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TOPOLOGICAL PROPERTIES OF INCOMPARABLE FAMILIES

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Abstract. We say that two sets $a, b \subseteq \omega$ are *incomparable* if both $a \setminus b$ and $b \setminus a$ are infinite. We study topological properties of families of mutually incomparable subsets of ω. We raise the question whether there may be an analytic maximal incomparable family and show that (1) it can not be K_σ, and (2) every incomparable family with the Baire property is meager. On the other hand, we show that a non-meager incomparable families exist in ZFC, while the existence of a non-null incomparable family is consistent. Finally, we show that there are maximal incomparable families which are both meager and null assuming either $\mathfrak{r} = \mathfrak{c}$ or the existence of a completely separable MAD family, in particular they exist if $\mathfrak{c} < \aleph_\omega$. Assuming CH, we can even construct a maximal incomparable family which is concentrated on a countable set, and hence of strong measure zero.

Introduction

This paper explores certain topological conditions that incomparable and maximal incomparable families of infinite subsets of ω may or may not fulfil. We mainly consider two orders over ω: the usual inclusion, and the almost-inclusion, that is, the relation $\subseteq^*$ such that $a \subseteq^* b$ iff $a \setminus b$ is finite. Hence, a and b are incomparable, if $a \setminus b$ and $b \setminus a$ are infinite; and a family $\mathcal{A}$ is incomparable, if each distinct pair of elements in $\mathcal{A}$ is incomparable. In order to avoid triviality, we consider incomparable families with more than one element. Incomparable families in Boolean algebras have been studied by Monk in [7]. There he asked if it is consistent that there is a maximal incomparable family of size strictly less than the size of the continuum. This question was answered negatively in [1].

Incomparable families as subsets of the power set $\mathcal{P}(\omega)$ are seen as subspaces of the Cantor space 2^ω, the product topology of the discrete space $2 = \{0, 1\}$. We have considered the following topological properties: Borel (analytic) complexity, Baire property and Lebesgue measurability. Several kinds of families of subsets of ω have been studied in the light of their definability. Ideals and filters are the most studied families, but some others have also been relevant such as independent[1] and almost disjoint[2] families. For example, Mathias [4] proved that there are no infinite analytic maximal almost disjoint (MAD) families. Törnquist [9] extended this result by proving that in the Solovay's model there are no infinite MAD families. On the other hand, Horowitz and Shelah [3] showed that, surprisingly, there is a Borel maximal family of *eventually different* functions in ω^ω.

Independent and almost disjoint families are incomparable; however, no independent or almost disjoint family is maximal incomparable. From these facts we may deduce some information about topological properties of incomparable families, for example, that there is

The research of the third author was partially supported by PAPIIT grants IN108014, and CONACYT grant 177758.

The fourth author was supported by PAPIIT grant IN106017.

[1]A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is *independent* if $\bigcap F_0 \setminus \bigcup F_1$ is inifinite for any pair of finite disjoint $F_0, F_1 \subseteq \mathcal{I}$.

[2]A family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is *almost disjoint* if $a \cap b$ is finite for any pair of distinct elements a, b of $\mathcal{A}$.

a perfect incomparable family. Actually independent and almost disjoint families may have different properties, for example, all almost disjoint families are meager and null (they are contained in the Borel ideal $\mathrm{Fin} \times \mathrm{Fin}$), while there are non-meager independent families, as Medini, Repovš and Zdomskyy proved in [5]. However, incomparable families may have different properties, for example, it is well known that maximal independent and maximal almost disjoint families may (consistently) have cardinality strictly less than $\mathfrak{c}$.

In this paper we prove that there are no maximal incomparable families which are F_σ. We also prove that there is a non-meager incomparable family (this proof is essentially different than Medini-Repovš-Zdomskyy's proof of this result for independent families), however, we show that independent families having the Baire property are meager. Moreover, we present three proofs of consistency of the existence of meager and null maximal incomparable families: assuming CH, assuming $\mathfrak{r} = \mathfrak{c}$, and assuming the existence of a completely separable MAD family. In the last section we repay a debt from [1], where we proved that assuming the existence of a completely separable MAD family, there is a maximal incomparable family which (with the obviously needed addition of ω as the root) is a maximal tree, by constructing such a family without any extra assumption.

1. Topological properties of incomparable families in $\mathcal{P}(\omega)/\mathrm{Fin}$

Probably the most interesting question about maximal incomparable families is whether they can be analytic or Borel. The question remains wide open as we were able to make only very partial progress by showing that there are not any which are F_σ, and that any incomparable family with the Baire property is meager.

Theorem 1.1. *Let G be a G_δ-set containing $\mathbb{Q}^* = \{a \subseteq \omega : |a| < \omega \vee |\omega \setminus a| < \omega\}$. Then there is an infinite set $y \in G$ incomparable with every $x \notin G$.*

Proof. Let us write $\mathcal{P}(\omega) \setminus G = \bigcup_n K_n$, where $\{K_n : n \in \omega\}$ is an increasing sequence of compact sets.

Lemma 1.2. *If $K \subseteq \mathcal{P}(\omega) \setminus \mathbb{Q}^*$ is compact, then for all $n \in \omega$ there is $n' > n$ such that for all $x \in K$*

$$x \cap [n, n') \neq \emptyset \text{ and } [n, n') \setminus x \neq \emptyset.$$

Proof of Lemma. Let us suppose otherwise, and let n_0 be such that for all $n \geq n_0$, there is $x_n \in K$ satisfying $x_n \cap [n_0, n) = \emptyset$ or $[n_0, n) \subseteq x_n$. By compactness of K, there is a subsequence x_{n_k} converging to some $x \in K$, and this x satisfies $x \cap [n_0, n) = \emptyset$ for all n, or $[n_0, n) \subseteq x$ for all n. Hence, x is finite or cofinite, which is a contradiction. □

By the lemma, there is a sequence of intervals $I_n \subseteq \omega$ satisfying:

- $\min I_0 = 0$,
- $\min I_{n+1} = \max I_n + 1$,
- $x \cap \mathsf{I}_n \neq \emptyset \neq I_n \setminus x$, for all $x \in K_n$.

Define $y = \bigcup_n I_{2n}$. Clearly, $y \in G$ and y is incomparable with all x in K_n, for all n. □

Corollary 1.3. *There are no maximal incomparable families which are F_σ.*

□

Definable incomparable families are not large in the sense of category, as the following theorem claims.

Theorem 1.4. *If $\mathcal{A}$ is a non-meager family satisfying the Baire property, then $\mathcal{A}$ is not incomparable.*

Proof. Let U be a non-empty open set so that $U \cap \mathcal{A}$ is comeager in U, and let $s \in 2^{<\omega}$ be such that $\langle s \rangle \subseteq U$. Let $\{U_n : n \in \omega\}$ be a decreasing sequence of dense open subsets of $\langle s \rangle$ with $\mathcal{A} \supseteq \bigcap_n U_n$. We now define x and y in $\bigcap_n U_n$ such that $x \subseteq y$. Recursively, define an increasing sequence n_k in ω, with restrictions on x and y in $n_k (= [0, n_k))$, as follows:

(1) $n_0 = \min\{m \geq |s| : \exists t \in 2^m : t \supseteq s \wedge \langle t \rangle \subseteq U_0\}$;
(2) $x \restriction n_0 = t = y \restriction n_0$, for a whitness t for n_0;
(3) if $k = 2j$, then let n_{k+1} be the minimal $m > n_k$ for which there is $r \in 2^m$ such that $r \supseteq y \restriction n_k{}^\frown 1$ and $\langle r \rangle \subseteq U_{j+1}$; in this case, define $y \restriction n_{k+1} = r$ and $x \restriction n_{k+1} = x^\frown 0^{n_{k+1} - n_k}$;
(4) if $k = 2j + 1$, then let n_{k+1} be the minimal $m > n_k$ for which there is $t \in 2^m$ such that $t \supseteq x \restriction n_k{}^\frown 0$ and $\langle t \rangle \subseteq U_{j+1}$; in this case, define $x \restriction n_{k+1} = t$ and $y \restriction n_{k+1} = y \restriction n_k{}^\frown 1^{n_{k+1} - n_k}$.

It is clear that $x \subseteq y$, and that for all $m \in \omega$, $\langle x \restriction n_{2m} \rangle \subseteq U_m$ and $\langle y \restriction n_{2m+1} \rangle \subseteq U_{m+1}$, which proves that $x, y \in \mathcal{A}$. $\square$

Corollary 1.5. *If $\mathcal{A}$ is a G_δ-incomparable family, then $\mathcal{A}$ is nowhere dense.*

Proof. If $\mathcal{A}$ was dense in a basic set $\langle s \rangle$ with $|s| = n$, then $\mathcal{A}' = \{A \setminus n : A \in \mathcal{A}\}$ would be a dense G_δ-incomparable family on $\omega = \omega \setminus n$, contradicting Theorem 1.4. $\square$

On the other hand, there are non-meager (and by Theorem 1.4 not definable) maximal incomparable families.

Theorem 1.6. *There exists a non-meager incomparable family.*

Proof. We will construct the non-meager incomparable family $\mathcal{A}$ by a recursion of length $\mathfrak{c}$, justified by the following result.

Lemma 1.7. *Let U be a dense G_δ-set and A an infinite subset of ω. Then, there exists a perfect almost-disjoint family P such that for all $B \in P$, $|A \setminus B| = |A \cap B| = |B \setminus A| = \aleph_0$ and $A \bigtriangleup B \in U$.*

Proof of Lemma. First, note that for a fixed $w \in 2^\omega$, the function $\varphi_w(x) = w \bigtriangleup x$ $(= x +_{mod\ 2} y)$ is an autohomeomorphism of 2^ω, and $\varphi_w^{-1} = \varphi_w$. Then, it is sufficient to prove that every dense G_δ-subset of 2^ω contains a perfect almost disjoint family.

Without lose of generality, we may assume that $U \subseteq [\omega]^\omega$. Let $\{U_n : n \in \omega\}$ be a decreasing family of open sets such that $U = \bigcap_n U_n$. For $s, t \in 2^{\leq\omega}$, let us denote by $s * t$, the maximal $r \in 2^{\leq\omega}$ so that $r \subseteq s \cap t$. Recursively, we now define a function φ from 2^ω to 2^ω and a sequence k_n satisfying the following:

(1) $\varphi(s) * \varphi(t) = \varphi(s * t)$, in particular, $s \subseteq t$ implies $\varphi(s) \subseteq \varphi(t)$,
(2) $|\varphi(s)^{-1}(1)| \geq |s|$,
(3) if $|s| = |t|$ and $\varphi(s)(i) = 1 = \varphi(t)(i)$, then $i < |\varphi(s * t)|$,
(4) $|\varphi(s)| = k_n$, for all $s \in 2^n$,
(5) for all n and all $s \in 2^n$, $\langle \varphi(s) \rangle \subseteq U_n$.

Before finalizing the construction, note that by (2), if $x \neq y \in 2^\omega$, then $\varphi(x)^{-1}(1) \cap \varphi(y)^{-1}(1) = \varphi(x * y)^{-1}(1)$, where $\varphi(x) := \bigcup_n \varphi(x \restriction n)$, proving that $\{\varphi(x)^{-1}(1) : x \in 2^\omega\}$ is almost disjoint; and by (5), $\varphi(x) \in U$, for all $x \in 2^\omega$.

Now let us do the construction. Let $\varphi(\emptyset)$ be so that $\langle\varphi(\emptyset)\rangle \subseteq U_0$. Assume $\varphi(s)$ has been defined for all $s \in 2^n$ satisfying (1), (2) and (3). Enumerate $2^n = \{s_j : j = 0, \ldots, 2^n - 1\}$. We now define an auxiliary function ψ from 2^{n+1} to $2^{<\omega}$ as follows. For $j = 0$, we take $\psi(s_0{}^\frown 1)$ as an extension r of $\varphi(s_0){}^\frown 1$ so that $\langle r\rangle \subseteq U_{n+1}$; and take $\psi(s_0{}^\frown 0)$ as an extension r of $\varphi(s_0){}^\frown 0^{|\psi(s_0{}^\frown 1)|-k_n}{}^\frown 1$ so that $\langle r\rangle \subseteq U_{n+1}$. If $j < 2^n - 1$, take $\psi(s_{j+1}{}^\frown 1)$ as an extension r of $\varphi(s_{j+1}){}^\frown 0^{|\psi(s_j{}^\frown 0)|-k_n}{}^\frown 1$ so that $\langle r\rangle \subseteq U_{n+1}$; and take $\psi(s_j{}^\frown 0)$ as an extension r of $\varphi(s_j){}^\frown 0^{|\psi(s_j{}^\frown 1)|-k_n}{}^\frown 1$ so that $\langle r\rangle \subseteq U_{n+1}$. Now, define $k_{n+1} = |\psi(s_{2^n-1}{}^\frown 0)|$ and then, for all $s \in 2^{n+1}$, define $\varphi(s) = \psi(s){}^\frown 0^{k_{n+1}-|\psi(s)|}$. It is clear that properties (1) to (5) are fulfilled. $\square$

We now return to the proof of the Theorem. Let $\{N_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the family of all meager F_σ-sets. Let us denote $[\omega]^\omega \setminus N_\alpha$ as U_α. Let A_0 be any infinite coinfinite element of U_0, and suppose $\{A_\beta : \beta < \alpha\}$ is an incomparable family so that $A_\beta \in U_\beta$ for all $\beta < \alpha$. Let P be a perfect almost-disjoint family so that $A_0 \vartriangle B \in U_\alpha$, for all $B \in P$. Note that, for each $\beta < \alpha$, there are no more than one $B \in P$ such that $A_0 \setminus A_\beta \subseteq^* B$, and analogously, there are no more than one $C \in P$ such that $A_\beta \setminus A_0 \subseteq^* C$. This is true since P is almost-disjoint. By cardinality, we may pick $B \in P$ such that $(A_0 \setminus A_\beta) \setminus B$ and $(A_\beta \setminus A_0) \setminus B$ are infinite, for all $\beta < \alpha$. Define $A_\alpha := A_0 \vartriangle B$. We now prove that A_α is incomparable with A_β for all $\beta < \alpha$. Just note that $A_\alpha \setminus A_0 = B \setminus A_0$, $A_0 \setminus A_\alpha = A_0 \cap B$, and for $0 < \beta < \alpha$, $A_\beta \setminus A_\alpha \supseteq (A_\beta \setminus A_0) \setminus B$ and $A_\alpha \setminus A_\beta \supseteq (A_0 \setminus A_\beta) \setminus B$. Then, $A_\alpha \setminus A_\beta$ and $A_\beta \setminus A_\alpha$ contain infinite sets, for all $\beta < \alpha$. $\square$

Using the trivial fact that the union of a chain of incomparable families is incomparable, it follows that there is a non-meager maximal incomparable family. However, by Theorem 1.4, there are no comeager incomparable families.

Considering now the Lebesgue measure of incomparable, and more specifically, independent families, we have the following result.

Theorem 1.8. *It is consistent that there is a non-null independent family, and consequently, that there is a non-null incomparable family.*

Proof. It is well known (see Miller's book [6]) that in the random real model, the set of reals given by the generic filter is a Sierpiński set, that is, it is an uncountable set whose intersection with any null set is countable. By adding $\aleph_1$ random reals, we have a Sierpiński set of independent sets, which is obviously non-null. $\square$

On the other hand, under CH, $\mathfrak{r} = \mathfrak{c}$[3], or assuming that there exists a completely separable MAD family, we can prove that there are meager and null (even at the same time) maximal incomparable families. Moreover, assuming CH, there is a maximal incomparable family which is of strong measure zero. Recall that a set $X \subseteq 2^\omega$ is *concentrated on* $\mathbb{Q}$ if $X \cap U \neq \emptyset$ for all G_δ-set U which contains $\mathbb{Q}$.

Theorem 1.9 (CH). *There is a maximal incomparable family which is concentrated on* $\mathbb{Q}(= \mathrm{Fin})$.

Proof. The following lemma justifies a recursive construction.

[3]Recall that $\mathfrak{r}$ is the minimal cardinality of a *reaping family*, i. e. a family $\mathcal{R}$ of infinite subsets of ω such that, for all inifinite A, there is $R \in \mathcal{R}$ such that $A \subseteq^* R$ or $A \cap R =^* \emptyset$

Lemma 1.10. *Let $\mathcal{A}$ be a countable incomparable family, U a G_δ-set containing $\mathbb{Q}$, and B an infinite subset of ω such that B is incomparable with A, for all $A \in \mathcal{A}$. Then there is an infinite subset C of B, which is in U and is incomparable with every $A \in \mathcal{A}$.*

Proof of Lemma. Let us enumerate $\mathcal{A} = \{A_n : n \in \omega\}$ and write $U = \bigcap_n U_n$ with $\{U_n : n < \omega\}$ a decreasing sequence of open sets containing $\mathbb{Q}$. We construct the following:

- an increasing sequence $k_n \in \omega$,
- a sequence of pairs of disjoint sets L_n and M_n in $[B]^{n+1}$, and
- an end-extension increasing sequence of finite subsets C_n of B,

satisfying that

1. $L_n = \{l_j : j \leq n\}$, $M_n = \{m_j : j \leq n\}$, and for each $j \leq n$, $l_j \in A_j$ and $m_j \notin A_j\}$,
2. $L_n \cup M_n \subseteq [k_{2n}, k_{2n+1})$,
3. $C_0 = \emptyset$, $C_{n+1} = C_n \cup M_n$,
4. $k_{2n+2} > \max(L_N \cup M_n)$ and $\langle \chi_{C_{n+1}} \restriction k_{2n+2} \rangle \subseteq U_n$.

It is possible to construct such sequences because of the incomparability of B with all A_j, and hence B has enough elements inside and outside of each A_j, allowing to fulfil conditions 1 and 2. Condition 3 is trivially satisfied and condition 4 is possible since each U_n is a neighbourhood of each finite set. Fix $C = \bigcup_n C_n$. Then, by 1, for all n, $C \setminus A_n$ and $A_n \setminus C$ have at least one element in each interval $[k_{2m}, k_{2m+1})$ for all $m \geq n$, which proves that C is incomparable with A_n. By 4, $C \in U$. □

Now we do the recursive construction. Let $\{U_\alpha : \alpha \in \omega_1\}$ be an enumeration of the family of all the G_δ-subsets of 2^ω, and $\{B_\alpha : \alpha \in \omega_1\}$ an enumeration of $[\omega]^\omega$. For each $\alpha \in \omega_1$, we define a set $C_\alpha \in [\omega]^\omega$ such that C_α is comparable with B_α, and if it is also comparable with C_β ($\beta < \alpha$), then $C_\alpha = C_\beta$. Let $C_0 = B_0$ and suppose C_β has been defined for all $\beta < \alpha$. If B_α is comparable with some C_β, let $C_\alpha = C_\beta$ for a minimal such β, and if not, by the previous lemma, there exists a C_α comparable with B_α, and incomparable with C_β for all $\beta < \alpha$, which is in the G_δ set $\bigcap_{\beta \leq \alpha} U_\beta$. Now, making $\mathcal{A} = \{C_\alpha : \alpha \in \omega_1\}$ we have finished, since every infinite subset of ω is comparable with some C_α, and for all α, there is $\beta \geq \alpha$ such that $C_\beta \in U_\alpha$, because of the non-maximality of countable incomparable families. □

Brendle and Flašková (Blobner)[4] defined the *generic existence number* $\mathfrak{ge}(\mathsf{I})$ of an ideal I as the minimal cardinality κ such that every filter base with cardinality less than κ can be extended to an I-ultrafilter $\mathcal{U}$, that is, an ultrafilter which satisfies that for every function $f \in \omega^\omega$, there is $U \in \mathcal{U}$ with $f''U \in \mathsf{I}$. The *cofinality* of an ideal is defined by $\mathsf{cof}(\mathsf{J}) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathsf{J} \wedge (\forall J \in \mathsf{J})(\exists B \in \mathcal{B}) J \subseteq B\}$. Brendle and Flašková proved that $\mathfrak{ge}(\mathsf{I}) = \min\{\mathsf{cof}(\mathsf{J}) : \mathsf{I} \subseteq \mathsf{J}\}$. Clearly, $\mathfrak{ge}(\mathsf{I}) \leq \mathsf{cof}(\mathsf{I})$.

Theorem 1.11. *There is an analytic tall ideal L such that for every L-positive set X, $\mathfrak{ge}(\mathsf{L} \restriction X) = \mathfrak{c}$.*

Proof. Let $\mathcal{C}$ be a perfect independent family and define L as the ideal generated by $\mathcal{C} \cup \{A \subseteq \omega : (\exists \mathcal{D} \in [\mathcal{C}]^\omega)(\forall D \in \mathcal{D})|A \cap D| < \omega\}$. Clearly, by its definition, L is an analytic ideal. Let us prove that $\mathfrak{ge}(\mathsf{L}) = \mathfrak{c}$. Let J be an ideal containing L. Then, every $J \in \mathsf{J}$ almost-contains just finitely many $C \in \mathcal{C}$. If $\mathcal{B} \in [\mathcal{C}]^{<\mathfrak{c}}$, then $|\{C \in \mathcal{C} : (\exists B \in \mathcal{B}) C \subseteq^* B\}| = |\mathcal{B}|$, and then there is $C \in \mathcal{C}$, which is not contained in any element of $\mathcal{B}$. Now, let us note that for

[4] Brendle and Flašková mention that this definition was done independently by Hong and Zhang [?].

every L-positive set X, the family $\mathcal{C} \upharpoonright X$ is a perfect independent family on X and $\mathsf{L} \upharpoonright X$ is generated by $(\mathcal{C} \upharpoonright X) \cup \{B \subseteq X : (\exists \mathcal{D} \in [\mathcal{C} \upharpoonright X]^\omega)(\forall D \in \mathcal{D})|B \cap D| < \omega\}$. By the previous argument in this proof, we are done. $\square$

Let us recall that from old results by Sierpiński and Kolmogorov, every ideal which satisfies the Baire Property is meager, and every Lebesgue measurable ideal is null. In particular, Borel and analytic ideals are meager and null.

Theorem 1.12. *If $\mathfrak{r} = \mathfrak{c}$, then there is a meager and null maximal incomparable family.*

Proof. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ an enumeration of $[\omega]^\omega$. We are going to recursively construct a maximal incomparable family $\{B_\alpha : \alpha < \mathfrak{c}\}$ included in the meager and null ideal L, as follows. For $\alpha = 0$, let B_0 be some infinite subset of A_0 in L, and for all $0 < \alpha < \mathfrak{c}$ we consider three cases.

1. If there is $\gamma < \alpha$ such that $B_\gamma \subseteq^* A_\alpha$ or $A_\alpha \subseteq^* B_\gamma$, take $B_\alpha = B_\gamma$ for such γ.
2. If $A_\alpha \in \mathsf{L}$ and is not in the previous case, by our hypothesis, $\mathcal{D} = \{B_\gamma \cap A_\alpha : \gamma < \alpha\} \cap [A_\alpha]^\omega$ is not reaping on A_α, then we can pick $B_\alpha \subseteq A_\alpha$ with $|B_\alpha \cap B_\gamma| = \aleph_0 = |B_\alpha \setminus B_\gamma|$, for all $\gamma < \alpha$ such that $B_\gamma \cap A_\alpha$ is in $\mathcal{D}$. However, for all $\gamma < \alpha$, $B_\gamma \setminus A_\alpha$ is infinite and so, $B_\gamma \setminus B_\alpha$ is infinite. Finally, since B_α is infinite, for all $\gamma < \alpha$, $B_\alpha \setminus B_\gamma$ is infinite.
3. In any other case, note that $\{B_\gamma \cap A_\alpha : \gamma < \alpha\}$ is not a cofinal family of $\mathsf{L} \upharpoonright A_\alpha$, hence, there is a set B in $\mathcal{L} \upharpoonright A_\alpha$ such that $B \not\subseteq^* B_\gamma$, for all $\gamma < \alpha$. Now B is like A_α in case 2, then we can pick B_α in the same way, and it will be contained in A_α.

From this construction, it is clear that $\{B_\alpha : \alpha < \mathfrak{c}\}$ is an incomparable family contained in L, and every infinite subset of ω is comparable with some B_α. $\square$

Let us recall that if $\mathcal{A}$ is a MAD family, $\mathsf{I}(\mathcal{A})$ denotes the ideal generated by $\mathcal{A}$, and this ideal is tall and contained in a copy of the Borel ideal $\mathrm{Fin} \times \mathrm{Fin}$. A MAD family is *completely separable* if every $\mathsf{I}(\mathcal{A})$-positive set X contains an element (equivalently, $\mathfrak{c}$-many elements) of $\mathcal{A}$. Consequently, if $\mathcal{A}$ is a completely separable MAD family, then for every $\mathsf{I}(\mathcal{A})$-positive set X and every family $\mathcal{B} \subseteq \mathsf{I}(\mathcal{A})$ with $|\mathcal{B}| < \mathfrak{c}$, there is $A \in \mathcal{A}$ such that $A \subseteq X$ and $A \not\subseteq^* B$, for all $B \in \mathcal{B}$. The question of the existence of completely separable MAD families in ZFC is an old open question due to Erdös and Shelah [2]. They do exist in all known models of ZFC. In particular, Shelah [8] proved that assuming $2^{\aleph_0} < \aleph_\omega$ there are such families.

Theorem 1.13. *If a completely separable MAD family exists, then there is a meager and null maximal incomparable family.*

Proof. Let $\mathcal{A}$ be a completely separable MAD family and let $\{C_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $[\omega]^\omega$. For all $\alpha < \mathfrak{c}$, let B_α be an infinite subset of ω recursively chosen as follows:

- if C_α is comparable with some B_β ($\beta < \alpha$), then make $B_\alpha = B_\beta$,
- if C_α is not comparable with any B_β ($\beta < \alpha$) and $C_\alpha \in \mathsf{I}(\mathcal{A})$, then make $B_\alpha = A_\alpha$, and
- if C_α is $\mathsf{I}(\mathcal{A})$-positive and is not comparable with any B_β ($\beta < \alpha$), then choose $B_\alpha \in \mathcal{A}$ contained in C_α and such that B_α is not almost contained in any B_β for $\beta < \alpha$.

From this construction, it is clear that $\{B_\alpha : \alpha < \mathfrak{c}\}$ is a maximal incomparable family. It is meager and null because it is contained in $\mathsf{I}(\mathcal{A})$. $\square$

2. Final remarks and questions

We now construct a maximal incomparable famly which is also a maximal tree, as was promised in [1], but first we include some definitions and remarks.

Definition 2.1. Let $\mathbb{B}$ be a Boolean algebra. A subset T of $\mathbb{B}$ is a *tree* if $\langle T, \leq\restriction T \times T\rangle$ is a tree (initial segments are well ordered). Among trees, we consider the end-extension order.

Remark 2.2. A tree T on $\mathcal{P}(\omega)/\mathrm{Fin}$ is a maximal tree if and only if for every $B \in [\omega]^\omega$, either $C \subseteq^* B$ for some $C \in T$, or there are $C_0 \neq C_1 \in T$ such that $B \subseteq^* C_0 \cap C_1$.

Proposition 2.2 in [1] claims that assuming the existence of a completely separable MAD family, there is a maximal incomparable family which is also a maximal tree (with the obvious addition of the root ω). Actually this assumption is not needed, as the following result shows.

Theorem 2.3. *There is a maximal incomparable family which is also a maximal tree.*

Proof. It follows from a simple recursion based on the following result.

Lemma 2.4. *If $\mathcal{A}$ is an incomparable family with $|\mathcal{A}| < \mathfrak{c}$ and $B \in [\omega]^\omega$, then there is an incomparable family $\mathcal{A}' \supseteq \mathcal{A}$ with $|\mathcal{A}'| \leq |\mathcal{A}| + 2$ and such that either there is $A \in \mathcal{A}'$ with $A \subseteq^* B$, or there are $A_0 \neq A_1 \in \mathcal{A}'$ such that $B \subseteq^* A_0 \cap A_1$.*

Proof of Lemma. Take $\mathcal{A}' = \mathcal{A} \cup \{B\}$ if this set is incomparable. If there is $A \in \mathcal{A}$ such that $A \subseteq^* B$, or there are distinct $A_0, A_1 \in \mathcal{A}$ such that $B \subseteq^* A_0 \cap A_1$, then take $\mathcal{A}' = \mathcal{A}$. The remaining case is when there is a unique $\bar{A} \in \mathcal{A}$ such that $B \subseteq^* \bar{A}$. Let us use some almost disjoint family $\mathcal{C}$ of subsets of $\omega \setminus \bar{A}$ with cardinality $\mathfrak{c}$. Note that for each $A \in \mathcal{A} \setminus \{\bar{A}\}$, there is at most one $C \in \mathcal{C}$ such that $A \subseteq B \cup C$, since $\mathcal{C}$ is almost disjoint. Moreover, for all $A \in \mathcal{A} \setminus \{\bar{A}\}$ and all $C \in \mathcal{C}$ it is not the case that $B \cup C \subseteq A$. Since $|\mathcal{A}| < \mathfrak{c}$, there are $C_0 \neq C_1 \in \mathcal{C}$ such that $B \cup C_0$ and $B \cup C_1$ do not contain any $A \in \mathcal{A}$. Define $\mathcal{A}' = \mathcal{A} \cup \{B \cup C_0, B \cup C_1\}$. It is obvious that such $\mathcal{A}'$ satisfies the lemma. □

Now we display the recursion. Let $\{B_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all the infinite coinfinite subsets of ω, and by using the previous lemma, for all $\alpha < \mathfrak{c}$ we take an incomparable family $\mathcal{A}_\alpha$ such that either there is $A \in \mathcal{A}_\alpha$ with $A \subseteq^* B_\alpha$, or there are $A_0 \neq A_1 \in \mathcal{A}_\alpha$ so that $B_\alpha \subseteq A_0 \cap A_1$. In step 0 we define $\mathcal{A}_0 = \{B_0\}$, and for $0 < \alpha < \mathfrak{c}$, let $\mathcal{A}_\alpha = (\bigcup_{\beta<\alpha} A_\beta)'$, as in the lemma. Then $\bigcup_{\alpha<\mathfrak{c}} \mathcal{A}_\alpha$ is the maximal tree which is also a maximal incomparable family. □

Let us recall from [1] that the number $\mathfrak{tr}$ is defined as the minimal cardinality of a maximal tree $\mathcal{T}$ on $\mathcal{P}(\omega)/\mathrm{Fin}$, and in this same paper we proved that it is consistent with ZFC that $\mathfrak{tr} < \mathsf{non}(\mathcal{M})$[5]. In particular, this shows that, consistently, there are meager maximal trees. For more on maximal trees see [**?**]

We conclude with a list of some open questions. The most intriguing is the following:

Question 2.5. *Is there an analytic (or even Borel or G_δ) maximal incomparable family?*

We needed some mild extra assumptions to construct "small" maximal incomparable familes. They should exist in ZFC alone:

[5]Recall that $\mathsf{non}(\mathcal{M})$ is the minimal cardinalty of a non-meager subset of the Cantor space.

Question 2.6. *Is there, in* ZFC*, a meager and null maximal incomparable family? Is there a nowhere dense one?*

Question 2.7. *Can there be a co-null incomparable family? Is every Lebesgue-measurable incomparable family necessarily null?*

Question 2.8. *Is there, in* ZFC*, a meager (null) maximal tree? Is there an analytic or Borel one?*

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