TOPOLOGY PROCEEDINGS

Volume 39, 2012

Pages 235–250

http://topology.auburn.edu/tp/

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Electronically published on November 12, 2011

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

http://topology.auburn.edu/tp/

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Volume 39 (2012)
Pages 235-250

E-Published on November 12, 2011

ON RESOLVABILITY AND EXTRARESOLVABILITY

S. GARCIA-FERREIRA AND M. HRUŠÁK

Abstract. Several examples of resolvable spaces are presented. We construct a countable ω-resolvable space which is not extraresolvable, and a compact first countable extraresolvable space. These examples answer some questions that have been asked in the literature. It is also shown that $CH + 2^{\omega_1} < \aleph_\omega$ implies that if X is a strongly extraresolvable space, then $X \times \omega$ is strongly extraresolvable. We shall also give a condition on the filters on ω that is equivalent to strong extraresolvability of the corresponding Seq-like spaces, and we give another condition that implies hereditary strong extraresolvability of the Seq-like spaces. It is shown that in the standard Solovay model of ZF where every set of reals has the property of Baire, every countable space is resolvable and there is an irresolvable space in this model.

1. Introduction

Our spaces will be Tychonoff. In this article we consider some questions concerning resolvability and extraresolvability of topological Hausdorff spaces. Research in this area stems from the paper of E. Hewit [11] who calls a topological space X *resolvable* if X contains two disjoint dense subsets. The notion was extended in various ways in the past few decades. J. G. Ceder [2] introduced and studied the notion of κ-resolvability (X is *κ-resolvable* if it contains a family of κ-many pairwise disjoint sets) with particular emphasis on *maximally resolvable* spaces, i.e. spaces X which are $\Delta(X)$-resolvable, where $\Delta(X) = \min\{|U| : U$ is a non-empty open subset of $X\}$ denotes the *dispersion character* of X.

2010 *Mathematics Subject Classification.* Primary 54F99, 54G15, 54G20; Secondary 54B99, 54D80.

Key words and phrases. Resolvable space, irresolvable, extraresolvable space, hereditarily extraresolvable, $Seq(p)$, free filter, independent family, cardinal invariants of the continuum.

Research of the first author was supported by CONACyT grant no. 81368-F and PAPIIT grant no. IN-101508.

Research of the second author was supported by CONACyT grant no. 80355 and PAPIIT grant no. IN-102311.

According to V. I. Malykhin [15] a space X is *extraresolvable* if there is a family $\{D_\alpha : \alpha < \Delta(X)^+\}$ of dense subsets of X such that $D_\alpha \cap D_\beta$ is nowhere dense in X for distinct α and β, and finally following W. Comfort and S. Garcia-Ferreira, a space X is *strongly extraresolvable* if there is a family $\{D_\alpha : \alpha < \Delta(X)^+\}$ of dense subsets of X such that $|D_\alpha \cap D_\beta| < nwd(X)$, where $nwd(X)$ denotes the minimal cardinality of a somewhere dense subset of X. We remark that $\Delta(X) \leq 2^{2^{nwd(X)}}$, for every space X.

Let us mention some relationships between the concepts. It has been shown in [9] that every extraresolvable space is ω-resolvable and in [1] it is proved that $MA_{\sigma-centered}$ implies that every countable space of character less than $\mathfrak{c}$ is extraresolvable. The question of existence of a countable extraresolvable space which is not strongly extraresolvable was posed in [4] and a positive answer was given in [8] and also in [5], where the authors constructed, for each infinite cardinal number κ, a space X such that $|X| = nwd(X) = \Delta(X) = \kappa$. Maximally resolvable spaces which are not extraresolvable exist in profusion, for instance the real numbers, but most of the known examples are uncountable and satisfy $|X| > nwd(X)$. In the paper [5], W. W. Comfort and W. Hu constructed several examples of maximally resolvable, extraresolvable spaces which are not strongly extraresovable and $|X| = nwd(X)$. Several years later in the paper [13], for each infinite cardinal number κ, a space X is constructed so that it is maximally resolvable but not extraresolvable satisfying $|X| = nwd(X) = \kappa$. In [6], W. W. Comfort and W. Hu showed that any space satisfying $S(X) \leq \Delta(X)$ admits a Tychonoff expansion which is maximally resolvable but not extraresolvable. $S(X)$ denotes the Suslin number of a space X.

In the present paper, we give a construction of a countable resolvable space that is not extraresolvable by using a technique different from the one used in [13]. This example was presented by the second author at the 2003 Summer Topology Conference and its Applications, after many years we finally decided to published it. This example will be described in the second section. In the third section, we first address the question of the strong extraresolvablity of a countable free sum of a strongly extraresolvable space. We show that the answer to this question is positive under the assumption $CH + 2^{\omega_1} < \aleph_\omega$. The fourth section is devoted to answer affirmatively Question 27 of [17] by constructing a compact, first countable, strongly extraresolvable space in ZFC. In the fifth section we give two conditions on filters on ω, one of them equivalent to strong extraresolvability, and the other implying hereditary strong extraresolvability of the corresponding Seq-spaces. In the final section, we show that it is relatively consistent with ZF that every countable space is resolvable.

2. A countable, maximally resolvable, non-extraresolvable space

Several known examples in the theory of irresolvable spaces involve independent families. We present another such example here. We start with some basic notation and concepts.

For $A, B \in \mathcal{P}(\omega)$, let $A\Delta B = (A\setminus B)\cup(B\setminus A)$ and for $\mathcal{A} \subseteq \mathcal{P}(\omega)$, we let $\langle\mathcal{A}\rangle$ stand for the set of all non-empty Boolean combinations of elements of $\mathcal{A}$. All ideals on ω are assumed to contain the ideal $[\omega]^{<\omega}$. Given an ideal $\mathcal{I}$ on ω and $A, B \in \mathcal{P}(\omega)$, we write $A \subseteq_{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$. Hence, given $A, B \in \mathcal{P}(\omega)$ we put $A =_{\mathcal{I}} B$ if $A\Delta B \in \mathcal{I}$ (equivalently, there is $I \in \mathcal{I}$ such that $A \setminus I = B \setminus I$). It is known that $=_{\mathcal{I}}$ is an equivalent relationship between the elements of $\mathcal{P}(\omega)$. We say that a family $\mathcal{A} \subseteq [\omega]^{\omega}$ is $\mathcal{I} - independent$ provided that for every $A \in \langle\mathcal{A}\rangle$, we must have that $\bigcap A \neq_{\mathcal{I}} \emptyset$. It is clear every $\mathcal{I} - independent$ family is independent and every independent family is $[\omega]^{<\omega}$-independent.

The *upper density* of a subset $A \subseteq \omega$ is the number

$$\overline{d}(A) = \lim\sup_{n\to\infty} \frac{|A \cap n|}{n}.$$

In our construction, we shall use the *density ideal* on ω:

$$\mathcal{D} = \{A \subseteq \omega : \overline{d}(A) = 0\}.$$

Also we need the *reaping number* of a Boolean algebra $\mathbb{B}$:

$\mathfrak{r}(\mathbb{B})$ is the least cardinal κ such that there is $\mathcal{A} \in [\mathbb{B} \setminus \{0_{\mathbb{B}}\}]^{\kappa}$ such that for every $b \in \mathbb{B} \setminus \{0_{\mathbb{B}}\}$ there is $a \in \mathcal{A}$ such that either $a \leq_{\mathbb{B}} b$ or $a \wedge b = 0_{\mathbb{B}}$. It is shown in [19] that $\mathfrak{r}(\mathcal{P}(\omega)/\mathcal{D}) = \mathfrak{c}$. We are ready to construct our example.

Theorem 2.1. *There is an ω-resolvable, non-extraresolvable topology on a countable set.*

Proof. We start with a partition $\{A_n : n < \omega\}$ of ω into infinite pieces so that $\overline{d}(A_n) > 0$ for each $n \in \mathbb{N}$. We will construct a topology τ on ω so that each set A_n is τ-dense and so that if X is a τ-dense subset of ω, then there is an $n < \omega$ such that $X \cap A_n$ is not nowhere dense. In order to do this we shall construct a special $\mathcal{D}$-independent family $\mathcal{I}$ of subsets of ω such that:

i). $\mathcal{I} \cup \{A_n\}$ is $\mathcal{D}$-independent, for every $n < \omega$, and ii). for every $X \in [\omega]^\omega$ either:

a. there is $I \in \langle \mathcal{I} \rangle$ such that $I \cap X =_{\mathcal{D}} \emptyset$, or

b. there are $n < \omega$ and $I \in \langle \mathcal{I} \rangle$ such that $A_n \cap I \subseteq_{\mathcal{D}} X$.

We will proceed by transfinite induction. First, enumerate $[\omega]^\omega$ as $\{X_\alpha : \alpha < \mathfrak{c}\}$. Assume that $I_\alpha \subseteq \omega$ has been defined for each $\alpha < \beta < \mathfrak{c}$ so that

1). $\{I_\alpha : \alpha < \beta\} \cup \{A_n\}$ is $\mathcal{D}$-independent, for every $n < \omega$,

2). For every $\alpha < \beta$ one of the following statements holds:

a. There is an $I \in \langle \{I_\gamma : \gamma < \beta\} \rangle$ such that $I \cap X_\alpha =_{\mathcal{D}} \emptyset$.

b. There are an $n < \omega$ and $I \in \langle \{I_\gamma : \gamma < \beta\} \rangle$ such that $A_n \cap I \subseteq_{\mathcal{D}} X_\alpha$.

Put $\mathcal{I}_\beta = \{I_\alpha : \alpha < \beta\}$. Assume that both conditions a) and b) of clause ii) do not hold for X_β. Then,

$$\{A_n \cap I \cap (\omega \setminus X_\beta) : I \in \mathcal{I}_\beta\}$$

is a $\mathcal{D}$-independent family on A_n, for all $n < \omega$. Fix $n < \omega$. Observe that $\overline{d}(A_n \cap I \cap (\omega \setminus X_\alpha)) > 0$, for each $I \in \langle \mathcal{I}_\beta \rangle$. Since $\mathfrak{r}(\mathcal{P}(\omega)/\mathcal{D}) = \mathfrak{c}$, there is $B_n \in [\omega]^\omega$ such that

$$A_n \cap I \cap (\omega \setminus X_\beta) \not\subseteq_{\mathcal{D}} B_n \text{ and } A_n \cap I \cap (\omega \setminus X_\beta) \cap B_n \neq_{\mathcal{D}} \emptyset,$$

for all $I \in \langle \mathcal{I}_\beta \rangle$. Hence, we have that

$$\overline{d}(A_n \cap I \cap (\omega \setminus X_\beta) \cap (\omega \setminus B_n)) > 0 \text{ and } \overline{d}(A_n \cap I \cap (\omega \setminus X_\beta) \cap B_n) > 0,$$

for all $I \in \langle \mathcal{I}_\beta \rangle$. Put $I_\beta = (\bigcup_{n<\omega} A_n \cap B_n) \setminus X_\beta$. It is evident that $\{I_\alpha : \alpha \leq \beta\} \cup \{A_n\}$ is $\mathcal{D}$-independent, for every $n < \omega$ and $I_\beta \cap X_\beta = \emptyset$. Finally we get $\mathcal{I} = \{I_\alpha : \alpha < \mathfrak{c}\}$. It is clear that $\mathcal{I}$ satisfies clauses i) and ii).

Having fixed a family $\mathcal{I}$ as above, moreover, we may assume that $\mathcal{I}$ satisfies:

iii). For each $m, n < \omega$ exists $I \in \mathcal{I}$ such that $|I \cap \{m, n\}| = 1$.

This is easily accomplished by making finite changes to countably many elements of $\mathcal{I}$.

To define the topology τ on ω enumerate $\langle \mathcal{I} \rangle$ as $\{I_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{D}$ as $\{D_\alpha : \alpha < \mathfrak{c}\}$ allowing each element of $\mathcal{D}$ appearing infinitely many times in this enumeration. Then, we declare all elements of $\langle \{I_\alpha \setminus D_\alpha : \alpha < \mathfrak{c}\} \rangle$ be clopen. It is not hard to see that condition ii) may be replaced by

ii'). for every $X \in [\omega]^\omega$ either:

a. there is $I \in \langle \{I_\alpha \setminus D_\alpha : \alpha < \mathfrak{c}\} \rangle$ such that $I \cap X = \emptyset$, or

b. there are $n < \omega$ and $I \in \langle \{I_\alpha \setminus D_\alpha : \alpha < \mathfrak{c}\} \rangle$ such that $A_n \cap I \subseteq X_\alpha$.

Then (ω, τ) is a zero-dimensional Hausdorff (by iii)) countable space, which is ω-resolvable as all A_n are dense in X by requirement i).

To show that X is not extraresolvable let $\mathcal{A}$ be an uncountable collection of dense subsets of X. As every $X \in \mathcal{A}$ is dense, $X \cap I \neq \emptyset$ for every $I \in \langle\{I_\alpha \setminus D_\alpha : \alpha < \mathfrak{c}\}\rangle$. So, by ii'.b), there are $n_X < \omega$ and $I_X \in \langle\{I_\alpha \setminus D_\alpha : \alpha < \mathfrak{c}\}\rangle$ such that $A_{n_X} \cap I_X \subseteq X$. Then, we can find $m < \omega$ and an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $n_X = m$ for every $X \in \mathcal{B}$. Hence, for each $X \in \mathcal{B}$, $A_m \cap I_X \subseteq X$. So, $X \cap A_m$ contains an open subset of A_m for each $X \in \mathcal{B}$. As A_m is countable, hence satisfying the countable chain condition, there are two distinct $X, Y \in \mathcal{B}$ such that $X \cap Y \cap A_m$ contains an open subset of A_m. Consequently, $X \cap Y$ is not nowhere dense, so (ω, τ) is not extraresolvable. □

3. Extraresolvable spaces

It was proved in [4] that if X is extraresolvable, then $X \times \alpha$ is extraresolvable for all infinite cardinals α (here, α is equipped with the discrete topology). In the same paper, [4, Th. 4.1 (d')] the authors pointed out that if X is strongly extraresolvable, then $X \times \alpha$ is strongly extraresolvable for every $\alpha < nwd(X)$. Hence, if $nwd(X) > \omega$ and X is strongly extraresolvable, then so is $X \times \omega$. It is then natural to ask the following question:

Question 3.1. Is the free sum of infinitely many copies of a strongly extraresolvable space strongly extraresolvable ?

For the countable case the question was formulated in [8]. The following result from [8] gives a necessary condition for the strong extraresolvability of $X \times \alpha$.

Theorem 3.2. *If $X \times \alpha$ is strongly extraresolvable and $nwd(X) < \alpha \leq \Delta(X)^+$, then X is κ-resolvable for every cardinal $\kappa < \alpha$. In particular, if $X \times \Delta(X)^+$ is strongly extraresolvable, then X is maximally resolvable.*

We provide a partial answer to the question by showing that the answer is consistently yes.

Lemma 3.3. *Let X be a strongly extraresolvable space. If $\Delta(X)^\omega < \Delta(X)^+$, then $nwd(X) > \omega$ and $X \times \omega$ is strongly extraresolvable.*

Proof. Let $\{D_\alpha : \alpha < \Delta(X)^+\}$ be a family of dense subsets of X witnessing the strong extraresolvability of X, and let U be an open subset of X with size $\Delta(X)$. For each $\alpha < \Delta(X)^+$, choose a countably infinite subset C_α of $D_\alpha \cap U$. By assumption, there are only strictly less than $\Delta(X)^+$ countable subsets of U and hence we can find $\alpha \neq \beta < \Delta(X)^+$ such that $C_\alpha = C_\beta$. So $|D_\alpha \cap D_\beta| \geq \omega$. Therefore, $nwd(X) > \omega$. □

Theorem 3.4. $[CH + 2^{\omega_1} < \aleph_\omega]$ *If X is a strongly extraresolvable space, then $X \times \omega$ is strongly extraresolvable.*

Proof. The case when $\Delta(X) = \omega$ is fairly easily handled by diagonalization (see [8]). Assume that $nwd(X) = \omega$. Hence, $\Delta(X) \leq 2^{2^{nwd(X)}} = 2^{2^{\omega}} = 2^{\omega_1}$. According to Lemma 3.3, we must have that $\Delta(X)^{\omega} \geq \Delta(X)^{+}$. As CH implies that $\omega_n^{\omega} = \omega_n$, for all $n < \omega$, and $2^{\omega_1} < \aleph_{\omega}$, we must have that $\Delta(X) = \omega_n$, for some $n \in \omega$. Thus, $\Delta(X)^{\omega} = \Delta(X)$ which is a contradiction. Therefore, $nwd(X) > \omega$. □

A ZFC answer to the Question 3.1 is still unknown. Let us discuss some another partial positive answer to this question. We need the next easy lemma.

Lemma 3.5. *If $\{A_\alpha : \alpha < \kappa\}$ is an almost disjoint family of dense subsets of a space X, then $\{\bigcup_{n<\omega}[(A_\alpha \setminus n) \times n] : \alpha < \kappa\}$ is an almost disjoint family of dense subsets of the space $X \times \omega$.*

Theorem 3.6. *If X is strongly extraresolvable, hereditarily separable space, then $X \times \omega$ is strongly extraresolvable.*

Proof. By assumption, we have that $nwd(X) = \omega$. Let $\{A_\alpha : \alpha < \Delta(X)^+\}$ be a family of dense subsets witnessing the strong extraresolvability of X. Since X is hereditarily separable, we may assume that $A_\alpha \in [X]^{\omega}$ for every $\alpha < \Delta(X)^+$. By the previous lemma, we obtain that $\{\bigcup_{n<\omega}[(A_\alpha \setminus n) \times n] : \alpha < \Delta(X)^+\}$ is an almost disjoint family of dense subsets of the space $X \times \omega$. So, $X \times \omega$ is strongly extraresolvable. □

The question of the existence of an extraresolvable (strongly extraresolvable) space that is not maximally resolvable was formulated in [4, Q. 6.7(b)]. We know from [4] that every strongly extraresolvable space X is $cf(nwd(X))$-resolvable. For extraresolvable spaces, it is shown in [12] that for every infinite cardinal number κ there is an extraresolvable space of dispersion character κ that is not ω_1-resolvable. Next, we shall give a partial positive answer to the question assuming GCH.

Lemma 3.7. $[GCH]$ *For every strongly extraresolvable space X, $\Delta(X) = nwd(X)$.*

Proof. Assume GCH. Put $\kappa = nwd(X)$. We know that $\Delta(X) \leq 2^{2^{nwd(X)}} = 2^{2^{\kappa}}$ and hence $\Delta(X) \leq \kappa^{++}$. Suppose that $\Delta(X) > \kappa$. As $2^{\kappa} = \kappa^{+}$, we must have that $\Delta(X)^{\kappa} = \Delta(X)$. Let $\{D_\alpha : \alpha < \Delta(X)^+\}$ be a family of dense subsets of X witnessing the strong extraresolvability of X, and let U be an open subset of size $\Delta(X)$. Since $nwd(X) = \kappa$ and $|D_\alpha \cap U| \geq \kappa$, we can pick a set $C_\alpha \subseteq D_\alpha \cap U$ of size κ, for every $\alpha < \Delta(X)^+$. As $\Delta(X) > \kappa$, there would be distinct $\alpha, \beta < \Delta(X)^+$ such that $C_\alpha = C_\beta$, but this contradicts the strong extraresolvability of X. □

Lemma 3.8. *Let X be a strongly extraresolvable space. If $nwd(X)$ is a regular cardinal, then X is $nwd(X)$-resolvable.*

Proof. Let X be a strongly extraresolvable space and let $\kappa = nwd(X)$. Choose any witness $\{D_\alpha : \alpha < \kappa^+\}$ to the strong extraresolvability of X. For every $\alpha < \kappa$, we define

$$D'_\alpha = D_\alpha \setminus \bigcup_{\beta<\alpha} D_\beta.$$

Clearly, the sets D'_α's are disjoint. Now, let $\alpha < \kappa$. By the regularity of κ, we have that

$$|D_\alpha \cap (\bigcup_{\beta<\alpha} D_\beta)| \leq |\bigcup_{\beta<\alpha} D_\alpha \cap D_\beta| < \kappa.$$

Hence, $D_\alpha \cap (\bigcup_{\beta<\alpha} D_\beta)$ is nowhere dense and consequently D'_α is a dense subset of X. This shows that X is $nwd(X)$-resolvable. □

As a direct consequences of Lemma 3.7 and 3.8, we have the following corollary.

Corollary 3.9. [GCH] *Let X be a strongly extraresolvable space. If $nwd(X)$ is a regular cardinal, then X is maximally resolvable.*

4. A first countable, compact, strongly extraresolvable space

In this section, we shall affirmatively answer Question 27 of [17] (which originally was listed in [9, Q. 3.25] and was attributed to O. T. Alas) which asked whether there is a compact, first countable, strongly extraresolvable space in ZFC. In their paper the authors appeal to a fairly complicated construction of an S-space due to M. E. Rudin for a consistent example. We will show that a very nice space definable in ZFC alone, namely the space $[0,1]^\omega$ with the interval topology induced by the lexicographic order, provides an example.

For $f \neq g \in [0,1]^\omega$ denote by $f\Delta g$ the minimal $n \in \omega$ such that $f(n) \neq g(n)$. Define $f < g$ if $f(f\Delta g) < g(f\Delta g)$. Obviously $([0,1]^\omega, <)$ is a linearly ordered set. Let X be $[0,1]^\omega$ with the interval topology induced by $<$. Denote by $\mathbf{0}$ the constant zero function and by $\mathbf{1}$ the constant function 1. It is evident that $\mathbf{0}$ and $\mathbf{1}$ are the first element and the last element of $([0,1]^\omega, <)$, respectively. To describe the topology on $([0,1]^\omega, <)$ we need the following terminology:

Theorem 4.1. $X = ([0,1]^\omega, <)$ *is a first countable compact strongly extraresolvable space.*

Proof. First note that $X = ([0,1]^\omega, <)$ is dense in itself and $\Delta(X) = \mathfrak{c}$. In order to prove that it is compact it is enough to show that it is Dedekind complete (see [23, P. 17E]). To do that let A be a non-empty subset of $[0,1]^\omega$. To define the least upper bound of A we need to define inductively a function $s : \omega \to \omega$ as follows:

First, we let $s(0) = \sup\{g(0) : g \in A\}$, and for each positive $n < \omega$, we let $s(n) = \sup\{g(n) : g \in A \text{ and } \forall i < n(g(i) = s(n))\}$. Notice that $s(n)$ could not be defined in some cases. Indeed, suppose that there is $m < \omega$ such that $s(m)$ can be defined and $g(m) < s(m)$, for every $g \in A$ such that $g(n) = s(n)$ for all $n < m$. Then, in this case, we define

$$f(n) = \begin{cases} s(n) & \text{for } n \leq m \\ 0 & \text{otherwise.} \end{cases}$$

If this is not the case (that is, $s(n)$ exists for every $n < \omega$), then we define $f = s$. It is not difficult to prove that f is the lest upper bound of A. So, X is a compact space. For each $f \in X$ and for each $n < \omega$, we define

$$U(f,n) = \{g \in X : \forall i < n(f(i) = g(i)) \text{ and } \mid f(n) - g(n) \mid < \frac{1}{n}\}.$$

It is not difficult to see that each $U(f,n)$ is open and $\bigcap_{n<\omega} U(f,n) = \{f\}$ for every $f \in X$. As the character and pseudo-character coincides at each point of X (see [16]), our space X is then first countable. Notice that $w(X) = nwd(X) = \Delta(X) = \mathfrak{c}$. According to Theorem 2.3 of [4], we obtain that X is strongly extraresolvable. □

5. Seq spaces

First, we give some terminology which is very helpful to define the topology of the Seq spaces.

$FF(\omega)$ will denote the set of all free filters on ω. For $\mathcal{F} \in FF(\omega)$, we let $\mathcal{F}^+ = \{A \subseteq \omega : \forall F \in \mathcal{F}(A \cap F \neq \emptyset)\}$. Let $Seq = \bigcup_{n<\omega} \omega^n$. If $s \in Seq$ and $n < \omega$, then the *concatenation* of s and n is the function $s^\frown n = s \cup \{(dom(s), n)\}$. For a function $\delta : Seq \to FF(\omega)$, we define a topology τ_δ on Seq by defining $V \in \tau_\delta$ iff for every $s \in V$, $\{n < \omega : s^\frown n \in V\} \in \delta(s)$. It is well-known that $Seq(\delta) = (Seq, \tau_\delta)$ is an extremally disconnected, zero dimensional Hausdorff space for every function $\delta : Seq \to FF(\omega)$ (see [22]). It was shown in [8] that if $\delta(s)$ is a free ultrafilter on ω for each $s \in Seq$, then the space $Seq(\delta)$ is extraresolvable but not strongly extraresolvable. We shall give a necessary and sufficient condition to guarantee the strong extraresolvability of $Seq(\delta)$. We also give a condition that implies the hereditary strong extraresolvability of $Seq(\delta)$. To do that we need the following notions introduced by V. I. Malykhin in [15].

Definition 5.1. Let $\mathcal{F} \in FF(\omega)$.

(1) $\mathcal{F}$ is called *small* if there exists an uncountable AD-family[1] $\mathcal{A}$ of subsets of ω such that $\mathcal{A} \subseteq \mathcal{F}^+$.
(2) $\mathcal{F}$ is called *everywhere small* if whenever $A \in \mathcal{F}^+$ there exists an uncountable AD-family of subsets of A contained in $\mathcal{F}^+$.

We have changed the terminology "big" and "everywhere big" originally used by Malykhin since the filters defined in 5.1 are somehow small in a set-theoretical sense.

Clearly each everywhere small filter is small. It is evident that an ultrafilter cannot be small. Observe that the Frechét filter on ω is the easiest example of an everywhere small filter. More generally, given an uncountable AD-family, we can find a free filter on ω so that the AD-family witnesses the smallness of the filter and the filter is everywhere small. Indeed, if $\mathcal{A}$ is an uncountable AD-family, then $\mathcal{F}_{\mathcal{A}} = \{F \subseteq \omega : \forall A \in \mathcal{A}(F \subseteq^* A)\}$ is an everywhere small filter, where $A \subseteq^* B$ means that $A \setminus B$ is finite. These filters are examples of sequential[2] filters (several basic properties of these filters appear in the paper [10]) which have convergent sequences. This is one of many reasons that we have called them small. Adding a free ultrafilter and a sequential filter, we get a big filter which is not everywhere big. Indeed, let $A \in [\omega]^\omega$ be with infinite complement. If p is a free ultrafilter on ω containing A, then the filter $\mathcal{F} = \{E \cup F : E$ is a cofinite subset of $\omega \setminus A, F \subseteq A$ and $F \in p\}$ is a small filter which is not everywhere small.

As we mention above, if the function δ takes only ultrafilters, then the space $Seq(\delta)$ cannot be strongly extraresolvable,

We shall need to introduce the basic open sets of $Seq(\delta)$. For $s \in Seq$ we put $c(s) = \{t \in Seq : s \subseteq t\}$. If $f \in \prod_{t \in c(s)\setminus\{s\}} \delta(t)$, then we let $c_0(s,f) = \{s\}$ and, for each positive $n < \omega$, we inductively define

$$c_n(s,f) = \{t^\frown n : t \in c_{n-1}(s,f) \text{ and } n \in f(t)\}.$$

And then $c(s,f) = \bigcup_{n<\omega} c_n(s,f)$ is a basic open of the space $Seq(\delta)$ that contains s, for each $s \in Seq$ and for each $f \in \prod_{t \in c(s)\setminus\{s\}} \delta(t)$.

To prove our first theorem we need some lemmas.

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[1]A family $\mathcal{A}$ of infinite subsets of ω is called *almost disjoint* (AD) if for distinct $A, B \in \mathcal{A}$ we have that $| A \cap B | < \omega$.

[2]A free filter $\mathcal{F}$ on ω is called *sequential* if for each $E \in \mathcal{F}^+$ there is an infinite subset A of E such that $A \setminus F$ is finite for every $F \in \mathcal{F}$.

Lemma 5.2. *If D is dense in $Seq(\delta)$, then for every $s \in Seq$ and for every $f \in \prod_{t \in C(s)\setminus\{s\}} \delta(t)$ there is $t \in c(s,f)$ such that*

$$\forall F \in \delta(t)(F \cap \{n < \omega : t^\frown n \in D\} \neq \emptyset).$$

Proof. Suppose that there are $s \in Seq$ and $f \in \prod_{t \in C(s)\setminus\{s\}} \delta(t)$ such that for every $t \in c(s,f)$ there is $F_t \in \delta(t)$ with $F_t \cap \{n < \omega : t^\frown n \in D\} = \emptyset$. Now, we define $g \in \prod_{t \in C(s)\setminus\{s\}} \delta(t)$ as

$$g(t) = \begin{cases} f(t) \cap F_t & \text{if } t \in c(s,f) \\ f(t) & \text{otherwise.} \end{cases}$$

Observe that $c(s,g) \subseteq c(s,f)$. Then, there is $t \in c(s,g) \cap D$. Choose $r \in c(s,g)$ and $k \in g(r)$ so that $t = r^\frown k$. Then, by definition, $k \in g(r) = f(r) \cap F_r$, but this is impossible since $t = r^\frown k \in D$. $\square$

Lemma 5.3. *Let $\delta : Seq \to FF(\omega)$ be a function. If $Seq(\delta)$ admits a dense subset D such that:*

(1) *$\delta(s)$ is small for every $s \in D$; and*
(2) *For every $s \in D$ there is an AD-family $\mathcal{A}_s$ of size ω_1 which can be indexed as $\{A(s,\nu) : \nu < \omega_1\}$ such that*
 a. *$\mathcal{A}_s$ witnesses the smallness of $\delta(s)$; and*
 b. *for every $\nu, \mu < \omega_1$, the set $\{s \in D : A(s,\nu) \cap A(s,\mu) \neq \emptyset\}$ is finite,*

then $Seq(\delta)$ is strongly extraresolvable.

Proof. Suppose that $Seq(\delta)$ has a dense subset D satisfying all the conditions stated in clauses 1 and 2. For $\nu < \omega_1$, we define $D_\nu = \{s^\frown n : n \in A(s,\nu) \text{ and } s \in D\}$. It is evident that D_ν is dense in $Seq(\delta)$, for all $\nu < \omega_1$. By clause *b*, we get that $D_\nu \cap D_\mu$ is finite whenever $\nu < \mu < \omega_1$. Thus, $\{D_\nu : \nu < \omega_1\}$ witnesses the strong extraresolvability of $Seq(\delta)$. $\square$

We need to introduce some notions concerning the topology of the Stone-Čech compactification $\beta(\omega)$ of ω with the discrete topology. We identify $\beta(\omega)$ with the set of all ultrafilters on ω, and ω^* will denote the set of all free ultrafilters on ω. For $A \subseteq \omega$, we let $\hat{A} = \{p \in \beta(\omega) : A \in p\}$ and, if $A \in [\omega]^\omega$, we let $A^* = \{p \in \omega^* : A \in p\}$. We know that $\{\hat{A} : A \subseteq \omega\}$ and $\{A^* : A \in [\omega]^\omega\}$ are bases consisting of clopen sets for $\beta(\omega)$ and ω^*, respectively.

There is a one-to-one correspondence between the free filters on ω and the non-empty closed subsets of ω^*:

$$\mathcal{F} \in FF(\omega) \longrightarrow M_{\mathcal{F}} = \bigcap_{F \in \mathcal{F}} \hat{F}, \text{ and}$$

$$C \subseteq \omega^* \text{ closed} \longrightarrow \mathcal{F}_C = \{F \subseteq \omega : C \subseteq \hat{F}\}.$$

We know that $\mathcal{F}_{M_\mathcal{F}} = \mathcal{F}$ and $C = M_{\mathcal{F}_C}$, for every $\mathcal{F} \in FF(\omega)$ and for every closed subset $C \subseteq \omega^*$.

Now, let $\mathcal{F} \in FF(\omega)$ and suppose that $\mathcal{A} = \{A_\nu : \nu < \omega_1\}$ is an AD-family such that $\mathcal{A} \subseteq \mathcal{F}^+$. Then, we have that $M_\mathcal{F} \cap A_\nu^* \neq \emptyset$, for every $\nu < \omega_1$. Hence, for each $\nu < \omega_1$, we choose $p_\nu \in M_\mathcal{F} \cap A_\nu^*$, and put $\mathcal{G} = \mathcal{F}_{\overline{\{p_\nu : \nu < \omega_1\}}}$. Clearly, $\mathcal{F} \subseteq \mathcal{G}$ and $A_\nu \in \mathcal{G}^+$ for all for each $\nu < \omega_1$. Observe that $\mathcal{A} \subseteq \mathcal{G}^+$ and $\mathcal{G}^+ \subseteq \mathcal{F}^+$. In the proof of the next Lemma, we will replace a filter $\mathcal{F}$ by the filter $\mathcal{G}$.

Lemma 5.4. *Let $\{\mathcal{F}_n : n < \omega\}$ be a countable subset of $FF(\omega)$. Suppose that, for every $n < \omega$, there is an uncountable AD-family $\mathcal{A}_n$ such that $\mathcal{A}_n \subseteq \mathcal{F}_n^+$. Then, for every $n < \omega$ and for every $\nu < \omega_1$, there is $A(n,\nu) \in [\omega]^\omega$ such that*

i. *$A(n,\nu)$ is almost contained in an element of $\mathcal{A}_n$, for every $n < \omega$ and for every $\nu < \omega_1$;*
ii. *$\{A(n,\nu) : \nu < \omega_1\}$ is an AD-family, for every $n < \omega$;*
iii. *$A(n,\nu) \in \mathcal{F}_n^+$, for every $n < \omega$ and for every $\nu < \omega_1$; and*
iv. *the set $\{n < \omega : A(n,\nu) \cap A(n,\mu) \neq \emptyset\}$ is finite, for every $\nu, \mu < \omega_1$.*

Proof. For each $n < \omega$, let $\mathcal{G}_n \in FF(\omega)$ be the filter constructed as in the comment above by using the filter $\mathcal{F}_n$. We will use the filters $\{\mathcal{G}_n : n < \omega\}$ rather than the filters $\{\mathcal{F}_n : n < \omega\}$. It is evident that $\mathcal{A}_n \subseteq \mathcal{G}_n^+$, for each $n < \omega$. By removing only countably many elements of each $\mathcal{A}_n$, we may assume that

($*$) for every $n < \omega$, we have that the set $\{A \in \mathcal{A}_m : A^* \cap M_{\mathcal{G}_n} \neq \emptyset\}$ is either empty or uncountable, for all $m < \omega$.

We shall proceed by transfinite induction. Suppose that for every $n < \omega$ and for every $\nu < \theta < \omega_1$ we have defined $A(n,\nu) \in [\omega]^\omega$ so that

i. $A(n,\nu)$ is almost contained in an element of $\mathcal{A}_n$, for every $n < \omega$ and for every $\nu < \theta$;
ii. $\{A(n,\nu) : \nu < \theta\}$ is an AD-family, for every $n < \omega$;
iii. $A(n,\nu) \in \mathcal{F}_n^+$, for every $n < \omega$ and for every $\nu < \theta$; and
iv. the set $\{n < \omega : A(n,\nu) \cap A(n,\mu) \neq \emptyset\}$ is finite, for every $\nu, \mu < \theta$.

By ($*$), for each $n < \omega$, we may find $A_n \in \mathcal{G}_n$ and $p_n \in A_n^* \cap M_{\mathcal{G}_n}$ such that $p_n \notin [\bigcup_{\nu<\theta} \bigcup_{m<\omega} A(m,\nu)^*] \cup [\bigcup_{m<n} A_m^*]$, for each $n < \omega$. Now, fix a bijection $f : \omega \to \theta$. Then, we define

$$A(n,\theta) = A_n \setminus [(\bigcup_{k<n} A_k) \cup (\bigcup_{i,j\leq n} A(i, f(j)))],$$

for every $n < \omega$. Observe that $p_n \in A(n,\theta)^*$, for every $n < \omega$. Suppose that there is $\nu < \theta$ such that $A(n_i,\nu) \cap A(n_i,\theta) \neq \emptyset$, for every $i < \omega$, and $n_i \neq n_j$ whenever $i < j < \omega$. Let $k < \omega$ be such that $f(k) = \nu$.

If $k < n_i$, then we have that $A(n_i, \nu) \cap A(n_i, \theta) = \emptyset$, but this is a contradiction. Therefore, $\{n < \omega : A(n, \nu) \cap A(n, \theta) \neq \emptyset\}$ is finite, for every $\nu < \theta$. □

Theorem 5.5. *Let $\delta : Seq \to FF(\omega)$ be a function. Then, $Seq(\delta)$ is strongly extraresovable if and only if the set $D = \{s \in Seq : \delta(s)$ is small$\}$ is dense in $Seq(\delta)$.*

Proof. Necessity. Let $\{D_\nu : \nu < \omega_1\}$ witness the strong extraresolvability of $Seq(\delta)$. Fix $s \in Seq$ and $f \in \prod_{t \in C(s)\setminus\{s\}} \delta(t)$. For each $\nu < \omega_1$, we define $S_\nu = \{t \in c(s,f) : \forall F \in \delta(t)(F \cap \{n < \omega : t^\frown n \in D_\nu\} \neq \emptyset)\}$, which is not void by Lemma 5.2. Since Seq is countable there must be $t \in c(s,f)$ and $I_t \in [\omega_1]^{\omega_1}$ such that $\forall F \in \delta(t) \forall \nu \in I_t (F \cap \{n < \omega : t^\frown n \in D_\nu\} \neq \emptyset)$. For each $\nu \in I_t$, we define $A(t, \nu) = \{n < \omega : t^\frown n \in D_\nu\}$. Clearly, $\{A(t, \nu) : \nu \in I_t\}$ witnesses the smallness of $\delta(t)$ and we have that $t \in c(s,f)$. Therefore, $D = \{s \in Seq : \delta(s)$ is small $\}$ is a dense subset of $Seq(\delta)$. In conclusion, for every $s \in D$, we have found $I_s \in [\omega_1]^{\omega_1}$ and an uncountable AD-family $\{A(s,\nu) : \nu \in I_s\}$ witnessing the smallness of $\sigma(s)$ such that if $\nu, \mu < \omega_1$, then $\{s \in D : \nu, \mu \in I_s$ and $A(s,\nu) \cap A(s,\mu) \neq \emptyset\}$ is finite.

Sufficiency. Suppose that $Seq(\delta)$ admits a dense subset D such that $\delta(s)$ is small, for every $s \in D$. For each $s \in D$, choose an uncountable AD-family $\mathcal{A}_s$ witnessing the smallness of $\delta(s)$. Enumerate D as $\{s_n : n < \omega\}$. For each $n < \omega$, let $\mathcal{G}_{s_n} \in FF(\omega)$ be the filter constructed as in the comment above by using the filter $\delta(s_n)$, and define $\gamma : Seq \to FF(\omega)$ by

$$\gamma(s) = \begin{cases} \mathcal{G}_s & \text{for } s \in D \\ \delta(s) & \text{for } s \in Seq \setminus D. \end{cases}$$

We then have that $\mathcal{G}_s$ traces on every element of $\mathcal{A}_s$, for every $s \in D$. According to Lemma 5.4, for every $n < \omega$ and for every $\nu < \theta < \omega_1$, we may find $A(s_n, \nu) \in [\omega]^\omega$ such that

i. $A(s_n, \nu)$ is almost contained in an element of $\mathcal{A}_{s_n}$, for every $n < \omega$ and for every $\nu < \omega_1$;
ii. $\{A(s_n, \nu) : \nu < \omega_1\}$ is an AD-family, for every $n < \omega$;
iii. $A(s_n, \nu) \in \gamma(s_n)^+$, for every $n < \omega$ and for every $\nu < \omega_1$; and
iv. the set $\{s \in D : A(s,\nu) \cap A(s,\mu) \neq \emptyset\}$ is finite, for every $\nu, \mu < \omega_1$.

It then follows from Lemma 5.3 that $Seq(\gamma)$ is strongly extraresolvable. Hence, we obtain that $Seq(\sigma)$ is strongly extraresolvable. □

We now turn out to study the hereditary strong extraresolvability of the spaces $Seq(\sigma)$. For that we need the next lemma.

Lemma 5.6. *Let $\delta : Seq \to FF(\omega)$ be a function. If $X \subseteq Seq(\delta)$ is dense in itself, then for every $s \in X$ and for every $f \in \prod_{t \in C(s)\setminus\{s\}} \delta(t)$, there is $r \in C(s,f)$ such that $\{n < \omega : r^\frown n \in X\} \in \delta(r)^+$.*

Proof. Suppose that there are $s \in X$ and $f \in \prod_{t \in C(s) \setminus \{s\}} \delta(t)$ such that for every $r \in C(s,f)$ there is $F_r \in \delta(r)$ for which

i. $F_r \cap \{n < \omega : r^\frown n \in X\} = \emptyset$, for every $r \in C(s,f)$, and
ii. $F_r \subseteq f(r)$, for every $r \in C(s,f)$.

Now, we define $g \in \prod_{t \in C(s) \setminus \{s\}} \delta(t)$ by

$$g(r) = \begin{cases} F_r & \text{for } r \in C(s,f) \\ f(r) & \text{otherwise.} \end{cases}$$

Since $s \in C(s,g) \cap X \subseteq C(s,f) \cap X$ and X is dense in itself there is $t \in C(s,g) \cap X$ such that $t = r^\frown n$ for some $r \in C(s,g)$ and $n \in g(r) = F_r$, but this is a contradiction. □

Theorem 5.7. *Let $\delta : Seq \to FF(\omega)$ be a function such that $\delta(s)$ is everywhere small for every $s \in Seq$. Then, $Seq(\delta)$ is hereditarily strongly extraresolvable.*

Proof. Let $X \subseteq Seq(\delta)$ be dense in itself. Put $D = \{s \in Seq : \{n < \omega : s^\frown n \in X\} \in \delta(s)^+\}$. We know that D is an infinite set because of Lemma 5.6. Enumerate D as $\{s_n : n < \omega\}$ and, for each $n < \omega$, let $E_n = \{k < \omega : s_n^\frown k \in X\}$. By hypothesis, for every $n < \omega$, we may find an uncountable AD-family $\mathcal{A}_n$ consisting of infinite subsets of E_n so that $\mathcal{A}_n \subseteq \delta(s_n)^+$. By Lemma 5.4, for every $n < \omega$ and for every $\nu < \theta < \omega_1$, we may find $A(s_n, \nu) \in [E_n]^\omega$ such that

i. $A(s_n,\nu)$ is almost contained in an element of $\mathcal{A}_{s_n}$, for every $n < \omega$ and for every $\nu < \omega_1$;
ii. $\{A(s_n,\nu) : \nu < \omega_1\}$ is an AD-family, for every $n < \omega$;
iii. $\delta(s_n)$ traces on $A(s_n,\nu)$, for every $n < \omega$ and for every $\nu < \omega_1$; and
iv. for every $\nu, \mu < \omega_1$, the set $\{s \in D : A(s,\nu) \cap A(s,\mu) \neq \emptyset\}$ is finite.

Then, for every $\nu < \omega_1$, we define

$$D_\nu = \{s^\frown k : s \in D \ \text{ and } k \in A(s,\nu)\}.$$

It is easy to verify that $\{D_\nu : \nu < \omega_1\}$ witnesses the strong extraresolvability of the space X. □

Theorem 5.8. *Let $\delta : Seq \to FF(\omega)$ be a function. If $Seq(\delta)$ is hereditarily strongly extraresolvable, then*

$$D = \{s \in Seq : \delta(s) \ \textit{ is everywhere small}\}$$

is a dense subset of $Seq(\delta)$.

Proof. Suppose the contrary. That is, there are $s \in Seq$ and $f \in \prod_{t \in C(s)\setminus\{s\}} \delta(t)$ such that if $t \in C(s,f)$, then $\delta(t)$ is not everywhere small. For each $t \in C(s,f)$, choose $B_t \in [\omega]^\omega$ witnessing that $\delta(t)$ is not everywhere small. Define $X = \{s\}$ and $X_{n+1} = \{t^\frown n : n \in B_t \text{ and } t \in X_n\}$, for each $1 \leq n < \omega$. Clearly, the set $X = \bigcup_{n<\omega} X_n$ is dense in itself. By assumption, X is strongly extraresolvable. Then X admits an AD-family $\{D_\nu : \nu < \omega\}$ consisting of dense subsets. According to Lemma 5.6, the set $E_\nu = \{t \in c(s,f) : \{n \in B_t : t^\frown n \in D_\nu\} \in \delta(t)^+\}$ is not void, for every $\nu < \omega_1$. We then deduce that there are $r \in C(s,f)$ and $I \in [\omega_1]^{\omega_1}$ such that $\delta(r)$ traces on the set $A_\nu = \{n \in B_r : r^\frown n \in D_\nu\}$, for each $\nu \in I$. But this is a contradiction to the choice of B_r since $\{A_\nu : \nu \in I\}$ is an uncountable AD-family of infinite subsets of B_r. This proves the theorem. □

6. Irresolvability in models of ZF

Let us look at the complexity of the notion of irresolvability. It seems to require some form of the Axiom of Choice. Usual constructions deal with maximal topologies or maximal independent families, strongly utilizing the Kuratowski-Zorn lemma (see for instance [5], [7] and [12]). Indeed, the next theorem suggests that some version of the Axiom of Choice is necessary for the existence of an irresolvable space.

Lemma 6.1. [Countable Axiom of Choice] *The ideal of nowhere dense subsets of a countable irresolvable space is saturated*[3].

Proof. Assume that $X = (\omega, \tau)$ is a countable irresolvable space and let $\mathcal{I}$ denote the ideal of nowhere dense subsets of X. Without loss of generality, we may assume that X is hereditarily (w.r.t. open sets) irresolvable, since the union of resolvable spaces is resolvable ([3]). To see this we only need Countable Axiom of Choice as X is countable. Let $\mathcal{A}$ be an almost disjoint, mod $\mathcal{I}$, family of somewhere dense subsets of X. Assume that $\mathcal{A}$ (note that here uncountable simply means not countable) is uncountable. As X is hereditarily irresolvable, each $A \in \mathcal{A}$ has non-empty interior. Let $D = \{n < \omega : \exists A \in \mathcal{A}(n \in int(A))\}$. Pick for every $n \in D$ a set $A_n \in \mathcal{A}$ such that $n \in int(A_n)$. Only Countable Axiom of Choice was needed here. So, by almost disjointness of $\mathcal{A}$, $\mathcal{A} = \{A_n : n \in D\}$ is countable. □

Lemma 6.2. [Countable Axiom of Choice] *No meager ideal on ω is saturated.*

[3] An ideal $\mathcal{I}$ is called *saturated* if every almost disjoint (mod $\mathcal{I}$) family of subsets of $\mathcal{P}(\omega) \setminus \mathcal{I}$ is countable.

Proof. Let I be a meager ideal on ω. By the Jalali-Naini–Talagrand Theorem (see either [21, Th. 1] or [20, Th. 21]), which also requires only Countable Axiom of Choice, there is a partition $\{I_n : n < \omega\}$ of ω into finite sets such that

(*) For every $A \in \mathcal{I}$, the set $\{n : I_n \subseteq A\}$ is finite.

Let $\mathcal{B}$ be an uncountable almost disjoint (mod Fin) family of subsets of ω and let

$$\mathcal{A} = \{\bigcup_{n\in A} I_n : A \in \mathcal{B}\}.$$

No element of $\mathcal{A}$ is then an element of $\mathcal{I}$ by (*), yet for distinct $A, B \in \mathcal{A}$, $A \cap B$ is finite, hence $A \cap B \in \mathcal{I}$. So the family $\mathcal{A}$ witnesses that $\mathcal{I}$ is not saturated. $\square$

Theorem 6.3. *It is relatively consistent with ZF that every countable space is resolvable.*

Proof. Consider the standard Solovay model M of ZF where every set of reals has the property of Baire (for the construction of the model see for instance [18]). It is known that the Countable Axiom of Choice holds in this model. Suppose that X is a countable irresolvable space and let $\mathcal{I}$ be the ideal of nowhere dense subsets of X. According to Lemma 6.1, the ideal $\mathcal{I}$ is saturated. We may assume that the underlying set of the space X is ω. That is the ideal $\mathcal{I}$ is a subset of $\mathcal{P}(\omega)$ and via characteristic functions can be considered a subset of the Cantor set 2^ω. As a subset of the Cantor set the ideal $\mathcal{I}$ has the Baire property. By Theorem 1 of [21], it is meager. This, however, contradicts Lemma 6.2. Therefore, every countable space is resolvable in the model M. $\square$

We shall remark that in the model M considered in the proof of the previous theorem there is an irresolvable space. This space is the Ellentuck space. Let us recall the *Ellentuck topology* on $[\omega]^\omega$: Given $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$, $s < A$ means that $\max\{s\} < \min\{A\}$, and if $s < A$, then we let $[s, A] = \{B \in [\omega]^\omega\} : s \subseteq B \subseteq s \cup A\}$. The Ellentuck topology τ_E has the family $\{[s, A] : s \in [\omega]^{<\omega}, A \in [\omega]^\omega$ and $s < A\}$ as a basic base of open sets. It is evident that the partition relation $\omega \to (\omega)^\omega_2$ is equivalent to the fact that $([\omega]^\omega, \tau_E)$ is irresolvable.

The partition relation does not hold assuming the Axiom of Choice. On the other hand, it is consistent with ZF that it holds. It is true in the $L(\mathbb{R})$ of a collapse of an inaccessible cardinal to ω, the standard *Solovay Model*.

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Instituto de Matemáticas (UNAM), Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México
E-mail address: sgarcia@matmor.unam.mx
E-mail address: michael@matmor.unam.mx