RAMSEY TYPE PROPERTIES OF IDEALS

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Abstract. We study several combinatorial properties concerning (mostly definable) ideals on countable sets. In several cases, we identify critical ideals for such properties in the Katětov order. In particular, the ideal $R$ generated by the homogeneous sets with respect to the random graph is critical with respect to the Ramsey property. The question as to whether there is a tall definable Ramsey ideal is raised and studied. It is shown that no tall $F_\sigma$ ideal is Ramsey, while there is a tall co-analytic Ramsey ideal.

Introduction

The classical Ramsey’s theorem asserts that for every coloring $\varphi$ of unordered $n$-tuples of natural numbers in $m$ colors, there is an infinite subset $X$ of $\omega$ such that $X$ is $\varphi$-homogeneous (i.e. $|\varphi([X]^n)| = 1$). In this paper we study ideals on $\omega$ satisfying Ramsey’s theorem in the sense that the homogeneous set can be found positive with respect to the ideal (i.e. a set not belonging to $I$). More precisely, an ideal $I$ is said to be Ramsey($\omega$) if for every coloring $\varphi : [\omega]^2 \to 2$ there is an $I$-positive set $X$ which is $\varphi$-homogeneous, this property is also denoted by $\omega \rightarrow (1^+)^2$. A stronger property is the following: We say that $I$ is Ramsey if for every $I$-positive set $X$ and every coloring $\varphi : [X]^2 \to 2$ there is an $I$-positive subset $Y$ of $X$ which is $\varphi$-homogeneous. This property is denoted by $1^+ \rightarrow (1^+)^2$.

The most common examples of Ramsey ideals are the selective ideals (or Mathias’ happy families) [22] and, more generally, Farah’s semiselective ideals [7]. In fact, it is well known that selective implies semiselective and this implication is proper.

We will be interested in combinatorial properties of tall ideals (i.e. those ideals satisfying that every infinite set contains an infinite subset belonging to the ideal) and more specifically of definable tall ideals (definable as subsets of the Cantor cube $2^\omega$). As we shall see, for many combinatorial properties of ideals there are ideals (usually Borel of a low complexity) which are critical in the Katětov order $\leq_K$ with respect to the given property. For instance, we show that $\omega \rightarrow (1^+)^2$ if and only if $R \not\leq_K I$, where $R$ is the ideal generated by cliques and free sets in the random graph.

We will focus on definable Ramsey tall ideals because it is known that tall semiselective ideals are not definable (see section §1). The principal question considered here is the following: Is there a Borel (analytic) tall Ramsey ideal? We construct examples of definable
(even $F_\sigma$) ideals satisfying $\omega \to (I^+)^2_2$, but failing $I^+ \to (I^+)^2_2$. We also present an example of a co-analytic Ramsey tall ideal.

It is well known (probably due to Kunen) that Ramsey ultrafilters are exactly the dual filters of maximal ideals which are both $P$ and $Q$-ideals. Kunen’s proof actually shows that if $I$ is both a $P^+$ and $Q^+$-ideal then $I^+ \to (I^+)^2_2$. It is easy to see that a Ramsey ideal is necessarily $Q^+$ (Proposition 2.1) but not necessarily $P^+$ (Example 3.1). In fact, no definable tall ideal is both $P^+$ and $Q^+$ (Proposition 1.2). We will investigate properties like $P^+$, $Q^+$ and some natural variants of them.

There is a close connection between combinatorial and descriptive theoretic properties of an ideal $I$ and the forcing properties of the quotient $\mathcal{P}(\omega)/I$. In particular, in several places of the text we shall take advantage of this connection and consider properties of $\mathcal{P}(\omega)/I$ such as: proper, $\sigma$-closed, does not add reals. We believe it would be useful to investigate this connection in more depth.

The paper is organized as follows. In the first section we briefly revisit the notion of semiselectivity.

Section 2 is dedicated to $Q^+(\omega)$ and $Q^+$-ideals and related properties.

In section 3 we study $P^+$-ideals. We include a characterization of $P$-points using Katětov order (Theorem 3.6), which extends a result of Zapletal (Claim 2.4 in [28]). We use this to characterize definable $P^+$-ideals (Theorem 3.10) as those definable ideals which are indecomposable and locally $F_\sigma$. We study several natural variants of the $P^+$-property there.

Section 4 deals with variations of the Ramsey and Ramsey($\omega$) properties, their connections and their critical ideals in Katětov order.

In section 5 we compare the Ramsey-like properties with those introduced by Filipów, Mrożek, Reclaw and Szuca in [8], and answer two of their questions. We finish by listing some open questions.

We end the introduction by fixing some terminology. An ideal on a set $X$ is a non-empty family of subsets of $X$ closed under subsets and finite unions, which does not contain the set $X$. A subset of $X$ that is not in $I$ is a called $I$-positive. The collection of all $I$-positive sets is denoted by $I^+$. We will always assume that an ideal contains all finite subsets of $X$. When $X$ is countable, an ideal $I$ over $X$ can be regarded as an ideal over $\omega$ via a bijection between $X$ and $\omega$. Thus we will state our results for ideals on $\omega$.

An ideal $I$ is tall if every infinite subset $A$ of $\omega$ contains an infinite subset which is in $I$.

An ideal $I$ on $\omega$ is a $P$-ideal if for every countable family $\{I_n : n \in \omega\} \subseteq I$, there is $I \in I$ such that $I_n \setminus I$ is finite, for all $n$. $I$ is a $P^+$-ideal if for every decreasing sequence $\{X_n : n \in \omega\}$ of $I$-positive sets there is an $I$-positive set $X$ such that $X \subseteq^* X_n$, for all $n$. When this property holds only for decreasing sequences $\{X_n\}_n$ of $I$-positive sets such that $X_n \setminus X_{n+1} \in I$ for all $n$, we say that $I$ is a $P^+_{\text{tower}}$-ideal. If, moreover, the property holds only when $X_0 = \omega$ we will say that the ideal is $P^+_{\text{tower}}$.

$I$ is a $Q$-ideal if for every partition $\{J_n : n \in \omega\}$ of $\omega$ into finite sets, there is $I \in I$ such that $|J_n \setminus I| \leq 1$ for all $n$. $I$ is a $Q^+(\omega)$-ideal if for every partition $\{F_n : n \in \omega\}$ of $\omega$ into finite sets there is an $I$-positive set $Y$ such that $|Y \cap F_n| \leq 1$, for all $n$. $I$ is a $Q^+$-ideal if for every $I$-positive set $X$ and every partition $\{F_n : n \in \omega\}$ of $X$ into finite sets there is an $I$-positive set $Y \subseteq X$ such that $|Y \cap F_n| \leq 1$, for all $n$. Such sets $Y$ are called selectors.

One of the main tools for our study of combinatorial properties of ideals is the Katětov order $\leq_K$. Given ideals $I$ and $J$ on $\omega$ we say that $I \leq_K J$ if there is a function $f : \omega \to \omega$
such that \( f^{-1}[I] \in J \) for all \( I \in l \). If \( f \) is finite-to-one then we write \( l \leq_{KB} J \) and say that \( l \) is Katětov-Blass below \( J \).

1. Selectivity and semiselectivity

The purpose of this short section is to make precise something we said in the introduction about definable tall ideals. In particular, we want to explain why when dealing with definable tall ideals one has to investigate the Ramsey property directly, rather than the more common notions of selectivity and semiselectivity.

We start by recalling a result of Mathias

**Theorem 1.1** (Mathias [22], Theorem 2.12). Let \( U \) be an ultrafilter on \( \omega \). Then, \( U \) is selective\(^1\) if and only if \( U \cap l \neq \emptyset \) for every analytic tall ideal \( l \) on \( \omega \).

The following fact is probably due to Mathias (see [7, 22, 27]). We present a short proof for the convenience of the reader.

**Proposition 1.2.** There are no tall analytic ideals which are both \( P^+ \) and \( Q^+ \).

**Proof.** Let \( l \) be an analytic ideal on \( \omega \), and suppose that \( l \) is both a \( P^+ \) and a \( Q^+ \)-ideal. Then, being \( P^+ \), the forcing \( \mathcal{P}(\omega)/l \), forcing equivalent with \( (l^+, \subseteq) \), is \( \sigma \)-closed, so it does not add new real numbers. Let \( G \) be a \( \mathcal{P}(\omega)/l \)-generic ultrafilter. We shall show now, that in \( V[G] \), \( U = \{ X \subseteq \omega : [X]_l \in G \} \) is a selective ultrafilter. To that end let \( \varphi : [\omega]^2 \rightarrow 2 \) be a coloring in \( V[G] \). Again, as \( \mathcal{P}(\omega)/l \) is \( \sigma \)-closed, \( \varphi \in V \). As \( P^+ \) together with \( Q^+ \) entail \( l^+ \rightarrow (l^+_1)^2 \), the set of \( l \)-positive \( \varphi \)-homogeneous sets is dense in \( \mathcal{P}(\omega)/l \), hence the generic filter contains a \( \varphi \)-homogeneous set. Note that a selective filter is necessarily an ultrafilter.

To finish the proof it suffices to note that the generic selective ultrafilter is contained in \( l^+ \), hence is disjoint from \( l \) which remains tall and analytic as no reals were added, contradicting (in \( V[G] \)) the theorem 1.1. \( \square \)

Recall that an ideal \( l \) is semiselective [7], if whenever \( D_i \) are dense open subsets of the partial order \( (l^+, \subseteq) \), for \( i \in \omega \), and \( A \) is a \( l \)-positive set, there is \( D \subseteq A \) in \( l^+ \) such that \( D/n \in D_n \) for all \( n \in D \). A subset \( C \subseteq [\omega]^\omega \) is said to be \( l \)-Ramsey, if for all \( A \in l^+ \) there is \( B \subseteq A \) in \( l^+ \) such that \( [B]^\omega \cap C = \emptyset \) or \( [B]^\omega \subseteq C \). It follows directly from the definition that an ideal \( l \) is itself \( l \)-Ramsey if and only if \( l \upharpoonright X \) is not tall for all \( X \in l^+ \). Farah [7] showed that every analytic set is \( l \)-Ramsey for any semiselective ideal \( l \). In particular, this shows that there are no tall analytic semiselective ideals. In fact, there are stronger results. The following result implies that, under suitable set theoretic assumptions, every subset of \( [\omega]^\omega \) is \( l \)-Ramsey for any semiselective ideal \( l \), and therefore \( l \) is not tall.

**Theorem 1.3** ([5]).

1. Assume \( ADR \). If \( l \) is a semiselective ideal, then every subset of \( [\omega]^\omega \) is \( l \)-Ramsey.

2. If \( ZFC \) is consistent with the existence of a weakly compact cardinal, then so is the statement that for every semiselective ideal \( l \) every subset of \( [\omega]^\omega \) from \( L(\mathbb{R}) \) is \( l \)-Ramsey.

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\(^1\)Recall that an ultrafilter is **selective** if and only if the dual ideal is Ramsey, if and only if the dual ideal is both a \( P \)-ideal and a \( Q \)-ideal.

\(^2\)Usually, a stronger condition is required but this is enough for our purposes.
In [20] a result similar to (1) was proved. We observe that part (1), together with the comments above, answers question 0.3 of [20].

2. $Q^+(\omega)$ and $Q^+$-ideals

The following result was also mentioned in the introduction.

**Proposition 2.1.** If $l$ satisfies $1^+ \longrightarrow (1^+)_2^2$ then $l$ is a $Q^+$-ideal.

**Proof.** Suppose that $l$ is not $Q^+$. Let $X$ be an $l$-positive set and let $\{F_n : n \in \omega\}$ be a partition of $X$ into finite sets every selector of which is in $l$. Define a coloring $\varphi : [X]^2 \to 2$ by $\varphi\{k, m\} = 0$ if and only if there is $n$ such that $k, m \in F_n$. Any $\varphi$-homogeneous set is finite or is a selector for $\{F_n : n \in \omega\}$ and thus $\varphi$-homogeneous sets are in $l$. \hfill $\Box$

Consider the following eventually different ideals:

$$\mathcal{ED} = \{A \subseteq \omega \times \omega : (\exists n)(\forall m \geq n)(|\{(m, k) : k < \omega\} \cap A| \leq n)\},$$

and $\mathcal{ED}_{fin}$ - the restriction of $\mathcal{ED}$ to the set $\Delta = \{(n, m) \in \omega \times \omega : m \leq n\}$. The ideal $\mathcal{ED}_{fin}$ is (up to Katetov-equivalence), the unique ideal generated by the selectors of some partition of $\omega$ into finite sets $\{I_n : n \in \omega\}$ such that $\limsup_n |I_n| = \infty$. The ideal $\mathcal{ED}_{fin}$ probably first appeared in [21] and is critical for the $Q^+$-property.

**Theorem 2.2.** Let $l$ be an ideal on $\omega$. Then

1. $l$ is a $Q^+(\omega)$-ideal if and only if $l \not\leq_{KB} \mathcal{ED}_{fin}$, and
2. $l$ is a $Q^+$-ideal if and only if $l \nmid X \not\leq_{KB} \mathcal{ED}_{fin}$ for all $l$-positive set $X$.

**Proof.** (1) Let us suppose that $\mathcal{ED}_{fin} \leq_{KB} l$, and let $f : \omega \to \Delta$ be the witnessing function. Let $\Delta_n = \Delta \cap \{n\} \times \omega$. Then, the family $\{f^{-1}(\Delta_n) : n \in \omega\}$ is a partition of $\omega$ into finite sets and every selector of the partition belongs to $l$.

On the other hand, if $l$ is not a $Q^+(\omega)$-ideal then there is a partition $\langle F_n : n \in \omega\rangle$ of $\omega$ into finite sets such that every selector belongs to $l$. Note that such a partition must have elements of arbitrarily large finite cardinality, otherwise $l$ would be improper. Take a sequence of sets $\langle G_n : n \in \omega\rangle$ such that $G_n \in [F_n]^n$ for some $i_{n-1} < i_n \in \omega$ and $X = \bigcup_n G_n \in l^+$. Then there is a copy of $\mathcal{ED}_{fin}$ inside of $l \upharpoonright X$.

(2) Analogous to (1). \hfill $\Box$

A family $\mathcal{A}$ of infinite subsets of $\omega$ is $\omega$-hitting if for every infinite partition $\langle X_n : n \in \omega\rangle$ of $\omega$ into infinite sets, there is $A \in \mathcal{A}$ such that $A \cap X_n$ is infinite, for all $n \in \omega$. Similarly, $\mathcal{A}$ is $\omega$-splitting if for every countable family $\mathcal{X}$ of infinite subsets of $\omega$ there is $A \in \mathcal{A}$ such that $|A \cap X| = |X \setminus A| = \aleph_0$ for all $X \in \mathcal{X}$. It is easy to see that an ideal on $\omega$ is $\omega$-hitting if and only if it is $\omega$-splitting. The following was proved as theorem 3.3 in [11] for Borel ideals:

**Corollary 2.3.** For any analytic ideal $l$ the following conditions are equivalent

1. $l$ is a $Q^+(\omega)$-ideal,
2. $\mathcal{ED}_{fin} \not\leq_{KB} l$,
3. $l$ is not $\omega$-hitting ideal,

The extension to analytic ideals can be deduced from a theorem of Otmar Spinas: Theorem 1.2 in [26] claims that every analytic $\omega$-splitting family contains a closed $\omega$-splitting family. Let $l$ be an analytic $\omega$-splitting ideal, and let $K$ be a closed $\omega$-splitting family contained in
the ideal \( l \). The function \( \varphi \) given by \( \varphi(X) = \min \{|B| : B \subseteq K \& X \subseteq \bigcup B\} \) is a lower semicontinuous submeasure (lscsm)\(^3\) and then, \( l' = \text{Fin}(\varphi) \) is \( F_\sigma \), \( \omega \)-splitting and \( l' \leq_{KB} l \). By the Borel case, \( \mathcal{E}D \leq_{KB} l \).

3. \( P^+ \)-ideals

We begin this section by showing that \( l^+ \longrightarrow (l^+)^2_2 \) does not imply \( P^+ \).

**Example 3.1.** A tall non \( P^+ \)-ideal satisfying \( l^+ \longrightarrow (l^+)^2_2 \).

Let \( \{A_n : n \in \omega\} \) be a sequence of MAD families, such that for each \( A \in A_n, A_{n+1} \upharpoonright A \) is an infinite MAD family in \( \mathcal{P}(A) \). Let \( l_n = \{I \subseteq \omega : |\{A \in A_n : |I \cap A| = \aleph_0\}| < \aleph_0\} \), and let \( l = \bigcap_{n \in \omega} l_n \).

As countable intersection of tall ideals is tall, \( l \) is tall. Let us prove that \( l \) satisfies \( l^+ \longrightarrow (l^+)^2_2 \). First, note that \( l^+ = \bigcup_{n \in \omega} l_n^+ \). By a result of Mathias (MAD families are happy, see [22]), \( l_n^+ \longrightarrow (l_n^+)^2_2 \) holds for each \( n \). Let \( \varphi : [X]^2 \rightarrow 2 \) be a coloring. If \( X \in l^+ \) then there is \( n \in \omega \) such that \( X \in l_n^+ \) and so, there is \( Y \in l_n^+ \cap \mathcal{P}(X) \) (and so in \( l^+ \)) which is \( \varphi \)-homogeneous.

To see that \( l \) is not a \( P^+ \)-ideal take a sequence \( \langle A_n : n \in \omega \rangle \) such that \( A_n \in A_n, A_n \supseteq A_{n+1}, \) and \( A_n \in l_n \setminus l_{n+1} \) for all \( n \in \omega \). Hence, if \( A \subseteq^* A_n \) for all \( n \in \omega \) then \( A \in l_n \) for all \( n \) and so, \( A \in l \). \( \square \)

A somewhat similar example appears in Farah [7, Example 2.8].

**Remark 3.2.** Recall that a MAD family \( A \) is completely separable if for every \( l(A) \)-positive set \( X \) there is an element \( A \) of \( A \) contained in \( X \).\(^4\) If the MAD families used in previous example are completely separable then \( \mathcal{P}(\omega)/l \cong \text{Coll}(\omega, 2^\omega) \).

**Proof.** By complete separability, \( \bigcup_{n \in \omega} A_n \) is dense in \( l^+ \). \( \square \)

In this section we will investigate the relation between \( P^+, F_\sigma \) and maximal \( P \)-ideals. The following lemma was probably first proved by Just and Krawczyk [14]. We include a proof for the sake of completeness. We recall Mazur’s characterization of \( F_\sigma \)-ideals [23, Lemma 1.2] which claims that for every \( F_\sigma \)-ideal \( l \), there is a lscsm \( \varphi \) on \( \omega \) such that \( l = \text{Fin}(\varphi) \).

**Lemma 3.3** ([14]). Every \( F_\sigma \)-ideal is a \( P^+ \)-ideal.

**Proof.** Let \( \varphi \) be a lscsm on \( \omega \) such that \( l = \text{Fin}(\varphi) \), and let \( \langle X_n : n \in \omega \rangle \) be a decreasing sequence of \( l \)-positive sets. For every \( n \in \omega \), pick a finite set \( F_n \subseteq X_n \) such that \( \varphi(F_n) \geq n \). Then \( X = \bigcup_{n \in \omega} F_n \) is a pseudointersection of \( \langle X_n : n \in \omega \rangle \) and \( \varphi(X) = \infty \). \( \square \)

**Corollary 3.4.** If \( l \) is a tall \( F_\sigma \)-ideal then there is an \( l \)-positive set \( X \) such that \( l \upharpoonright X \) is \( \omega \)-splitting (on \( X \)).

**Proof.** By 1.2, a tall \( F_\sigma \)-ideal \( l \) is not \( Q^+ \), so, by 2.2, there is an \( l \)-positive set \( X \) such that \( l \upharpoonright X \geq_{KB} \mathcal{E}D_{fin} \), i.e., \( l \upharpoonright X \) is \( \omega \)-splitting on \( X \). \( \square \)

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\(^3\)A lscsm \( \varphi \) is a function from \( \mathcal{P}(\omega) \) to \([0, \infty]\) so that \( \varphi(\emptyset) = 0, \varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B) \) and \( \varphi(A) = \lim_{n \to \infty} \varphi(A \cap n) \). It is well known that \( \text{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\} \) is always an \( F_\sigma \)-ideal.

\(^4\)The question whether completely separable MAD families exist in ZFC is an old open problem of Erdős and Shelah [6]. Shelah [24] recently proved that there is a completely separable MAD family assuming \( 2^{\aleph_0} < \aleph_2 \).
It is perhaps worth noticing that a similar result is not true for ideals of higher complexity. In particular, the ideal $\text{nwd}$ of nowhere dense subsets of the rationals is an $F_{\sigma\delta}$ ideal, no positive restriction of which is $\omega$-hitting.

On the other hand, Farah [7] showed that every analytic ideal which is $Q^+$ and whose quotient is $\omega$-distributive is not tall. Thus an analogous argument, as in the proof of Corollary 3.4, shows that there is an $l$-positive set $X$ such that $I \upharpoonright X$ is $\omega$-hitting (on $X$) for every tall analytic ideal $I$ such that the quotient $P(\omega)/I$ is $\omega$-distribute (does not add reals).

### 3.1. $P^+$, extendability to $F_\sigma$, and P-points

In this segment we investigate which Borel ideals can be extended to $F_\sigma$-ideals. Claude Laflamme [17] showed that this is sufficient for destructibility of the given ideal by an $\omega^\omega$-bounding forcing.\(^5\) It turns out this problem has a close connection to the $P^+$-property.

An easy result on extendability is the following lemma.

**Lemma 3.5 (CH).** If $I$ is a $P^+$-ideal then there is a maximal $P$-ideal $J$ containing $I$.

**Proof.** We will find a $P$-point $U$ such that $I^* \subseteq U$. Let $\langle X_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of $P(\omega)$. For each $\alpha < \omega_1$, choose $U_\alpha \subseteq U^+$ such that $U_\alpha \subseteq U_\beta$ for all $\beta < \alpha$, and either $U_\alpha \subseteq X_\alpha$ or $U_\alpha \cap X_\alpha = \emptyset$. This is not difficult because if $\langle U_\beta : \beta < \alpha \rangle$ is a decreasing sequence in $I^+$ then there is $V \subseteq I^+$ such that $V \subseteq U_\beta$ for all $\beta < \alpha$ and then we can define $U_\alpha = V \cap X_\alpha$ if this set belongs to $I^*$, or $U_\alpha = V \setminus X_\alpha$ if not. Hence, $\langle U_\alpha : \alpha < \omega_1 \rangle$ is a base for an ultrafilter contained in $I^*$, and consequently such ultrafilter contains $I^*$, and since this sequence is $\subseteq^*$-decreasing, the filter generated is a $P$-point. \(\square\)

An important ideal for our considerations is the ideal $\text{Fin} \times \text{Fin}$:

$$\text{Fin} \times \text{Fin} = \{ A \subseteq \omega \times \omega : \exists n \forall m \geq n |A \cap \{m\} \times \omega| < \aleph_0 \}.$$  

Laczkovich and Reclaw [19] proved that there are no Borel $P^+$-ideals (in particular, there are no $F_\sigma$-ideals) Katětov above $\text{Fin} \times \text{Fin}$. The following characterization of $P$-points is essentially due to Zapletal [28]:

**Theorem 3.6 ([28]).** Let $\mathcal{U}$ be a free ultrafilter on $\omega$. Then the following are equivalent:

1. $\mathcal{U}$ is a $P$-point.
2. For every analytic tall ideal $I$ disjoint from $\mathcal{U}$ there is an $F_\sigma$-ideal $J$ disjoint from $\mathcal{U}$ containing $I$.

**Proof.** Suppose that $\mathcal{U}$ is not a $P$-point. Let $\langle U_n : n \in \omega \rangle$ be a strictly $\subseteq$-decreasing sequence of elements of $\mathcal{U}$ without a pseudointersection in $\mathcal{U}$. Letting $X_n = U_n \setminus U_{n+1}$ defines a partition $\{X_n : n \in \omega\}$ of $\omega$ into $\mathcal{U}$-small sets such that every set $A$ satisfying $|A \cap X_n| < \aleph_0$ for all $n$ is in $\mathcal{U}^*$. By tallness of $I$ for any $n \in \omega$ there is an infinite set $I_n \in I$ contained in $X_n$. Let $J$ be the ideal generated by $I \cup \{A \subseteq \omega : (\forall n \in \omega)(|A \cap I_n| < \aleph_0)\}$. Hence $J$ contains a copy of $\text{Fin} \times \text{Fin}$ and is contained in $\mathcal{U}^*$, so $J$ cannot be extended to an $F_\sigma$-ideal disjoint from $\mathcal{U}$. The other direction is Claim 2.4 in [28]. \(\square\)

Notice the similarity of the result with Theorem 1.1. The theorem has the following immediate corollary:

**Corollary 3.7.** For any analytic ideal $I$ the following conditions are equivalent:

\(^5\)A forcing $P$ destroys an ideal $I$ on $\omega$ if $P$ adds a new real number $x$ such that $x \cap I$ is finite for all ground model element $I$ of $I$.  

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(1) there is an $F_\sigma$-ideal $J$ containing $l$,
(2) there is a $P^+$-ideal $K$ containing $l$
and assuming there are enough $P$-points (e.g. assuming CH), (1) and (2) are equivalent to
(3) there is a maximal $P$-ideal $L$ containing $l$.

Proof. (1) implies (2) follows from Lemma 3.3, (2) implies (3) from Lemma 3.5 and the rest
follows from Theorem 3.6.

The $P^+$ property as such does not have a critical ideal in the Katětov order since the ideal
$L = (\text{Fin} \times \text{Fin}) \oplus \mathcal{E}^6$ is Katětov-equivalent to $\mathcal{E}^6$ and the second one is a $P^+$-ideal while
the first one is not.

We will say that an ideal $I$ on $\omega$ is decomposable if there is an infinite partition $\{X_n : n \in \omega\}$
of $\omega$ into $I$-positive sets such that for every $A \subseteq \omega$,
$$A \in I \text{ if and only if } (\forall n \in \omega)(A \cap X_n \in I).$$
Such a partition will be called an $I$-decomposition. We will say that $I$ is hereditarily decomposable
if for every $I$-positive set $X$, $I \upharpoonright X$ is decomposable. We call an ideal $I$ indecomposable
if it is not decomposable. We will say that an ideal $I$ on $\omega$ is $\sigma$-closed, if the quotient forcing
$\mathcal{P}(\omega)/I$ is $\sigma$-closed.

Theorem 3.8. Let $I$ be an ideal. Then
(1) $I$ is a $P^+$-ideal if and only if $I$ is $P^+_{\text{tower}}$ and $\sigma$-closed.
(2) $I$ is $\sigma$-closed if and only if $I$ is indecomposable.
(3) $I$ is not $P^+_{\text{tower}}(\omega)$ if and only if $\text{Fin} \times \text{Fin} \leq_K I$.
(4) $I$ is not $P^+_{\text{tower}}$ if and only if $I \upharpoonright X$ for some $X \in I^+$.

Proof. (1) and (2) are straightforward and left to the reader.

(3) Suppose $I$ is not $P^+_{\text{tower}}(\omega)$ and let $X_n$ be a decreasing sequence of $I^+$-positive sets such
that $X_0 = \omega$ and $X_n \setminus X_{n+1} \in I$ for all $n$ and without a $I^+$-positive diagonalization. Let $(a_n)$
be an enumeration of $X_0$. Define $f : \omega \to \omega \times \omega$ by $f(a_n) = (m,n)$, if $a_n \in X_m \setminus X_{m+1}$.
Then $f$ shows that $\text{Fin} \times \text{Fin} \leq_K I$. The other direction is easy as $\text{Fin} \times \text{Fin}$ is clearly not
$P^+_{\text{tower}}(\omega)$.

Clause (4) follows directly from (3). \qed

Building on the work contained here, the first author and J. Verner in [13] showed that

Theorem 3.9 ([13, Theorem 2.5]). Let $I$ be an analytic ideal such that $\mathcal{P}(\omega)/I$ does not add
reals. Then the $\mathcal{P}(\omega)/I$ generic filter is a $P$-point if and only if ideal $I$ is locally $F_\sigma$, i.e. for
every $X \in I^+$ there is a $Y \subseteq X$, $Y \in I^+$ such that $I \upharpoonright Y$ is $F_\sigma$.

A similar argument can be used to characterize definable $P^+$ filters as follows:

Theorem 3.10. An analytic ideal $I$ is $P^+$ if and only if it is indecomposable and locally $F_\sigma$.

Proof. First, assume $I$ is analytic and $P^+$. Then it is indecomposable by theorem 3.8. We
shall show that it is locally $F_\sigma$. Let $X$ be an $I$-positive set, and let $U$ be the $I^+$-generic
ultrafilter on $\omega$ containing $X$. Then, in $V[U]$, $U$ is a $P$-point disjoint from $I$. This follows
from the fact that $I$ is $P^+$ and, in particular, $\mathcal{P}(\omega)/I \simeq I^+$ is $\sigma$-closed. By theorem 3.6 there

---

\footnote{For two ideals $I$ and $J$, the sum is defined by $I \oplus J = \{A \subseteq (\omega \times \{0\}) \cup (\omega \times \{1\}) : \{n : (n,0) \in A\} \in I$ and \{m : (m,1) \in A\} \in J\}.}
is, in \( V[U] \) an \( F_\sigma \) ideal \( J \) such that \( I \subseteq J \) and \( J \cap U = \emptyset \). Again as \( \mathcal{P}(\omega)/l \) is \( \sigma \)-closed, and \( F_\sigma \) ideals are coded by reals, \( J \) is in \( V \), and there is a \( Y \in U \), without loss of generality \( Y \subseteq X \), and \( Y \in I^+ \), such that \( Y \models "I \subseteq J \) and \( J \cap U = \emptyset" \). To finish the argument it suffices to see that \( I \upharpoonright Y = J \upharpoonright Y \). If not there is a \( Z \subseteq Y \), \( Z \in J \setminus l \). Then, however, \( Z \in I^+ \) and \( Z \models "Z \in U \cap J" \), which is a contradiction.

Now assume that \( I \) is indecomposable and locally \( F_\sigma \). Let \( \langle X_n : n \in \omega \rangle \) be a decreasing sequence of \( I \)-positive sets. As \( I \) is indecomposable, there is a \( Y \in I^+ \) such that \( Y \setminus X_n \in I \) for all \( n \in \omega \). Now, let \( Z \subseteq Y \) be \( I \)-positive and such that \( I \upharpoonright Z \) is \( F_\sigma \). Then \( \langle Z \cap X_n : n \in \omega \rangle \) is a decreasing sequence of \( I \upharpoonright Z \)-positive sets, which has a \( I \upharpoonright Z \)-positive pseudointersection \( X \subseteq Z \) by lemma 3.3. Then \( X \) is an \( I \)-positive pseudointersection of \( \langle X_n : n \in \omega \rangle \), hence \( I \) is \( P^+ \).

While there are locally \( F_\sigma \) ideals which are not \( P^+ \) (consider e.g. the ideal \( \emptyset \times I \), for any tall \( F_\sigma \) ideal \( l \)), in [13, Example 2.6] the authors construct tall Borel ideals of arbitrarily high Borel complexity which are locally \( F_\sigma \). It can be easily checked that ideals presented there are, in fact, \( P^+ \) ideals.

The close relationship between \( F_\sigma \) ideals and the \( P^+ \) property will be further examined in the next section.

### 3.2. More \( P^+ \)-type properties.

Following [18], given a family \( \mathcal{X} \) of infinite subsets of \( \omega \), we call a tree \( T \subseteq ([\omega]^{<\omega})^{<\omega} \) a \( \mathcal{X} \)-tree of finite sets if for each \( s \in T \) there is an \( X_s \in \mathcal{X} \) such that \( s \upharpoonright a \in T \) for each \( a \in [X_s]^{<\omega} \). An ideal \( I \) on \( \omega \) is a \( P^+_\text{tree} \)-ideal if every \( I^+ \)-tree of finite sets has a branch whose union is in \( I^+ \).

The next result ([12]) shows that for definable ideals this strengthening of the \( P^+ \)-property coincides with being \( F_\sigma \).

#### Theorem 3.11.

An analytic ideal \( I \) is \( P^+_\text{tree} \) if and only if it is \( F_\sigma \).

It is easy to verify that for all ideals (Borel or not) \( P^+_\text{tree} \) implies \( P^+ \) implies \( \sigma \)-closed. The reverse implications are not true. \( \text{Fin} \times \text{Fin} \) is a \( \sigma \)-closed non-\( P^+ \)-ideal and the ideal \( I_0 \) defined below is a \( P^+ \)-ideal that is not a \( P^+_\text{tree} \)-ideal.

For each \( f \in 2^{<\omega} \), let us denote \( A_f = \{ f \upharpoonright n : n \in \omega \} \). Then, \( I_0 \) is defined as the ideal on \( 2^{<\omega} \) generated by the family of sets \( A_f \) where \( f \) is not eventually zero. The ideal \( I_0 \) is \( F_{\sigma^0} \), non-\( F_\sigma \) and is not tall.

Going back to the problem of characterizing when an ideal \( I \) can be extended to an \( F_\sigma \)-ideal, it seems that the following ideal may play a critical role. Let \( \text{conv} \) be the ideal on \( \mathbb{Q} \cap [0,1] \) generated by the convergent (in \( [0,1] \)) sequences of rational numbers\(^7\). It is easy to see that \( \text{conv} \leq_K \text{Fin} \times \text{Fin} \), in fact, every \( \text{conv} \)-positive set \( X \) contains a \( \text{conv} \)-positive set \( Y \) such that \( \text{conv} \upharpoonright Y \simeq \text{Fin} \times \text{Fin} \).

The following theorem characterizes those ideals which are Katětov above the ideal \( \text{conv} \).

#### Theorem 3.12.

For any ideal \( I \) on \( \omega \) the following are equivalent:

1. \( I \geq_K \text{conv} \),
2. there is a linear order \( \sqsubseteq \) for \( \omega \) such that \( (\omega, \sqsubseteq) \) is order-isomorphic to \( \mathbb{Q} \cap [0,1] \) and every increasing and every decreasing sequence with respect to \( \sqsubseteq \) is in \( I \),

\(^7\)In other words, we can define \( \text{conv} \) as the ideal of all subsets of \( \mathbb{Q} \cap [0,1] \) such that the Cantor-Bendixon’s derivative of its closure in \( [0,1] \) is finite.
(3) there is a topology $\tau$ on $\omega$ such that $(\omega, \tau)$ is homeomorphic to $Q \cap [0,1]$ and every $\tau$-convergent sequence (in $[0,1]$) is in $l$, and

(4) there is a countable family $\mathcal{X} \subseteq [\omega]^{\omega}$ such that for every $Y \in l^+$ there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_0$.

Proof. (2 $\to$ 1) Such an isomorphism from $\omega$ into $Q \cap [0,1]$ is a Katětov reduction between $l$ and $\text{conv}$ since every convergent sequence can be written as the union of an increasing and a decreasing sequence.

(3 $\to$ 2) Such homeomorphism between $\omega$ and $Q \cap [0,1]$ induces an order $\sqsubseteq$ on $\omega$ isomorphic to the order of $Q \cap [0,1]$; and all the $\sqsubseteq$-increasing and all the $\sqsubseteq$-decreasing sequences are in $l$ as they are $\tau$-convergent in $[0,1]$.

(4 $\to$ 3) Let $\mathcal{X}$ be such family. We can suppose that $\mathcal{X}$ separates points (i.e., for each pair $\{n, m\}$ there is $X \in \mathcal{X}$ such that $|X \cap\{m, n\}| = 1$), is closed under complements, and every non-empty Boolean combination of its elements is infinite; if not, we can recursively replace $\mathcal{X} = \{X_n : n \in \omega\}$ with $\{X'_n : n \in \omega\}$, where $X'_0 = X_0$ and $X'_{n+1}$ is an infinite set $D$ which is a Boolean combination of $\{X'_k : k \leq n\}$ such that $X'_{n+1} =^* D$, if such $D$ exists, otherwise, $X'_{n+1} = X_{n+1}$.

The topology generated by $\mathcal{X}$ is then homeomorphic to the subspace topology of $Q \cap [0,1]$, and no $Y \in l^+$ can be covered by finitely many $\tau$-convergent sequences; given $Y \in l^+$ one can recursively construct a family of $\tau$-clopen sets $\{C_s : s \in 2^{< \omega}\}$ such that $\{C_s : s \in 2^n\}$ is pairwise disjoint for all $n$, $C_{s+1} \cup C_{s+1} = C_s$ and $Y \cap C_s$ is infinite for all $s \in 2^{< \omega}$.

(1 $\to$ 4) Let $C$ be a countable base of the topology of $Q \cap [0,1]$ consisting of clopen sets, and let $f$ be a witness to $\text{conv} \not\leq_K l$. Let $\mathcal{X} = \{f^{-1}[C] : C \in C\}$. Let $I \subseteq \omega$ be such that for every $X \in \mathcal{X}$, $I \cap X$ is finite or $I \setminus X$ is finite. Then, for every basic set $C \in C$, $f''I$ is almost-contained in $C$, or it is almost-contained in $Q \cap [0,1] \setminus C$. Then $f''I$ is a convergent sequence of $Q$, hence $l \subseteq f^{-1}[f''I] \subseteq l$.

Recall that a boolean algebra $B$ is $(\omega, 2)$-distributive, if for every sequence $\{b_n : n \in \omega\} \subseteq B$ and every $b \in B^+$, there is $0 < a < b$ such that $a < b_n$ or $a < b_n^*$, for all $n$. Thus $\mathcal{P}(\omega)/l$ is $(\omega, 2)$-distributive, if given $l$-positive sets $Y, \{X_n : n \in \omega\}$, there is an $l$-positive set $X \subseteq Y$ such that $Y \setminus X_n \in l$ or $Y \setminus X_n \in l$, for all $n$. As in Theorem 3.12 (4), the following holds.

**Theorem 3.13.** $\text{conv} \not\leq_K l \mid X$ for all $l$-positive sets $X$ if and only if $l$ is $P_{\text{tower}}^+$ and $\mathcal{P}(\omega)/l$ is $(\omega, 2)$-distributive.

Proof. Suppose that $1 \mid X \not\leq_K \text{conv}$ for all $X \in l^+$, and let $\{X_n : n \in \omega\}$ be a family of $l$-positive sets. By Theorem 3.12(4), there is an $l$-positive set $Y \subseteq X$ such that for all $n$, $|Y \setminus X_n| < \aleph_0$, or $|Y \setminus X_n| < \aleph_0$. In particular, $Y \setminus X_n \in l$, or $Y \setminus X_n \in l$ for all $n$, proving that $\mathcal{P}(\omega)/l$ is $(\omega, 2)$-distributive. By Theorem 3.8 and the fact that $\text{conv} \leq_K \text{Fin} \times \text{Fin}$, it follows that $l$ is $P_{\text{tower}}^+$.

Now suppose that $\mathcal{P}(\omega)/l$ is $(\omega, 2)$-distributive and $l$ is $P_{\text{tower}}^+$. Let $X$ be an $l$-positive set, and let $\{X_n : n \in \omega\}$ be a family of infinite subsets of $X$. Define $Y_n, n \in \omega$, by $Y_n = X_n$, if $X_n \in l^+$, otherwise let $Y_n = \omega \setminus X_n$. By the $(\omega, 2)$-distributivity, there is an $l$-positive $Z$ subset of $X$ such that for all $n \in \omega$, $Z \setminus Y_n \in l$ or $Z \cap Y_n \in l$. For $n \in \omega$, let $Z_n = Z \cap Y_n$ if $Z \setminus Y_n \in l$, and $Z_n = Z \setminus Y_n$ otherwise. Note that $Z_n \setminus Z_m \in l$ for all $n < m$, hence we can assume that the sequence $\{Z_n : n \in \omega\}$ is decreasing. Since $l$ is $P_{\text{tower}}^+$, there is an $l$-positive set $W$ such that $W \subseteq Z_n$ for all $n$. Notice that $Z_n \subseteq X_n$ or $Z_n \subseteq \omega \setminus X_n$ for all $n$, then (4) in 3.12 fails for $1 \mid X$ and we are done. \hfill $\square$
Corollary 3.14. If \( I \) is an \( F_\sigma \)-ideal on \( \omega \) then \( I \not\subseteq_K \text{conv} \).

In the light of the previous results we conjecture the following

Conjecture 3.15. If \( I \) is a Borel ideal then either there is an \( I \)-positive set \( X \) such that \( I \upharpoonright X \geq_K \text{conv} \) or there is an \( F_\sigma \)-ideal \( J \) containing \( I \).

Equivalently, Let \( I \) be a Borel \( P^+_{\text{tower}} \)-ideal such that \( \mathcal{P}(\omega)/I \) is \((\omega,2)\)-distributive. Can \( I \) be extended to an \( F_\sigma \) ideal?

Decomposability of ideals gives a criterion for ideals to be Katětov-above \( \text{conv} \).

Lemma 3.16. If there is a family \( \{X_n : n \in \omega \} \) of \( I \)-decompositions such that (1) \( X_{n+1} \) refines \( X_n \) and (2) all pseudointersections of decreasing chains \( \langle A_n : n \in \omega \rangle \) such that \( A_n \in X_n \) are in \( I \), then \( I \geq_K \text{conv} \).

Proof. We will prove that \( X = \bigcup_{n \in \omega} X_n \) is a family as in 3.12(4). Let \( Y \) be an \( I \)-positive set. As each \( X_n \) is an \( I \)-decomposition, for each \( n \), there is an \( A \in X_n \) such that either \( Y \subseteq^* A \), or \( |Y \cap A| = |Y \setminus A| = \aleph_0 \). Note that for some \( n \in \omega \) the second possibility holds, as otherwise \( Y \) would be a pseudointersection of the sequence \( \langle A_n : n \in \omega \rangle \), where \( A_n \) is the (unique) element of \( X_n \) almost containing \( Y \), which is a contradiction.

Lemma 3.17. Let \( I \) be an ideal on \( \omega \) such that \( \mathcal{P}(\omega)/I \) is proper and adds a new real. Then there is an \( I \)-positive set \( X \) such that \( I \upharpoonright X \geq_K \text{conv} \).

Proof. We will work with the forcing \( I^+ \) instead of \( \mathcal{P}(\omega)/I \) again. Let \( \dot{r} \) be an \( I^+ \) name for a new element of \( 2^\omega \), and pick a family \( \{A_n : n \in \omega \} \) of maximal antichains in \( I^+ \) so that \( A_{n+1} \) refines \( A_n \) and, for all \( n \in \omega \), any condition in \( A_n \) decides \( \dot{r} \upharpoonright n \). Properness of \( I^+ \) implies that there is an \( X \in I^+ \) such that \( B_n = \{A \in A_n : A \cap X \in I^+ \} \) is countable for all \( n \in \omega \).

Recursively, one can refine each \( B_n \) into a pairwise disjoint family of \( I \)-positive sets \( X_n \) which is a family of decompositions as in previous lemma, as follows: Enumerate \( B_0 = \{B_j : j < \omega \} \), and let \( X_j = B_j \setminus \bigcup_{i < j} B_i \) for all \( j \in \omega \). Then \( X_0 = \{X_j : j < \omega \} \) is a refinement of \( B_0 \) which is an \( I \upharpoonright X \)-decomposition. Construct \( X_{n+1} \) by using the same argument for an enumeration of the family \( \{A \cap X \in I^+ : A \in B_{n+1}, X \in X_n \} \). Note that if \( X_n \) is an \( I^+ \)-decomposition then \( X_{n+1} \) is an \( I^+ \)-decomposition. Finally, let \( Y \) be an \( I \)-positive subset of \( X \). We claim that there is \( n \in \omega \) and there are \( A_0 \neq A_1 \in X_n \) such that \( |Y \cap A_0| = |Y \cap A_1| = \aleph_0 \). Since \( \dot{r} \) is a name for a new real, there is an \( n \) such that \( Y \) does not decide \( \dot{r} \upharpoonright n \). That is \( Y \) is compatible with at least two distinct elements of \( X_n \) and the conclusion follows.

Finally, our best approximation to the Conjecture 3.15 is the following result.

Theorem 3.18. Let \( I \) be a Borel ideal such that \( \mathcal{P}(\omega)/I \) is proper. Then, either there is an \( I \)-positive set \( X \) such that \( \text{conv} \leq_K I \upharpoonright X \), or there is an \( F_\sigma \)-ideal \( J \) containing \( I \).

Proof. By the previous lemma, if \( \mathcal{P}(\omega)/I \) adds a new real then we are done. Suppose that \( \mathcal{P}(\omega)/I \) does not add new reals. Let \( U \) be the \( I \)-generic ultrafilter. If \( U \) were a P-point (in \( V[U] \)) then by Claim 2.4 in [28] (see Theorem 3.7) there is an \( F_\sigma \)-ideal \( J \supseteq I \) disjoint from \( U \).

As the forcing \( \mathcal{P}(\omega)/I \) does not add new reals, \( J \) is in \( V \), and \( I \subseteq J \).

If \( U \) is not a P-point then there is a strictly \( C^* \)-decreasing sequence \( X = \langle U_n : n \in \omega \rangle \) of elements of \( U \) without pseudointersections in \( U \). Define

\[
D = \{Y \in I^+ : (\forall n \in \omega)(Y \subseteq^* U_n) \lor (\exists n \in \omega)(Y \cap U_n = \emptyset)\}.
\]
Since \( \mathcal{X} \) has no pseudointersections in \( U \), \( D \) is not dense, and so, there is \( Z \in \mathbb{I}^+ \) such that every \( \mathbb{I}^- \)-positive subset \( Y \) of \( Z \) is not in \( D \). Let us see that \( \mathbb{I} \upharpoonright Z \geq_K \text{Fin} \times \text{Fin} \). Note that for any \( n \in \omega \), \( Z \setminus U_n \in \mathbb{I} \), but \( Z \not\subseteq^* U_n \). For every \( n \in \omega \) define a set \( I_n \in \mathbb{I} \) as follows: 
\[ I_0 = Z \setminus U_0, \quad I_{n+1} = Z \cap U_n \setminus U_{n+1}. \]
Each \( I_n \) is an infinite element of \( \mathbb{I} \upharpoonright Z, \quad \bigcup_{n \in \omega} I_n = Z \) and if \( A \subseteq Z \) is such that \( A \cap I_n \) is finite for all \( n \in \omega \) then \( A \in \mathbb{I} \). Hence \( \mathbb{I} \upharpoonright Z \geq_K \text{Fin} \times \text{Fin} \).

It should be remarked that in the proof of the theorem, we did not use properness in the case when \( \mathcal{P}(\omega)/\mathbb{I} \) does not add new reals, so the question remains open for Borel \( \mathbb{I} \) such that \( \mathcal{P}(\omega)/\mathbb{I} \) is not proper and adds new reals, in other words, when the ideal \( \mathbb{I} \) is \( \mathbb{P} \_\text{tower} \), the algebra \( \mathcal{P}(\omega)/\mathbb{I} \) is \( (\omega,2) \)-distributive but not \( \omega \)-distributive.

4. Idealized Ramsey Theorem

We now turn our attention to the study of the Ramsey(\( \omega \)) and Ramsey properties.

4.1. A critical ideal for Ramsey(\( \omega \)). The random graph on \( \omega \) can be defined as follows: Let \( \{X_n : n \in \omega\} \) be an independent family of subsets of \( \omega \) such that \( n \in X_m \) if and only if \( m \in X_n \), for all \( n, m \in \omega \) (getting such a family is easy by making finite changes to elements of any countable independent family). The set \( E = \{\{n, m\} : m \in X_n\} \) is the set of vertices of the random graph. The random graph satisfies the following property: Given \( F \) and \( G \) disjoint finite subsets of \( \omega \) there is \( k < \omega \) such that \( \{\{k, l\} : l \in F\} \subseteq E \) and \( \{\{k, l\} : l \in G\} \cap E = \emptyset \). As an easy consequence, given a countable graph \( \langle \omega, G \rangle \), there is a subset \( X \subseteq \omega \) such that \( \langle \omega, G \rangle \cong \langle X, E \upharpoonright X \rangle \). This can be proved recursively using the above property.

The Random graph ideal \( \mathcal{R} \) is the ideal generated by the cliques and free sets in the random graph \( \langle \omega, E \rangle \).

Clearly \( \mathcal{R} \) is a tall \( F_\sigma \)-ideal. It is critical with respect to the \( \omega \longrightarrow (\mathbb{I}^+)^2_2 \) property in the Katětov order, as the following result shows.

**Theorem 4.1.** Let \( \mathbb{I} \) be an ideal on \( \omega \). Then, \( \omega \longrightarrow (\mathbb{I}^+)^2_2 \) if and only if \( \mathbb{I} \not\geq_K \mathcal{R} \).

**Proof.** Suppose \( \mathcal{R} \leq_K \mathbb{I} \). Let \( f \in \omega^\omega \) be such that for every \( R \in \mathcal{R} \), \( f^{-1}(R) \in \mathbb{I} \). Define a coloring \( \varphi \) by
\[
\varphi(n, m) = \begin{cases} 
0 & \text{if } (f(n), f(m)) \in E \\
1 & \text{otherwise}.
\end{cases}
\]
If \( A \subseteq \omega \) is \( \varphi \)-homogeneous in color 0 (color 1 is analogous) then \( \{f(n), f(m)\} \in E \) for all \( n \neq m \in A \). That is \( f''A \) is a clique and consequently \( f''A \in \mathcal{R} \), and so, \( A \subseteq f^{-1}(f''A) \in \mathbb{I} \).

Now, suppose that \( \omega \longrightarrow (\mathbb{I}^+)^2_2 \) fails, and let \( \varphi : \omega \to 2 \) be the witnessing coloring. Let \( G = \varphi^{-1}(1) \), and consider the graph \( \langle \omega, G \rangle \). By the universality of the random graph mentioned above, there is a set \( X \subseteq \omega \) and a function \( f : \omega \to X \) such that \( f : (\omega, G) \cong (X, E \cap [X]^2) \).

It is easy to see that \( f \) is a Katětov reduction. \( \square \)

4.2. Local minimality of \( \mathcal{R} \). The following is an immediate corollary of the last theorem:

**Corollary 4.2.** \( \mathbb{I} \longrightarrow (\mathbb{I}^+)^2_2 \) if and only if \( \mathcal{R} \not\leq_K \mathbb{I} \upharpoonright X \) for all \( X \in \mathbb{I}^+ \).
This result has immediate connection with the question of existence of a locally K-minimal tall Borel ideal\(^8\) (see [10]). In fact, it is not known whether \(\mathcal{R}\) is locally K-minimal. This is, of course, equivalent asking whether no tall Borel ideal is Ramsey. We have been able to check this manually for all tall Borel ideals known to us but the general question remains open. We shall show that \(\mathcal{R}\) is locally K-minimal among \(F_{\sigma}\) ideals, while there is a coanalytic Ramsey ideal.

We shall need the fact that \(\text{conv}\) is not Ramsey.

**Lemma 4.3.** \(\omega \rightarrow (\text{conv}^+)_2^2\).

**Proof.** We will use a trick due to W. Sierpiński. Let \(\subseteq\) be an order of \(\mathbb{Q}\) of type \(\omega\), and define \(\psi : [\mathbb{Q}]^2 \rightarrow 2\) by \(\psi\{q, r\} = 0\) if and only if \(q \subseteq r \iff q < r\) (here \(<\) denotes the usual order of \(\mathbb{Q}\)). Every \(\psi\)-homogeneous set is a \(<\)-monotone sequence and so it is in \(\text{conv}\). \(\square\)

It is immediate from the previous lemma and Theorem 4.1 that \(\text{conv} \geq_K \mathcal{R}\).

**Theorem 4.4.** Let \(l\) be a tall ideal on \(\omega\) which is analytic, and \(\mathcal{P}(\omega)/l\) does not add reals, or \(\mathcal{P}(\omega)/l\) is proper and adds a real. Then there is an \(l\)-positive set \(X\) such that \(l \upharpoonright X \geq_K \mathcal{R}\).

**Proof.** Suppose that \(l\) is a tall analytic ideal such that \(\mathcal{P}(\omega)/l\) does not add new reals. Aiming toward a contradiction assume furthermore that for every \(l\)-positive set \(X\) and for every \(f : [\omega]^2 \rightarrow 2\) there is a \(Y \subseteq X\) which is \(l\)-positive and \(f\)-homogeneous.

Let \(U\) be an \(l^+\)-generic ultrafilter. Then \(U\) is a selective ultrafilter with respect to \(V\) (that is, for every coloring of \([\omega]^2\) in \(V\) there is \(U \in U\) which is homogeneous for such partition). Since \(l^+\) does not add new colorings of \([\omega]^2\), \(U\) is also selective in \(V[G]\). However, in \(V[G]\), \(U \cap 1 = \emptyset\), contradicting Mathias’ theorem 1.1.

Now, if \(\mathcal{P}(\omega)/l\) is proper and adds a real, then by Lemma 3.17, there is \(X \in l^+\) such that \(l \upharpoonright X \geq_K \text{conv}\), and then, by Lemma 4.3 we are done. \(\square\)

A large class of ideals having \(\mathcal{R}\) as a local minimum in the Katětov order is the class of \(F_{\sigma}\) ideals.

**Theorem 4.5.** For every \(F_{\sigma}\) tall ideal \(l\) there is an \(l\)-positive \(X\) such that \(l \upharpoonright X \geq_K \mathcal{R}\).

**Proof.** By Proposition 1.2, \(l\) is not \(Q^+\), then, by Theorem 2.2, there is an \(l\)-positive set \(X\) such that \(l \upharpoonright X \geq_{KB} \mathcal{E}D_{\text{Fin}} \geq_{KB} \mathcal{R}\). \(\square\)

Next we provide an example of a co-analytic Ramsey tall ideal. The basic idea is similar to the one used in [7, Example 2.8]. The following lemma says that Ramsey’s theorem holds uniformly, in the sense that there is a Borel function that selects an homogeneous set for a given coloring. This result may be known, but we did not find it in the literature.

**Lemma 4.6.** There is a Borel function \(F : [\omega]^\omega \times 2[\omega]^2 \rightarrow [\omega]^\omega\) such that \(F(A, \varphi)\) is a \(\varphi\)-homogeneous infinite subset of \(A\).

**Proof.** We will verify that the standard proof of Ramsey’s theorem produces the required function. Let \(A\) be an infinite subset of \(\omega\), and let \(\varphi\) be a 2-coloring of \([\omega]^2\).

(i) For each \(i \in 2\) and \(m \in A\) consider the following subset of \(A\):

\[A_i^m = \{n \in A : \varphi\{m, n\} = i\}\].

\(^8\)A tall Borel ideal \(J\) is locally \(K\)-minimal if for any tall Borel ideal \(l\) there is an \(l\)-positive set \(X\) such that \(J \leq_K l \upharpoonright X\).
Define by recursion a sequence \( \langle i(m) : m \in A \rangle \) and sets \( B_m \), for \( m \in A \), as follows: Let \( \{m_k : k \in \omega\} \) be the increasing enumeration of \( A \). Let \( i(m_0) = 0 \), if \( A^0 \) is infinite; otherwise, put \( i(m_0) = 1 \). Let \( B_{m_0} = A^{i(m_0)}_m \). For the inductive step, let \( i(m_{k+1}) = 0 \) if \( A^k \cap B_m \) is infinite; otherwise, let \( i(m_{k+1}) = 1 \). Let
\[
B_{m_{k+1}} = A^{i(m_{k+1})}_{m_{k+1}} \cap B_m.
\]
Notice that if \( r \in B_{m_k} \), then \( \varphi\{m_k, r\} = i(m_k) \). The function that maps \( (A, \varphi) \) to the sequence \( \langle B_m : m \in A \rangle \) is Borel.

(ii) There is a Borel map that assigns a pseudointersection to each decreasing sequence \( \langle B_m : m \in A \rangle \) of infinite subsets of \( A \), i.e., it assigns an infinite subset \( B \) of \( A \) such that \( B \subseteq B_m \) for all \( m \in A \).

(iii) Given a decreasing sequence \( \langle B_m : m \in A \rangle \) of infinite subsets of \( A \) and its pseudointersection \( B \), define an increasing sequence \( \langle a_k : k \in \omega \rangle \subseteq \omega \) as follows: Let \( m_0 = \min A \), and \( a_0 = \min \{p \in B : B \setminus [0, p) \subseteq B_{m_0}\} \). Then recursively on \( k \in \omega \)
\[
a_{k+1} = \min \{p \in B : p > a_k \& B \setminus [0, p) \subseteq B_{a_k}\}.
\]
Notice that \( a_{k+2} \in B_{a_k} \) for all \( k \in \omega \). The function that maps \( B \) and \( \langle B_m : m \in A \rangle \) to the sequence \( \langle a_k : k \in \omega \rangle \) is Borel.

Now we construct an homogeneous set for \( \varphi \). Let \( D = \{a_{2k} : k \in \omega \} \). Then
\[
D \setminus m \subseteq B_m
\]
for all \( m \in D \). Thus for \( p, q \in D \) with \( p < q \), we have that \( q \in B_p \) and therefore \( \varphi\{p, q\} = i(p) \).
Finally, let \( D_j \) be the set of all \( p \in D \) such that \( i(p) = j \). If \( D_0 \) is infinite, let \( H = D_0 \), otherwise put \( H = D_1 \). Then \( H \) is homogeneous for \( \varphi \). Let \( F(A, \varphi) = H \). Then \( F \) is the required function.

\[ \square \]

**Lemma 4.7.** There is a continuous function \( \psi : [\omega]^{\omega} \times 2^\omega \to [\omega]^{\omega} \) such that for every infinite \( A \subseteq \omega \), the collection \( \{\psi(A, x) : x \in 2^\omega\} \) is an almost disjoint family of infinite subsets of \( A \). Moreover, for all infinite \( A \subseteq \omega \) there is an infinite \( B \subseteq A \) such that \( B \cap \psi(A, x) = \emptyset \) for all \( x \in 2^\omega \).

**Proof.** Consider \( D_x = \{x \mid n : n \in \omega\} \) for each \( x \in 2^\omega \). Then \( \{D_x : x \in 2^\omega\} \) is an almost disjoint family of subsets of \( 2^{<\omega} \). Notice that the map that sends \( x \) to \( D_x \) is continuous. Any bijection between \( \omega \) and \( 2^{<\omega} \) allows one to get the almost disjoint family over \( \omega \), which will be also denoted \( \{D_x : x \in 2^\omega\} \). For each infinite \( A \subseteq \omega \) let \( \{e_A(m) : m \in \omega\} \) be the increasing enumeration of \( A \). Let \( \psi(A, x) = \{e_A(2m) : m \in D_x\} \). Let \( B = \{e_A(2m+1) : m \in D_x\} \). \( \square \)

**Theorem 4.8.** There is a co-analytic Ramsey tall ideal.

**Proof.** Let \( \varphi : 2^\omega \to 2^{[\omega]^2} \) be a continuous surjection. Note that the range of \( \varphi \) is the collection of all 2-colorings of \( [\omega]^2 \). Let \( \psi \) and \( F \) be the functions given by lemma 4.7 and lemma 4.6, respectively. Define \( A_s \) for each \( s \in (2^\omega)^{<\omega} \) as follows. Let \( x \in 2^\omega \),
\[
A_{(x)} = F(\psi(\omega, x), \varphi(x)),
\]
\[
A_{s^*x} = F(\psi(A_s, x), \varphi(x)).
\]
Then for each $s$, the collection $\{A_{s,x} : x \in 2^\omega\}$ is an almost disjoint family of subsets of $A_s$ such that for each coloring $\varphi(x)$ of $A_s$, $A_{s,x}$ is $\varphi(x)$-homogeneous. Let

$$C_1 = \{F(\psi(x), \varphi(x)) : x \in 2^\omega\} = \{A(x) : x \in 2^\omega\}$$

$$C_{n+1} = \{F(A(x), \varphi(x)) : (A, x) \in C_n \times 2^\omega\} = \{A_{s,x} : |s| = n, x \in 2^\omega\}.$$ 

Since $F$ is Borel, then each $C_n$ is analytic. Finally, let

$$G \in \mathcal{H} \iff (\exists n < \omega)(\exists D \in \mathcal{C}_n) D \subseteq^* G.$$ 

We will show that $I = \mathcal{P}(\omega) \setminus \mathcal{H}$ is a co-analytic Ramsey tall ideal. It is clear that $\mathcal{H}$ is analytic, hence $I$ is co-analytic.

Now we show that $I$ is an ideal. Suppose $G \cup K \in \mathcal{H} = I^+$. Let $n < \omega$ and $D \in \mathcal{C}_n$ be such that $D \subseteq^* G \cup K$. Consider the following coloring: $c\{n, m\} = 1$ if and only if $\{n, m\} \subseteq G$. Let $x \in 2^\omega$ be such that $\varphi(x) = c$. Let $E = F(\psi(D, x), \varphi(x))$. Then $E \in \mathcal{C}_{n+1}$, and it is $c$-homogeneous. If $E$ is 1-homogeneous, then $E \subseteq G$ and if $E$ is 0-homogeneous, then $E \cap G$ has at most one point. Since $E \subseteq D \subseteq^* G \cup K$, then $E \subseteq^* G$ or $E \subseteq^* K$. That is either $G$ or $K$ is 1-positive.

To see that $I$ is Ramsey, let $G \in \mathcal{H}$ and $D \in \mathcal{C}_n$ be such that $D \subseteq G$. Let $c$ be a coloring of $[\omega]^2$. Then $c = \varphi(x)$, for some $x$, and $F(\psi(D, x), \varphi(x))$ is a $c$-homogeneous infinite subset of $G$ in $\mathcal{H}$.

Finally, let us see that $I$ is tall. Fix $A \in \mathcal{H}$. Then there is $n < \omega$ and $D \in \mathcal{C}_n$ such that $D \subseteq A$. By lemma 4.7, there is $B \subseteq D$ infinite such that $\psi(D, x) \cap B = \emptyset$ for all $x \in 2^\omega$. We claim that $B \notin \mathcal{H}$. In fact, let $n < m$ and $E \in \mathcal{C}_m$. Towards a contradiction, suppose $E \subseteq^* B$. As $\mathcal{C}_m$ is a.d., then there is $x$ such that $E \subseteq^* \psi(D, x)$, which is impossible. □

**Remark 4.9.** We do not know whether an ideal constructed as in the proof of theorem 4.8 can be Borel.

### 4.3. Weaker partition properties

The partition properties defined below are weak versions of the Ramsey properties defined previously. We study them for their own sake, but also as tools to describe Borel ideals satisfying $\omega \rightarrow (1^+)^2_2$.

**Definition 4.10.** Let $I$ be an ideal on $\omega$. We will say that $I$ satisfies:

(a) $\omega \rightarrow (\omega, 1^+)^2_2$ if for every coloring $\varphi : [\omega]^2 \rightarrow 2$ there is an infinite 0-homogeneous set, or there is a 1-positive 1-homogeneous set.

(b) $\omega \rightarrow (\omega, 1^+)^2_2$ if for every coloring $\varphi : [\omega]^2 \rightarrow 2$, either for every $m < \omega$ there is a 0-homogeneous set $X$ of size $m$, or there is a 1-positive 1-homogeneous set.

(c) $1^+ \rightarrow (\omega, 1^+)^2_2$ if for any 1-positive set $Y$ and any coloring $\varphi : [Y]^2 \rightarrow 2$ there is an infinite 0-homogeneous set, or there is a 1-positive 1-homogeneous set.

(d) $1^+ \rightarrow (\omega, 1^+)^2_2$ if for any 1 positive set $Y$ and any coloring $\varphi : [Y]^2 \rightarrow 2$, either for every $m < \omega$ there is a 0-homogeneous set $X$ of size $m$, or there is a 1-positive 1-homogeneous set.
The following diagram shows the trivial implications between these properties:

\[
\begin{align*}
\omega \rightarrow (1^+)^2 & \implies \omega \rightarrow (\omega,1^+)^2 \implies \omega \rightarrow (\omega,1^+)^2 \\
I^+ \rightarrow (1^+)^2 & \implies I^+ \rightarrow (\omega,1^+)^2 \implies I^+ \rightarrow (\omega,1^+)^2
\end{align*}
\]

In order to study these properties, we introduce the following definition:

**Definition 4.11.** Given an l-positive set \(X\) and a sequence \(\langle X_i : i \in \omega \rangle\) of subsets of \(X\) we say that \(\langle X_i : i \in \omega \rangle\) is an **l-tower** if \(X \setminus X_i \in I\) for all \(i \in \omega\).

A set \(D \subseteq X\) is a **diagonal** of an l-tower \(\langle X_i : i \in \omega \rangle\) if \(D \setminus (i + 1) \subseteq X_i\) for all \(i \in D\).

We say that an ideal \(I\) contains an **l-tower** if there is an l-tower such that all of its diagonals are in the ideal.

**Theorem 4.12.** If an ideal \(I\) on \(\omega\) does not contain an \(l\)-tower then \(l^+ \rightarrow (\omega,1^+)^2\).

**Proof.** Suppose the conclusion fails, and let \(X \in \mathcal{P}(\omega)\) and let \(\psi : [X]^2 \rightarrow 2\) be a coloring such that for some \(m < \omega\) there is no \(Y \subseteq [X]^m\) homogeneous in color 0, and there is no \(Y \in I^+\) homogeneous in color 1. Take an l-positive \(\mathcal{X} \subseteq X\) such that this \(m\) becomes minimal. Define \(X_a = \{y \in \mathcal{X} : \psi(\{a,y\}) = 1\}\) for \(a \in \mathcal{X}\). We claim that \(\mathcal{X} \setminus X_a \in I\) for all \(a \in \mathcal{X}\). Suppose \(Z = \mathcal{X} \setminus X_a \in I^+\). By minimality of \(m\), there is a \(Z' \in [Z]^{m-1}\) homogeneous in color 0, then \(Z' \cup \{a\}\) is also homogeneous of color 0 and has \(m\) elements, a contradiction.

Therefore, \(\mathcal{X} \setminus X_a \in I\) for all \(a \in \mathcal{X}\). Let \(\{a_i\}\) be an enumeration of \(\mathcal{X}\) and \(\mathcal{X}_{a_i} = \bigcap_{j \leq i} X_{a_j}\).

Then \(\langle \mathcal{X}_{a_i} : a \in \mathcal{X} \rangle\) is an l-tower, and any diagonal of it is homogeneous in color 1, so by our assumption is in \(I\). Thus the ideal contains an l-tower. \(\square\)

An example of an ideal \(I\) that does not contain a l-tower is \(\text{nwd}\), the ideal of nowhere dense subsets of \(Q\). Hence, by 4.12, \(\text{nwd}^+ \rightarrow (\omega,\text{nwd}^+)^2\).

**Lemma 4.13.** \(\text{nwd}\) does not contain a \(\text{nwd}\)-tower.

**Proof.** Suppose that \(\langle X_i : i \in \omega \rangle\) is a \(\text{nwd}\)-tower. Without loss of generality \(X_{i+1} \subseteq X_i\). Let \(\{U_i : i < \omega\}\) be a base for the topology of \(Q\).

We will construct a positive diagonal \(D = \{d_i : i < \omega\}\) recursively. Let \(d_0 \in E\) be arbitrary, and suppose \(d_i\) was found. Let \(\sqsubseteq\) be an order for \(Q\) of type \(\omega\). If there is \(d \sqsubseteq d_i\) with \(d \in X_{d_i} \cap U_i\), then take \(d_{i+1} = d\), otherwise take arbitrary \(d_{i+1} \sqsubseteq d_i\) with \(d_{i+1} \in X_{d_i}\).

By the construction \(D\) is a diagonal of the tower. As \(X\) is somewhere dense, say dense in an open set \(U\), then \(D\) is also dense in \(U\), hence \(D \in \text{nwd}^+\). \(\square\)

However, there are also ideals \(I\) satisfying \(l^+ \rightarrow (\omega,1^+)^2\) which do contain l-towers, as we shall see next. For the rest of the paper we will adopt the following notation: given a set \(A \subseteq \omega \times \omega\) and \(n \in \omega\) we denote \(A_n = \{m \in \omega : (n,m) \in A\}\).

**Theorem 4.14.** The ideal \(\mathcal{ED}_{fin}\) has the following properties:

1. It contains an \(\mathcal{ED}_{fin}\)-tower.
2. \(\mathcal{ED}_{fin}^+ \rightarrow (\omega,\mathcal{ED}_{fin}^+)^2\).
3. \(\omega \not\rightarrow (\omega,\mathcal{ED}_{fin}^+)^2\).
Proof. (1) Let \( X_{(n,m)} = \{(k,l) \in \Delta : n < k\} \) for \((n,m) \in \Delta\). Then \( \{X_{(n,m)} : (n,m) \in \Delta\} \) is a \( \mathcal{E}\mathcal{D}_{fin}\)-tower. Let \( f : \omega \to \Delta \) be a bijection such that if \( f(n) = (k,l) \) and \( f(n') = (k',l') \) and \( n < n' \), then \( k \leq k' \). If \( D \setminus \{f(0), \ldots, f(n)\} \subseteq X_{f(n+1)} \), then \( D \) is a selector and thus belongs to \( \mathcal{E}\mathcal{D}_{fin} \).

(2) Let a coloring \( \psi : [\Delta]^2 \to 2 \) be given and suppose that there is a bound for the size of the 0-homogeneous sets. Let \( \Delta_n = \Delta \cap (\{n\} \times \omega) \).

Claim. Let \( B \) be an infinite subset of \( \omega \), \( m_0 \not\in B \) and \( A_m \subseteq \Delta_m \), for \( m \in B \), with \( |A_m| \) unbounded. Let \( C \subseteq \Delta_{m_0} \) non-empty. For all \( n \in \omega \) there are a set \( a \subset C \) of size at least \( |C|/2 \), an infinite set \( B' \subseteq B \), an \( l < 2 \) and \( A'_m \subseteq A_m \), for \( m \in B' \), with \( |A'_m| \) unbounded such that \( \psi(\{x,y\}) = l \) for all \( x \in a \) and \( y \in \bigcup_{m \in B'} A'_m \).

Proof of the Claim: Let \( \{x_i : i < p\} \) be an enumeration of \( C \). For each \( m \in B \), choose \( \epsilon_m < 2 \), \( A'_m \subseteq A_m \) finite of size at least \( |A_m|/2 \) such that \( \psi(\{x_i, y\}) = \epsilon_m \) for all \( y \in A'_m \).

Now choose \( B_0 \subseteq B \) infinite and \( l_0 \) such that \( \epsilon_m = l_0 \) for all \( m \in B_0 \). We can repeat this construction and recursively find sets \( B_i, A^i_m \) for \( m \in B_i \) and \( l_i < 2 \) such that for all \( i \leq p \)

1. \( B_{i+1} \subseteq B_i \) are infinite sets,
2. \( A^i_m \subseteq A'_m \) and \( A^i_{m+1} \) has size at least \( |A^i_m|/2 \) for all \( m \in B_{i+1} \),
3. \( \psi(\{x_i, y\}) = l_i \) for all \( y \in \bigcup_{m \in B_i} A^i_m \).

Let \( l < 2 \) and \( D \subseteq \{0,1, \ldots, p-1\} \) be a set of size at least \( p/2 \) such that \( l_i = l \) for all \( i \in D \). Then take \( a = \{x_i : i \in D\} \), \( B' = B_{p-1} \) and \( A'_m = A^p_m \) for \( m \in B' \). By the construction \( |A'_m| \geq |A_m|/2^p \) for \( m \in B' \), and the proof of the claim is complete.

Now, using the claim recursively we construct:

1. An increasing sequence \( \{m_i : i \in \omega\} \subseteq \omega \),
2. for each \( i \in \omega \) a finite set \( a_i \subseteq \Delta_{m_i} \) of size \( i+1 \),
3. a color \( l_i < 2 \) and
4. a decreasing sequence \( \{D_i : i \in \omega\} \) of subsets of \( \Delta \) each satisfying that the sequence \( \{|(D_i)_m| : m \in \omega\} \) is unbounded.

such that \( \psi(\{x,y\}) = l_i \) for all \( x \in a_i \) and \( y \in D_i \).

For each \( m \in \omega \), pick a subset \( A_m \subseteq \Delta_m \) which is \( \psi \)-homogeneous and of the largest possible size. By Ramsey’s theorem, \( |A_m| \) is unbounded, and by our assumption, we can find an infinite subset \( B \subseteq \omega \) such that \( A_m \) is homogeneous of color 1 for all \( m \in B \). Now we will use the claim. Let \( m_0 \) be the least element of \( B \), \( C = A_{m_0} \) and \( n = 1 \), by the claim there is \( a_0 \subseteq A_{m_0} \) of size 1, \( l_0 < 2 \), \( B' \subseteq B \setminus \{m_0\} \) infinite and \( A'_m \subseteq A_m \) for each \( m \in B' \) with \( |A'_m| \) unbounded such that \( \psi(\{x,y\}) = l_0 \) for all \( x \in a_0 \) and \( y \in \bigcup_{m \in B'} A'_m \). Let \( D_0 = \bigcup_{m \in B'} A'_m \).

In step \( i+1 \), let \( m_{i+1} = \min\{m : (D_i)_m \neq \emptyset\} \), and apply the claim to \( C = (D_i)_m \), \( B = \{m > m_{i+1} : (D_i)_m \neq \emptyset\} \), \( A_m = (D_i)_m \) for \( m \in B \) and \( n = i+1 \), to obtain \( B' \subseteq B \), \( a_{i+1} \subseteq C \), \( A'_m \subseteq A_m \) for \( m \in B' \) and \( l_i < 2 \) as in the conclusion of the claim. Put \( D_{i+1} = \bigcup_{m \in B'} A'_m \).

Finally, let \( E \subseteq [\omega]^\omega \) and \( l < 2 \) be such that \( l_i = l \) for all \( i \in E \). If \( l = 1 \), then \( \bigcup_{i \in E} a_i \) is an \( \mathcal{E}\mathcal{D}_{fin} \)-positive \( \psi \)-homogeneous set of color 1. Now we observe that \( l \) cannot be 0, in fact, suppose not and let \( z_i \in a_i \) for \( i \in E \). Then \( \{z_i : i \in E\} \) is a \( \psi \)-homogeneous infinite set of color 0 which contradicts our assumption.
(3) Let \( \varphi : \Delta \to 2 \) be given by \( \varphi(\{(n_1, m_1), (n_2, m_2)\}) = 0 \) if and only if \( n_1 = n_2 \). It is clear that every 0-homogeneous set is contained in a column, and every 1-homogeneous is a partial selector of columns and so 1-homogeneous are in \( \mathcal{E} \mathcal{D}_{\text{fin}} \).

In fact, every ideal \( I \geq_{KB} \mathcal{E} \mathcal{D}_{\text{fin}} \) fails to satisfy the property \( \omega \longrightarrow (\omega, 1^+)^2_2 \) since every such ideal admits a partition of \( \omega \) into finite sets such that every selector of the partition is in \( I \), and so, a coloring analogous to the one for \( \mathcal{E} \mathcal{D}_{\text{fin}} \) witnesses that \( I \) does not satisfy \( \omega \longrightarrow (\omega, 1^+)^2_2 \). In other words, \( \omega \longrightarrow (\omega, 1^+)^2_2 \) implies \( Q^+(\omega) \). However, the reverse implication does not hold, since \( \text{Fin} \times \text{Fin} \) is a \( Q^+(\omega) \)-ideal but \( \omega \rightarrow (\omega, \text{Fin} \times \text{Fin}^+)^2_2 \).

Now we will present a method for constructing Ramsey(\( \omega \)) ideals. Given an ideal \( I \) on \( \omega \), the ideal \( \tilde{I} \) is defined as follows.

\[
\tilde{I} = \{ A \subseteq \omega \times \omega : \exists k \in \omega (\forall i < k (A)_i \in I) \land (\forall i > k (|A|)_i < k)\}.
\]

It is clear that if \( I \) is a Borel ideal then \( \tilde{I} \) is a Borel ideal too. In fact, if \( I \) is \( F_\sigma \) then so is \( \tilde{I} \).

**Lemma 4.15.** Suppose \( I^+ \longrightarrow (< \omega, 1^+)^2_2 \). Then for every \( X \in I^+ \) and every coloring \( \varphi : [X]^2 \to 2 \), there is an \( I \)-positive \( \varphi \)-homogeneous \( Y \subseteq X \), or for every \( n \in \omega \) and each \( i < 2 \) there is a \( \varphi \)-homogeneous set of color \( i \) of size \( n \).

**Proof.** Apply the hypothesis to both \( \varphi \) and \( 1 - \varphi \).

**Theorem 4.16.** If \( I^+ \longrightarrow (< \omega, 1^+)^2_2 \) then \( \omega \longrightarrow (\tilde{I}^+)^2_2 \).

**Proof.** Let a coloring \( \psi : [\omega \times \omega]^2 \to 2 \) be given. Aiming toward a contradiction assume that there is no homogeneous \( X \in \tilde{I}^+ \).

**Claim.** Given \( C \in I^+ \), \( B \subseteq \omega \) infinite, \( A_m \in I^+ \) for \( m \in B \), \( m_0 \notin B \) and \( n \in \omega \), there are a finite set \( a \subseteq \{m_0\} \times C \) of size \( n \), \( B' \subseteq B \) infinite, \( l < 2 \) and \( A'_m \subseteq A_m \) in \( I^+ \) for \( m \in B' \) such that \( \psi(\{x, y\}) = l \) for all \( x \in a \) and \( y \in a \cup \bigcup_{m \in B'} \{m\} \times A'_m \) with \( y \neq x \).

**Proof of the claim:** Let \( \{c_i : i \in \omega\} \) be an enumeration of \( C \). Put \( x_0 = (m_0, c_0) \). For each \( m \in B \), choose \( \epsilon_m < 2 \), and \( A^0_m \subseteq A_m \) in \( I^+ \) such that \( \psi(\{x_0, y\}) = \epsilon_m \) for all \( y \in \{m\} \times A^0_m \).

Now choose \( B_0 \subseteq B \) infinite and \( l_0 \in 2 \) such that \( \epsilon_m = l_0 \) for all \( m \in B_0 \). We can repeat this construction and recursively find sets \( B_i, A^i_m \) for \( m \in B_i \) and \( l, i < 2 \) such that for all \( i \in \omega \)

- (i) \( B_{i+1} \subseteq B_i \) are infinite sets,
- (ii) \( A^i_m \in I^+ \) for \( m \in B_i \) and \( A^{i+1}_m \subseteq A^i_m \) for all \( m \in B_{i+1} \),
- (iii) \( \psi(\{x, y\}) = l \) for all \( y \in \bigcup_{m \in B_i} \{m\} \times A^i_m \).

Let \( l < 2 \) and \( D \subseteq \omega \) be an infinite set such that \( l_i = l \) for all \( i \in D \) and let \( C' = \{c_i : i \in D\} \in I^+ \). Since \( \{m_0\} \times C' \) is in \( \tilde{I}^+ \) and there are no such \( \psi \)-homogeneous sets, then by Lemma 4.15 (applied to the coloring of \( C' \) given by the restriction of \( \psi \) to \( \{m_0\} \times C' \)), there is \( a \subseteq \{m_0\} \times C' \) of size \( n \) which is \( \psi \)-homogeneous of color \( l \). Finally, let \( k \) be the maximum of the \( j \)'s such that \( x_j \in a \), then put \( B' = B_k \) and \( A'_m = A^k_m \) for \( m \in B' \). We have completed the proof of the claim.

Now, using the claim recursively we construct:

- (i) An increasing sequence \( \{m_i : i \in \omega\} \subseteq \omega \),
(ii) for each \( i \in \omega \) a finite set \( a_i \subseteq \{ m_i \} \times \omega \) of size \( i + 1 \),
(iii) a color \( l_i < 2 \), and
(iv) a decreasing sequence \( \{ D_i : i \in \omega \} \) of subsets of \( \omega \times \omega \) each satisfying that
\[
| \{ m : (D_i)_m \in 1^+ \} | = \aleph_0.
\]
such that \( \psi(\{ x, y \}) = l_i \) for all \( x \in a_i \) and \( y \in D_i \).

Let \( m_0 = 0 \). By the claim applied to \( B = \omega \setminus \{ 0 \} \), \( A_m = \omega \) for \( m \in B \), \( C = \omega \), and \( n = 1 \),
there is \( a_0 \subseteq \{ m_0 \} \times \omega \) of size 1, \( l_0 < 2 \), \( B' \subseteq B \) infinite and an \( l \) positive set \( A'_m \subseteq A_m \)
for each \( m \in B' \) such that \( \psi(\{ x, y \}) = l_0 \) for all \( x \in a_0 \) and \( y \in \bigcup_{m \in B'} \{ m \} \times A'_m \).
Let \( D_0 = \bigcup_{m \in B'} \{ m \} \times A'_m \).

In step \( i + 1 \), let \( m_{i+1} = \min \{ m : (D_i)_m \in 1^+ \} \) and apply the claim to \( C = (D_i)_{m_{i+1}} \),
\( B = \{ m > m_{i+1} : (D_i)_m \in 1^+ \} \), \( A_m = (D_i)_m \) for \( m \in B \) and \( n = i + 1 \), to obtain
\( B' \subseteq B \), \( a_{i+1} \subseteq A'_m \subseteq A_m \) for \( m \in B' \) and \( l_i < 2 \) as in the conclusion of the claim. Put
\( D_{i+1} = \bigcup_{m \in B'} \{ m \} \times A'_m \).

Finally, let \( E \subseteq [\omega]^{\omega} \) be such that \( l_i = l_j \) for all \( i, j \in E \). Then \( \bigcup_{i \in E} a_i \) is an \( I \)-positive
homogeneous set for \( \psi \), which contradicts our assumption.

An immediate consequence of the 4.12, 4.13, 4.14 and 4.16 is the following.

**Corollary 4.17.** \( \omega \rightarrow (nwd^+)_2 \) and \( \omega \rightarrow (\bar{\mathcal{ED}}_{fin}^+)_2 \).

Next we list the partition properties of the ideal \( \mathcal{ED} \).

**Theorem 4.18.** The ideal \( \mathcal{ED} \) has the following properties.

1. \( \omega \nrightarrow (\mathcal{ED}^+)_2 \).
2. \( \mathcal{ED}^+ \nrightarrow (\omega, \mathcal{ED}^+)_2 \).
3. \( \omega \rightarrow (\omega, \mathcal{ED}^+)_2 \).
4. \( \mathcal{ED}^+ \rightarrow (\omega, \mathcal{ED}^+)_2 \).

**Proof.** For (1) and (2), use a coloring analogous to the one appearing in the proof of part (3) of Theorem 4.14, moreover, for (2) use the fact that \( \Delta \in \mathcal{ED}^+ \).

(3) The argument is similar to the one used in the proof of Theorem 4.16. Let a coloring
\( \psi : [\omega \times \omega]^2 \rightarrow 2 \) be given, and suppose that there are no infinite 0-homogeneous sets.

**Claim.** Let \( C \) and \( B \) be infinite subsets of \( \omega \), \( m_0 \not\in B \) and \( A_m \) with \( m \in B \), a collection of
infinite subsets of \( \omega \). For all \( n \in \omega \) there are a finite set \( a \subseteq \{ m_0 \} \times C \) of size \( n \), \( B' \subseteq B \)
infinite, \( l < 2 \) and \( A'_m \subseteq A_m \) infinite for \( m \in B' \) such that \( \psi(\{ x, y \}) = l \) for all \( x \in a \) and \( y \in \bigcup_{m \in B'} \{ m \} \times A'_m \).

**Proof of the claim:** Let \( \{ c_i : i \in \omega \} \) be an enumeration of \( C \). Put \( x_0 = (m_0, c_0) \). For each \( m \in B \), choose \( \epsilon_m < 2 \), and \( A^0_m \subseteq A_m \) infinite such that
\( \psi(\{ x_0, y \}) = \epsilon_m \) for all \( y \in \{ m \} \times A^0_m \). Now choose \( B_0 \subseteq B \) infinite and \( l_0 < 2 \) such that \( \epsilon_m = l_0 \) for all \( m \in B_0 \).
Repeat this construction, and recursively find sets \( B_i, A^i_m \) for \( m \in B_i \) and \( l_i < 2 \) such that
for all \( i \in \omega \)

(i) \( B_{i+1} \subseteq B_i \) are infinite sets,
(ii) \( A'_m \) infinite for all \( m \in B_i \) and \( A^{i+1}_m \subseteq A'_m \) for all \( m \in B_{i+1} \), and
(iii) \( \psi(\{ x, y \}) = l_i \) for all \( y \in \bigcup_{m \in B_i} \{ m \} \times A^i_m \).
Lemma 4.19. If \( H \) isomorphism) such that for every graph \( G \), the result follows from Theorem 4.14.

Given a graph \( G \), the girth of \( G \) is defined as the minimal length of a cycle contained in \( G \). Erdös' lemma claims that for any \( k < \omega \) there is a graph \( H \) with girth greater than \( k \) and chromatic number greater than \( k \).

Now aiming for a contradiction, suppose that \( A_1, A_2, \ldots, A_k \) are mutually disjoint free sets with respect to \( E_n \) such that \( \bigcup_{i=0}^k A_i = \omega \), and let \( H \) be a graph with girth greater than
$l = \max\{k, n\}$. By the property of $E_n$, there is $X \subseteq \omega$ such that $\langle \omega, H \rangle \cong \langle X, E_n \upharpoonright X \rangle$. However, there is a coloring for $E_n$ in $k$ colors, and then, the chromatic number of $H$ is not greater than $l$. Since this happens to all graphs with girth greater than $l$, we contradict Erdős’ lemma.

The ideal $R_n$ is critical with respect to the $n$-Ramsey property.

**Theorem 4.21.** Let $I$ be an ideal on $\omega$. Then $I$ is $n$-Ramsey if and only if $I \not\subseteq K R_n$.

**Proof.** If $I$ is not $n$-Ramsey then there is a coloring $\varphi : [\omega]^2 \to 2$ such that there is no 0-homogeneous set of cardinality $n$ and every 1-homogeneous set is in $I$. Let $H = \varphi^{-1}\{0\}$. Note that $H$ does not have complete subgraphs of cardinality $n$, and so, there is an isomorphism $f : \langle \omega, H \rangle \to \langle A, E_n \upharpoonright A \rangle$ for some $A \subseteq \omega$. The function $f$ then witnesses $I \not\supseteq K R_n$, since for every $E_n$-free set $B$, $f^{-1}(B)$ is 1-homogeneous respect to $\varphi$, and so, $f^{-1}(B) \in I$.

On the other hand, let $f$ be a witness to $I \supseteq K R_n$, i.e., a function such that for every $G_n$-free set $f^{-1}(A) \in I$, and define $\varphi : [\omega]^2 \to 2$ by $\varphi(\{m, n\}) = 0$ iff $f(m) \neq f(n)$ and $\{f(m), f(n)\} \in E_n$. If $A$ is a 0-homogeneous set with $|A| = n$ then $f[A]$ is a complete subgraph of $G_n$, a contradiction. If $A$ is a 1-homogeneous set then $f[A] \in R_n$ and so, $A \subseteq f^{-1}(f[A]) \in I$.

We do not know of any ideal satisfying $I^+ \to (\omega, I^+)_2$ and $I^+ \not\to (I^+)_2$. This is the only arrow in the diagram at the beginning of this section of which we do not know that it does not reverse. Let us mention that in [3] it was shown the two partition relations are equivalent for maximal ideals (i.e. ultrafilters). The proof essentially amounts to showing that $I^+ \to (\omega, I^+)_2$ implies that the ideal is $P^+_\text{tower}$ and clearly also $Q^+$.  

## 5. Monotonicity and Ramsey property

In [8] the authors gave the following definitions:

**Definition 5.1.** Let $I$ be an ideal on $\omega$. Then

1. $I$ is **Mon** (monotone) if for any sequence $S = \langle x_n : n \in \omega \rangle$ of real (equivalently rational) numbers there is an $I$-positive set $X$ such that $S \upharpoonright X$ is monotone.
2. $I$ is **$h$-Mon** (hereditarily monotone) if $I \upharpoonright A$ is Mon for all $A \in I^+$.
3. $I$ is **Fin-BW** (Fin-Bolzano-Weierstrass) if for any bounded sequence $S = \langle x_n : n \in \omega \rangle$ of real numbers there is an $I$-positive set $X$ such that $S \upharpoonright X$ is convergent.
4. $I$ is **locally selective** if for every partition $\{A_n : n \in \omega\}$ of $\omega$ into sets from $I$ there is an $I$-positive set $S$ such that $|A_n \cap S| \leq 1$.

It is easy to see that Ramsey($\omega$) implies Mon and Mon implies Fin – BW. Moreover,

1. $I$ is Fin – BW if and only if $I \not\subseteq K \text{conv}$ and
2. $I$ is locally selective if and only if $I \not\subseteq K \mathcal{E} \mathcal{D}$.

Lemma 3.9 in [8] claims that if $I$ is Mon then is locally selective. The authors of [8] remark that the summable ideal $1^n \omega$ is Fin – BW but is not Mon and they ask if every Mon ideal satisfies $\omega \longrightarrow (I^+)^n_2$, for all $n \in \omega$. The following is an easy consequence of their results.

**Theorem 5.2.** If $I$ is Mon then $I \not\subseteq K \text{conv}$ and $I \not\subseteq K \mathcal{E} \mathcal{D}$.

On the other hand we have a criterion for monotonicity of ideals.
\textbf{Theorem 5.3.} Let $I$ be an ideal on $\omega$. If $I \nec K \text{ conv}$ and $I \cap X \nec K \mathcal{ED}$ for all $l$-positive set $X$ then $I$ is Mon.

\textit{Proof.} Given $f : \omega \to \mathbb{Q}$ there is $A \in l^+$ such that $f''A$ is a convergent sequence, since $I \nec K \text{ conv}$. Consider the partition $\{f^{-1}\{m\} : m \in f''A\}$ of $A$. If there is $m \in f''A$ such that $f^{-1}\{m\} \in l^+$ we are done. If not, there is an $l$-positive subset $B$ of $A$ such that $f \upharpoonright B$ is one to one, since $I \nec K \mathcal{ED}$. Let $l$ be the limit of $f \upharpoonright B$. Then either $B_0 = B \cap f^{-1}(-\infty, l) \in l^+$ or $B_1 = B \cap f^{-1}(l, \infty) \in l^+$. Let us suppose the first case (the other case is analogous). Let $\{b_k : k < \omega\}$ be the increasing enumeration of $B_0$. Let $k_0 = 0$ and $k_1$ be such that $f(b_k) > f(b_0)$ for all $k \geq k_1$; and for every $j \geq 1$ let $k_{j+1}$ be such that $f(b_k) > \max\{f(b_i) : i < k_j\}$ for all $k \geq k_{j+1}$. For any $i < \omega$ define the family $C_i = \{b_k : k_i \leq k \leq k_{j+1}\}$. Then $\{C_i : i < \omega\}$ is a partition of $B$ in finite sets. Since $I \cap B \nec K \mathcal{ED}_{fin}$ there is an $l$-positive subset $D$ of $B$ such that $|C \cap C_i| \leq 1$ for all $i < \omega$. Now, either $D_0 = \bigcup_{i<\omega}(C \cap C_{2i}) \in l^+$ or $D_1 = \bigcup_{i<\omega}(C \cap C_{2i+1}) \in l^+$, and $f \upharpoonright D_0$ and $f \upharpoonright D_1$ are both increasing sequences.

In [8] the authors ask about colorings with more than two colors. An immediate corollary of the theorem is the following:

\textbf{Corollary 5.4.} If $I$ is $(\omega, 2)$-distributive and $I \cap X \nec K \mathcal{ED}$ for all $l$-positive $X$ then $\omega \longrightarrow (l^+)^2_k$ for all $k$.

Recently Kwela in [16] described a monotone $F_\sigma$ ideal $I$ such that $\omega \longrightarrow (l^+)^2_2$. We present an example of an $F_\sigma$-ideal for which the number of colors matters. This answers a question from [8]).

\textbf{Theorem 5.5.} $\omega \longrightarrow (\mathcal{ED}^+)^2_2$ but $\omega \nec (\mathcal{ED}^+)^2_3$.

\textit{Proof.} By corollary 4.18, $\omega \longrightarrow (\omega, \mathcal{ED}^+)^2_3$, and by 4.16, $\omega \longrightarrow (\mathcal{ED}^+)^2_2$. Let us consider the following coloring $\varphi : [\omega \times \omega \times \omega]^3 \to 3$ given by

$$\varphi\{(n_1, n_2, n_3), (m_1, m_2, m_3)\} = \begin{cases} 0 & \text{if } n_1 = m_1 \text{ and } n_2 = m_2 \\ 1 & \text{if } n_1 = m_1 \text{ and } n_2 \neq m_2 \\ 2 & \text{if } n_1 \neq m_1 \end{cases}$$

Let $A$ be a subset of $\omega \times \omega \times \omega$. If $A$ is 0-homogeneous set then the first projection of $A$ is in $\mathcal{ED}$ because second projection of $A$ is contained in a column of $\omega \times \omega$; if $A$ is 1-homogeneous then the first projection of $A$ is in $\mathcal{ED}$ because second projection of $A$ is contained in a selector of columns in $\omega \times \omega$. Finally, if $A$ is 2-homogeneous, then $A$ is contained in a selector of the columns of the product $\omega \times (\omega \times \omega)$. In the three cases, $A \in \mathcal{ED}$. \hfill \square

6. Questions

The main question considered but not answered in the paper is the following

\textbf{Question 6.1.} Is there a Borel tall ideal $I$ on $\omega$ satisfying $I^+ \longrightarrow (l^+)^2_2$?

The question is equivalent to:

\textbf{Question 6.2.} Is $R$ locally $\leq K$-minimal among Borel tall ideals?
A positive answer to the second question would be a positive answer to half of the following more general question:

**Question 6.3.** Is there a (locally) $\leq_K$-minimal ideal $I$ among Borel ideals?

A related question is the following:

**Question 6.4.** Does every tall Borel ideal $I$ contain an $F_\sigma$ tall ideal? Does this hold locally, i.e. Is there for every tall Borel ideal $I$ a set $X \in I^+$ such that $I \upharpoonright X$ contains a tall $F_\sigma$ ideal (on $X$)?

A result of M. Laczkovich and I. Reclav [19] shows that for every Borel ideal $I$ either $I \geq_K \text{Fin} \times \text{Fin}$ or there is a $F_\sigma$ set $E$ such that $I \subseteq E$ and $E \cap I^* = \emptyset$. We wish to know if the $F_\sigma$ hypothesis could be weakened to $F_{\sigma\delta}$ in order to replace “set” with “ideal”. Solecki [25, Corollary 1.5] has shown that for any $F_{\sigma\delta}$ ideal $I$ there is a $F_\sigma$ set $F$ such that $I \subseteq F$ and $F \cap I^* = \emptyset$.

**Question 6.5.** Does every Borel ideal $I$ satisfy that either $I \geq_K \text{Fin} \times \text{Fin}$ or there is an $F_{\sigma\delta} -$ ideal $J$ such that $I \subseteq J$?

**Question 6.6.** Is there an ideal such that $I^+ \to (\omega, I^+)^2_2$ and $I^+ \not\to (I^+)^2_2$? Is there Borel such ideal?

Another promising line of research deals with higher dimensions, that is Ramsey type properties of ideals considering colorings of $n$-tuples for $n > 2$. A standard “stepping up” type argument shows that $I^+ \to (I^+)^2_2$ implies the stronger statement $I^+ \to (I^+)^n_k$ for all $n, k > 0$. In [3], Baumgartner and Taylor in effect showed that $I^+ \to (4, I^+)^3_2$ is equivalent to $I^+ \to (I^+)^2_2$, which shows that higher dimensions do behave differently from dimension 2.

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**References**


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