0,1,2\}^{i+1}]^2$ by (1) and (2) of Theorem 8.

If $\gamma < \beta < \alpha$ we have $c(\gamma,\beta) = c(\gamma,\alpha)$ with the possible exception for $\gamma < \beta$ in the set $D(\beta,\alpha) = \{\delta < \beta \mid c(\delta,\beta) \leq c(\beta,\alpha)\}$ by (3) of Theorem 8. $D(\beta,\alpha)$ is moreover finite by (4) of Theorem 8. So almost all summands in the definition of Y_β appear literally in the definition of Y_α. The remaining summands of Y_β are $X_\gamma \setminus \bigcup_{i \leq c(\gamma,\beta)} [\{0,1,2\}^i]^2$ for $\gamma \in D(\beta,\alpha)$. Each of them is almost equal to a summand of Y_α of the form $X_\gamma \setminus \bigcup_{i \leq c(\gamma,\alpha)} [\{0,1,2\}^i]^2$ for $\gamma \in D(\beta,\alpha)$ which proves that $Y_\beta \subseteq^* Y_\alpha$ that is we have (3) which completes the proof of the theorem.

□

An example of the use of the partition of X_αs above into A_α and B_α is given in the following proposition which has found an application in [10].

Proposition 10. *There are families $(X'_\alpha, Y'_\alpha, \alpha < \omega_1)$ of subsets of $\mathbb{N}$ and bijections $f_\alpha : \mathbb{N} \times \mathbb{N} \to X'_\alpha$ such that*

(1) $X'_\beta \cap X'_\alpha =^* \emptyset$ for all $\beta < \alpha < \omega_1$,
(2) $Y'_\beta \subseteq^* Y'_\alpha$ for all $\beta < \alpha < \omega_1$,
(3) $X'_\beta \subseteq^* Y'_\alpha$ for all $\beta < \alpha < \omega_1$,
(4) $X'_\alpha \cap Y'_\alpha = \emptyset$ for all $\alpha < \omega_1$,

(5) *For every* $\alpha < \omega_1$ *and every* $k \in \mathbb{N}$ *for all but finitely many* $\beta < \alpha$ *there are* $m_1 < ... < m_k$ *and* $n_1 < ... < n_k$ *such that*

$$f_\alpha(i, n_j) = f_\beta(j, m_i)$$

for all $1 \leq i, j \leq k$.

Proof. Consider a pairwise disjoint family $\{I_l \mid l \in \mathbb{N}\}$ of finite subsets $\mathbb{N}$ where $I_l = \{l_{i,j} \mid 1 \leq i, j \leq l\} \cup \{r_l\}$. Define $X'_\alpha = \bigcup\{I_l \mid l \in X_\alpha\}$ and $Y'_\alpha = \bigcup\{I_l \mid l \in Y_\alpha\}$ where X_α, Y_α satisfy Theorem 9. It is clear that (1) - (4) are satisfied. Put $X''_\alpha = \bigcup\{I_l \setminus \{r_l\} \mid l \in X_\alpha\}$. Now for $\alpha < \omega_1$ let A_α and B_α be as in Theorem 9 and define recursively in $l \in X_\alpha$ for elements of $I_l \setminus \{r_l\}$ an injection $h_\alpha : X''_\alpha \to \mathbb{N} \times \mathbb{N}$ in such a way that if $l \in A_\alpha$, then there are $m_1 < ... < m_l$ such that $h_\alpha(l_{i,j}) = (j, m_i)$ for all $1 \leq i, j \leq l$, and if $l \in B_\alpha$, then there are $n_1 < ... < n_l$ such that $h_\alpha(l_{i,j}) = (i, n_j)$ for all $1 \leq i, j \leq l$. Now use the elements $\{r_l \mid l \in X_\alpha\}$ to extend h_α to a bijection $h'_\alpha : X'_\alpha \to \mathbb{N} \times \mathbb{N}$ and define $f_\alpha = (h'_\alpha)^{-1}$. Note that (6) of Theorem 9 gives $l > k$ such that $l \in A_\beta \cap B_\alpha$, and so (5) follows. □

We may note several interesting properties of the almost disjoint family $(X_\alpha \mid \alpha < \omega_1)$ from Theorem 9.

Corollary 11. *There is an almost disjoint family* $\mathcal{A}$ *which is inseparable (Luzin) but for every countable* $\mathcal{B} \subseteq \mathcal{A}$, *the families* $\mathcal{B}$ *and* $\mathcal{A} \setminus \mathcal{B}$ *can be separated.*

Proof. As countable almost disjoint families can be separated, it is enough to separate the initial fragment $\{X_\beta \mid \beta < \alpha\}$ from the remaining part $\{X_\beta \mid \beta \geq \alpha\}$. Our family from Theorem 9 of course has such separation Y_α, so it is enough to note that it is inseparable. For this note that Theorem 9 (5) implies that given $\alpha < \omega_1$ and $k \in \mathbb{N}$ for all but finitely many $\beta < \alpha$ we have $\max(X_\beta \cap X_\alpha) > k$. This condition implies the inseparability of the family in the standard way as in the case of the Lusin family (cf. [15]). □

Corollary 12. *There is a Luzin family* $(X_\alpha \mid \alpha < \omega_1)$ *such that whenever* $\mathcal{X} \subseteq \omega_1$ *is uncountable, councountable, then there is a a Hausdorff gap* $(A^{\mathcal{X}}_\alpha, B^{\mathcal{X}}_\alpha)_{\alpha<\omega_1}$ *for which* $((X_\alpha \mid \alpha \in \mathcal{X}), (X_\alpha \mid \alpha \in \omega_1 \setminus \mathcal{X}))$ *is its almost disjoint refinement.*

Proof. Take the families $(X_\alpha \mid \alpha < \omega_1)$ and $(Y_\alpha \mid \alpha < \omega_1)$ from Theorem 9. Using the nonexistence of countable gaps in $\wp(\mathbb{N})/Fin$ for each $\alpha < \omega_1$ we can recursively construct separation C^X_α of $(X_\beta \mid \beta \in \mathcal{X} \cap \alpha)$ and $(X_\beta \mid \beta \in \alpha \setminus \mathcal{X})$ i.e., such $C^X_\alpha \subseteq \mathbb{N}$ that

- $X_\beta \subseteq^* C^X_\alpha$, if $\beta \in \alpha \cap \mathcal{X}$,
- $X_\beta \cap C^X_\alpha =^* \emptyset$, if $\beta \in \alpha \setminus \mathcal{X}$.
- $C^X_\beta \cap Y_\beta \subseteq^* C^X_\alpha$, if $\beta < \alpha$,
- $(Y_\beta \setminus C^X_\beta) \cap C^X_\alpha =^* \emptyset$, if $\beta < \alpha$.

Putting $A^X_\alpha = C^X_\alpha \cap Y_\alpha$, $B^X_\alpha = Y_\alpha \setminus C^X_\alpha$ we obtain a Hausdorff gap. □

4. $\mathbb{R}$-embeddability in the Cohen model

The Cohen forcing $\mathbb{C}$ consists of elements of $\mathbb{N}^{<\omega}$ and is ordered by reverse inclusion. By the *Cohen model* we mean the model obtained by adding ω_2-Cohen reals with finite supports to a model of the Generalized Continuum Hypothesis (GCH). Given $X \subseteq \omega_2$ we define $\mathbb{C}_X$ as the forcing adding Cohen reals (with finite supports) indexed by X. The following lemma is well known:

Lemma 13 (Continuous reading of names for Cohen forcing)**.** *If $\dot{A}$ is a $\mathbb{C}$-name for a subset of $\mathbb{N}$, then there is a pair $\left(\langle\mathcal{B}_n\rangle_{n\in\mathbb{N}}, F\right)$ such that*

(1) each $\mathcal{B}_n \subseteq \mathbb{N}^{<\omega}$ is a maximal antichain.
(2) if $s \in \mathcal{B}_{n+1}$ then there is $t \in \mathcal{B}_n$ such that $t \subseteq s$.
(3) $F : \bigcup\limits_{n\in\mathbb{N}} \mathcal{B}_n \longrightarrow 2$.
(4) If $c \in \mathbb{N}^{\omega}$ is Cohen over V, then

$$\dot{A}[r] = \{n \mid \exists m ((r \restriction m) \in \mathcal{B}_n \ \&\ F(c \restriction m) = 1)\}.$$

Here by $\dot{A}[c]$ we denote the evaluation of the name $\dot{A}$ using the generic real c. If the conditions (1) - (4) hold, we will say that $\left(\langle\mathcal{B}_n\rangle_{n\in\mathbb{N}}, F\right)$ codes $\dot{A}$.

As a warm-up we present a direct proof of a result of Dow and Hart from [8] which was obtained there using an ingenious axiomatization of $\wp(\mathbb{N})/Fin$ in the Cohen model.

Theorem 14 ([8])**.** *In the Cohen model, every almost disjoint family of size ω_2 is ω_2-anti Lusin.*

Proof. It is enough to show that in the Cohen model, every almost disjoint family of size ω_2 contains two subfamilies of size ω_2 that are separated. Let $\mathcal{A} = \{\dot{A}_\alpha \mid \alpha \in \omega_2\}$ be a $\mathbb{C}_{\omega_2}$-name for an almost disjoint family. Since $\mathbb{C}_{\omega_2}$ has the countable chain condition, for every $\alpha \in \omega_2$, we can find $S_\alpha \in [\omega_2]^{\omega}$ such that each $\dot{A}_\alpha$ is, in fact, a $\mathbb{C}_{S_\alpha}$-name. By CH and the Δ-system lemma, (see [18] Lemma III.6.15) we can find $X \in [\omega_2]^{\omega_2}$ such that $\{S_\alpha \mid \alpha \in X\}$ forms a Δ-system with root $R \in [\omega_2]^{\omega}$.

We may further assume that the root R is the empty set (if this is not the case, we simply move to the intermediate model obtained by forcing with $\mathbb{C}_R$). Since $\mathbb{C}_{S_\alpha}$ is a forcing notion equivalent to $\mathbb{C}$, we may assume that for each $\alpha \in X$, $\dot{A}_\alpha$ is a $\mathbb{C}_{\{\alpha\}}$-name. Since V is a model of CH, we can find $X_1 \in [X]^{\omega_2}$ and a pair $\left(\langle\mathcal{B}_n\rangle_{n\in\omega}, F\right)$ that codes every $\dot{A}_\alpha$. In other words, each $\dot{A}_\alpha$ is forced to be equal to

$$\{n \mid \exists m ((\dot{c}_\alpha \restriction m) \in \mathcal{B}_n \ \&\ F(\dot{c}_\alpha \restriction m) = 1)\}$$

(where $\dot{c}_\alpha$ is the name of the α^{th}-Cohen real). Since $\mathcal{A}$ is forced to be an almost disjoint family, there are $s, t \in \mathbb{N}^{<\omega}$ such that:

(1) s and t are incomparable nodes of the same length,
(2) there are no m, s', t' with the following properties:
  (a) $m > |s|, |t|$.
  (b) $s', t' \in \mathcal{B}_m$.
  (c) $s \subseteq s'$, $t \subseteq t'$.
  (d) $F(s') = F(t') = 1$.

(In fact, every pair of incomparable nodes can be extended to a pair of nodes satisfying these properties). In $V[G]$, define families $\mathcal{C}_0$ and $\mathcal{C}_1$ by

$$\mathcal{C}_0 = \{\dot{A}_\alpha[c_\alpha] \mid \alpha \in X_1 \wedge s \subseteq c_\alpha\}$$

and

$$\mathcal{C}_1 = \{\dot{A}_\alpha[c_\alpha] \mid \alpha \in X_1 \wedge t \subseteq c_\alpha\}.$$

It is easy to see that both families are of size ω_2 and are separated by $\bigcup\{A \setminus m \mid A \in \mathcal{C}_0\}$. □

A stronger statement: "Every almost disjoint family of size continuum contains an $\mathbb{R}$-embedabble subfamily of size continuum" is consistent but it is false in the Cohen model. We will prove the latter in the rest of this section and the former in the next section.

By $\mathbb{T}$ we denote the set of all finite trees $T \subseteq \mathbb{N}^{<\omega}$ such that all maximal nodes of T have fixed the same height, we denote this common value by $ht(T)$. Given a tree $T \subseteq \mathbb{N}^{<\omega}$ we define $[T]^{2,=} = \{\{s,t\} \in [T]^2 \mid |s| = |t|\}$.

Definition 15. *Define $\mathbb{P}$ as the collection of all triples $p = (T_p, R_p, \phi_p)$ that satisfy the following properties:*

1. *$T_p \in \mathbb{T}$.*
2. *$R_p \subseteq [T_p]^{2,=}$.*
3. *If $\{s,t\} \in R_p$ and $\{s',t'\} \in [T_p]^{2,=}$ is such that $s \subseteq s'$ and $t \subseteq t'$ then $\{s',t'\} \in R_p$.*
4. *$\phi_p : T_p \longrightarrow 2$.*
5. *There is no $\{s,t\} \in R_p$ such that $\phi_p(s) = \phi_p(t) = 1$.*

Given $p, q \in \mathbb{P}$ we say $p \leq_{\mathbb{P}} q$ if $T_q \subseteq T_p$, $R_q = R_p \cap [T_q]^{2,=}$, $\phi_q \subseteq \phi_p$.

Since $\mathbb{P}$ is a countable partial order, it is a forcing notion equivalent to the Cohen forcing. We define $\dot{\phi}_{gen}$ to be equal to $\bigcup\{\phi_p \mid p \in \dot{\mathbb{G}}\}$ (where $\dot{\mathbb{G}}$ is the name for a generic filter of $\mathbb{P}$). It is easy to see that $\dot{\phi}_{gen}$ is forced to be a function from $\mathbb{N}^{<\omega}$ to 2.

Definition 16. *We define $\mathbb{U}$ as the set of all sequences $(p, \langle s_\alpha \rangle_{\alpha \in F})$ with the following properties:*

1. *$p \in \mathbb{P}$.*
2. *$F \in [\omega_2]^{<\omega}$.*
3. *$s_\alpha \in T_p$ for every $\alpha \in F$ (where $p = (T_p, R_p, \phi_p)$).*

We define $(p, \langle s_\alpha \rangle_{\alpha \in F}) \leq (q, \langle t_\alpha \rangle_{\alpha \in G})$ if the following conditions hold:

1. *$p \leq_{\mathbb{P}} q$.*
2. *$G \subseteq F$.*
3. *$t_\alpha \subseteq s_\alpha$ for every $\alpha \in G$.*

It is easy to see that $\mathbb{U}$ is forcing equivalent to $\mathbb{C}_{\omega_2}$. Moreover, $\mathbb{U}$ is forcing equivalent to first forcing with $\mathbb{P}$ and then adding ω_2-Cohen reals. Given $\alpha < \omega_2$ we define $\dot{A}_\alpha$ to be the set $\{n \mid \dot{\phi}_{gen}(\dot{c}_\alpha \upharpoonright n) = 1\}$ (where $\dot{c}_\alpha$ is the name for the α-th Cohen real). It is easy to see that $\dot{\mathcal{A}} = \{\dot{A}_\alpha \mid \alpha < \omega_2\}$ is forced to be an almost disjoint family of size ω_2.

Theorem 17. *In the Cohen model, there is an almost disjoint family of size ω_2 that does not contain uncountable $\mathbb{R}$-embeddable subfamilies.*

Proof. Since $\mathbb{U}$ is forcing equivalent to $\mathbb{C}_{\omega_2}$, we can think of the Cohen model as the model obtained after forcing with $\mathbb{U}$ over a model of the Continuum Hypothesis. Let $\mathcal{A}$ be the almost disjoint family that was defined above. We argue by contradiction, so assume that there is $\dot{\mathcal{B}} = \{\dot{A}_{\dot{\alpha}_\xi} \mid \xi \in \omega_1\}$ and $\dot{f}$ such that $\dot{f}$ is forced to be an embedding $\Psi(\dot{\mathcal{B}})$ into 2^ω as in Lemma 2 (6). For every $\xi \in \omega_1$, we may find $r_\xi = (p_\xi, \langle s^{r_\xi}_\eta \rangle_{\eta \in F_\xi}) \in \mathbb{U}$ and β_ξ with the following properties:

1. $r_\xi \Vdash \dot{\alpha}_\xi = \beta_\xi$.
2. $\beta_\xi \in F_\xi$.

(3) $s^{r_\xi}_{\beta_\xi} {}^\frown 0$, $s^{r_\xi}_{\beta_\xi} {}^\frown 1 \in T_{p_\xi}$ (where $p_\xi = (T_{p_\xi}, R_{p_\xi}, \phi_{p_\xi})$).

By the Δ-system lemma, (see [18] Lemma III.2.6) we may find $p \in \mathbb{P}$, $R \in [\omega_2]^{<\omega}$, $W \in [\omega_2]^{\omega_1}$ and $s \in \mathbb{N}^{<\omega}$ with the following properties:

(1) $p_\xi = p$ for every $\xi \in W$.
(2) $\{F_\xi \mid \xi \in W\}$ forms a Δ-system with root R.
(3) $s^{r_\xi}_\eta = s^{r_{\xi'}}_\eta$ for every $\xi, \xi' \in W$ and $\eta \in R$.
(4) $s = s^{r_\xi}_{\beta_\xi}$ for every $\xi \in W$.

It is easy to see that $\{r_\xi \mid \xi \in W\} \subseteq \mathbb{U}$ is a centered set. Let $\{H_\alpha \mid \alpha \in \omega_1\} \subseteq [W]^2$ be a pairwise disjoint family. For every $\alpha \in \omega_1$ we find $r'_\alpha = (p'_\alpha, \langle u^{r'_\alpha}_\eta \rangle_{\eta \in F'_\alpha}) \in \mathbb{U}$, t_α and z_α with the following properties:

(1) $r'_\alpha \leq r_{\xi_1}, r_{\xi_2}$ where $H_\alpha = \{\xi_1, \xi_2\}$ and $\xi_1 < \xi_2$.
(2) $s^\frown 0 \subseteq u^{r'_\alpha}_{\beta_{\xi_1}}$.
(3) $s^\frown 1 \subseteq u^{r'_\alpha}_{\beta_{\xi_2}}$.
(4) t_α, $z_\alpha \in \mathbb{N}^{<\omega}$ are incompatible.
(5) $r'_\alpha \Vdash t_\alpha \subseteq \dot{f}(\dot{A}_{\beta_{\xi_1}}) \wedge z_\alpha \subseteq \dot{f}(\dot{A}_{\beta_{\xi_2}})$.

The last condition (5) can be obtained since $\dot{f}$ is forced to be injective when restricted to $\dot{\mathcal{B}}$ as in Lemma 2 (6). Once again, we can find $W_0 \in [\omega_2]^{\omega_1}$, $p' \in \mathbb{P}$, $s_0, s_1 \in \mathbb{N}^{<\omega}$, $R'' \in [\omega_2]^{<\omega}$, t, z such that for every $\alpha \in W_0$ the following holds:

(1) $p'_\alpha = p'$.
(2) $t_\alpha = t$ and $z_\alpha = z$.
(3) $\{F'_\alpha \mid \alpha \in W_0\}$ forms a Δ-system with root R'.
(4) $u^{r'_\alpha}_\eta = u^{r'_\delta}_\eta$ for every $\alpha, \delta \in W_0$ and $\eta \in R'$.
(5) $s_0 = u^{r'_\alpha}_{\beta_{\xi_1}}$ and $s_1 = u^{r'_\alpha}_{\beta_{\xi_2}}$ for every $\alpha \in W_0$ where $H_\alpha = \{\xi_1, \xi_2\}$.

Once again, the set $\{r'_\alpha \mid \alpha \in W_0\} \subseteq \mathbb{U}$ is centered. Let $\mathcal{M}$ be a countable elementary submodel of some $\mathsf{H}(\kappa)$ (where κ is a sufficiently big cardinal) containing all objects that have been defined so far. Let $\gamma \in \mathcal{M} \cap W_0$ and $\delta \in W \setminus \mathcal{M}$. Find $m \in \mathbb{N}$ such that $s^\frown m \notin T_{p'}$, let $\widehat{s}$ be a sequence extending $s^\frown m$ such that $|\widehat{s}| = |s_0| = |s_1|$. Then we find $\widehat{r} = (\widehat{p}, \langle y^{\widehat{r}}_\eta \rangle_{\eta \in F})$ with the following properties:

(1) $\widehat{r} \leq r'_\gamma, r_\delta$.
(2) $T_{p'} \cup \{\widehat{s}\} \subseteq T_{\widehat{p}}$ (where $\widehat{p} = (T_{\widehat{p}}, R_{\widehat{p}}, F_{\widehat{p}})$).
(3) $F = F'_\gamma \cup F_\delta$.
(4) $y^{\widehat{r}}_{\beta_\delta} = \widehat{s}$.
(5) $\{s_0, \widehat{s}\}, \{s_1, \widehat{s}\} \notin R_{p'}$.

We claim that $\widehat{r}$ forces that $\dot{f}[\dot{A}_{\beta_\delta}]$ has infinitely many elements below t and infinitely many elements below z, this will be a contradiction. Let $\widehat{r}_1 \leq \widehat{r}$ and $k \in \mathbb{N}$, it will be enough to prove that we can extend $\widehat{r}_1$ to a condition that forces that there is $l > k$ such that l is in $\dot{A}_{\beta_\delta}$ and its image under $\dot{f}$ will be an extension of t whose height is bigger than k (the case of z is similar). Let $\alpha \in \mathcal{M} \cap W_0$ such that $supp(\widehat{r}_1) \cap M$ and $F'_\alpha \setminus R'$ are disjoint. Let $\widehat{r}_2$ be the greatest lower bound of $\widehat{r}_1 \cap \mathcal{M}$ and r'_α, note that $\widehat{r}_2 \in \mathcal{M}$. Let $e \in \omega$ such that $s_0 {}^\frown e$ has not been used and let v be extending $s_0 {}^\frown e$ such that $|v| = \left| s^{\widehat{r}_1}_{\beta_\delta} \right|$ and $\widehat{r}_3$ such that $s^{\widehat{r}_3}_{\beta_{\xi_1}} = v$ (where $H_\alpha = \{\xi_1, \xi_2\}$) and $\{v, s^{\widehat{r}_1}_{\beta_\delta}\} \notin R_{\widehat{r}_3}$. Since $\widehat{r}_3, \dot{f} \in \mathcal{M}$ we can find $\widehat{r}_4 \in \mathcal{M}$ such that $\widehat{r}_4 \leq \widehat{r}_3$ and $l > k$ such that $\widehat{r}_4 \Vdash l \in \dot{A}_{\beta_{\xi_1}} \wedge t \subseteq \dot{f}(l)$. Since the support of $\widehat{r}_4$ is contained in $\mathcal{M}$, then it is compatible with $\widehat{r}_1$. Since $\{v, s^{\widehat{r_1}}_{\beta_\delta}\} \notin R_{\widehat{r_3}}$, we can find a common extension that forces that l is in $\dot{A}_{\beta_\delta}$. $\square$

The above family clearly does not have ω_1-controlled $\mathbb{R}$-embedding property but a much stronger fact concerning ω_1-controlled $\mathbb{R}$-embedding property can be proved in the Cohen model.

Theorem 18. *In the Cohen model, no uncountable almost disjoint family $\mathcal{A}$ has ω_1-controlled $\mathbb{R}$-embedding property.*

Proof. Let $\{c_\alpha \mid \alpha < \omega_2\}$ be the sequence of Cohen reals generating the Cohen model. Let $\mathcal{F}$ be an uncountable almost disjoint family. For every $A \in \mathcal{A}$ there is a countable $X_A \subseteq \omega_2$ such that $A \in V[\{c_\alpha \mid \alpha \in X_A\}]$. Define $\phi : \mathcal{A} \to 2^\omega$ by $\phi(A) = c_{\alpha_A}$ where $\alpha_A \notin X_A$ and all α_A's are distinct.

Suppose that $f : \mathbb{N} \to 2^\omega$. There is a countable $Y \subseteq \omega_2$ such that $f \in V[\{c_\alpha \mid \alpha \in Y\}]$. As $\mathcal{A}$ is uncountable, there is $A \in \mathcal{A}$ such that $\alpha_A \notin Y$, so $\alpha_A \notin X_A \cup Y$. Hence $\lim_{n\in A} f(n) \neq c_{\alpha_A} = \phi(A)$, proving that $\mathcal{A}$ does not have ω_1-controlled property. $\square$

Remark 19. *The above proofs remains valid for any finite support product of not less than 2^ω c.c.c. forcings in place of the Cohen forcing.*

5. $\mathbb{R}$-embeddability in the Sacks model

By the *Sacks model* we mean the model obtained by adding ω_2-Sacks reals (with countable support) to a model of the GCH. Recall that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if every node of p can be extended to a splitting node. We denote by $\mathbb{S}$ the collection of all Sacks tree and we order it by inclusion. Given $\alpha \leq \omega_2$ we denote by $\mathbb{S}_\alpha$ the countable support iteration of $\mathbb{S}$ of length α. We will now prove that in the Sacks model, every almost disjoint family of size continuum contains an $\mathbb{R}$-embeddable family of the same size. We will need to recall some important notions and results on Sacks forcing. For more of this forcing notion the reader may consult [4], [12] and [20].

Definition 20. *Let $\alpha \leq \omega_2$, $n, m \in \mathbb{N}$.*

(1) Given $p, q \in \mathbb{S}$ we say that $(p, m) \leq (q, n)$ if the following holds:
  - *(a) $p \leq q$.*
  - *(b) $n \leq m$.*
  - *(c) $q_n = p_n$.*
  - *(d) If $n < m$ then for every $s \in q_n$ there are distinct $t_0, t_1 \in p_m$ such that $s \subseteq t_0, t_1$.*

(2) Given $p, q \in \mathbb{S}_\alpha$ and $F \in [\alpha]^{<\omega}$ we say that $(p, m) \leq_F (q, n)$ if the following holds:
  - *(a) $p \leq q$.*
  - *(b) $n \leq m$.*
  - *(c) if $\beta \in F$ then $p \restriction \beta \Vdash (p(\beta), m) \leq (q(\beta), n)$.*

We will often use the following result:

Lemma 21 (Fusion lemma [4])**.** *Let $\alpha \leq \omega_2$ and $\{(p_i, F_i, n_i) \mid i \in \mathbb{N}\}$ be a family such that for every $i \in \mathbb{N}$ the following holds:*

(1) $p_i \in \mathbb{S}_\alpha$.

(2) $F_i \in [\alpha]^{<\omega}$.
(3) $F_i \subseteq F_{i+1}$.
(4) $n_i < n_{i+1}$.
(5) $(p_{i+1}, n_{i+1}) \leq_{F_i} (p_i, n_i)$.
(6) $\bigcup_{j\in\mathbb{N}} F_j = \bigcup_{j\in\mathbb{N}} supp\,(p_j)$

Define p *such that* $supp\,(p) = \bigcup_{j\in\mathbb{N}} supp\,(p_j)$ *and if* $\beta \in supp\,(p)$ *then* $p\,(\beta)$ *is a* $\mathbb{S}_\beta$*-name for the intersection of* $\{p_i\,(\beta) \mid \beta \in supp\,(p_i)\}$. *Then* $p \in \mathbb{S}_\alpha$ *and* $p \leq p_i$ *for every* $i \in \mathbb{N}$.

If $p \in \mathbb{S}$ and $s \in 2^{<\omega}$ we define $p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}$. Note that p_s is a Sacks tree if and only if $s \in p$.

Definition 22. *Let* $p \in \mathbb{S}_\alpha$, $F \in [supp\,(p)]^{<\omega}$ *and* $\sigma : F \longrightarrow 2^n$. *We define* p_σ *as follows:*

(1) $supp\,(p_\sigma) = supp\,(p)$.
(2) Letting $\beta < \alpha$ *the following holds:*
  (a) $p_\sigma\,(\beta) = p\,(\beta)$ *if* $\beta \notin F$.
  (b) $p_\sigma\,(\beta) = p\,(\beta)_{\sigma(\beta)}$ *if* $\beta \in F$.

Similar to previous situation, p_σ is not necessarily a condition of $\mathbb{S}_\alpha$. We will say that $\sigma : F \longrightarrow 2^n$ is *consistent with* p if $p_\sigma \in \mathbb{S}_\alpha$. A condition p is (F, n)*-determined* if for every $\sigma : F \longrightarrow 2^n$ either σ is consistent with p or there is $\beta \in F$ such that $\sigma \restriction (F \cap \beta)$ is consistent with p and $(p \restriction \beta)_{\sigma\restriction(F\cap\beta)} \Vdash \sigma\,(\beta) \notin p\,(\beta)$.

We say that $p \in \mathbb{S}_\alpha$ is *continous* if for every $F \in [supp\,(p)]^{<\omega}$ and for every $n \in \mathbb{N}$ there are G and m such that the following holds:

(1) $G \in [supp\,(p)]^{<\omega}$.
(2) $F \subseteq G$.
(3) $n < m$.
(4) p is (G, m)-determined.

We will need the following lemmas:

Lemma 23 ([4])**.** *Let* $p \in \mathbb{S}_\alpha, n \in \mathbb{N}$ *and* $F \in [supp\,(p)]^{<\omega}$. *There is* $(q, m) \leq_F (p, n)$ *such that* q *is* (F, n)*-determined.*

Lemma 24 ([12])**.** *For every* $p \in \mathbb{S}_\alpha$ *there is a continous* $q \leq p$ *such that* q *is continous.*

Let p be a continuous condition. We say that $\{(F_i, n_i, \Sigma_i) \mid i \in \omega\}$ is *a representation of* p if the following holds:

(1) $F_i \in [supp\,(p)]^{<\omega}$, $n_i \in \omega$.
(2) $F_i \subseteq F_{i+1}$ and $n_i < n_{i+1}$.
(3) $supp\,(p) = \bigcup_{i\in\mathbb{N}} F_i$.
(4) p is (F_i, n_i)-determined for every $i \in \omega$.
(5) Σ_i is the set of all $\sigma : F_i \longrightarrow 2^{n_i}$ such that σ is consistent with p.

Note that if $\{(F_i, n_i, \Sigma_i) \mid i \in \omega\}$ is a representation of p and $f : \omega \longrightarrow \omega$ is an increasing function, then $\left\{\left(F_{f(i)}, n_{f(i)}, \Sigma_{f(i)}\right) \mid i \in \mathbb{N}\right\}$ is also a representation of p. It is also easy to see that if p is continuous with representation $\{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and $\sigma \in \Sigma_i$, then p_σ is also a continuous condition. Given a continuous condition

$p \in \mathbb{S}_\alpha$ and $R = \{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ a representation of p, we define $[p]_R$ as the set of all $\langle y_\beta \rangle_{\beta \in supp(p)} \in (2^\omega)^{supp(p)}$ such that for every $i \in \omega$ the function $\sigma : F_i \longrightarrow 2^{n_i}$ given by $\sigma(\beta) = y_\beta \restriction n_i$ belongs to Σ_i.

Lemma 25. *Let $p \in \mathbb{S}_\alpha$ be a continuous condition. If $R = \{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and $R' = \{(G_i, m_i, \Pi_i) \mid i \in \mathbb{N}\}$ are two representations of p, then $[p]_R = [p]_{R'}$.*

Proof. We argue by contradiction, assume that there is $\overline{y} = \langle y_\beta \rangle_{\beta < a} \in [p]_R \setminus [p]_{R'}$. Since $\overline{y} \notin [p]_{R'}$ there must be $i \in \omega$ such that the function $\sigma : G_i \longrightarrow 2^{m_i}$ given by $\sigma(\beta) = y_\beta \restriction m_i$ is not in Π_i, i.e. σ is not consistent with p. Since p is (G_i, m_i)-determined, there is $\beta \in G_i$ such that $\sigma \restriction (G_i \cap \beta)$ is consistent with p but $p_{\sigma \restriction (G_i \cap \beta)} \Vdash \sigma(\beta) \notin p(\beta)$. Let $j \in \omega$ such that $G_i \subseteq F_j$ and $m_i < n_j$. Since $\overline{y} \in [p]_R$ we know that the function $\tau : F_j \longrightarrow 2^{n_j}$ given by $\tau(\xi) = y_\xi \restriction n_j$ is consistent with p. It is clear that $p_{\tau \restriction (F_j \cap \beta)} \leq p_{\sigma \restriction (G_i \cap \beta)}$ and $\sigma(\beta) \subseteq \tau(\beta)$ so $p_{\tau \restriction (F_j \cap \beta)}$ forces that $\tau(\beta)$ is not in $p(\beta)$, which contradicts the fact that τ is consistent with p. □

In light of the previous result, we will omit the subscript and only write $[p]$ to refer to $[p]_R$ where R is any representation of p. It is easy to see that if $p \in \mathbb{S}_\alpha$ is a continuous condition then $[p]$ is a compact set and $p \Vdash \overline{s}_{gen} \restriction supp(p) \in [p]$ (where $\overline{s}_{gen}$ is the sequence of generic reals). Let $S \in [\omega_2]^\omega$ and $\sigma : F \longrightarrow 2^{<\omega}$ where $F \in [S]^{<\omega}$, we define $\langle \sigma \rangle_S$ as the set $\{\langle y_\beta \rangle_{\beta \in S} \in (2^\omega)^S \mid \forall \beta \in F\, (\sigma(\beta) \subseteq y_\beta)\}$. Note that this is family of sets are the basis for the topology of $(2^\omega)^S$. The following result is well known:

Lemma 26 (Continuous reading of names for Sacks forcing)**.** *Let $\alpha < \omega_2$, $p \in \mathbb{S}_\alpha$ and $\dot{x}$ be a $\mathbb{S}_\alpha$-name such that $p \Vdash \dot{x} \in [\omega]^\omega$. There is a continuous condition $q \leq p$ and a continuous function $F : [q] \longrightarrow [\mathbb{N}]^\omega$ such that $q \Vdash F(\dot{s}_{gen} \restriction supp(q)) = \dot{x}$ (where $\dot{s}_{gen}$ is the name for the generic real).*

We will need the following notion:

Definition 27. *Let $\mathcal{C}, \mathcal{D}$ be two subfamilies of $\wp(\mathbb{N})$. We say that the pair $(\mathcal{C}, \mathcal{D})$* is decisive *if one of the following two conditions hold:*

1. *(1) Either $c \cap d$ is infinite for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$ or*
2. *(2) $c \cap d$ is finite for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$.*

Note that if the second alternative holds and $\mathcal{C}$ and $\mathcal{D}$ are both compact, then there is an m such that $c \cap d \subseteq m$ for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Lemma 28. *Let p, q be two continous conditions in $\mathbb{S}_\alpha$ such that $supp(p) = supp(q)$ and $F : (2^\omega)^{supp(p)} \longrightarrow [\mathbb{N}]^\omega$ a continuous function. There are $p', q' \in \mathbb{S}_\alpha$ such that the following holds:*

(1) $p' \leq p$ and $q' \leq q$.
(2) $supp(p) = supp(q) = supp(p') = supp(q')$.
(3) The pair $(F[[p']], F[[q']])$ is decisive.

Proof. We proceed by cases, the first case is that there are $p' \leq p$, $q' \leq q$ with $supp(p) = supp(q) = supp(p') = supp(q')$ and $m \in \mathbb{N}$ such $F(\overline{y}) \cap F(\overline{z}) \subseteq m$ for every $\overline{y} \in [p']$ and $\overline{z} \in [q']$.

In this case it is clear that the pair $(F[[p']], F[[q']])$ is decisive. The second case is that for every $p' \leq p$, $q' \leq q$ with $supp(p) = supp(q) = supp(p') = supp(q')$ and $m \in \mathbb{N}$ there are $\overline{y} \in [p']$, $\overline{z} \in [q']$ and $k > m$ such that $k \in F(\overline{y}) \cap F(\overline{z})$.

Let $supp\,(p) = \{\alpha_n \mid n \in \mathbb{N}\}$. We will now recursively build the two sequences $\{(p^n, m_n, F_n) \mid n \in \mathbb{N}\}$ and $\{(q^n, k_n, G_n) \mid n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$ the following holds:

(1) $p^0 = p$ and $q^0 = q$, $F_0 = G_0 = \emptyset$.
(2) $F_n \in [supp\,(p)]^{<\omega}$, $F_n \subseteq F_{n+1}$ and $\alpha_n \in F_{n+1}$.
(3) $G_n \in [supp\,(q)]^{<\omega}$, $G_n \subseteq G_{n+1}$ and $\alpha_n \in G_{n+1}$.
(4) $m_0 = k_0 = 0$.
(5) $m_n < m_{n+1}$ and $k_n < k_{n+1}$.
(6) p^n and q^n are continous conditions.
(7) $supp\,(p^n) = supp\,(q^n) = supp\,(p)$.
(8) $\left(p^{n+1}, m_{n+1}\right) \leq_{F_{n+1}} (p^n, m_n)$ and $\left(q^{n+1}, k_{n+1}\right) \leq_{G_{n+1}} (q^n, k_n)$.
(9) p^{n+1} is (F_{n+1}, m_{n+1})-determined and q^{n+1} is (G_{n+1}, k_{n+1})-determined.
(10) For every $\sigma : F_n \longrightarrow 2^{m_n}$ and $\tau : G_n \longrightarrow 2^{k_n}$ if σ is consistent with p^n and τ is consistent with q^n then there is $l > n$ such that $l \in F(\overline{y}) \cap F(\overline{z})$ for every $\overline{y} \in [p^n_\sigma]$ and $\overline{z} \in [q^n_\tau]$.

Assume we are at step $n+1$. Since both p^n and q^n are continuous conditions, we can find $F_{n+1}, G_{n+1}, m_{n+1}$ and k_{n+1} with the following properties:

(1) $F_n \cup \{\alpha_n\} \subseteq F_{n+1}$ and $G_n \cup \{\alpha_n\} \subseteq G_{n+1}$.
(2) $m_n < m_{n+1}$, $k_n < k_{n+1}$.
(3) p^n is (F_{n+1}, m_{n+1})-determined and q^n is (G_{n+1}, k_{n+1})-determined.

Let $W = \{(\sigma_i, \tau_i)\}_{i<u}$ enumerate all pairs (σ, τ) for which $\sigma : F_{n+1} \longrightarrow 2^{m_{n+1}}$ and $\tau : G_{n+1} \longrightarrow 2^{k_{n+1}}$. We recursively find a sequence $\left\{\left(p^i_1, q^i_1\right) \mid i < u+1\right\}$ such that for every $i < u$ the following holds:

(1) $p^n = p^0_1$ and $q^n = q^0_1$.
(2) $\left(p^{i+1}_1, m_{n+1}\right) \leq_{F_{n+1}} \left(p^i_1, m_{n+1}\right)$ and $\left(q^{i+1}_1, k_{n+1}\right) \leq_{G_{n+1}} \left(q^i_1, k_{n+1}\right)$.
(3) p^i_1 and q^i_1 are continous.
(4) $supp\left(p^i_1\right) = supp\left(q^i_1\right) = supp\,(p)$.
(5) If σ_i is consistent with p^{i+1}_1 and τ_i is consistent with q^{i+1}_1 then there is $l > n$ such that $l \in F(\overline{y}) \cap F(\overline{z})$ for every $\overline{y} \in [\left(p^{i+1}_1\right)_{\sigma_i}]$ and $\overline{z} \in [\left(q^{i+1}_1\right)_{\tau_i}]$.

Assume we are at step i. In case either σ_i is not consistent with p^i_1 or τ_i is not consistent with q^i_1 we simply define $p^{i+1}_1 = p^i_1$ and $q^{i+1}_1 = q^i_1$. Assume σ_i is consistent with p^i_1 and τ_i is consistent with q^i_1. By the hypothesis, there are $l > n$, $\overline{y} \in [\left(p^i_1\right)_{\sigma_i}]$ and $\overline{z} \in [\left(q^i_1\right)_{\tau_i}]$ such that $k \in F(\overline{y}) \cap F(\overline{z})$. Since F is a continous function, we can find p^{i+1}_1 and q^{i+1}_1 with the following properties:

(1) σ_i is consistent with p^{i+1}_1.
(2) τ_i is consistent with q^{i+1}_1.
(3) For every $\overline{y_1} \in [\left(p^{i+1}_1\right)_{\sigma_i}]$ and $\overline{z_1} \in [\left(q^{i+1}_1\right)_{\tau_i}]$ it is the case that $k \in F(\overline{y}) \cap F(\overline{z})$.
(4) p^{i+1}_1 and q^{i+1}_1 are continous.
(5) $\left(p^{i+1}_1, m_{n+1}\right) \leq_{F_{n+1}} \left(p^i_1, m_{n+1}\right)$ and $\left(q^{i+1}_1, k_{n+1}\right) \leq_{F_n} \left(q^i_1, k_{n+1}\right)$.
(6) $supp\left(p^i_1\right) = supp\left(q^i_1\right) = supp\,(p)$.

We then define $p^{n+1} = p^{u+1}_1$ and $q^{n+1} = q^{u+1}_1$.

Let p' and q' be the respective fusion sequences. It is easy to see that $F[\overline{c}] \cap F[\overline{e}]$ is infinite for every $\overline{c} \in [p']$ and $\overline{e} \in [q']$. □

Note that if p is continous and $\beta = min\left(supp\left(p\right)\right)$ then we may assume that $p\left(\beta\right)$ is a real Sacks tree (not only a name).

Proposition 29. *Let $p \in \mathbb{S}_\alpha$ be a continuous condition, $F : [p] \longrightarrow [\mathbb{N}]^\omega$ a continuous function and $\beta = min\left\{supp\left(\alpha\right)\right\}$. Then there are $q \in \mathbb{S}_\alpha$ with representation $\left\{\left(F_i, m_i, \Sigma_i\right) \mid i \in \mathbb{N}\right\}$ such that the following holds:*

(1) $q \leq p$.
(2) $supp\left(q\right) = supp\left(p\right)$.
(3) $F_0 = \left\{\beta\right\}$.
(4) For every $i \in \mathbb{N}$ the following holds: for every $\sigma, \tau \in \Sigma_i$ such that $\sigma\left(\beta\right) \neq \tau\left(\beta\right)$, the pair $\left(F\left[\left[q_\sigma\right]\right], F\left[\left[q_\tau\right]\right]\right)$ is decisive.

Proof. Let $supp\left(p\right) = \left\{\alpha_n \mid n \in \mathbb{N}\right\}$ with $\alpha_0 = \beta$. We recursively build a sequence $\left\{\left(p^n, m_n, F_n\right) \mid n \in \mathbb{N}\right\}$ with the following properties:

(1) $p^0 = p$.
(2) $F_0 = \left\{\beta\right\}$ and $m_0 = 0$.
(3) Each p^n is continuous and $supp\left(p^n\right) = supp\left(p\right)$.
(4) $F_n \in \left[supp\left(p\right)\right]^{<\omega}$ and $\alpha_n \in F_n$.
(5) $\left(p^{n+1}, m_{n+1}\right) \leq_{F_n} \left(p^n, m_n\right)$.
(6) $m_n < m_{n+1}$.
(7) For every $\sigma, \tau : F_n \longrightarrow 2^{m_n}$ such that $\sigma\left(\beta\right) \neq \tau\left(\beta\right)$ and both are consistent with p^n, the pair $\left(F\left[\left[p^n_\sigma\right]\right], F\left[\left[p^n_\tau\right]\right]\right)$ is decisive.

Assume we are at step n. We first find F_{n+1} and $m_{n+1} > m_n$ such that $F_n \cup \left\{\alpha_n\right\} \subseteq F_{n+1}$ and p^n is $\left(F_{n+1}, m_{n+1}\right)$-determined. Let W be the set of all pairs $\left(\sigma, \tau\right)$ such that $\sigma, \tau : F_{n+1} \longrightarrow 2^{m_{n+1}}$, $\sigma\left(\beta\right) \neq \tau\left(\beta\right)$ and both are consistent with p^n. Enumerate $W = \left\{\left(\sigma_i, \tau_i\right) \mid i \leq l\right\}$. We recursively build $\left\{q_i \mid i \leq l\right\}$ with the following properties:

(1) Each q_i is $\left(F_{n+1}, m_{n+1}\right)$-determined and continuous.
(2) $supp\left(q_i\right) = supp\left(p\right)$.
(3) $\left(q_0, m_n\right) \leq_{F_{n+1}} \left(p^n, m_{n+1}\right)$.
(4) $\left(q_{i+1}, m_{n+1}\right) \leq_{F_n} \left(q_i, m_{n+1}\right)$ for $i < l$.
(5) For each $i \leq l$ one of the following conditions hold:
  (a) Either σ_i or τ_i is not consistent with q_i or
  (b) the pair $(F[[\left(q_i\right)_{\sigma_i}]], F[[\left(q_i\right)_{\tau_i}]])$ is decisive.

Assume we are at step $i < l$. In case that σ_{i+1} or τ_{i+1} is not consistent with q_i we simply define $q_{i+1} = q_i$. We now assume both σ_{i+1} and τ_{i+1} are consistent with q_i. By applying the previous lemma to $\left(q_i\right)_{\sigma_i}$ and $\left(q_i\right)_{\tau_i}$ we obtain r_0, r_1 continous conditions with the following properties:

(1) $r_0 \leq \left(q_i\right)_{\sigma_i}$.
(2) $r_1 \leq \left(q_i\right)_{\tau_i}$.
(3) $supp\left(r_0\right) = supp\left(r_1\right) = supp\left(p\right)$.
(4) the pair $(F[[r_0]], F[[r_1]])$ is decisive.

We now define the r to be a Sacks tree with the following properties:

(1) $r_{\sigma_{i+1}(0)} = r_0\left(\beta\right)$.
(2) $r_{\tau_{i+1}(0)} = r_1\left(\beta\right)$.
(3) $r_s = q_i\left(\beta\right)_s$ for $s \in q_i\left(\beta\right)_{m_n}$ and $s \notin \left\{\sigma_{i+1}\left(\beta\right), \tau_{i+1}\left(\beta\right)\right\}$.

Let $\dot{u}$ be a $\mathbb{S}$-name with the following properties:

(1) $r_0 \upharpoonright \left(\beta + 1\right) \Vdash \dot{u} = \left\langle r_0\left(\xi\right)\right\rangle_{\xi > \beta}$.

(2) $r_1 \restriction (\beta+1) \Vdash \dot{u} = \langle r_1(\xi) \rangle_{\xi > \beta}$.
(3) $r' \Vdash$ "$\dot{u} = \langle q_i(\xi) \rangle_{\xi > \beta}$" for every $r' \leq r \restriction (\beta+1)$ that is incompatible with both $r_0 \restriction (\beta+1)$ and $r_1 \restriction (\beta+1)$.

Let $q_{i+1} = r^\frown \dot{u}$. It is easy to see that q_{i+1} has the desired properties. Finally, we define $p^{n+1} = q_l$. The fusion has the desired properties. □

Let a be a countable subset of ω_2. We can define $\mathbb{S}_a$ as a countable support iteration of Sacks forcing with domain a. Clearly, $\mathbb{S}_a$ is isomorphic to $\mathbb{S}_\delta$ where δ is the order type of a. Note that if $p \in \mathbb{S}_{\omega_2}$ is a continuous condition, then it can be seen as a condition of $\mathbb{S}_{supp(p)}$. With this remark, it is easy to prove the following:

Proposition 30. *Let $p \in \mathbb{S}_\alpha$ be a continuous condition that has a representation $\{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and $F : [p] \longrightarrow [\mathbb{N}]^\omega$ a continuous function. Let α^* be the order type of $supp(p)$ and $\pi : supp(p) \longrightarrow \alpha^*$ be the (unique) order isomorphism. There are $q \in \mathbb{S}_{\alpha^*}$ and a continuous function $H : [q] \longrightarrow [\mathbb{N}]^\omega$ with the following properties:*

(1) $supp(q) = \alpha^*$.
(2) The set $\{(\pi[F_i], n_i, \pi\Sigma_i) \mid i \in \mathbb{N}\}$ is a representation of q (where $\pi\Sigma_i = \{\pi\sigma \mid \sigma \in \Sigma_i\}$).
(3) If $\overline{\pi} : (2^\omega)^{supp(p)} \longrightarrow (2^\omega)^{\alpha^*}$ denotes the natural homeomorphism induced by π. then $\overline{\pi} \restriction [p]$ is an homeomorphism and $F = H\overline{\pi}$.

We will say that (p, F) and (q, H) are *isomorphic* if the previous conditions hold.

Theorem 31. *In the Sacks model, every almost disjoint family of size ω_2 contains an $\mathbb{R}$-embedabble subfamily of size ω_2.*

Proof. Let $\dot{\mathcal{A}} = \left\{ \dot{A}_\alpha \mid \alpha \in \omega_2 \right\}$ be a $\mathbb{S}_{\omega_2}$-name for an almost disjoint family. For every $\alpha < \omega_2$ we choose a pair (p_α, F_α) with the following properties:

(1) p_α is a continuous condition.
(2) $F_\alpha : [p_\alpha] \longrightarrow [\mathbb{N}]^\omega$ is a continuous function.
(3) $p_\alpha \Vdash F_\alpha(\overline{r}_{gen} \restriction supp(p_\alpha)) = \dot{A}_\alpha$

By the Δ-system lemma, we can assume that $\{supp(p_\alpha) \mid \alpha \in \omega_2\}$ forms a delta system with root $R \in [\omega_2]^\omega$. Let $\delta \in \omega_2$ such that $R \subseteq \delta$. By a pruning argument, we may assume that $R = supp(p_\alpha) \cap \delta$ for every $\alpha < \omega_2$. Since $\mathbb{S}_{\omega_2}$ has the ω_2-chain condition, there is $p \in \mathbb{S}_{\omega_2}$ such that p forces that the set $\{\alpha \mid p_\alpha \in \dot{G}\}$ will have size ω_2 (where $\dot{G}$ is the name of the generic filter). Note that we may assume that $p \in \mathbb{S}_\delta$ (by increasing δ if needed).

Let $G_0 \subseteq \mathbb{S}_\delta$ be a generic filter such that $p \in G_0$. We will now work in $V[G_0]$. Let $W = \{\alpha \mid (p_\alpha \restriction \delta) \in G_0\}$ which has size ω_2 by the nature of p. For every $\alpha \in W$, let p'_α be the $\mathbb{S}_\delta$-name such that $p_\alpha = (p_\alpha \restriction \delta)^\frown p'_\alpha$. Note that we may view each $p'_\alpha[G_0]$ as a condition of $\mathbb{S}_{\omega_2}$ where $supp(p'_\alpha[G_0]) = supp(p_\alpha) \setminus \delta$. Let $\overline{r} = \langle r_\beta \rangle_{\beta < \delta}$ be the generic sequence of reals added by G_0. We can now define $H_\alpha : [p'_\alpha[G_0]] \longrightarrow [\mathbb{N}]^\omega$ given by $H_\alpha(\langle y_\beta \rangle) = F_\alpha((\overline{r} \restriction supp(p_\alpha))^\frown \langle y_\beta \rangle)$ which is a continous function. By a previous lemma, for each $\alpha \in W$ we can find a continous condition q_α and $\{(F_i^\alpha, m_i^\alpha, \Sigma_i^\alpha) \mid i \in \omega\}$ a representation of q_α with the following properties:

(1) $q_\alpha \leq p'_\alpha[G_0]$.
(2) $supp(q_\alpha) = supp(p'_\alpha[G_0])$.
(3) $F_0^\alpha = \{\beta_\alpha\}$ where $\beta_\alpha = min(supp(p'_\alpha[G_0]))$.

(4) For every $i \in \omega$ the following holds: for every $\sigma, \tau \in \Sigma_i^\alpha$ such that $\sigma(\beta_\alpha) \neq \tau(\beta)$, the pair $(H_\alpha[[(q_\alpha)_\sigma]], H_\alpha[[(q_\alpha)_\tau]])$ is decisive.

Let α^* be the order type of $supp(q_\alpha)$. For each $\alpha \in W$ we find $q_\alpha^* \in \mathbb{S}_{\alpha^*}$ and $H_\alpha^* : [q_\alpha^*] \longrightarrow [\mathbb{N}]^\omega$ such that (q_α, H_α) and (q_α^*, H_α^*) are isomorphic. We can then find find γ, $q^* \in \mathbb{S}_\gamma$ with representation $\{(F_i, m_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and a continous function $H : [\gamma] \longrightarrow [\mathbb{N}]^\omega$ such that the set $W' \subseteq W$ consisting of all α such that $\alpha^* = \gamma$, $q_\alpha^* = q^*$ and $H_\alpha^* = H$ has size ω_2.

We first note that for every $i \in \mathbb{N}$ the following holds: for every $\sigma, \tau \in \Sigma_i$ such that $\sigma(0) \neq \tau(0)$, the pair $(H[[q_\sigma^*]], H[[q_\tau^*]])$ is decisive, furthermore, $H(\overline{y}) \cap H(\overline{z})$ is finite for every $\overline{y} \in q_\sigma^*$ and $\overline{z} \in q_\tau^*$. It is decisive since (q_α, H_α) and (q^*, H) are isomorphic, the second part of the claim follows since any pair of conditions indexed by elements of W' have disjoint supports (and $\mathcal{A}$ is forced to be an almost disjoint family).

Given $s \in q^* \cap 2^{n_i}$ let $B_s = \bigcup \{\cup H[[q_\sigma]] \mid \sigma \in \Sigma_i \wedge \sigma(0) = s\}$. Note that if s and t are two different elements of $q^* \cap 2^{n_i}$ then B_s and B_t are almost disjoint. Let $T = \{B_s \mid s \in q^* \cap 2^{n_i} \wedge i \in \mathbb{N}\}$.

Note that if $\alpha \in W'$ then $q_\alpha \Vdash \dot{A}_\alpha \subseteq \bigcap_{s \subseteq \dot{r}_{\beta_\alpha}} B_s$ where $\dot{r}_{\beta_\alpha}$ denotes the name of the β_α-generic real. It follows by genericity that $\mathcal{A}$ will contain an $\mathbb{R}$-embeddable subfamily of size ω_2. $\square$

The rest of this section is devoted to the study of the controlled version of the $\mathbb{R}$-embeddability in the Sacks model. In Theorem 41 we obtain the maximal possible ω_1-controlled embedding property since no family of size $\mathfrak{c}$ can have $\mathfrak{c}$-controlled $\mathbb{R}$-embedding property by Theorem 6.

Definition 32. *$e : 2^\omega \to 2^\omega$ is the function satisfying $e(x)(n) = x(2n)$ for every $n \in \omega$.*

Lemma 33. *Let $u \subseteq 2^{<\omega}$ be in $\mathbb{S}$ and $H : [u] \to 2^{\mathbb{N}}$ be a homeomorphism. Let $\alpha < \omega_2$. Whenever $p \in \mathbb{S}_{\omega_2}$ is such that $p \restriction \alpha \Vdash p(\alpha) = \check{u}$ and $p \restriction \alpha \Vdash \dot{x} \in 2^\omega$ for an $\mathbb{S}_\alpha$-name $\dot{x}$, then there is an $\mathbb{S}_\alpha$-name $\dot{q}$ such that $(p \restriction \alpha)^\frown \dot{q} \in \mathbb{S}_{\alpha+1}$, $(p \restriction \alpha)^\frown \dot{q} \leq p \restriction (\alpha+1)$ and*

$$(p \restriction \alpha)^\frown \dot{q} \Vdash e \,\check{\circ}\, H(\dot{s}_\alpha) = \dot{x},$$

In particular $(p \restriction \alpha)^\frown \dot{q} \Vdash \check{e}(\dot{s}_\alpha) = \dot{x}$, if $p(\alpha) = 1_{\mathbb{S}}$.

Proof. Define $\dot{q}$ to be an $\mathbb{S}_\alpha$-name for the set

$$\{y \restriction n \mid y \in [u],\ \forall k \in \omega\ H(y)(2k) = x(k), n \in \omega\}.$$

This is an $\mathbb{S}_\alpha$-name for a perfect subtree of u and so $(p \restriction \alpha)^\frown \dot{q} \in \mathbb{S}_{\alpha+1}$, $(p \restriction \alpha)^\frown \dot{q} \leq p \restriction (\alpha+1)$. We also have $(p \restriction \alpha)^\frown \dot{q} \Vdash \dot{s}_\alpha \in \dot{q}$ and $e(H(z)) = x$ for every $z \in [q]$, so the lemma follows. $\square$

Lemma 34. *Let $\beta < \delta < \omega_2$ and suppose that $p \in \mathbb{S}_{\delta+1} \subseteq \mathbb{S}_{\omega_2}$ and $\dot{F}$ is an $\mathbb{S}_\delta$-name for a continuous function from 2^ω onto 2^ω such that $F^{-1}[\{x\}] \cap [p(\delta)]$ is perfect for every $x \in 2^\omega$ in any forcing extension. There is an $\mathbb{S}_\delta$-name $\dot{r}$ such that $p \restriction \delta^\frown \dot{r} \leq p$ and*

$$p \restriction \delta^\frown \dot{r} \Vdash \dot{F}(\dot{s}_\delta) = \dot{s}_\beta.$$

Proof. Let $\dot{q}$ be an $\mathbb{S}_\delta$-name for the set

$$\bigcap_{u \in \mathbb{G}_\delta} F^{-1}[[u(\beta)]] \cap [p(\delta)] = F^{-1}[\bigcap_{u \in \mathbb{G}_\delta} [u(\beta)]] \cap [p(\delta)] = F^{-1}[\{s_\beta\}] \cap p(\delta).$$

It is a name for a perfect set, as preimages of singletons under F are perfect in $p(\delta)$ in any forcing extension. Let $\dot{r}$ be such a name that $[r] = q$. So $p \restriction \delta^\frown \dot{r} \in \mathbb{S}_{\delta+1}$. Also $q \subseteq [p(\delta)]$, so $p \restriction \delta^\frown \dot{r} \leq p$. If $z \in q$, then $F(z) = \dot{s}_\beta$. But $p \restriction \delta^\frown \dot{r} \Vdash \dot{s}_\delta \in [\dot{r}] = \dot{q}$, so the lemma follows.

□

Definition 35. *$c_1 : \mathbb{S} \to 2^\omega$ is the following coding of perfect subtrees of $2^{<\omega}$ by the reals. Let $\tau : \mathbb{N} \to 2^{<\omega}$ be any fixed bijection. Then given $p \in \mathbb{S}$ we define $c_1(p)(n) = 1$ if and only if $\tau(n) \in p$. c_2 will denote the decoding function i.e.,*

$$c_2(x) = \{\tau(n) \mid x(n) = 1,\ n \in \omega\}.$$

Definition 36. *Let $\{U_n \mid n \in \mathbb{N}\}$ be a fixed bijective enumeration of all clopen subsets of 2^ω. Suppose that $p \in \mathbb{S}$. Define $F_p : 2^\omega \to 2^\omega$ as follows: First by recursion define a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ such that n_0 is minimal satisfying $U_{n_0} \cap [p] \neq \emptyset \neq [p] \setminus U_{n_0}$ and both U_{n_0} and $2^\omega \setminus U_{n_0}$ are intervals in the lexicographical order on 2^ω. Given $n_0, ..., n_k$ for $k \in \mathbb{N}$ let n_{k+1} be minimal such that $n_{k+1} > n_k$ and the following conditions hold for every $\sigma \in 2^{k+2}$:*

(1) $\bigcap_{0\leq i\leq k+1} U_{n_i}^{\sigma(i)}$ is an interval in the lexicographical order on 2^ω,

(2) $\bigcap_{0\leq i\leq k+1} U_{n_i}^{\sigma(i)} \cap [p] \neq \emptyset$,

(3) diam$(\bigcap_{0\leq i\leq k+1} U_{n_i}^{\sigma(i)} \cap [p]) \leq (2/3)^{k+1}$,

where $V^1 = V$ and $V^0 = 2^\omega \setminus V$. Finally for $x \in 2^\omega$ and $i \in \omega$ we define

$$F_p(x)(i) = \chi_{U_{n_{2i}}}(x).$$

Lemma 37. *Let $p \in \mathbb{S}$. $F_p^{-1}[\{x\}] \cap [p]$ is perfect for any $x \in 2^\omega$ in any forcing extension.*

Proof. The conditions (1) - (3) of Definition 36 guarantee the property in the statement of the lemma, but they are preserved by any forcing. □

Lemma 38. *The function $f : 2^\omega \times 2^\omega \to 2^\omega$ defined as*

$$f(x, y) = F_{c_2(x)}(y)$$

is continuous.

Proof. Let $\varepsilon > 0$. Let $\{U_n \mid n \in \mathbb{N}\}$ and $\{U_{n_i} \mid i \in \mathbb{N}\}$ be as in Definition 36. Let $i_0 \in 2\mathbb{N}$ be such that $\Sigma_{i=i_0}^\infty 1/2^i < \varepsilon/2$. Given p there is $m \in \omega$ such that if p, p' are perfect subsets of 2^ω such that $c_1(p) \restriction m = c_1(p') \restriction m$, then the constructions of $\{U_{n_i} \mid i < i_0\}$ for p and p' agree. It follows that if x_n is sufficiently close to x, then $|F_{c_2(x_n)}(z) - F_{c_2(x)}(z)| < \varepsilon/2$ (i.e., $F_{c_2(x_n)}$ converges uniformly to $F_{c_2(x)}$). So

$$|F_{c_2(x)}(y) - F_{c_2(x_n)}(y_n)| = |F_{c_2(x)}(y) - F_{c_2(x)}(y_n) + F_{c_2(x)}(y_n) - F_{c_2(x_n)}(y_n)| \leq$$
$$\leq |F_{c_2(x)}(y) - F_{c_2(x)}(y_n)| + |F_{c_2(x)}(y_n) - F_{c_2(x_n)}(y_n)| < \varepsilon$$

if $|y - y_n|$ and $|x - x_n|$ are sufficiently small by the continuity of $F_{c_2(x)}$ and the above-mentioned uniform convergence. □

Theorem 39. *The following statement is true in the Sacks model: Suppose that $\{x_\xi \mid \xi < \omega_2\} \subseteq 2^\omega$ is a set of distinct reals and $\{y_\xi \mid \xi < \omega_2\} \subseteq 2^\omega$. Then there is a continuous $g : 2^\omega \to 2^\omega$ and $X \subseteq \omega_2$ of cardinality ω_1 such that $g(x_\xi) = y_\xi$ for all $\xi \in X$. In fact, there is a ground model continuous $\phi : 2^\omega \times 2^\omega \to 2^\omega$ such that $\phi(x_\xi, s_\delta) = y_\xi$ for all $\xi \in X$ and some $\delta < \omega_2$.*

Proof. As CH holds in intermediate models we may assume that there are strictly increasing $\{\beta_\theta \mid \theta < \omega_2\}$, conditions $p_\theta \in \mathbb{S}_{\beta_\theta} \subseteq \mathbb{S}_{\omega_2}$ and $\mathbb{S}_{\beta_\theta}$-names $\dot{x}_\theta, \dot{y}_\theta$ for x_θ and y_θ respectively where $\theta < \omega_2$ such that $p_\theta \Vdash \dot{x}_\theta \notin V^{\mathbb{S}_{\theta+1}}$. Using the CH in the ground model we can apply the stationary Δ-system lemma[1] for countable sets and obtain a stationary $A \subseteq \{\alpha \in \omega_2 : cf(\alpha) = \omega_1\}$ such that $\{supp(p_\theta) \mid \theta \in A\}$ forms a Δ-system with root $\Delta \subseteq \omega_2$ and all the conditions agree on Δ.

We can use the result of [11] to find continuous $h_\theta : 2^\omega \to 2^\omega$ and $q_\theta \leq p_\theta$ such that $q_\theta \Vdash \check{h}_\theta(\dot{x}_\theta) = \dot{s}_\theta$, for all $\theta \in A$. Use the pressing down lemma finding a stationary $A' \subseteq A$ such that there is $\alpha < \omega_2$ with $supp(q_\theta) \cap \theta \subseteq \alpha$ for all $\theta \in A'$. We will work for the rest of the proof in $V^{\mathbb{S}^\alpha}$ which will be treated as the ground model. By passing to a subset of A' of cardinality ω_2 and renaming the q_θ's we may assume that

$$p_\theta \Vdash \check{h}(\dot{x}_\theta) = \dot{s}_\theta, \tag{1}$$

for a fixed continuous $h : 2^\omega \to 2^\omega$ and all $\theta \in A'$ and $p(\theta)$ is a fixed perfect tree $u \subseteq 2^{<\omega}$ and the supports of p_θs for $\theta \in A'$ are pairwise disjoint and $\min(supp(p_\theta)) = \theta$ for all $\theta \in A'$. Also fix a homeomorphism $H : [u] \to 2^\omega$. Construct a strictly increasing $\{\theta_\xi \mid \xi < \omega_1\}$ such that $\theta_\xi < \beta_{\theta_\xi} < \theta_{\xi'}$ for all $\xi < \xi' < \omega_1$. Relabel the involved objects as $p_\xi := p_{\theta_\xi}$, $\alpha_\xi := \theta_\xi$, $\beta_\xi := \beta_{\theta_\xi}$, $\dot{x}_\xi := \dot{x}_{\theta_\xi}$, $\dot{y}_\xi := \dot{y}_{\theta_\xi}$. Let $\delta < \omega_2$ be $\sup\{\alpha_\xi : \xi < \omega_1\} = \sup\{\beta_\xi \dot{\xi} < \omega_1\}$.

We will work with the iteration $\mathbb{S}_{\delta+1}$. In the model $V^{\mathbb{S}_{\delta+1}}$ g is defined by

$$g(x) = e \circ f(e \circ H(h(x)), s_\delta),$$

where f is as in Lemma 38. By (1) it is enough to prove that given $p \in \mathbb{S}_{\delta+1}$ and $\xi < \omega_1$ there is $p' \leq p$, $p' \in \mathbb{S}_{\delta+1}$ and $\xi < \xi' < \omega_1$ such that

$$p' \Vdash \dot{f}(e \circ H(\dot{s}_{\alpha_{\xi'}}), \dot{s}_\delta) = \dot{s}_{\beta_{\xi'}}, \quad e(\dot{s}_{\beta_{\xi'}}) = \dot{y}_\xi. \tag{2}$$

Let $\xi < \xi' < \omega_1$ be such that the support of $p \restriction \delta$ is included in $\alpha_{\xi'}$, so we can assume that $p \restriction \delta \in \mathbb{S}_{\alpha_{\xi'}}$ and so $p(\delta)$ is an $\mathbb{S}_{\alpha_{\xi'}}$-name. As $supp(p_{\xi'}) \subseteq [\alpha_{\xi'}, \beta_{\xi'})$, the conditions p and $p_{\xi'}$ are compatible. Let $p'' \in \mathbb{S}_{\delta+1}$ be obtained from p by replacing 1 by $p_{\xi'}(\alpha)$ on any $\alpha \in [\alpha_{\xi'}, \beta_{\xi'})$ so that $p'' \leq p, p_{\xi'}$ and $p''(\alpha_{\xi'}) = u$. Now to obtain the desired $p' \leq p''$ we will modify p'' on $\alpha_{\xi'}, \beta_{\xi'}$ and δ using Lemmas 33 and 34.

By Lemma 33 there is an $\mathbb{S}_{\alpha_{\xi'}}$-name $\dot{q}$ such that $(p'' \restriction \alpha_{\xi'})^\frown \dot{q} \in \mathbb{S}_{\alpha_{\xi'}+1}$, $(p'' \restriction \alpha_{\xi'})^\frown \dot{q} \leq p'' \restriction (\alpha_{\xi'} + 1)$ and

$$(p'' \restriction \alpha_{\xi'})^\frown \dot{q} \Vdash \check{e} \circ \check{H}(\dot{s}_{\alpha_{\xi'}}) = \check{c}_1(p(\delta)). \tag{3}$$

Since $p''(\beta_{\xi'}) = 1$ and $y_{\xi'}$ is an $\mathbb{S}_{\beta_{\xi'}}$-name by the last part of Lemma 33 there is an $\mathbb{S}_{\beta_{\xi'}}$-name $\dot{o}$ such that $(p'' \restriction \beta_{\xi'})^\frown \dot{o} \in \mathbb{S}_{\beta_{\xi'}+1}$, $(p'' \restriction \beta_{\xi'})^\frown \dot{o} \leq p'' \restriction (\beta_{\xi'} + 1)$ and

$$(p'' \restriction \beta_{\xi'})^\frown \dot{o} \Vdash \check{e}(\dot{s}_{\beta_{\xi'}}) = \dot{y}_{\xi'}. \tag{4}$$

[1]By the stationary Δ-system lemma we will mean the following lemma: given a family $\{X_\theta \mid \theta < \omega_2\}$ of countable subsets of ω_2 there is a stationary set $A \subseteq \{\alpha \in \omega_2 \mid cf(\alpha) = \omega_1\}$ such that $\{X_\theta \mid \theta \in A\}$ forms a Δ-system. One can prove it as follows: Take regressive $f : \{\theta < \omega_2 \mid cf(\theta) = \omega_1\} \to \omega_2$ given by $f(\theta) = \sup(X_\theta \cap \theta)$. Use the pressing down lemma obtaing a stationary $A' \subseteq A$ where f is constantly equal to θ_0. By CH and the ω_1-additivity of the nonstationary ideal on ω_2 there is a stationary $A'' \subseteq A'$ such that $X_\theta \cap \theta_0$ is constant for $\theta \in A''$. Consider $g : \omega_2 \to \omega_2$ given by $g(\theta) = \sup\{\sup(X_\eta) \mid \eta \leq \theta\}$. Let $A \subseteq A''$ be the intersection of A'' with the club consisting of the ordinals bigger than θ_0 and closed under g. A is the required set.

In $V^{\mathbb{S}_{\beta_{\xi'}}}$ consider the continuous function $F_{p(\delta)}$ as defined in Definition 36. Apply Lemma 34 whose hypothesis is satisfied by Lemma 37 finding an $\mathbb{S}_\delta$-name $\dot{r}$ such that $p'' \restriction \delta^\frown \dot{r} \leq p''$ and

$$p'' \restriction \delta^\frown \dot{r} \Vdash \dot{F}_{p(\delta)}(\dot{s}_\delta) = \dot{s}_{\beta_{\xi'}}. \tag{5}$$

Define $p' \leq p$ in $\mathbb{S}_{\delta+1}$ by replacing in p''

- u by $\dot{q}$ on the $\alpha_{\xi'}$-th coordinate,
- 1 by $\dot{o}$ on the $\beta_{\xi'}$-th coordinate,
- $p(\delta)$ by $\dot{r}$ on the δ-th coordinate.

It follows that $p' \in \mathbb{S}_\delta$, $p' \leq p'' \leq p$ and $p' \restriction (\alpha_{\xi'}+1) \leq (p'' \restriction \alpha_{\xi'})^\frown \dot{q}$, $p' \restriction (\beta_{\xi'}+1) \leq (p'' \restriction \alpha_{\xi'})^\frown \dot{o}$ and $p' \restriction (\delta+1) \leq (p'' \restriction \delta)^\frown \dot{r}$.

Note that (5) and (3) gives that

$$p' \Vdash \dot{F}_{c_2(\check{e} \circ \check{H}(s_{\alpha_{\xi'}}))}(\dot{s}_\delta) = \dot{F}_{p(\delta)}(\dot{s}_\delta) = \dot{s}_{\beta_{\xi'}}.$$

which together with (4) gives the required (2). □

Remark 40. *It is proved in* [11] *that under the hypothesis of Proposition 39 there is a continuous* $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *and either there is* $X \subseteq \omega_2$ *of cardinality* ω_2 *such that* $g(x_\xi) = y_\xi$ *or* $g(y_\xi) = x_\xi$*. Note that if* $x_\xi = s_\xi$ *and* $y_\xi = s_{\xi+1}$*, where* s_ξ *denotes the* ξ*-th Sacks real for* $\xi < \omega_2$*, then there is there is no continuous* $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ *such that* $g(x_\xi) = y_\xi$ *for* ω_2*-many* $\xi < \omega_2$*. This follows from the fact that any continuous function is coded in some intermediate model.*

Theorem 41. *In the Sacks model every almost disjoint family of cardinality* ω_2 *has the* ω_1*-controlled embedding property.*

Proof. Work in the Sacks model. Let $\mathcal{A}$ be any almost disjoint family of cardinality $2^\omega = \omega_2$ and $\phi : \mathcal{A} \to 2^\omega$ any function. By Theorem 31 and Lemma 2 and Remark 3 there is a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ of cardinality ω_2 and a function $f : \mathcal{A}' \to 2^\omega$, such that the limits $x_A = \lim_{n \in A} f(n)$ exist for each $A \in \mathcal{A}'$ and are different for distinct $A \in \mathcal{A}'$. By Theorem 39 there is a subfamily $\mathcal{B} \subseteq \mathcal{A}'$ of cardinality ω_1 and a continuous $g : 2^\omega \to 2^\omega$ such that $g(x_A) = \phi(A)$ for all $A \in \mathcal{B}$. By the continuity of g we have $\phi(A) = g(x_A) = \lim_{n \in A} g(f(n))$ for all $A \in \mathcal{B}$. So $f' : \mathbb{N} \to 2^\omega$ given by $g \circ f$ witnesses the ω_1-controlled embedding property for $\mathcal{A}$ and ϕ.

□

6. An application: Abelian subalgebras of Akemann-Doner C*-algebras

The application of our combinatorial results from the previous sections presented here is related to noncommutative C*-algebras defined by C. Akemann and J. Doner in [1] with the help of almost disjoint families. Let us recall these constructions. We consider the C*-algebra M_2 of all complex 2×2 matrices with the usual operations like in linear algebra and with the linear operator norm. In this section $\mathbb{C}$ will stand for the field of complex numbers. By $\ell_\infty(M_2)$ we denote the C*-algebra of all norm bounded sequences from M_2 with the supremum norm and the coordinatewise operations. By $c_0(M_2)$ we denote the C*-subalgebra of $\ell_\infty(M_2)$ consisting of sequences of matrices whose norms converge to zero.

For $\theta \in [0, 2\pi)$ define a 2×2 complex matrix of a rank one projection by

$$p_\theta = \begin{bmatrix} \sin^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{bmatrix}.$$

Given $A \subseteq \mathbb{N}$ and $\theta \in [0, 2\pi)$ define $P_{A,\theta} \in \ell_\infty(M_2)$ by

$$P_{A,\theta}(n) = \begin{cases} 0 & n \notin A \\ p_\theta & n \in A \end{cases}$$

Given an almost disjoint family $\mathcal{A} \subseteq \wp(\mathbb{N})$ and a function $\phi : \mathcal{A} \to [0, 2\pi)$ the Akemann-Donner algebra $AD(\mathcal{A}, \phi)$ is the subalgebra of $\ell_\infty(M_2)$ generated by $c_0(M_2)$ and $\{P_{A,\phi(A)} \mid A \in \mathcal{A}\}$. As the distances between $P_{A,\theta}$ and $P_{A',\theta'}$ are at least one for infinite and distinct $A, A' \subseteq \mathbb{N}$ and any $\theta, \theta' \in [0, 2\pi)$, such algebras are nonseparable if $\mathcal{A}$ is uncountable. Clearly if $\mathcal{A}$ is uncountable and $\phi : \mathcal{A} \to [0, 2\pi)$ is constantly equal to θ, then $AD(\mathcal{A}, \phi)$ contains the nonseparable commutative C*-algebra isomorphic to $C_0(\Psi(\mathcal{A}))$ of all complex valued continuous functions on $\Psi(\mathcal{A})$ vanishing at infinity because $P_\theta^2 = P_\theta = P_\theta^*$ since it is a projection. However, as Akemann and Doner proved under CH, one can choose $\mathcal{A}$ so that for every injective $\phi : \mathcal{A} \to (0, \pi/6)$ the algebra $AD(\mathcal{A}, \phi)$ has no nonseparable commutative subalgebra. In [5] the hypothesis of CH was removed by showing that a ZFC Luzin family $\mathcal{A}$ is sufficient for this result of Akemann and Doner. We have the following two lemmas implicitly from [1, 5]:

Lemma 42. *Suppose that $\mathcal{A}$ is an almost disjoint family and $\phi : \mathcal{A} \to [0, 2\pi)$. If there is $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ and $f : \mathbb{N} \to [0, 2\pi)$ such that $\lim_{n\in B} f(n) = \phi(n)$ for every $B \in \mathcal{B}$, then $AD(\mathcal{A}, \phi)$ contains a commutative C*-subalgebra of density κ.*

Proof. First define $P_f \in \ell_\infty(M_2)$ by $P_f(n) = p_{f(n)}$. For $B \in \mathcal{B}$ define $R_B \in \ell_\infty(M_2)$ by $R_B(n) = P_f\chi_B(n)$, where χ_B is the characteristic function of B. The hypothesis about f implies that $R_B - P_{B,\phi(B)} \in c_0(M_2)$ and so R_B is in $AD(\mathcal{A}, \phi)$. The algebra generated by $\{R_B \mid B \in \mathcal{B}\}$ is commutative isomorphic to $C_0(\Psi(\mathcal{B}))$ and of density κ as required. □

Lemma 43. *Let $c \in \mathbb{R}$ be such that $\|P_0 - P_\theta\| < 1/4$ for $\theta \in [0, c]$. Suppose that $\mathcal{A}$ is an almost disjoint family and that $\phi : \mathcal{A} \to [0, c]$ is such that for no $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ there is $f : \mathbb{N} \to [0, c]$ such that $\lim_{n\in A} f(n) = \phi(A)$ for every $A \in \mathcal{B}$. Then $AD(\mathcal{A}, \phi)$ does not contain any commutative C*-subalgebra of density κ.*

Proof. This is a slight modification of an argument from [1] and modified in [5]. Let $\rho : \{P_\theta \mid \theta \in [0, 1/4]\} \to [0, 1/4]$ be defined by $\rho(P_\theta) = \theta$. ρ is a continuous map from a closed subset of the unit ball $\mathbb{B}_1$ of M_2 into $[0, 1/4]$. Use the Tietze extension theorem to find a continuous $\eta : \mathbb{B}_1 \to [0, 1/4]$ which extends ρ.

Suppose that $\mathcal{C}$ is a commutative subalgebra of $AD(\mathcal{A}, \phi)$ whose density is κ. As in [1] and [5], in a slightly different language, it follows from simultaneous diagonalization of commuting matrices that there are rank one projections $q(n) \in M_2$ such that $a(n)q(n) = q(n)a(n)$ for each $n \in \mathbb{N}$ and each $a \in \mathcal{C}$ and we may assume that $\|q(n) - P_0\|^2 \leq 1/2$ by (2.1.) of [5]. It is easy to note that for each element a of $AD(\mathcal{A}, \phi)$ the limit $lim_{n\in A} a(n)$ exists and is equal to a multiple of $p_{\phi(A)}$. The density of $\mathcal{C}$ being κ means that there is $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ such that for each $B \in \mathcal{B}$ there is $a_B \in \mathcal{C}$ such that the limit $lim_{n\in B} a_B(n)$ exists and is equal $z_B p_{\phi(B)}$ for a nonzero complex number z_B. By the compactness of the unit ball in M_2 for each infinite $B' \subseteq B$ there is an infinite $B'' \subseteq B'$ such that $\lim_{n\in B''} q(n) = q'$ exists, and so it needs to be a rank one projection which commutes with $lim_{n\in A} a_B(n)$ which is $z_B p_{\phi(B)}$, so $p_{\phi(B)}$ and q' commute but $\|q' - p_{\phi(B)}\| \leq 1/\sqrt{2} + 1/4 < 1$ and so $q' = p_{\phi(B)}$ (see e.g. Lemma 3 of [5]). This means that actually $\lim_{n\in B} q(n)$ exists and is equal to $p_{\phi(B)}$ for each $B \in \mathcal{B}$. By the continuity of η we have $\lim_{n\in B} \eta(q(n)) = \eta(p_{\phi(B)}) = \phi(B)$. Define $f : \mathbb{N} \to [0, 1/4]$ by $f(n) = \eta(q(n))$. So $\lim_{n\in B} f(n) = \phi(B)$ for every $B \in \mathcal{B}$ contradicting the hypothesis on $\mathcal{A}$. □

As corollaries we obtain:

Theorem 44. *In ZFC, for every almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ there is $\phi : \mathcal{A} \to [0, 2\pi)$ such that the Akemann-Doner C*-algebra $AD(\mathcal{A}, \phi)$ of density $\mathfrak{c}$ has no commutative subalgebras of density $\mathfrak{c}$.*

Proof. Fix an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$. By Theorem 6 there is $\phi : \mathcal{A} \to \mathbb{R}$ such that for no $\mathcal{B} \subseteq \mathcal{A}$ of cardinality $\mathfrak{c}$ there is $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. By applying a continuous injective mapping we may assume that $\mathbb{R}$ is replaced by $[0, c]$, where $c \in \mathbb{R}$ is like in Lemma 43. Now Lemma 43 implies that $AD(\mathcal{A}, \phi)$ has no commutative subalgebras of density $\mathfrak{c}$.

□

Theorem 45. *It is consistent that every Akemann-Doner algebra of density $\mathfrak{c}$ contains a nonseparable commutative subalgebra.*

Proof. We claim that the above statement holds in the Sacks model. By Theorem 41 given any almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ and a functions $\phi : \mathcal{A} \to \mathbb{R}$ there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\lim_{n\in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. It follows form Lemma 42 that $AD(\mathcal{A}, \phi)$ contains a nonseparable commutative subalgebra. □

Theorem 46. *Let $c \in \mathbb{R}$ be such that $\|P_0 - P_\theta\| < 1/4$ for $\theta \in [0, c]$. It is consistent with the negation of* CH *that for every almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ there is $\phi : \mathcal{A} \to [0, c]$ such that the Akemann-Doner algebra $AD(\mathcal{A}, \phi)$ of density $\mathfrak{c}$ has no nonseparable commutative subalgebra.*

Proof. Work in the Cohen model. Fix an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$. By Theorem 18 there is $\phi : \mathcal{A} \to \mathbb{R}$ such that for no uncountable $\mathcal{B} \subseteq \mathcal{A}$ there is $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. By applying a continuois mapping we may assume that the interval $\mathbb{R}$ is replaced by $[0, c]$, where $c \in \mathbb{R}$ is like in Lemma 43. Now Lemma 43 implies that $AD(\mathcal{A}, \phi)$ has no commutative nonseparable subalgebras. □

Theorem 47. *Let $c \in \mathbb{R}$ be such that $\|P_0 - P_\theta\| < 1/4$ for $\theta \in [0, c]$. It is consistent with the negation of* CH *that there is an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ such that for every $\phi : \mathcal{A} \to [0, c]$ the Akemann-Doner algebra $AD(\mathcal{A}, \phi)$ of density $\mathfrak{c}$ has no nonseparable commutative subalgebra.*

Proof. Work in the Cohen model. Let $\mathcal{A}$ be an almost disjoint family of cardinality $\mathfrak{c}$ from Theorem 17. By Theorem 17 for no $\phi : \mathcal{A} \to \mathbb{R}$ there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ and $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. Now Lemma 43 implies that $AD(\mathcal{A}, \phi)$ has no commutative nonseparable subalgebras. □

These results complete earlier result of [5] that there is in ZFC an almost disjoint family $\mathcal{A}$ (any inseparable family) such that for every $\phi : \mathcal{A} \to [0, c)$ the Akemann-Doner algebra of density ω_1 has no nonseparable commutative subalgebra.

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