OPEN PROBLEMS ON COUNTABLE DENSE HOMOGENEITY

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Abstract. We survey recent development in research on countable dense homogeneity with special emphasis on open problems.

1. Introduction

Unless stated otherwise, all spaces under discussion are separable and metrizable.

A separable space X is *countable dense homogeneous* (CDH) if, given any two countable dense subsets D and E of X, there is a homeomorphism $f\colon X \to X$ such that $f[D] = E$. This is a classical notion that can be traced back to the works of Cantor [16], Brouwer [14], and Fréchet [36]. Examples of CDH-spaces are the Euclidean spaces ([14, 36]), the Hilbert cube ([34]) and the Cantor set. In fact, every strongly locally homogeneous (SLH[1]) Polish space is CDH, as was shown by Bessaga and Pełczyński [13] (see also [12, 4, 32, 37]).

The term *countable dense homogeneous* was coined by Bennett in his seminal paper [12] where the fundamental themes of the subject were introduced by either proving a theorem, or by asking an insightful question: its relation with other notions of homogeneity such as strong local homogeneity and n-homogeneity, issues dealing with products, open subspaces and connectedness. The subject was subsequently popularized and advanced greatly by the efforts of Fitzpatrick [25, 26, 27, 28, 29, 30, 31].

Several surveys on homogeneity and countable dense homogeneity have been written [31, 28, 6]. In [28], Fitzpatrick and Zhou presented the most interesting open problems on countable dense homogeneity at the time. Now we are almost 25 years later in time. Since there is ongoing interest in the subject, it seems appropriate to see what has remained of their problems and which new ones have emerged. As Fitzpatrick and Zhou remarked, the problems range in flavor from the geometric to the set theoretic. This remains very true of the subject today.

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[1]Recall that a space X is *strongly locally homogeneous* [33] if it has a basis $\mathscr{B}$ of open sets such that for every $U \in \mathscr{B}$ and every $x, y \in U$ there is a autohomeomorphism h of X such that $h(x) = y$ and h restricts to the identity on $X \setminus U$.

2. Fitzpatrick and Zhou's Problems

First we recall the problems from [28] and review the progress made.

Problem 1. *Is every connected* CDH *metric space* SLH*?*

This question was implicitly asked in van Mill [61]. In 1990, there were two known examples of spaces that are CDH but not SLH. The first one, due to Fitzpatrick and Zhou [27], is Hausdorff but not regular, and the second one, due to Simon and Watson [91] is regular but not completely regular. The problem was finally solved by van Mill [64]. He showed that there is a connected, Polish, CDH-space X that is not SLH. In fact, any homeomorphism of X that is the identity on some non-empty open subset of X is the identity on all of X.

An important special case of Problem 1 remains unsolved. Ungar [87, 86] proved that that CDH metric continua are n-homogeneous[2] for all n. Kennedy [49] showed that a 2-homogeneous metric continuum X must be SLH, provided that X admits a nontrivial homeomorphism that is the identity on some nonempty open set. Whether every 2-homogeneous metric continuum admits such a homeomorphism remains an open problem.

Problem 1'. *Is every* CDH *metric continuum* SLH*?*

It was claimed by Ungar [87] that every dense open subset of a locally compact CDH-space is again CDH. The proof is, however, incomplete. This prompted Fitzpatrick and Zhou to ask the following:

Problem 2. *If X is* CDH *and metric and U is open in X, must U be* CDH*?*

And more specifically:

Problem 2'. *If X is connected,* CDH*, and metric, and U is an open, connected set in X, must U be homogeneous? If U is homogeneous, is it necessarily* CDH*?*

Problem 2 and the first part of Problem 2' were answered by van Mill [65]: there is a CDH Polish space with a dense connected open subset that is rigid. Here a space is called *rigid* if the identity function is the only autohomeomorphism. The question whether every dense open subset of a locally compact CDH-space is again CDH remains unsolved.

There are more problems which deal with connectedness issues.

Problem 3. *If X is a* CDH*, connected, Polish space, must X be locally connected?*

This is known to have an affirmative answer in the case when X is also locally compact (Fitzpatrick [25]), but the only examples we know of CDH connected metric spaces are also locally connected. Partial results were recently obtained by van Mill in [70].

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[2]For a space X we let $\mathscr{H}(X)$ denote its group of homeomorphisms. According to Burgess [15], a space X is *n-homogeneous* if given two sets $F, E \subseteq X$ each having exactly n-many elements, there is a homeomorphism $h \in \mathscr{H}(X)$ such that $h[F] = E$. The space X is *strongly n-homogeneous* if given two sequences $\{a_i : i < n\}$, $\{b_i : i < n\}$ of points in X both of length n, there is a homeomorphism $h \in \mathscr{H}(X)$ such that $h(a_i) = b_i$ for every $i < n$.

Problem 4. *For which 0-dimensional subsets X of $\mathbb{R}$ is X^ω homogeneous?* CDH*?*

The first part of this problem was answered by Lawrence [53], and further extended by Dow and Pearl [20] who proved that X^ω is homogeneous for every zero-dimensional first countable space X. See also Motorov [75] and van Engelen [22] for earlier partial results.

The second part of the question is still being investigated though important partial results were obtained: First Fitzpatrick and Zhou in [29] showed that $\mathbb{Q}^\omega$ is not CDH (here $\mathbb{Q}$ denotes the space of rational numbers) and then Hrušák and Zamora-Avilés [45] extended their result by showing that every space X such that X^ω is CDH is a Baire space. They solved the problem for Borel zero-dimensional spaces by showing that a Borel CDH space must be Polish (see also [67, 43]), and, in particular, for Borel $X \subseteq 2^\omega$, the space X^ω is CDH if and only if X is G_δ. This raised the question whether there can be a non-G_δ set X such that X^ω is CDH. This problem stimulated research on topological properties of filters (viewed as subspaces of 2^ω) treated in a series of papers [57, 40, 42, 51, 78] where it was established that for a filter $\mathscr{F}$ on ω the following properties are equivalent: (1) $\mathscr{F}$ is CDH, (2) $\mathscr{F}^\omega$ is CDH, and (3) $\mathscr{F}$ is a non-meager P-filter[3]. Quite recently Medini in [55] constructed a ZFC example of a set $X \subseteq 2^\omega$ which is not Polish yet X^ω is CDH. It is interesting to note that his example is consistently analytic, while it is also consistent that there are no analytic examples.

Problem 5. *Is the ω^{th} power of the Niemytzki plane homogeneous?*

Observe that the Niemytzki plane is not metrizable. This problem is open.

Problem 6. *Does there exist a* CDH *metric space that is not Polish?*

Problem 6'. *Is there an absolute example of a* CDH *metric space of cardinality ω_1?*

Both Problems 6 and 6' were solved in the affirmative by Farah, Hrušák and Martínez Ranero [24]. They proved that there exists a CDH-subset of $\mathbb{R}$ of size $\aleph_1$. Their example is a so-called λ-*set*, i.e., a space in which every countable set is G_δ. Note that every countable CDH-space is discrete, hence $\aleph_1$ is the first cardinal where anything of CDH-interest can happen. Since $\mathbb{R}$ is CDH and has size $\mathfrak{c}$, it is an interesting open problem what can happen for cardinals greater than $\aleph_1$ but below $\mathfrak{c}$. It was shown recently in Hernandez-Gutiérrez, Hrušák and van Mill [41] that for every cardinal κ such that $\omega_1 \leq \kappa \leq \mathfrak{b}$[4] there exists a CDH subset of $\mathbb{R}$ of size κ. More about this in the next section.

To recapitulate: Problems 1, 2, 2' (first part), 4 (first part), 6 and 6' are solved, while parts of problems 2', and 4', and problems 1', 3, and 5 remain open. We shall return to some of these questions in more detail in what follows.

[3]Recall that a filter $\mathscr{F}$ is a *P-filter* if given a sequence $\{F_n : n \in \omega\}$ of elements of $\mathscr{F}$ there is an $F \in \mathscr{F}$ such that $F \setminus F_n$ is finite for every $n \in \omega$.

It is known that $\mathscr{F}^\omega$ is homeomorphic to a non-meager P-filter whenever $\mathscr{F}$ is a non-meager P-filter (Hernández-Gutiérrez and Hrušák [40]).

It is a major open problem in set-theory whether non-meager P-filters exist [46]. They do exist in all known models of set theory, and their possible non-existence implies the existence of measurable cardinals.

[4]Recall that $\mathfrak{b} = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \omega^\omega \ \forall g \in \omega^\omega \ \exists f \in \mathscr{F} \ \exists^\infty n \in \omega \ f(n) > g(n)\}$ is a combinatorially defined cardinal number whose value may be any regular cardinal between ω_1 and $\mathfrak{c}$.

3. New list of some new and some old problems

3.1. CDH and related notions of homogeneity.

A topological space X is *homogeneous*[5] if for every $x, y \in X$ there is an $h \in \mathscr{H}(X)$ such that $h(x) = y$. Every CDH space is a topological sum of clopen homogeneous subspaces, in particular, every connected CDH space is homogeneous [12, 26]. It was noted by van Mill in [61] that for zero-dimensional spaces one can show more, as every homogeneous such space is SLH (or, equivalently, representable [8]). As mentioned in the introduction, all complete SLH-spaces are CDH. Now this fails for non-complete spaces, e.g. $\mathbb{Q}^\omega$ is SLH but not CDH. However, the connection becomes more interesting if one considers connected spaces (see Problem 1). van Mill [64] showed that there is a connected, Polish, CDH-space X that is not SLH, while the problem remains open for continua:

Problem 7. *Is every (locally) compact connected* CDH*-space* SLH*?*

We do not know to whom this problem belongs. By the result of Kennedy [49], for continua this is equivalent to asking whether every 2-homogeneous metric continuum admits a homeomorphism that is the identity on some non-empty open set. On the other hand, there are even Baire SLH connected spaces which are not CDH [61, 79].

Making use of the celebrated theorem of Effros [21], Ungar [87, 86] (see also [63, 66, 44]) proved a fundamental result on CDH spaces by showing that a sufficiently connected (no finite set separates) locally compact space X is CDH if and only if it is n-homogeneous for all $n \in \omega$ if and only if it is strongly n-homogeneous for all $n \in \omega$. van Mill in [69] extended his result by proving that a connected, CDH-space is n-homogeneous for every n, and strongly 2-homogeneous when locally connected[6]. He also presented an example of a connected CDH-space which is not strongly 2-homogeneous.

Problem 8 (van Mill [69])**.** *Is there a connected* CDH*-space which is not strongly* 2*-homogeneous?*

Moreover, van Mill in [68] showed that Ungar's theorem works level-by-level, i.e. if a space X is CDH and no set of size $n-1$ separates it, then X is strongly n-homogeneous, and more importantly, showed that there is a Polish space that is strongly n-homogeneous for every $n \in \omega$ which is not CDH.

3.2. Open subsets of CDH spaces.

Ungar's theorem was originally stated as a corollary to the fact that every open dense subset of a locally compact CDH-space is CDH the proof of which is incomplete and remains open:

Problem 9 (Ungar)**.** *Is there a locally compact* CDH*-space with a dense open non-CDH-subspace?*

In [65] van Mill constructs a Polish space X (a convex subspace of ℓ^2) with a dense open rigid subspace, hence showing that there are Polish such spaces.

[5]The notion was introduced by Sierpiński in [82].

[6]Note that $\mathbb{R}$ is CDH but not strongly 3-homogeneous.

Several questions considering open subsets of CDH spaces were formulated in [91], one interesting problem persists:

Problem 10 (Watson-Simon [91]). *Are open subsets of* CDH *continua* CDH*? (homogeneous)?*

3.3. **Baire and meager-in-itself** CDH**-spaces.** It was noted by Fitzpatrick and Zhou in [29] that every CDH-space can be decomposed into two clopen parts, one Baire and the other meager-in-itself, and that a metric meager-in-itself CDH-space is is a λ-set[7]. Hernández-Gutiérrez, Hrušák and van Mill in [41] showed that there is a meager-in-itself metric space of size κ if and only of there is a λ-set of size κ. In particular, for every cardinal κ such that $\omega_1 \leq \kappa \leq \mathfrak{b}$ there exists a CDH subset of $\mathbb{R}$ of size κ. It is consistent with the continuum arbitrarily large that there is a CDH set of reals of size κ for every $\kappa < \mathfrak{c}$, while it is also consistent with the continuum arbitrarily large that the only sizes of CDH metric spaces are ω_1 and $\mathfrak{c}$. These results were further extended by Medvedev in [59].

Very recently Hrušák and van Mill [44] showed that the existence of a connected CDH metric space which is meager-in-itself is independent of ZFC. The construction is a nice combination of techniques of infinite-dimensional topology and set-theory, which, however, left them wondering about the following:

Problem 11. *Is there consistently a connected* CDH $X \subseteq \mathbb{R}^2$ *which is meager-in-itself?*

For more on countable dense homogeneity of subspaces of the plane see [38, 80, 79]

While the possible sizes of meager-in-themselves metric spaces have been determined, the same is not true for Baire CDH spaces.

Problem 12. *Is there consistently a crowded* CDH *set* $X \subseteq \mathbb{R}$ *of size less than* $\mathfrak{c}$ *which is Baire?*

It was shown recently in Hernández-Gutiérrez, Hrušák and van Mill [41] that there is CDH-subset of $\mathbb{R}$ of size $\mathfrak{c}$ which is Baire but not Polish. Consistent examples were before constructed by Fitzpatrick and Zhou in [29] and by Baldwin and Beaudoin [7]. In fact, their examples were Bernstein sets[8].

Problem 13 (Baldwin and Beaudoin [7]). *Is there a Bernstein subset of* $\mathbb{R}$ *which is* CDH*?*

3.4. **Products of** CDH**-spaces and filters.** Products of CDH-spaces need not be CDH. It was first observed by Kuperberg, Kuperberg and Transue in [52] that the square of the Menger curve M is not even 2-homogeneous, while Bennett [12], building on work of Anderson [2, 3], showed that M is CDH. A simple example is $\mathbb{R} \times 2^\omega$, which is obviously not 2-homogeneous. To see that it is not CDH, consider a countable dense subset which intersects some component of X in exacly one point, and one that lacks this property.

[7]A space X is called a *λ-set* if every countable subset of X is G_δ in X. A subset X of a Polish space Y is a *λ'-set* if every countable modification of X in Y is a λ-set, i.e. if every countable subset A of Y is G_δ in $X \cup A$.

[8]A subset X of $\mathbb{R}$ is called *Bernstein* if X and $\mathbb{R} \setminus X$ intersect every Cantor set in $\mathbb{R}$.

The situation changes if one considers only zero-dimensional spaces. In [45] it is shown that Borel zero-dimensional CDH spaces are productively CDH. Using $\mathsf{MA}(\sigma-\text{centered})$, Medini in [56] constructed a homogeneous CDH Bernstein set whose square is not CDH.

Problem 14 (Medini [56])**.** *Is there (in* ZFC*) a zero-dimensional* CDH *space* X *such that* X^2 *is not* CDH*?*

Observe that zero-dimensionality is essential in this problem. To see this, just consider the topological sum of the reals and the Cantor set. Hernández-Hrušák-van Mill [41] noted that there is a zero-dimensional CDH $X \subseteq \mathbb{R}$ such that X^ω is not CDH. However, the following seems to be open, even omitting the requirement of zero-dimensionality.

Problem 15 (Medini [56])**.** *For which* $n \in \omega$ *is there a* CDH *space* X *such that* X^i *is* CDH *for all* $i \leq n$ *while* X^{n+1} *is not* CDH*?*

On the other hand, there are spaces X which are not CDH while X^2 is (see [60]).

Problem 16 (Medini [56])**.** *For which* $n \in \omega$ *is there a* CDH *space* X *such that* X^i *is not* CDH *for all* $i \leq n$ *while* X^{n+1} *is* CDH*? Can* X *be a continuum?*

Problem 17 (Medini [56])**.** *Is there a zero-dimensional space* X *that is not* CDH *while* X^2 *is* CDH*? Can* X *be rigid?*

Medini [56] also raises the question whether any of such spaces can be definable (e.g. analytic or co-analytic)?

The question, whether there is a subset X of $\mathbb{R}$ which is not Polish while X^ω is CDH, was raised by Hrušák and Zamora in [45]. The problem was first solved assuming MA by Medini and Milovich [57] who proved that under MA there is a free ultrafilter $\mathscr{U}$ on ω such that $\mathscr{U}^\omega$ is CDH. This was extended by Hernández and Hrušák [40] by showing that for every every non-meager P-filter $\mathscr{F}$ both $\mathscr{F}$ and $\mathscr{F}^\omega$ are CDH. Finally Kunen, Medini and Zdomskyy [51] showed that the reverse implication also holds. The following remains a major open problem in combinatorial set theory:

Problem 18 (see [46])**.** *Is there a non-meager P-filter?*

Recently Medini in [55] constructed a non-Polish set of reals X such that X^ω is CDH. Assuming $\mathsf{MA}+\neg\mathsf{CH}+\ \omega_1^V = \omega_1^L$ his example can be made analytic, while ZFC proves that there are no co-analytic examples and determinacy implies there are no analytic examples. He, in fact, showed that X^ω is CDH for every $X \subseteq 2^\omega$ such that $2^\omega \setminus X$ is a λ'- set in 2^ω.

While both examples (Medini's and a non-meager P-filter) are very differently placed in 2^ω - one is possibly homeomorphic to its complement in 2^ω, while the complement of the other contains no perfect set - they both contain many copies of 2^ω. In fact, this is a necessary requirement as Hernández [39] showed that any crowded space X such that X^ω is CDH must contain a copy of the Cantor set. An interesting consequence of this result is that X^ω is never CDH when X is a Bernstein set. In particular, X being completely Baire and zero-dimensional is not a sufficient condition for the countable dense homogeneity of X^ω. We conclude this discussion by noting that Medini and Zdomskyy in [58] studied properties which are candidates for characterizing when X^ω is CDH.

Countable dense homogeneity of products of manifolds with boundary was studied by Yang in [92].

3.5. **Baumgartner's Axiom and κ-CDH.** The celebrated theorem of Baumgartner [11] shows that (assuming the Proper Forcing Axiom PFA) every two $\aleph_1$-dense[9] sets of reals are order isomorphic[10]. The proof of this influential result in set theory, which was for instance an important part of Moore's proof [71] of the consistent existence of a five element basis for uncountable linear orders, actually proves more: It shows that given two families $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ of pairwise disjoint countable dense sets there is a homeomorphism $h \in \mathscr{H}(\mathbb{R})$ such that $h[A_\alpha] = B_\alpha$ for every $\alpha < \omega_1$.

Inspired by this fact, let us call a space X κ-CDH if given two families $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ of pairwise disjoint countable dense sets there is a homeomorphism $h \in \mathscr{H}(X)$ such that $h[A_\alpha] = B_\alpha$ for every $\alpha < \kappa$. Following [85] we denote by $BA(X, \kappa)$ the formally weaker property that given two κ-dense subsets A, B of a space X there is a homeomorphism $h \in \mathscr{H}(X)$ such that $h[A] = B$.

It was shown by Abraham and Shelah in [1] that Baumgartner's result $BA(\mathbb{R}, \aleph_1)$ is independent of MA. The famous problem of Baumgartner, whether $BA(\mathbb{R}, \aleph_2)$ is consistent, has attracted a lot of attention recently [72] and apparently has been solved recently in the affirmative by Neeman [76] who is in the process of writing its very complex proof. It is very much an open problem, what happens for cardinals above $\aleph_2$.

Problem 19 (Baumgartner [11]). *Is $BA(\mathbb{R}, \aleph_3)$ consistent?*

It turns out that dimension 1 is by far the most complicated case here. Steprāns and Watson in [85] showed that, assuming $\mathsf{MA}_\kappa(\sigma\text{−centered})$ (1) every n-dimensional manifold, for $n > 1$, is κ-CDH, and consequently (2) $\mathbb{R}^n$ for $n > 1$, is κ-CDH, while Baldwin and Beaudoin [7] proved that under the same hypothesis 2^ω is κ-CDH.

The properties $BA(X, \kappa)$ and the space X being κ-CDH are different in general, as was shown by Steprāns and Watson [85, p. 309]. They in fact proved the interesting fact that "$\mathbb{R}$ is ω_1-CDH" is false. This inspires the following open problems:

Problem 20.

(1) *Does $\mathbb{R}^n$ being κ-CDH imply $\mathbb{R}^m$ is κ-CDH for $n, m > 1$?*
(2) *Does $\mathbb{R}$ being κ-CDH imply $\mathbb{R}^n$ is κ-CDH for all $n > 1$ and $\kappa > \aleph_1$?*
(3) *Does $\mathbb{R}^n$ being κ-CDH imply $\mathbb{R}^n$ is λ-CDH for $\kappa > \lambda$?*

It can be easily deduced from the Bessaga and Pełczyński [13] theorem that every locally compact SLH-space (such as $\mathbb{R}^n$) is ω-CDH. What is not clear what happens if the space is not locally compact. The following problems were asked in a private conversation by Matty Rubin:

Problem 21. *Is every Polish CDH space ω-CDH?*

[9] A subset A of a space X is κ-dense if $|A \cap U| = \kappa$ for every non-empty open $U \subseteq X$.

[10] See also his paper [10] preceding [11]. In that paper the consistency of every two $\aleph_1$-dense sets of reals are isomorphic with MA is proved. Hence this statement, unlike PFA, does not require large cardinals for its consistency.

Problem 22 (Rubin)**.** *Is there a (Polish) space X which is* CDH *but not* 2-CDH*?*

An interesting reletively recent paper on the subject is Kunen [50], where the result of Baumgartner is improved to get absolute continuity of the map and its inverse.

3.6. **Nearly CDH spaces.** Hrušák and van Mill in [43] considered the following weakening of countable dense homogeneity: Call a topological space X $\frac{1}{\kappa}$-CDH if there are exactly κ-many *types* of countable dense subsets of X, i.e. κ-many equivalence classes of the family of all countable dense sets of X modulo an equivalence relation defined by $D \simeq D'$ if and only if there is an $h \in \mathscr{H}(X)$ such that $h[D] = D'$.

The authors were unaware of work on the topic by Kennedy [47, 48] who has much earlier considered the same kind of issues. They would like to use this as an opportunity to give credit where credit is due. In [47] it is shown that a space which is $\frac{1}{n}$-CDH for some $n \in \omega$ has only countably many orbits, all but finitely many of them open, while in [48] the same result is extended to $\frac{1}{\omega}$-CDH spaces, and it is proved there that a homogeneous continuum with at most countably many $\simeq$-classes is CDH. These results were, in retrospect, extended in [43] published more than 30 years later: A locally compact space with at most countably many $\simeq$-classes has an open dense CDH subspace whose complement is countable and fixed by all autohomeomorphisms of X. If, moreover, the space has only n-many $\simeq$-classes then the complement of the open dense CDH subspace fixed by all autohomeomorphisms of X has size at most $n-1$. It is not clear whether the same is true for all Polish spaces:

Problem 23. *Does every Polish $\frac{1}{\omega}$-CDH space contain a dense open CDH subspace?*

Or, say,

Problem 24. *Is there a $\frac{1}{2}$-CDH Polish space without a dense open CDH subspace?*

As an extension of a theorem of Hrušák and Zamora-Áviles [45], it is shown in [43] that every Borel space which is not Polish is $\frac{1}{\mathfrak{c}}$-CDH. It is also shown in [43] that every Polish space has either at most countably many $\simeq$-equivalence classes or $\mathfrak{c}$-many (and for each such possibility there is a locally compact example), or possibly $\aleph_1$-many. As of now it is not known whether there is such a Polish (or even locally compact) space, a problem equivalent to the Topological Vaught's Conjecture [88, 74]:

Problem 25. *Is there a (Polish, locally compact) space which is $\frac{1}{\omega_1}$-CDH?*

Under $\mathfrak{u} < \mathfrak{g}$ (which holds in Miller's model), one can prove the following dichotemy: every filter is either CDH or $\frac{1}{\mathfrak{c}}$-CDH (see Theorem 23 in Kunen, Medini and Zdomskyy [51]). But it is not known whether this holds in ZFC.

Problem 26 (Kunen, Medini and Zdomskyy [51, Question 5])**.** *Is it consistent that there exists a $\frac{1}{\kappa}$-CDH filter for a cardinal κ such that $1 < \kappa < \mathfrak{c}$?*

The next problem is rather *ad hoc.* It asks whether there are $\frac{1}{2}$-CDH spaces of an essentially different nature than the locally compact ones:

Problem 27. *Is there a homogeneous $\frac{1}{2}$-CDH subset of $\mathbb{R}$?*

We conclude this section by including a question of Medini [56]:

Problem 28 (Medini [56]). *For which cardinals κ is there a zero-dimensional* CDH *space X such that X^2 is $\frac{1}{\kappa}$-*CDH*?*

3.7. **Countable dense homogeneity with special maps.** In the same 1895 volume of Mathematische Annalen where Cantor published the first result on CDH-spaces - his topological characterization of $\mathbb{R}$ - appeared the paper of Stäckel [83] in which it is shown that given two countable dense subsets of the complex plane $\mathbb{C}$ there is an analytic function h sending one into the other. One of the early results, due to Franklin [35] showed that Cantor's argument can be strengthened to show that the witnessing homeomorphism is an analytic function. The early research concentrated on extensions of Stäckel's results using methods of complex analysis (see [91]: Erdős [23] asked if given two countable dense subsets of $\mathbb{C}$ there is an entire function sending one onto the other. This was answered in the affirmative by Maurer [54], while Barth and Schneider [9] extended his result by showing that such a function can also map the complement of one of the dense sets to the complement of the other. Sato-Rankin [81] and Nienhuys and Thiemann [77] independently showed that given two countable dense subsets of the real there is an entire transcendental function (in $\mathbb{C}$) the restriction of which to $\mathbb{R}$ is a homemomorphism mapping one of the dense sets to the other. Dobrowolski [18] and Morayne [73] conluded this line of research by proving that for any two countable and dense subsets A, B of $\mathbb{C}^n$ for $n \geq 2$ there exists an analytic measure preserving diffeomorphism $F\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $F[A] = B$. Again the case $n = 1$ is singular, as the theorem fails there - the only analytic diffeomorphisms from $\mathbb{C}$ onto $\mathbb{C}$ are linear. Perhaps the final result in this line of research was provided by Dobrowolski [19] who showed that given a (separable) Banach space X, a bijection $\Phi\colon E \to E$ with a nowhere dense set of fixed ponts, and A, B countable dense subsets of E there is a real-analytic diffeomorphism $h\colon E \to E$ such that $h^{-1} \circ \Phi \circ h[A] = B$.

Zamora [93] and Dijkstra [17] studied countable dense homogeneity of metric spaces with isometries and "almost isometries", i.e functions such that for some arbitrarily small $\varepsilon > 0$

$$1 - \varepsilon < \frac{d(h(x), h(y))}{d(x, y)} < 1 + \varepsilon.$$

They call a metric spaces X *Lipschitz countable dense homogeneous (Lipschitz-*CDH*)* if given $\varepsilon > 0$ and A, B countable dense subsets of E there is an ε-almost isometry h such that $h[A] = B$. Zamora [93] showed that the Euclidean spaces are Lipschitz-CDH. Dijkstra [17] extended her result greatly by showing (1) every Lipschitz-SLH space[11] is Lipschitz-CDH and (2) proving that every (separable) Banach space is Lipschitz-SLH.

Problem 29 (Dijkstra [17]). *Is the Hilbert cube with its standard metric Lipschitz-CDH?*

3.8. **Non-metrizable CDH spaces.** *In this section, all spaces are Tychonoff.*

While most of the research on CDH-spaces concentrates on separable metric ones, a fair portion of it deals with general separable spaces. Arhangel'skii and van Mill in [5] showed

[11] Lipschitz-SLH is the obvious modification of SLH, requiring that the corresponding maps be almost isometries.

among other things that every CDH space has size at most $\mathfrak{c}$ and that, assuming CH every compact CDH space is first countable. In the process, they realized that at the moment no ZFC example of a compact non-metrizable CDH space was known (there were several consistency results, in particular, it was shown by Steprāns and Zhou [84] that both 2^{ω_1} and $[0,1]^{\omega_1}$ are CDH under $\mathsf{MA}+\neg\mathsf{CH}$)

It was shown afterwards by Hernández-Gutiérrez, Hrušák and van Mill [41] that for every cardinal κ such that there is a λ'-set of size κ there exists a compact CDH-space of weight κ, hence, in particular, there is a compact CDH-space of weight ω_1 in ZFC. This, however, leaves the following problem open:

Problem 30 (Hernández-Hrušák-van Mill [41])**.** *Is there (in* ZFC*) a compact* CDH *space of weight* $\mathfrak{c}$*?*

Arhangel'skii and van Mill in [5] showed that the *double arrow* space is not CDH. It was believed at the time that a trivial modification of it may be CDH. These hopes were trashed by Hernández [39] who showed that none of the trivial modifications is CDH, and also that the double arrow space itself is quite far from being CDH, as it is in fact $\frac{1}{\mathfrak{c}}$-CDH. Inspired by his results we ask:

Problem 31. *Is there a compact* CDH*-space which does not contain a copy of the Cantor set?*

Also the space constructed in [41] is zero-dimensional.

Problem 32 (Hernández-Hrušák-van Mill [41])**.** *Is there a non-metrizable* CDH *continuum in* ZFC*?*

Steprāns and Zhou [84] studied countable dense homogeneity of separable non-metrizable manifolds. In particular, they showed that any separable manifold of weight $<\mathfrak{b}$ is CDH, while there is a non-CDH separable manifold of weight $\mathfrak{c}$. Watson in [89] showed that consistently there is a separable, non-CDH manifold of weight $<\mathfrak{c}$. The following question is a bold reformulation of a question of Watson in [90]:

Problem 33. *Is* $\mathfrak{b}$ *the minimal weight of a separable, non-*CDH *manifold?*

If not, is there (as Watson asks) a nice combinatorial characterization of the minimal weight of a separable, non-CDH manifold?

References

[1] U. Avraham and S. Shelah, *Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic*, Israel J. Math. **38** (1981), no. 1-2, 161–176.
[2] R. D. Anderson, *A characterization of the universal curve and a proof of its homogeneity*, Ann. of Math. **67** (1958) 313–324.
[3] R. D. Anderson, *One dimensional continuous curves and a homogeneity theorem*, Ann. of Math. **68** (1958) 1–16.
[4] R. D. Anderson, D. W. Curtis and J. van Mill, *A fake topological Hilbert space*, Trans. Amer. Math. Soc. **272**, (1982), 311–321.

[5] A. V. Arhangel'skii and J. van Mill, *On the cardinality of countable dense homogeneous spaces*, Proc. Amer. Math. Soc. **141**, (2013), 4031–4038.
[6] A. V. Arhangel'skii and J. van Mill, *Topological homogeneity*, in Recent Progress in General Topology III. (K. P. Hart, J. van Mill, and P. Simon, eds.), Atlantis Press, Paris, 2014, 1–68.
[7] S. Baldwin, R. E. Beaudoin, *Countable dense homogeneous spaces under Martin's axiom*, Israel J. Math. **65 no. 2**, (1989), 153–164.
[8] J. W. Bales, *Representable and strongly locally homogeneous spaces and strongly n-homogeneous spaces*, Houston J. Math. **2** (1976), no. 3, 315–327.
[9] K. F. Barth, W. J. Schneider, *Entire functions mapping arbitrary countable dense sets and their complements onto each other*, J. London Math. Soc. **4** (1971/72), 482–488.
[10] J. E. Baumgartner, *All $\aleph_1$-dense sets of reals can be isomorphic*, Fund. Math. **79** (1973), 101–106.
[11] J. E. Baumgartner, *Applications of the proper forcing axiom*, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984), 913–955.
[12] R. B. Bennett, *Countable dense homogeneous spaces*, Fund. Math. **74** (1972) 189–194.
[13] C. Bessaga and A. Pełczyński, *The estimated extension theorem homogeneous collections and skeletons, and their application to the topological classification of linear metric spaces and convex sets*, Fund. Math. **69** (1970), 153–190.
[14] L. E. J. Brouwer, *Some remarks on the coherence type η*, Proc. Akad. Amsterdam **15** (1913), 1256-1263.
[15] C. E. Burgess, *Some theorems on n-homogeneous continua*, Proc. Amer. Math. Soc. **5**, (1954) 136–143.
[16] G. Cantor, *Baiträge zur Begründung der transfiniten Mengenlehre*, Math. Ann. **46** (1895) 481–512.
[17] J. J. Dijkstra, *Homogeneity properties with isometries and Lipschitz functions*, Rocky Mountain J. Math. **40** (2010), no. 5, 1505–1525.
[18] T. Dobrowolski, *On smooth countable dense homogeneity*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **24** (1976), 627–634.
[19] T. Dobrowolski, *A conjugated version of the countable dense homogeneity*, Topology Appl. **156** (2009), no. 4, 795–802.
[20] A. Dow and E. Pearl, *Homogeneity in powers of zero-dimensional first-countable spaces*, Proc. Amer. Math. Soc. **125** (1997), 2503–2510.
[21] E. G. Effros, *Transformation groups and C^*-algebras*, Ann. of Math. **81** (1965), 38–55.
[22] F. van Engelen, *On the homogeneity of infinite products*, Topology Proc. **17** (1992), 303–315.
[23] P. Erdős, *Some unsolved problems*, Michigan Math. J. **4** (1957), 291–300.
[24] I. Farah, M. Hrušák, and C. Martínez Ranero, *A countable dense homogeneous set of reals of size $\aleph_1$*, Fund. Math. **186** (2005), 71–77.
[25] B. Fitzpatrick, Jr., *A note on countable dense homogeneity*, Fund. Math. **75** (1972), 33–34.
[26] B. Fitzpatrick, Jr. and N. F. Lauer, *Densely homogeneous spaces. I*, Houston J. Math. **13** (1987), 19–25.
[27] B. Fitzpatrick, Jr. and H.-X. Zhou, *Densely homogeneous spaces. II*, Houston J. Math. **14** (1988), 57–68.
[28] B. Fitzpatrick, Jr. and H. X. Zhou, *Some open problems in densely homogeneous spaces*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland Publishing Co., Amsterdam, 1990, pp. 252–259.
[29] B. Fitzpatrick, Jr. and H.-X. Zhou, *Countable dense homogeneity and the Baire property*, Topology Appl. **43** (1992), 1–14.
[30] B. Fitzpatrick, J. M. S. White, and H.-X. Zhou, *Homogeneity and σ-discrete sets*, Proceedings of the Symposium on General Topology and Applications (Oxford, 1989). Topology Appl. **44** (1992), no. 1-3, 143–147.
[31] B. Fitzpatrick, Jr. and H.-X. Zhou, *A survey of some homogeneity properties in topology*, Papers on general topology and related category theory and topological algebra (New York, 1985/1987), 28–35, Ann. New York Acad. Sci., **552**, New York Acad. Sci., New York, 1989.

[32] P. Fletcher. and R. A. McCoy, *Conditions under which a connected representable space is locally connected*, Pacific J. Math. **51**, (1974) 433–437.
[33] L. R. Ford, Jr., *Homeomorphism groups and coset spaces* , Trans. Amer. Math. Soc. **77** (1954), 490–497.
[34] M. K. Fort, Jr., *Homogeneity of infinite products of manifolds with boundary*, Pacific J. Math. **12** (1962), 879–884.
[35] P. Franklin, *Analytic transformations of everywhere dense point sets*, Trans. Amer. Math. Soc. **27** (1925), 91–100.
[36] M. Fréchet, *Les dimensions dun ensemble abstrait*, Math. Ann. **68** (1910), 145–168.
[37] J. de Groot, *Topological Hilbert space and the drop-out effect*, Rapport ZW 1969-016, Mathematisch Centrum, Amsterdam, 1969.
[38] W. R. Heath and W. L. Saltsman, *A Jones space that is not countable dense homogeneous*, Topology Appl. **52** (1993), no. 3, 237–240.
[39] R. Hernández-Gutiérrez, *Countable dense homogeneity and the double arrow space*, Topology Appl. **160** (2013), no. 10, 1123–1128.
[40] R. Hernández-Gutiérrez, M. Hrušák, *Non-meager P-filters are countable dense homogeneous*, Colloq. Math. **130** (2013), 281–289.
[41] R. Hernández-Gutiérrez, M. Hrušák, and J. van Mill, *Countable dense homogeneity and λ-sets*, Fund. Math. **226** (2014), 157–172.
[42] R. Hernández-Gutiérrez and P. Szeptycki, *Some observations on filters with properties defined by open covers*, Comment. Math. Univ. Carolin. **56**, (2015), no.3, 355–364.
[43] M. Hrušák, and J. van Mill, *Nearly countable dense homogeneous spaces*, Canad. J. Math. **66** (2014), 743–758.
[44] M. Hrušák and J. van Mill, *The existence of a connected meager in itself CDH space is independent of ZFC*, Proc. Amer. Math. Soc., to appear.
[45] M. Hrušák, and B. Zamora-Áviles, *Countable dense homogeneity of definable spaces*, Proc. Amer. Math. Soc. **133** (2005), 3429–3435.
[46] W. Just, A. R. D. Mathias, K. Prikry, and P. Simon, *On the existence of large P-ideals*, J. Symbolic Logic **55** (1990), 457–465.
[47] J. Kennedy, *A weaker form of countable dense homogeneity*, Houston J. Math. **3 no. 1** (1977), 89–97.
[48] J. Kennedy, *More on a weaker form of countable dense homogeneity*, Houston J. Math. **10** (1984), no 4, 507–524.
[49] J. Kennedy, *A condition under which 2-homogeneity and representability are the same in continua*, Fund. Math. **121** (1984), 89–98.
[50] K. Kunen, *Homeomorphisms with small twist*, Preprint, 2012, pp. 1-31.
[51] K. Kunen, A. Medini and L. Zdomskyy, *Seven characterizations of non-meager P-filters*, Fund. Math **231** (2015), no. 2, 189–2018.
[52] K. Kuperberg, W. Kuperberg and W. R. R. Transue, *On the 2-homogeneity of Cartesian products*, Fund. Math **110** (1980), no. 2, 131–134.
[53] L. B. Lawrence, *Homogeneity in powers of subspaces of the real line*, Trans. Amer. Math. Soc **350** (1984), no. 8, 3055–3064.
[54] W. D. Maurer, *Conformal equivalence of countable dense sets*, Proc. Amer. Math. Soc. **18** (1967), 269–270.
[55] A. Medini, *Countable dense homogeneity in powers of zero-dimensional definable spaces*, Canad. Math. Bull. **58** (2015), no.2, 334–349.
[56] A. Medini, *Products and countable dense homogeneity*, Top. Proc. **46** (2015), 135-143.
[57] A. Medini and D. Milovich, *The topology of ultrafilters as subspaces of* 2^ω, Topology Appl. **159** (2012), no. 5, 1318–1333.

[58] A. Medini and L. Zdomskyy, *Between Polish and completely Baire*, Arch. for Math. Logic **54** (2015), no. 1-2, 231–245.
[59] S. V. Medvedev, *Metrizable DH-spaces of the first category*, Topology Appl. **179** (2015), 171–178.
[60] J. van Mill, *A rigid space X for which $X \times X$ is homogeneous; an application of infinite-dimensional topology*, Proc. Amer. Math. Soc. **83** (1981), no.3, 597–600.
[61] J. van Mill, *Strong local homogeneity does not imply countable dense homogeneity*, Proc. Amer. Math. Soc. **84** (1982), 143–148.
[62] J. van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Publishing Co., Amsterdam, 2001.
[63] J. van Mill, *A note on the Effros Theorem*, Amer. Math. Monthly. **111** (2004), 801–806.
[64] J. van Mill, *On countable dense and strong local homogeneity*, Bull. Pol. Acad. Sci. Math. **53** (2005), 401–408.
[65] J. van Mill, *A countable dense homogeneous space with a dense rigid open subspace*, Fund. Math. **201** (2008), 91–98.
[66] J. van Mill, *Homogeneous spaces and transitive actions by Polish groups*, Israel J. Math. **165** (2008), 133–159.
[67] J. van Mill, *Analytic groups and pushing small sets apart*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5417–5434.
[68] J. van Mill, *On countable dense and strong n-homogeneity*, Fund. Math. **214** (2011), no. 3, 215–239.
[69] J. van Mill, *On countable dense and n-homogeneity*, Can. Math. Bull. **56** (2013), 860–869.
[70] J. van Mill, *Countable dense homogeneous rimcompact spaces and local connectivity*, Filomat **29** (2015), no. 1, 179–182.
[71] J. T. Moore, *A five element basis for the uncountable linear orders*, Ann. of Math. (2), **163** (2006), no. 2, 669–688.
[72] J. T. Moore and S. Todorcevic, *Baumgartner's isomorphism problem for $\aleph_2$-dense suborders of $\mathbb{R}$*, Archive for Math. Logic, to appear.
[73] M. Morayne, *Measure preserving analytic diffeomorphisms of countable dense sets in $\mathbb{C}^n$ and $\mathbb{R}^n$*, Colloq. Math. **52** (1987), 93–98.
[74] M. Morley, *The number of countable models*, J. Symbolic Logic **35** (1970), 14–18.
[75] D. B. Motorov, *Homogeneity and π-networks*, Moscow Univ. Math. Bull. **44** (1989), no.4, 45–50.
[76] I. Neeman, *http://settheory.mathtalks.org/appalachian-set-theory-workshop-itay-neeman-april-9-2016/*
[77] J. W. Nienhuys and J. G. F. Thiemann, *On the existence of entire functions mapping countable dense sets onto each othe*r, Indag. Math. **38** (1976), 331-334.
[78] D. Repovš, L. Zdomskyy and S. Zhang, *Countable dense homogeneous filters and the Menger covering property*, Fund. Math **224** (2014), no. 3, 233–240.
[79] W. L. Saltsman, *Concerning the existence of a connected, countable dense homogeneous subset of the plane which is not strongly locally homogeneous*, Topology Proc. **16** (1991), 137–176.
[80] W. L. Saltsman, *Concerning the existence of a nondegenerate connected, countable dense homogeneous subset of the plane which has a rigid open subset*, Topology Proc. **16** (1991), 177–183.
[81] D. Sato and S. Rankin, *Entire functions mapping countable dense subsets of the reals onto each other monotonically*, Bull. Austral. Math. Soc. **10** (1974), 67-70.
[82] W. Sierpiński, *Sur une propriété topologique des ensembles d enombrables denses en soi*, Fund. Math. 1 (1920), 11–16.
[83] P. Stäckel, *Ueber aritmetishe Eigenschaften analytischer Functionen*, Math. Ann. **46** (1895) 513–520.
[84] J. Steprāns and H. X. Zhou, *Some results on CDH spaces. I*, Topology Appl. **28** (1988), 147–154.
[85] J. Steprāns and S. Watson, *Homeomorphisms of manifolds with prescribed behaviour on large dense sets*, Bull. London Math. Soc., **19** (1987), 305–310.
[86] G. S. Ungar, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. **212** (1975), 393–400.
[87] G. S. Ungar, *Countable dense homogeneity and n-homogeneity*, Fund. Math. **99** (1978), 155–160.

[88] R. L. Vaught, *Denumerable models of complete theories*, In: Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw,1959), Pergamon, Oxford; Państwowe Wydawnictwo Naukowe, Warsaw, 1961, 303–321.

[89] S. Watson, *The homogeneity of small manifolds*, Topology Proc. **13** (1988), no. 2, 365–369.

[90] S. Watson, *Problems I wish I could Solve*, Open Problems in Topology (J. van Mill and G. M. Reed, eds.), North-Holland Publishing Co., Amsterdam, 1990.
1990, pp. 38–76.

[91] S. Watson and P. Simon, *Open subspaces of countable dense homogeneous spaces*, Fund. Math. **141** (1992), 101–108.

[92] Z. Yang, *Homogeneity of infinite products of manifolds with boundary*, Papers on general topology and applications (Brookville, NY, 1990), 194–208, Ann. New York Acad. Sci., **659**, New York Acad. Sci., New York, 1992.

[93] B. Zamora-Áviles, *Espacios numerablemente densos homogéneos*, Master's Thesis (in Spanish), UMSNH, Mexico, (2003).

[94] H.-X. Zhou, *Two applications of set theory to homogeneity*, Questions Answers Gen. Topology **6** (1988), no. 1, 49–56.

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