Cardinal Invariants of Strongly Porous Sets

Osvaldo Guzmán
Michael Hrušák
Arturo Martínez-Celis

In this work we study the cardinal invariants of the ideal of strongly porous sets on ${}^{\omega}2$. We prove that $\mathsf{add}(\mathbf{SP}) = \omega_1$, $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$ and that it is consistent that $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$, answering questions of [5]. We also find a connection between the strongly porous sets on ${}^{\omega}2$ and the Martin number for σ-linked forcings, and we use this connection to construct a model where all the Martin numbers for σ-k-linked forcings are mutually different.

03E17, 28A75; 03E35, 03E50

1 Introduction

The notion of porosity is a concept of smallness. Intuitively, a subset of a metric space is porous if it has holes that are big in some sense. The study of σ-porous sets began in 1967 in [4] and since then, many applications have been found. One of these applications can be found in [8], where the authors proved that, given a Banach space X with a separable dual and a continuous convex function f on X, the set of points which f is not Fréchet differentiable is σ-porous. Other applications can be found in [2], [6], [9], [10] and [14].

We shall study the notion of strong porosity: Given a metric space $\langle X, d\rangle$, a subset $A \subseteq X$ is *strongly porous* if there is a $p > 0$ such that for any $x \in X$ and any $0 < r < 1$, there is $y \in X$ such that $B_{p\cdot r}(y) \subseteq B_r(x) \setminus A$. In this paper we will refer to strongly porous sets as porous sets. We shall work mostly with porous sets in ${}^{\omega}2$: We will say that a set $A \subseteq {}^{\omega}2$ is n-porous if for every $s \in {}^{<\omega}2$ there is a $t \in {}^{n}2$ such that $\langle s^\frown t\rangle \cap A = \emptyset$. By $s^\frown t$ we denote the concatenation of s followed by t, and by $\langle s\rangle$ we denote the cone of s, that is $\langle s\rangle = \{f \in {}^{\omega}2 : s \subseteq f\}$. It can be shown (see [5]) that a set $A \subseteq {}^{\omega}2$ is porous if and only if there is an $n \in \omega$ such that A is n-porous. A set A in a metric space $\langle X, d\rangle$ is σ-porous if and only if it is σ-lower porous (see [13]), where A is *lower porous* if for every $x \in X$ there exists $\rho_x > 0$ and $r_{0_x} > 0$ such that for any $0 \leq r \leq r_{0_x}$ there is $y \in X$ such that $B_{\rho_x\cdot r}(y) \subseteq B_r(x) \setminus A$. Another classical notion of porosity is upper porosity: A set A in a metric space $\langle X, d\rangle$ is *upper porous* if for every $x \in X$ there is $\rho_x > 0$ and a sequence $r_n \to 0$ such that for every $n \in \omega$ there is $y_n \in X$ such that $B_{\rho \cdot r_n}(y_n) \subseteq B_{r_n}(x) \setminus A$. We will denote the σ-ideal generated by porous sets on ${}^{\omega}2$ by $\mathbf{SP}$, the σ-ideal generated by n-porous sets by $\mathbf{SP}_n$, and the σ-ideal generated by upper porous sets by $\mathbf{UP}$. A detailed survey of the different types of porosity can be found in [13].

Cardinal invariants of these σ-ideals have been studied in [3], [5], [10], [11] and [12]. Recall that, given a σ-ideal $\mathcal{I}$ over a set X, the following are the standard cardinal invariants of $\mathcal{I}$:

$$\begin{aligned}
\mathsf{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\},\\
\mathsf{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \neq X\},\\
\mathsf{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\},\\
\mathsf{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I}(\exists A \in \mathcal{A}(B \subseteq A))\}.
\end{aligned}$$

In [5] the authors proved that the cardinal invariants of the σ-ideal of lower porous sets in the real line are the same as the cardinal invariants of $\mathbf{SP}$. The authors proved that $\mathsf{non}(\mathbf{SP}) < \mathfrak{m}_{\sigma\text{-centered}}$ is consistent, that $\mathsf{cov}(\mathbf{SP}) > \mathsf{cof}(\mathcal{N})$ is consistent, and that $\mathsf{cov}(\mathbf{SP}) < \mathfrak{c}$ is consistent, where $\mathfrak{m}_{\sigma\text{-centered}}$ is the smallest cardinal where the Martin's axiom for σ-centered forcings fails and $\mathcal{N}$ is the ideal of sets of Lebesgue measure zero. In contrast with these results, there are some analogue inequalities that holds for $\mathbf{UP}$. In [12], M. Repický proved that $\mathsf{non}(\mathbf{UP}) \geq \mathfrak{m}_{\sigma\text{-centered}}$ and $\mathsf{cov}(\mathbf{UP}) \leq \mathsf{cof}(\mathcal{N})$ holds. He also proved [10] that $\mathsf{non}(\mathbf{UP}) \geq \mathsf{add}(\mathcal{N})$ and in [3], J. Brendle proved that $\mathsf{add}(\mathbf{UP}) = \omega_1$ and $\mathsf{cof}(\mathbf{UP}) = \mathfrak{c}$ holds. In [5], the authors asked if those last three inequalities hold for the $\mathbf{SP}$ ideal. In this work we give an answer to that question: We show that $\mathsf{add}(\mathbf{SP}) = \omega_1$, $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$ and that it is consistent that $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$.

Given $k \in \omega$ and a forcing notion $\mathbb{P}$ a subset $A \subseteq \mathbb{P}$ is *k-linked* if for every collection $\{a_i : i \in k\}$ of k elements of A, there is an $a \in \mathbb{P}$ stronger than each a_i, that is, for every $i \in k$, $a \leq a_i$. $\mathbb{P}$ is *σ-k-linked* if $\mathbb{P}$ is the countable union of k-linked subsets of $\mathbb{P}$. We will denote $\mathfrak{m}_k$ the *Martin number for* σ-k-linked forcings, that is, the smallest cardinal κ such that there is a σ-k-linked forcing $\mathbb{P}$ and κ $\mathbb{P}$-dense subsets of $\mathbb{P}$ such that no filter of $\mathbb{P}$ intersects them all.

If X, Y are sets, then ${}^{Y}X$ is the set of all functions from Y to X and ${}^{<\omega}X = \bigcup_{n\in\omega} {}^{n}X$. If $T \subseteq {}^{<\omega}X$ is a tree, then by $[T]$ we denote the set of branches of T, that is, $[T] = \{f \in {}^{\omega}X : \forall n \in \omega(f \restriction n \in T)\}$. For $\mathsf{end}(T)$ we will denote the end nodes of T, that is the nodes of T without extensions. If $\sigma, s \in {}^{<\omega}2$, then we will denote that σ is an initial segment of s by $\sigma \sqsubseteq s$. In our forcing notation, the stronger conditions are the smaller ones. For everything else, our notation follows [1].

2 Additivity and cofinality.

The main goal of this section is to prove that $\mathsf{add}(\mathbf{SP}) = \omega_1$ and $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$. We will use the following notion.

Definition 2.1 *Let $k \in \omega$. A tree $T \subseteq {}^{<\omega}2$ is a k-*porous tree *if for every $s \in {}^{<\omega}2$ there is $t \in {}^{k}2$ such that $s^\frown t \notin T$.*

Note that $A \subseteq {}^{\omega}2$ is k-porous if and only if there is a k-porous tree T such that $[T]$ contains A.

Theorem 2.2 *There is a family $\{T_f : f \in {}^{\omega}2\}$ of 2-porous trees such that for every $X \in \mathbf{SP}$, the set $\{f \in {}^{\omega}2 : [T_f] \subseteq X\}$ is countable.*

Proof We will construct the family $\{T_f : f \in {}^{\omega}2\}$ as follows: For every $a \subseteq {}^{<\omega}2$ such that $|a| = 2^n$, let $\varphi_a : a \to {}^{n}2$ be a bijective function. For every $i \in \omega$, let $\psi_i : \{a \subseteq {}^{i}2 : \exists k \in \omega(|a| = 2^k)\} \to \omega \setminus \{0\}$ be an injective function. If $a \subseteq {}^{i}2$ and $|a| = 2^k$, define

$$\sigma_a = \langle 0, \underbrace{1, \ldots, 1}_{2\psi_i(a) \text{ times}}, 0\rangle.$$

For each $\sigma \in {}^{<\omega}2$, we will recursively define a finite tree T_σ as follows: $T_\emptyset = \{\emptyset\}$ and if T_σ is defined, then

$$T_{\sigma^\frown i} = \{s \in {}^{<\omega}2 : \exists t \in \mathsf{end}(T_\sigma)(\exists j \in \omega(\exists a \subseteq {}^{|\sigma|+1}2$$
$$(|a| = 2^j \wedge \sigma^\frown i \in a \wedge s \sqsubseteq t^\frown \sigma_a^\frown \varphi_a(\sigma^\frown i))))\} \cup \{s \in {}^{<\omega}2 : \exists t \in \mathsf{end}(T_\sigma)(s \sqsubseteq t^\frown \langle 1, 1\rangle)\}.$$

It is easy to see that, for each $\sigma \in {}^{<\omega}2$, T_σ is a finite 2-porous tree and that if $\sigma \sqsubseteq \tau$, then $T_\sigma \subseteq T_\tau$. For each $f \in {}^{\omega}2$, define $T_f = \bigcup_{n\in\omega} T_{f\restriction n}$. It follows easily that each T_f is a 2-porous tree.

We will show that the family $\{T_f : f \in^{\omega} 2\}$ is the family we were looking for: Let $X \in \mathbf{SP}$. Without loss of generality we will assume that $X = \bigcup_{i\in\omega}[T_i]$, where T_i is an $i+1$-porous tree. We must show that the set $B = \{f \in {}^{\omega}2 : [T_f] \subseteq X\}$ is countable: For each $s, t \in {}^{<\omega}2$ and each $n \in \omega$, define $B_{s,t,n} = \{f \in {}^{\omega}2 : t \sqsubseteq f, s \in T_t \wedge [T_f] \cap \langle s\rangle \subseteq [T_n]\}$. We will see that $B \subseteq \bigcup_{s,t\in{}^{<\omega}2, n\in\omega} B_{s,t,n}$: If f is such that $f \in B$, then $[T_f] \subseteq \bigcup_{n\in\omega}[T_i]$. Using the Baire Category Theorem we can find $s \in T_f$ and $n \in \omega$ such that $[T_f] \cap \langle s \rangle \subseteq [T_n]$. Find $k \in \omega$ such that $s \in T_{f \restriction k}$. It follows that $f \in B_{s,f\restriction k,n}$. To finish the proof we will see that each $B_{s,t,n}$ has at most $2^{n+1}-1$ elements: Suppose this is not the case and let $s,t \in {}^{<\omega}2$, $n \in \omega$ and $\{f_i\}_{i<2^{n+1}} \subseteq B_{s,t,n}$. Extend s to σ such that $\sigma \in \mathsf{end}(T_t)$. Let $j \in \omega$ be such that the set $a = \{f_i \restriction j : i < 2^{n+1}\}$ has 2^{n+1} elements and let

$$s = \sigma^\frown \langle \underbrace{1, \ldots, 1}_{2\cdot(j-|t|)\text{ times}} \rangle^\frown \sigma_a.$$

The tree T_n is $n+1$-porous, so there is a $\tau \in 2^{n+1}$ such that $s^\frown \tau \notin T_n$. Find $k < 2^{n+1}$ such that $\varphi_a(f_k|j) = \tau$ and observe that $s^\frown\tau = s^\frown\varphi_a(f_k|j) \in T_{f_k}$. As a consequence, $[T_{f_k}] \cap \langle t \rangle \nsubseteq [T_n]$, but this contradicts the fact that $f_k \in B_{s,t,n}$. This implies that each $B_{s,t,n}$ is finite, and therefore B is countable. $\square$

We can now prove the main result of this section.

Corollary 2.3 $\mathsf{add}(\mathbf{SP}) = \omega_1, \mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$.

Proof Let $\{T_f : f \in {}^\omega 2\}$ be the family given by the theorem above. If $H \subseteq {}^\omega 2$ is an uncountable set, then the set $\bigcup\{[T_f] : f \in H\} \notin \mathbf{SP}$. As a consequence, $\mathsf{add}(\mathbf{SP}) = \omega_1$. On the other hand, if $\kappa < \mathfrak{c}$ and if $\{X_\alpha : \alpha < \kappa\} \subseteq \mathbf{SP}$, then there is an $f \in {}^\omega 2$ such that, for every $\alpha < \kappa$, $[T_f] \nsubseteq X_\alpha$ and therefore $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$. $\square$

Observe that this last proof can be used to show that $\mathsf{add}(\mathbf{SP}_n) = \omega_1 = \mathsf{add}(\mathbf{SP})$ and $\mathsf{cof}(\mathbf{SP}_n) = \mathfrak{c} = \mathsf{cof}(\mathbf{SP})$. In section 4 and 5 we will see that the behavior of the uniformity number and the covering number is more complicated.

3 Uniformity number

In this section we will prove the consistency of $\mathsf{non}(\mathbf{SP}) < \mathsf{add}(\mathcal{N})$. We will also develop some tools that we will use later in this paper. We will need the following concept, inspired in the usual concept of a k-Sacks tree in ${}^{<\omega}k$.

Definition 3.1 *Let $k \in \omega$. A tree $T \subseteq {}^{<\omega}k$ is a k-anti-Sacks tree if for every $s \in T$ there is $i < k$ such that $s^\frown\langle i \rangle \notin T$. We will denote by $\mathbf{AS}_k$ the σ-ideal generated by the branches of k anti-Sacks trees.*

This notion corresponds to the analogue of the notion of 1-porous tree in ${}^{<\omega}k$ and it is closely related to the k-Sacks forcing. Recall that a *k-Sacks tree T* is a tree on ${}^{<\omega}k$ such that for every $s \in T$, there is a $t \in T$ such that, for every $i < k$, $t^\frown i \in T$. The k-Sacks forcing $\mathbb{S}_k$ is the collection of all k-Sacks trees ordered by reverse inclusion. It is well-known that the k-Sacks forcing is equivalent to $\mathrm{Borel}({}^{\omega}k)/\mathbf{AS}_k$. There is a natural connection between ${}^{k}2$-anti-Sacks trees and k-porous sets given by the following argument:

Let $\varphi_k : 2^k \to {}^{k}2$ be a bijective function. Let $\psi_k : {}^{\omega}(2^k) \to {}^{\omega}2$ defined by $\psi_k(x) = \varphi_k(x|1)^\frown \varphi_k(x|2)^\frown \ldots$ Clearly $A \in \mathbf{AS}_{2^n}$ if and only if $\psi_n(A) \in \mathbf{SP}_n$. As a consequence, the ideals $\mathbf{AS}_{2^k}$ and $\mathbf{SP}_k$ share the same cardinal invariants, i.e. $\mathrm{add}(\mathbf{SP}_k) = \mathrm{add}(\mathbf{AS}_{2^k})$, $\mathrm{cov}(\mathbf{SP}_k) = \mathrm{cov}(\mathbf{AS}_{2^k})$, $\mathrm{non}(\mathbf{SP}_k) = \mathrm{non}(\mathbf{AS}_{2^k})$ and $\mathrm{cof}(\mathbf{SP}_k) = \mathrm{cof}(\mathbf{AS}_{2^k})$. Using a similar argument to the ones we gave in the last section, it is possible to show that $\mathsf{add}(\mathbf{AS}_k) = \omega_1$ and that $\mathsf{cof}(\mathbf{AS}_k) = \mathfrak{c}$. Alternatively, a proof of this fact can be found in [7].

We shall introduce a notion that we will use to keep $\mathsf{non}(\mathbf{SP})$ small in a forcing extension.

Definition 3.2 *Let $\mathbb{P}$ be a forcing notion and let $A \subseteq 2^\omega$ be such that $A \notin \mathbf{AS}_k$. We say that $\mathbb{P}$* strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A *if for every $\mathbb{P}$-name $\dot{X}$ of a k-anti-Sacks tree there is a $Y \in \mathbf{AS}_k$ such that, for every $x \in A$, if $x \notin Y$ then $\mathbb{P} \Vdash$ "$x \notin \dot{X}$". We will say that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ if $\mathbb{P}$* strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ *in ${}^{\omega}k$.*

It is easy to see that, if $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A, then $\Vdash$ "$\mathbb{P}$"$A \notin \mathbf{AS}_n$ and if $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$, then $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_n)$ in A for every $A \subseteq {}^{\omega}k$. The following lemma will show the connecion between strongly preserving $\mathsf{non}(\mathbf{AS}_k)$ and preserving $\mathsf{non}(\mathbf{SP})$ as a small cardinal.

Lemma 3.3 *Suppose that a forcing notion $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ for every $k \in \omega$, then $\mathbb{P} \Vdash$ "${}^{\omega}2 \cap V \notin \mathbf{SP}$".*

Proof Let $\{\dot{C}_i : i \in \omega\}$ be a collection of $\mathbb{P}$-names such that each $\dot{C}_i$ is a name for an i-porous set. For each $i \in \omega$, there is a collection $\{C^i_j : j \in \omega\}$ of i-porous sets such that if $x \in {}^{\omega}2$ and $x \notin \bigcup_{j\in\omega} C^i_j$, then $\mathbb{P} \Vdash$ "$x \notin \dot{C}_i$" (this is achieved using that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_{2^n})$ and the properties of the function ψ_k defined above). If $x \notin \bigcup\{C^i_j : i,j \in \omega\}$, then $\mathbb{P} \Vdash$ "$x \notin \bigcup\{\dot{C}_i : i \in \omega\}$". As a consequence, $\mathbb{P} \Vdash$ "${}^{\omega}2 \notin \mathbf{SP}$". $\square$

The next lemma shows that there is a connection between porous sets and σ-k-linked forcings.

Lemma 3.4 *Let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is σ-k-linked, then $\mathbb{P}$ strongly preserves $\mathbf{AS}_k$ in ${}^{\omega}k$.*

Proof Let $\{\mathbb{P}_i : i \in \omega\} \subseteq \mathbb{P}$ be a sequence of k-linked subsets such that $\mathbb{P} = \bigcup_{i\in\omega} \mathbb{P}_i$. Let $\dot{A}$ be a $\mathbb{P}$-name of an k-anti-Sacks tree. Define $T_n \subseteq {}^{\omega}k$ as follows:

$$T_n = \{s \in {}^{<\omega}k : \exists p \in \mathbb{P}_n(p \Vdash \text{“}s \in \dot{A}\text{"})\}$$

We claim that, for each $n \in \omega$, T_n is a k-anti-Sacks tree. Suppose this is not the case, so there is an $s \in T_n$ such that, for every $i \in k$, $s^\frown i \in T_n$. For every $i \in k$, we can pick a condition $p_i \in \mathbb{P}_n$ such that $p_i \Vdash$ “$s^\frown i \in \dot{A}$". Let $p \in \mathbb{P}$ be such that, for every $i \in k$, $p \leq p_i$. Then $p \Vdash$ “$\forall i \in k(s^\frown i \in \dot{A})$". This contradicts the fact that $\dot{A}$ is a $\mathbb{P}$-name of a k-anti-Sacks tree.

To conclude the proof, note that for every $x \in {}^{\omega}k$, if $p \Vdash$ “$x \in [\dot{A}]$", then $x \in [T_n]$, where n is such that $p \in \mathbb{P}_n$. □

The lemma above is optime in the sense that, for each k, there is a σ-$k-1$-linked forcing $\mathbb{P}_k$ such that $\mathbb{P}_k \Vdash$ “${}^{\omega}k \cap V \in \mathbf{AS}_k$" and therefore $\mathbb{P}_k$ does not strongly preserve $\mathbf{AS}_k$. This will be proved in the next section.

We shall show that the property of strongly preserve $\mathsf{non}(\mathbf{AS}_k)$ is preserved along finite support iterations.

Lemma 3.5 *Let $A \subseteq {}^{\omega}k$ and let $\mathbb{P} = \{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \kappa\}$ be a finite support iteration of c.c.c. forcings such that $\mathbb{P}_\alpha \Vdash$ “$\dot{\mathbb{Q}}_\alpha$ strongly preserves $\mathbf{AS}_k$ in A", then $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A.*

Proof We will proceed by induction over κ. It is easy to see that the lemma holds for succesor ordinals, and if κ has uncountable cofinality we can use a standard reflection argument to show that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in A, so it is enough to show that the lemma holds for $\kappa = \omega$: let $\dot{T}$ be a $\mathbb{P}$-name of a k-anti-Sacks tree. For each $n \in \omega$, let $\dot{T}_n$ be a $\mathbb{P}_n$-name for the following set.

$$T_n = \{s \in {}^{\omega}k \cap V : \mathbb{P}_{(n,\omega)} \Vdash \text{“}s \in \dot{X}\text{"}\}.$$

It is easy to see that each $\dot{T}_n$ is name for a k-anti-Sacks tree. Now we use that each $\mathbb{P}_n$ strongly preserves $\mathsf{non}(\mathbf{AS})$ to find a family $\{T_i^j : i,j \in \omega\}$ such that, for each $n \in \omega$, if $x \in A$ and If $x \in A$ and $x \notin \bigcup_{i\in\omega}[T_i^n]$, then $\mathbb{P} \Vdash$ “$x \notin [\dot{T}_n]$". It is easy to see that the set $Y = \bigcup\{[T_i^j] : i,j \in \omega\}$ is the set we are looking for. □

For constructing the model we are looking for, we will use the amoeba forcing $\mathbb{A}$ with the following presentation:

$$\mathbb{A} = \{B \in \text{Borel}(2^\omega) : \mu(B) > \frac{1}{2}\}$$

Borel(2^ω) represents the collection of Borel subsets of the Cantor space and μ is the standard Lebesgue measure over 2^ω. The order is given by $A \leq B$ if and only if $A \subseteq B$. The following lemma is well-known.

Lemma 3.6 *The amoeba forcing is σ-n-linked for every $n \in \omega$.*

Proof Let $n \in \omega$. For every clopen C in 2^ω, define

$$\mathbb{A}_C = \{A \in \mathbb{A} : \mu(C \backslash A) < \frac{1}{n} \cdot (\mu(C) - \frac{1}{2})\}$$

We will show that $\mathbb{A} = \bigcup\{\mathbb{A}_C : C \text{ is a clopen on } 2^\omega\}$: Let $A \in \mathbb{A}$ and let $\varepsilon > 0$ such that $\mu(A) = \frac{1}{2} + \varepsilon$. Find an open set $U \subseteq 2^\omega$ such that $A \subseteq U$ and $\mu(U \backslash A) < \frac{\varepsilon}{n}$. Now find a clopen set $C \subseteq U$ such that $\mu(C) > \frac{1}{2} + \varepsilon$. Then

$$\mu(C \backslash A) < \mu(U \backslash A) < \frac{\varepsilon}{n} = \frac{1}{n} \cdot (\frac{1}{2} + \varepsilon - \frac{1}{2}) < \frac{1}{n} \cdot (\mu(C) - \frac{1}{2}).$$

Therefore $A \in \mathbb{A}_C$. Now we must show that, for every clopen set $C \subseteq 2^\omega$, the intersection K of an arbitrary family $\{A_j : j \in n\} \subseteq \mathbb{A}_C$ is an element of $\mathbb{A}$. This is a consecuence of the following calculations:

$$\mu(C) \leq \mu(K) + \sum_{j \in n} \mu(C \backslash A_j) < \mu(K) + \frac{1}{n} \cdot (\sum_{j \in n} \mu(C) - \frac{1}{2}) = \mu(K) + \mu(C) - \frac{1}{2}.$$

As a consecuence, $\frac{1}{2} < \mu(K)$. Therefore $K \in \mathbb{A}$. $\square$

We are ready to prove the main result of this section. The method of the proof was suggested to us by J. Brendle.

Theorem 3.7 *If* ZFC *is consistent, then* ZFC $+ \operatorname{non}(\mathbf{SP}) < \operatorname{add}(\mathcal{N})$ *is consistent.*

Proof Start with a model V such that $V \models CH$. Let $\mathbb{P} = \{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2\}$ be a finite support iteration of the amoeba forcing. It follows from the lemmas above that $\mathbb{P}$ strongly preserves $\operatorname{non}(\mathbf{AS}_k)$ for every $k \in \omega$ and therefore $\mathbb{P} \Vdash$ "$2^\omega \cap V \notin \mathbf{SP}$". As a consequence, we have that $V[G] \models \operatorname{non}(\mathbf{SP}) = \omega_1$. It is a known fact (see [1]) that $V[G] \models \operatorname{add}(\mathcal{N}) = \omega_2$, hence $V[G] \models \operatorname{non}(\mathbf{SP}) < \operatorname{add}(\mathcal{N})$. $\square$

4 The Martin numbers of σ-k-linked forcings

It is easy to see that $\mathfrak{m}_2 \leq \mathfrak{m}_3 \leq \dots$ and, for each $k > 1$, it is possible to get the consistency of $\mathfrak{m}_k < \mathfrak{m}_{k+1}$ by forcing with a finite support iteration of σ-$k+1$-linked forcings over a model of CH. In this section we will construct a model where all the Martin numbers $\mathfrak{m}_i$ are mutually different. In this model, all the cardinals $\mathsf{non}(\mathbf{AS}_i)$ will be different all at once (as a consequence, the cardinals $\mathsf{non}(\mathbf{SP}_i)$ will differ from each other). We will use the following forcing notions. Given $k > 2$ let

$$\mathbb{P}_k = \{\langle s, F\rangle : \begin{array}{l} \textbf{(a) } s \text{ is a finite } k\text{-anti-Sacks trees of height } \mathsf{ht}(s), \\ \textbf{(b) } F \in [{}^{\omega}k]^{<\omega}, \text{ and } \lceil F \restriction \Delta_F \rceil \text{ is a finite } k\text{-anti-Sacks tree,} \\ \textbf{(c) } s \subseteq \lceil F \restriction \Delta_F + 1 \rceil \}, \end{array}$$

where $F \restriction k = \{f \restriction k : f \in F\}$, $\lceil F \rceil = \{s \in {}^{<\omega}k : \exists f \in F(s \subseteq F)\}$ and $\Delta_F = \min\{n \in \omega : |F \restriction n| = |F|\}$. The order is defined by $\langle s', F'\rangle \leq \langle s, F\rangle$ if and only if $s \subseteq s'$ and $F \subseteq F'$. We will be using the following proposition.

Proposition 4.1 *Given a $k > 2$, $\mathbb{P}_k \Vdash$ "${}^{\omega}k \cap V \in \mathbf{AS}_k$".*

Proof It is easy to see that, for every $f \in {}^{\omega}k$ and $n \in \omega$, the following sets are dense in $\mathbb{P}$:

$$D_f = \{\langle s, F\rangle \in \mathbb{P}_k : \exists \sigma \in {}^{<\omega}k(\sigma^\frown f \restriction (\omega \setminus |\sigma|) \in F)\},$$

$$E_n = \{\langle s, F\rangle \in \mathbb{P}_k : \Delta_F > n \wedge s = F \restriction \Delta_F + 1\}.$$

If $G \subseteq \mathbb{P}_k$ is a filter meeting all these dense sets, then, using that the sets E_n are dense, it follows that $T = \bigcup\{s : \exists F(\langle s, F\rangle \in G)\}$ is a k-anti-Sacks tree. If $\sigma \in {}^{\omega}k$ and if $C[\sigma] = \{\sigma^\frown x \restriction (\omega \setminus |\sigma|) : x \in [T]\}$, then, using that the D_f are dense, it follows that ${}^{\omega}k \cap V \subseteq \bigcup\{C[\sigma] : \sigma \in {}^{<\omega}k\} \in \mathbf{AS}_k$. $\square$

The last proposition together with the Lemma 3.4 implies that $\mathbb{P}_k$ is not σ-k-linked. In contrast of this last observation, we have the following proposition.

Proposition 4.2 *For each $k > 1$, $\mathbb{P}_{k+1}$ is σ-k-linked.*

Proof For every s, t finite k-anti-Sacks tree of height $\mathsf{ht}(s), \mathsf{ht}(t)$ respectively, define

$$P(s,t) = \{\langle s, F\rangle \in \mathbb{P}_{k+1} : \mathsf{ht}(t) > \Delta_F \wedge F \restriction \mathsf{ht}(t) = t\}.$$

It is easy to see that $\mathbb{P}_{k+1} = \bigcup\{P(s,t) : s, t \text{ are finite } k \text{ anti-Sacks trees }\}$. We will show that every $P(s,t)$ is k-linked: Let $\{\langle s, F_i\rangle : i < k\} \subseteq \mathbb{P}(s, F)$ and let $F = \bigcup_{i<k} F_i$.

We must show that $\langle s, F\rangle \in \mathbb{P}_{k+1}$. The properties **(a)** and **(c)** are easily verified, so the only thing left to do is to show that $\lceil F \restriction \Delta_F \rceil$ is a k-anti-Sacks tree: Let $s \in \lceil F \restriction \Delta_F \rceil$. If $|s| < \mathsf{ht}(t)$, then, because $F \restriction \mathsf{ht}(t) = t$, it is possible to find an $i \in k$ such that $s^\frown\langle i\rangle \notin F \restriction \Delta_F + 1$. If $|s| \geq \mathsf{ht}(t)$, then, for every $i < k$, s only has (at most) one inmediate succesor in F_i and therefore it is always possible to find a $j \in k$ such that $s^\frown\langle j\rangle \notin F \restriction \Delta_F + 1$. □

From these last two propositions we get the following result.

Corollary 4.3 *For each $k > 1$, $\mathfrak{m}_k \leq \mathsf{non}(\mathbf{AS}_{k+1})$.*

Proof It follows easily from the proof of the proposition 4.1 and the last proposition. □

For the proof of the main theorem we will need the following notion.

Definition 4.4 *Given a regular cardinal κ, $k \in \omega$ and $L \in [{}^\omega k]^\kappa$, we will say that L is $\langle \kappa, \mathbf{AS}_k\rangle$-*Luzin *if $\mathbf{AS}_k \restriction L = [L]^{<\kappa}$.*

Recall that Cohen reals are added at every limit step of countable cofinality of a finite support iteration of arbitrary length. One common application of Cohen reals is that they are used to construct Luzin sets with special properties. The following lemma is one of those applications.

Lemma 4.5 *Let κ be a regular cardinal, let $i > 2$ and let $\mathbb{L} = \langle \mathbb{L}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \kappa\rangle$ be a finite support iteration of length κ such that $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_i$", then $\mathbb{L} \Vdash$ "There is a $\langle \kappa, \mathbf{AS}_i\rangle$-Luzin set.".*

Proof Working in $V[G]$, let $L = \{f_\alpha : \alpha \in \kappa \wedge \alpha \text{ has countable cofinality }\}$ be a family of Cohen reals such that each f_α is added at the α-th stage of the iteration. Using the proposition 4.1, it is easy to show that $V[G] \models [L]^{<\kappa} \subseteq \mathbf{AS}_i \restriction L$. On the other hand, if $T \in V[G]$ is such that $V[G] \models T$ is an i anti-Sacks tree, then, by a standard reflection argument, there is an intermediate model such that $T \in V[G(\beta)]$. As a consequence, $V[G] \models \forall \gamma > \beta (f_\gamma \notin [T])$. This implies that $V[G] \models \mathbf{AS}_i \restriction L \subseteq [L]^{<\kappa}$. □

The following theorem is the main tool we will use to prove the main result of this section.

Theorem 4.6 *If* ZFC *is consistent, then* $\mathsf{ZFC} + \forall i \in \omega\ (\exists L_i\ (L_i$ *is* $\langle \aleph_i, \mathbf{AS}_i \rangle$*-Luzin*$))$ *is consistent.*

Proof Let $\mathbb{L} = \langle \mathbb{L}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \in \omega_\omega \rangle$ a finite support iteration of length ω_ω such that, for each $i > 1$ and each $\alpha \in [\omega_i, \omega_{i+1})$, $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{i+1}$" (for $\alpha < \omega_2$, $\mathbb{L}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \{\emptyset\}$"). Using the lemma above, for each $i > 2$, in $V[G_{\omega_i}]$ there is a $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin set L_i. The only thing left to do is to show that L_i remains $\langle \aleph_i, \mathbf{AS}_i \rangle$-Luzin in $V[G]$. Using that $\mathbb{L}$ is c.c.c. it is easy to see that, in $V[G]$, $[L_i]^{<\omega_i} \subseteq \mathbf{AS}_i \restriction L_i$, so we only need to show that $\mathbf{AS}_i \restriction L_i \subseteq [L_i]^{<\omega_i}$ holds in $V[G]$: First observe that $\mathbb{L}_{[\omega_i, \omega_\omega]}$ strongly preserves $\mathsf{non}(\mathbf{AS}_i)$ in L_i, so if $\dot{T}$ is a $\mathbb{L}_{[\omega_i, \omega_\omega]}$-name of a i-anti-Sacks tree, then, in $V[G_{\omega_i}]$, there is a $X \in \mathbf{AS}_i \restriction L_i$ such that $\mathbb{L}_{[\omega_i, \omega_\omega]} \Vdash$ "$[\dot{T}] \cap L_i \subseteq X$". Then it follows that $\mathbf{AS}_i \restriction L_i \subseteq [L_i]^{<\omega_i}$ holds in $V[G]$. □

The actual value of $\mathfrak{c}$ in the model above may depend on V. For example, if $V \models \mathsf{GCH}$, then it is easy to see that $V[G] \models \mathfrak{c} = \aleph_{\omega+1}$. The following lemma is the last tool we need to prove the main result of this section.

Lemma 4.7 *Let* κ *be a regular cardinal and let* $L \subseteq {}^\omega k$ *be an* $\langle \kappa, \mathbf{AS}_i \rangle$*-Luzin, if* $\mathbb{P}$ *is a forcing notion such that* $|\mathbb{P}| < \kappa$*, then* $\mathbb{P}$ *strongly preserves* $\mathsf{non}(\mathbf{AS}_i)$ *in* L*.*

Proof Let $\dot{A}$ be a $\mathbb{P}$-name of an i-anti-Sacks tree and let $\mathbb{P} = \{p_\alpha : \alpha \in \mu\}$. For each $\alpha \in \mu$ define $T_\alpha = \{s \in {}^{<\omega}k : p_\alpha \Vdash$ "$s \in \dot{A}$"$\}$. It follows that each T_α defines an i-anti-Sacks tree. If $Y = \bigcup \{[T_\alpha] \cap L : \alpha \in \mu\}$ then $Y \in \mathbf{AS}_k$. If $x \in L$ and $p_\alpha \Vdash$ "$x \in [\dot{A}]$", then $x \in [T_\alpha] \cap L \subseteq Y$. □

We are ready to prove the main result of this section.

Corollary 4.8 *If* ZFC *is consistent, then* $\mathsf{ZFC} + \forall k > 1\ (\mathfrak{m}_k = \mathsf{non}(\mathbf{AS}_{k+1}) = \aleph_k) + \mathsf{non}(\mathbf{SP}) = \aleph_{\omega+1}$ *is consistent.*

Proof Start with $V \models \forall i \in \omega\ (\exists L_i\ (L_i$ is $\langle \omega_i, \mathbf{AS}_i \rangle$-Luzin$)) + \mathfrak{c} = \aleph_{\omega+1}$. Using a standard bookkeeping argument, it is possible to construct a finite support iteration $\mathbb{P}$ of length $\omega_{\omega+1}$ of σ-k-linked forcings of size smaller than $\aleph_{k+1}$ (for every $k > 1$), such that any partial order which appears in an intermediate model is listed cofinally along the iteration. Now, using all the preservation lemmas we proved on this article, it is possible to show that, for every $k \in \omega$, $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ in L_k. If $G \subseteq \mathbb{P}$ is a generic filter over V, then $V[G] \models \mathsf{non}(\mathbf{AS}_k) \leq \aleph_{k+1}$. We note that, as each small σ-k-linked forcing appears in an intermediate model in the iteration, then

$V[G] \models \aleph_{k+1} \leq \mathfrak{m}_k$. As a consequence $V[G] \models \aleph_{k+1} = \mathfrak{m}_k = \mathsf{non}(\mathbf{AS}_k)$. To finish the proof, use the fact that $\mathsf{non}(\mathbf{SP})$ does not have countable cofinality and that, for every $n \in \omega$, $\mathsf{non}(\mathbf{SP}_n) \leq \mathsf{non}(\mathbf{SP})$ to show that $V[G] \models \mathsf{non}(\mathbf{SP}) = \mathfrak{c} = \aleph_{\omega+1}$. $\square$

It follows from $\mathbf{SP}_1 \subseteq \mathbf{SP}_2 \subseteq \mathbf{SP}_3 \subseteq \ldots$ that $\omega_1 = \mathsf{non}(\mathbf{SP}_1) \leq \mathsf{non}(_{SP})2 \leq \mathsf{non}(\mathbf{SP}_3) \leq \ldots \leq \mathsf{non}(\mathbf{SP})$ and we proved in the theorem above that each inequality can be diferent. It is important to remark that each one of these numbers are not comparable with $\mathfrak{m}_{\sigma\text{-centered}}$ (as a consequence, it is impossible to prove that $\mathsf{non}(\mathbf{AS}_{k+1}) = \mathfrak{m}_k$). An argument for this can be found in [5].

5 The covering number

We have some results about the covering number of the ideals mentioned in this article. It follows from the fact that $\mathbf{AS}_2 \subseteq \mathbf{AS}_3 \subseteq \ldots$ that $\mathsf{cov}(\mathbf{SP}) \leq \ldots \leq \mathsf{cov}(\mathbf{AS}_3) \leq \mathsf{cov}(\mathbf{AS}_2) = \mathfrak{c}$. We can show that every pair of these numbers can be different.

Proposition 5.1 *Let $k > 1$, if* ZFC *is consistent, then* $\mathsf{ZFC} + \mathsf{cov}(\mathbf{AS}_{k+1}) < \mathsf{cov}(\mathbf{AS}_k)$ *is consistent.*

Proof Let V be a model such that $V \models \mathsf{cov}(\mathbf{AS}_k) = \mathfrak{c} = \omega_2$. Let $\mathbb{P}$ be a finite support iteration of length ω_1 of the $\mathbb{P}_{k+1}$ forcing defined above and let $G \subseteq \mathbb{P}$ be a generic filter over V. It follows that $\mathbb{P}$ is an iteration of σ-k-linked forcing notions and therefore $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$. In $V[G]$, consider the family $C = \{V[G_\alpha] \cap {}^\omega k+1 : \alpha < \omega_1\}$. It is easy to see that $V[G] \models C \subseteq \mathbf{AS}_{k+1}$ and $V[G] \models \bigcup C = {}^\omega k+1$. As a consecuence we have that $V[G] \models \mathsf{cov}(\mathbf{AS}_{k+1}) = \omega_1$. On the other hand, if $\{\dot{T}_\alpha : \alpha \in \omega_1\}$ is a collection of $\mathbb{P}$-names for k-anti-Sacks trees, then we can use the fact that $\mathbb{P}$ strongly preserves $\mathsf{non}(\mathbf{AS}_k)$ to show that there is a collection $\{C_\alpha : \alpha \in \omega_1\} \subseteq \mathbf{AS}_k$ such that if $x \in {}^\omega k$ and $x \notin \bigcup\{C_\alpha : \alpha \in \omega_1\}$, then $\mathbb{P} \Vdash$ "$x \notin \bigcup_{\alpha\in\omega_1}[\dot{T}_\alpha]$". This, together with $V \models \mathsf{cov}(\mathbf{AS}_k) > \omega_1$, imply that $V[G] \models \mathsf{cov}(\mathbf{AS}_{k+1}) < \mathsf{cov}(\mathbf{AS}_k)$. $\square$

An alternate proof of this proposition follows from the results proven in [7]. A tree $T \subseteq {}^{<\omega}\omega$ is a *k-tree* if every $s \in T$ has at most k immediate succesors. A forcing notion $\mathbb{P}$ has the *k-localization property* if $\mathbb{P} \Vdash$ "$\forall f \in {}^\omega\omega(\exists T \in V(T$ *is a k-tree and* $f \in [T]))$". It is easy to see that if $\mathbb{P}$ has the k-localization property, then $\mathbb{P} \Vdash$ "$\bigcup(\mathbf{AS}_k \cap V) = {}^\omega k$". Let $\mathbb{S}_k = \{T \subseteq {}^{<\omega}k : \forall s \in T(\exists t \in T(\forall i \in k(s \sqsubseteq t \wedge t^\frown i \in T)))\}$ be the k-Sacks forcing ordered by inclusion. It turns out that $\mathbb{S}_k$ is forcing equivalent to $\mathrm{Borel}({}^\omega k)/\mathbf{AS}_k$ and that if $\mathbb{P}$ is the countable support iteration or the countable support product of length ω_2 of the forcing $\mathbb{S}_k$, then $\mathbb{P}$ has the k-localization property (see [7]). As a consequence, in the extension $\mathsf{cov}(\mathbf{AS}_k) = \omega_1$ and $\mathsf{cov}(\mathbf{AS}_{k-1}) = \omega_2$.

Obviously it is impossible to have the analogous of the corollary 4.8 for $\mathsf{cov}(\mathbf{SP}_n)$, this arises a natural question.

Question 5.2 *How many of the* $\mathsf{cov}(\mathbf{SP}_n)$ *can be separated at the same time?*

We do not know how to separate three of them. Another question we have is the following:

Question 5.3 *Is it possible to get the consistency of* $\mathsf{ZFC} + \forall k \in \omega(\mathsf{cov}(\mathbf{SP}) < \mathsf{cov}(\mathbf{SP}_n))$ *?*

We are also interested about the relationship between $\mathsf{non}(\mathbf{SP})$ and $\mathsf{cov}(\mathbf{SP})$. It follows from the fact that the Cohen forcing is σ-centered that, in the Cohen's model, $\mathsf{non}(\mathbf{SP}) < \mathsf{cov}(\mathbf{SP})$. However, we do not know if it is possible to construct a model where $\mathsf{non}(\mathbf{SP}) > \mathsf{cov}(\mathbf{SP})$.

Question 5.4 *Is* $\mathsf{non}(\mathbf{SP}) \leq \mathsf{cov}(\mathbf{SP})$ *?*

This question is strongly related to the question found in [7]. In this paper, the authors asked about the relation between $\mathsf{non}(\mathbf{AS}_i)$ and $\mathsf{cov}(\mathbf{AS}_i)$. We do not know if there is an $i > 2$ such that $\mathsf{non}(\mathbf{AS}_i) \leq \mathsf{cov}(\mathbf{AS}_i)$.

6 Acknowledgements

The authors would like to thank J. Brendle and J. Cancino for many helpful suggestions and hours of stimulating conversations. The authors were supported by CONACyT and PAPIIT. The first-listed author was supported by CONACyT scholarship 420090. The second-listed author was supported by a PAPIIT grant IN 102311 and CONACyT grant 177758. The third-listed author was supported by CONACyT scholarship 332652.

References

[1] **T Bartoszyński**, **H Judah**, *Set theory*, A K Peters, Ltd., Wellesley, MA (1995)On the structure of the real line

[2] **C L Belna**, **M J Evans**, **P D Humke**, *Symmetric and ordinary differentiation*, Proc. Amer. Math. Soc. 72 (1978) 261–267

[3] **J Brendle**, *The additivity of porosity ideals*, Proc. Amer. Math. Soc. 124 (1996) 285–290

[4] **E P Dolženko**, *Boundary properties of arbitrary functions*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967) 3–14

[5] **M Hrušák**, **O Zindulka**, *Cardinal invariants of monotone and porous sets*, J. Symbolic Logic 77 (2012) 159–173

[6] **J Lindenstrauss**, **D Preiss**, *On Fréchet differentiability of Lipschitz maps between Banach spaces*, Ann. of Math. (2) 157 (2003) 257–288

[7] **L Newelski**, **A Rosłanowski**, *The ideal determined by the unsymmetric game*, Proc. Amer. Math. Soc. 117 (1993) 823–831

[8] **D Preiss**, **L Zajíček**, *Fréchet differentiation of convex functions in a Banach space with a separable dual*, Proc. Amer. Math. Soc. 91 (1984) 202–204

[9] **S Reich**, **A J Zaslavski**, *Well-posedness and porosity in best approximation problems*, Topol. Methods Nonlinear Anal. 18 (2001) 395–408

[10] **M Repický**, *Porous sets and additivity of Lebesgue measure*, Real Anal. Exchange 15 (1989/90) 282–298

[11] **M Repický**, *Additivity of porous sets*, Real Anal. Exchange 16 (1990/91) 340–343

[12] **M Repický**, *Cardinal invariants related to porous sets*, from: "Set theory of the reals (Ramat Gan, 1991)", Israel Math. Conf. Proc. 6, Bar-Ilan Univ., Ramat Gan (1993) 433–438

[13] **L Zajíček**, *On σ-porous sets in abstract spaces*, Abstr. Appl. Anal. (2005) 509–534

[14] **A J Zaslavski**, *Well-posedness and porosity in optimal control without convexity assumptions*, Calc. Var. Partial Differential Equations 13 (2001) 265–293

Centro de Ciencas Matemáticas, UNAM, A.P. 61-3, Xangari, Morelia, Michoacán, 58089, México

Centro de Ciencas Matemáticas, UNAM, A.P. 61-3, Xangari, Morelia, Michoacán, 58089, México

Centro de Ciencas Matemáticas, UNAM, A.P. 61-3, Xangari, Morelia, Michoacán, 58089, México

oguzman@matmor.unam.mx, michael@matmor.unam.mx, arturo@matmor.unam.mx

http://matmor.unam.mx/~michael