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Forcing with quotients

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Abstract We study an extensive connection between quotient forcings of Borel subsets of Polish spaces modulo a σ-ideal and quotient forcings of subsets of countable sets modulo an ideal.

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1 Introduction

In recent years there has been a wave of interest in partial orders given as quotients. We will discuss two kinds of them: a σ-ideal I on a Polish space X comes with the quotient poset of I-positive Borel sets ordered by inclusion, denoted by P_I; and an ideal J on some countable set Y comes with the quotient poset of all J-positive sets ordered by inclusion, denoted by Q_J or $\mathrm{Power}(X)/J$ for an ideal J on a countable set X. The former turned out to be very close to traditional forcings adding a real, and they allow of a comprehensive theory [20]. From the forcing point of view, the latter are harder to understand [5]. In this paper we describe a close relationship between the two classes of posets. The connecting link is the following definition due to Brendle [3]:

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Definition 1.1 For a σ-ideal I on ω^ω the *trace ideal* $tr(I)$ on $\omega^{<\omega}$ is defined by $a \in tr(I) \leftrightarrow \{r \in \omega^\omega : \exists^\infty n\ r \restriction n \in a\} \in I$. Similarly for the Cantor space.

First we show that the quotient forcings P_I and $Q_{tr(I)}$ are very close for a large class of σ-ideals I described in the following definition:

Definition 1.2 Let I be a σ-ideal on a Polish space such that the forcing P_I is proper. The forcing P_I has the *continuous reading of names* if for every I-positive Borel set B and a Borel function $f : B \to 2^\omega$ there is an I-positive Borel set $C \subset B$ such that the function $f \restriction C$ is continuous.

Theorem 1.3 *Suppose that I is a σ-ideal on $X = \omega^\omega$. If P_I is a proper forcing with continuous reading of names, then $Q_{tr(I)}$ is a proper forcing as well and in fact Q is naturally isomorphic to a two step iteration of P and an $\aleph_0$-distributive forcing.*

This result makes it easy to generate and understand a large variety of quotients Q_J of ideals on ω. Our methods provide many ideals J for which these quotients are proper as well as examples of ideals for which the quotient forcings are improper. Restricting attention to σ-ideals on the Baire space as opposed to an arbitrary Polish space is both necessary and innocuous: necessary since it makes the definition of the trace ideal possible, and innocuous because every proper forcing with continuous reading of names has a presentation on the Baire space with the continuous reading of names—Claim 2.2.

Earlier results in this area include a note of Steprāns [18] on what in retrospect are trace ideals for a small class of forcings, a result of Balcar, Hernández, and Hrušák [1] regarding the properness of the factor Power($\mathbb{Q}$)/nowhere dense sets, and results of Steprāns and Farah concerning the properness of factors Power$(\omega)/J$ for various analytic P-ideals J. It should be noted that the trace ideals are analytic P-ideals only in the case the original forcing P_I had an exhaustive submeasure on it by a result of Solecki [17].

The second main theorem deals with the action of forcings on ideals on ω.

Definition 1.4 A forcing *destroys* an ideal K on ω if it introduces an infinite set $x \subset \omega$ such that every ground model element of the ideal K has a finite intersection with x.

In fact, what is destroyed is the *tallness* of the ideal K, where K is tall if every infinite set of natural numbers has an infinite subset in K; we suggested the terminology of the definition for its brevity. This definition is useful in the study of maximal almost disjoint families and their preservation under forcings. It turns out that for many forcings the class of ideals on ω which are destroyed can be simply understood in terms of the Katětov order $\leq_K$ [8], which is an interesting notion in itself:

Definition 1.5 If J, L are ideals on ω, we say that $L \leq_K J$ if there is a function $f : \omega \to \omega$ such that f-preimages of K-small sets are L-small. Similarly for ideals on other sets.

It is not difficult to see that if a forcing destroys an ideal L on ω then it destroys all $\leq_K$-smaller ideals. The paper [3] showed that for many particular forcings there is a critical ideal J such that the forcing destroys an ideal L if and only if $L \leq_K J$. It turns out that there is a simple general pattern:

Theorem 1.6 *If I is a σ-ideal on the Baire space and P_I is a proper forcing with the continuous reading of names and L is an ideal on ω then the following are equivalent*:

1. *there is a condition $B \in P_I$ such that $B \Vdash$"the ideal L is destroyed"*
2. *there is a $tr(I)$-positive set a such that $L \leq_K tr(I) \restriction a$.*

In fact, for many forcings used in practice the trace ideals are homogeneous in the sense that $tr(I) \restriction a \leq_K tr(I)$ in which case the second item of the theorem can be improved accordingly: $L \leq_K tr(I)$.

The notation used in the paper follows the set theoretic standard of [9]. If I is a σ-ideal on a Polish space X, the symbol P_I denotes the poset of I-positive Borel subsets of X ordered by inclusion. If J is an ideal on a countable set X, the symbol Q_J denotes the poset of J-positive subsets of X ordered by inclusion. For a tree $T \subset (2 \times \omega)^{<\omega}$ the symbol $[T]$ denotes the set of all its infinite branches and the symbol $p[T]$ its projection, that is the set of those $r \in 2^\omega$ such that there is $b \in \omega^\omega$ such that the pair r, b constitutes a branch through the tree T. The characteristic function of a set $a \subset \omega$ is denoted by $\chi(a)$. For a sequence $t \in 2^{<\omega}$ the symbol $[t]$ denotes the basic open subset of the space 2^ω determined by t. LC denotes the use of suitable large cardinal assumptions.

2 The continuous reading of names

We will begin with several simple observations on the continuous reading of names.

Claim 2.1 Let I be a σ-ideal on a Polish space X. The following are equivalent:

1. the forcing P_I has the continuous reading of names
2. for every I-positive Borel set B and a countable collection $\{D_n : n \in \omega\}$ of Borel sets there is an I-positive Borel set $C \subset B$ such that all sets $D_n \cap C$ are relatively open in C
3. for every I-positive Borel set B and every Borel function $f : B \to Y$ to a Polish space Y there is an I-positive Borel set $C \subset B$ such that $f \restriction C$ is continuous.

Proof (1)$\to$(2). Fix sets $B, D_n : n \in \omega$ and define a Borel function $f : B \to 2^\omega$ by $f(r)(n) = 1$ if $r \in D_n$. By the continuous reading of names there is an I-positive Borel set $C \subset B$ such that $f \restriction C$ is continuous. It is immediate that the sets $D_n \cap C$ must be relatively open in C.

(2)$\to$(3). Suppose that B is a Borel I-positive set and $f : B \to Y$ is a Borel function. For every basic open set O from some fixed countable basis for the space Y, let $D_O = f^{-1}O$. It is clear that D_O is a Borel set and if $C \subset B$ is any set such that all sets $D_O \cap C$ are relatively open in C, the function $f \restriction C$ must be continuous.

(3)$\to$(1). Trivial.

Claim 2.2 Every proper forcing of the form P_I with the continuous reading of names has a presentation on the Baire space ω^ω with the continuous reading of names.

Proof Suppose that X is a Polish space and I is a σ-ideal on it such that the forcing P_I is proper and has the continuous reading of names. There is a continuous bijection

$\pi : Y \to X$ between a closed subset Y of the Baire space and the space X [11, 7.9.] Let J be the σ-ideal on the Baire space generated by $\omega^\omega \setminus Y$ and the π-preimages of sets in the ideal I. Clearly, P_I is forcing isomorphic to P_J. We claim that P_J has the continuous reading of names. If $B \subset \omega^\omega$ is a Borel J-positive set and $f : B \to 2^\omega$ is a Borel function, consider the Borel I-positive set $B' \subset X$ given by $x \in B' \leftrightarrow \pi^{-1}(x) \in B$ and the function $f' : B' \to 2^\omega$ defined by $f'(x) = f(\pi^{-1}(x))$. Note that as f, π^{-1} are both Borel, so is the function f'. Use the CRN on P_I to find an I-positive Borel set $C' \subset B'$ such that the function $f' \restriction C'$ is continuous. A simple diagram-chasing argument shows that the function f is continuous on the J-positive set $C = \pi^{-1}C'$.

Most definable proper partial orderings have the continuous reading of names under a suitable representation.

Example 2.3 Cohen forcing has the continuous reading of names. Recall that the Cohen forcing is naturally represented as P_I where I is the ideal of meager sets. Now, in fact, for every Borel function $f : 2^\omega \to 2^\omega$ there is a meager set $C \in I$ such that the function $f \restriction 2^\omega \setminus C$ is continuous [11, 8.38.]. The Cohen forcing is the only forcing which satisfies this strengthening of the continuous reading of names. To show this, suppose that J is a σ-ideal on a Polish space X such that for every Borel function $f : X \to 2^\omega$ there is a set $C \in J$ such that the function $f \restriction X \setminus C$ is continuous. Then below some condition, the forcing P_J has a countable dense subset, and so is in the forcing sense equivalent to Cohen forcing. If not [2, Proposition 1.4], shows that one can refine the countable basis $\mathcal{O}$ of the space X into a collection of pairwise disjoint J-positive sets $B_O : O \in \mathcal{O}$ such that $B_O \subset O$. Define $f : X \to 2^\omega$ by $f(x)(n) = 1$ if $x \in B_{O_n}$ where O_n is the nth element of the basis $\mathcal{O}$ in some fixed enumeration. Suppose that $C \in J$ is a small set such that $f \restriction X \setminus C$ is continuous. This means that the set $B_{O_0} \setminus C$ is relatively open in $X \setminus C$, containing some basic open neighborhood $O_n \setminus C$. However, this is impossible since the set $B_{O_n} \subset O_n$ is a J-positive subset disjoint from B_{O_0}.

Example 2.4 [20, 2.2.3]. Every proper ω^ω-bounding poset P_I has the continuous reading of names.

Proof For simplicity assume that the underlying space of the ideal I is 2^ω. Suppose B is a Borel I-positive set and $f : B \to 2^\omega$ is a Borel function. Let $T \subset (2 \times 2 \times \omega)^{<\omega}$ be a tree which projects to the graph of the function f. By a standard absoluteness argument, $B \Vdash$"for some $\dot{s} \in 2^\omega, \dot{b} \in \omega^\omega$ the triple $\langle \dot{r}_{\text{gen}}, \dot{s}, \dot{b} \rangle$ constitutes a branch through the tree $\check{T}$." Since the forcing P_I is bounding, there is a condition $D \subset B$ which forces $\dot{b}$ to be pointwise dominated by some function $c \in \omega^\omega$. Let S be the subset of the tree T consisting of those sequences whose third coordinate is pointwise dominated by the function c. Then S is a finitely branching tree and

- $p[S]$ is a compact subset of the graph of the function f, so it is a graph of a continuous subfunction of f
- $C = pp[S]$ is a compact subset of the set B, D forces the generic real into $\dot{C}$ and therefore C is I-positive.

All in all, $C \subset B$ is an I-positive compact set on which the function f is continuous.

Example 2.5 If the ideal I is σ-generated by closed sets then the forcing P_I is proper and it has the continuous reading of names.

Proof The properness of the poset P_I is the contents of [20, Lemma 2.3.11]. For the continuous reading of names, let us first deal with the case of the σ-ideal generated by nowhere dense sets. It is a classical result [11, 8.38] that for every Polish space X and every Borel function $f : X \to \omega^\omega$ there is a comeager G_δ set $C \subset X$ such that f is continuous on it.

In the general case, suppose that B is a Borel I-positive set and $f : B \to \omega^\omega$ is a Borel function. By a result of Solecki [17], thinning out the set B we may assume that it is G_δ. Furthermore, removing all the I-small sets $O \cap B$ where O is a basic open set, we may assume that every basic open set O, $O \cap B \notin I \leftrightarrow O \cap B \neq 0$. Note that the latter operation preserves the fact that the set B is G_δ. It now immediately follows that then every closed set in the ideal I is nowhere dense in the set B. Since the set B is G_δ, it is Polish in the relative topology, and so every set $C \subset B$ comeager in B must be positive in the ideal I. By the first paragraph of the proof, there must be a comeager G_δ set $C \subset B$ such that the function f is continuous on it.

These two classes of examples include many forcings used in practice, such as the Cohen, Solovay, or Miller reals. In other situations, the continuous reading of names has to be checked carefully.

Example 2.6 The Laver forcing in the natural presentation has the continuous reading of names. This is a folklore knowledge, and it follows from Example 3.7 in this paper.

Example 2.7 The *Steprāns forcing* [19] in the natural presentation does not have the continuous reading of names. Here the Steprāns forcing P_I is obtained from a Borel function $f : 2^\omega \to 2^\omega$ which cannot be decomposed into countably many continuous functions by considering the ideal I σ-generated by the sets on which the function f is continuous. The poset P_I is proper and up to the forcing equivalence does not depend on the initial choice of the function f—see [20, 2.3.49].

It is interesting to note that in a slightly different presentation the ideal associated with Steprāns forcing is generated by closed sets and therefore does have the continuous reading of names. Let us describe this different presentation.

We will need a definite example of a Borel function which cannot be decomposed into countably many continuous functions, due to Pawlikowski [4]. Consider the space $\omega+1$ equipped with the order topology, the space $(X, \sigma) = (\omega+1)^\omega$ with the product topology, and the *Pawlikowski function* $P : X \to \omega^\omega$ defined by $P(r)(n) = r(n) + 1$ if $r(n) \in \omega$ and $P(r)(n) = 0$ if $r(n) = \omega$. This function cannot be decomposed into countably many continuous functions and it is in a sense a minimal such example [16].

Let I be the σ-ideal on the space X generated by the sets B on which the function P is continuous. Clearly the poset P_I does not have the continuous reading of names as witnessed by the function P. However, the function P turns out to be the only obstacle. Namely, if the space X is equipped with the smallest Polish topology $\tau \supset \sigma$ which makes the function P continuous and generates the same Borel structure, the σ-ideal I is generated by τ-closed sets and so the forcing P_I has the continuous reading of names in this new presentation. An outline of the easy argument: the topology τ is the product topology on X with $\omega + 1$ viewed as a discrete space. If $B \subset X$ is a set such that $P \restriction B$ is continuous with respect to the topology σ and C is the τ-closure of the set B, then $P \restriction C$ is continuous with respect to σ as well. If U, V are basic open subsets of (X, σ) and ω^ω respectively such that $P''(B \cap U) \subset V$ then $P''(C \cap U) \subset V$ as well.

Definition 2.8 Let J be an ideal on ω. The *Prikry forcing* $P(J)$ for the ideal J is defined as the set of all pairs $\langle t, a\rangle$ where $t \subset \omega$ is a finite set, $a \subset \omega$ is a set in the ideal J, and $\langle u, b\rangle \leq \langle t, a\rangle$ if $t \subset u, a \subset b$ and $a \cap u \setminus t = 0$. We will refer to the union of the first coordinates of conditions in the generic filter as the generic subset of ω, and denote it by $\dot{a}_{gen}$.

Example 2.9 Let J be an ideal on ω. The forcing $P(J)$ has the continuous reading of names if and only if J is a P-ideal.

Proof Let I be the σ-ideal on 2^ω associated with the forcing $P(J)$, namely I is the collection of those sets $B \subset 2^\omega$ for which it is outright forced that $\chi(\dot{a}_{gen}) \notin B$. Thus the poset P_I is in the forcing sense equivalent to the poset $P(J)$, with a canonical correspondence between the respective generic objects.

First suppose that J is not a P-ideal, as witnessed by a countable collection $\{a_n : n \in \omega\}$ of sets in the ideal such that no set in the ideal contains each of them modulo a finite set. Consider the Borel function $f : 2^\omega \to 2^\omega$ defined by $f(r)(n) =$the parity of the size of the set $\{m \in a_n : r(m) = 1\}$. The function f is defined on an I-large set, and we claim that it cannot be reduced to a continuous function on an I-positive Borel set.

Suppose that B is an I-positive Borel set, and $\langle t, b\rangle \Vdash$"$\chi(\dot{a}_{gen}) \in \dot{B}$". Thinning out the set B we may assume that it consists only of functions r such that $\forall m \in \omega\ (t(m) = 1 \to r(m) = 1$ and $m \in b \to r(m) = 0)$. Let n be such that the set $a_n \setminus b$ is infinite. It is not difficult to see that both sets $\{r \in B : f(r)(n) = 0\}$ and $\{r \in B : f(r)(n) = 1\}$ are dense in the set B, and therefore the function f cannot be continuous on B.

Now suppose that J is a P-ideal, $B \notin I$ is a Borel set, and $f : B \to \omega^\omega$ is a Borel function. Let M be a countable elementary submodel of a large enough structure containing the ideal J, let $a \subset \omega$ be a set in the ideal J which modulo finite contains all sets in $J \cap M$, and for every number n consider the sets $C_n = \{r \in B : r$ is M-generic for P_I and for all $k > n, r(k) = 1 \to k \notin a\}$. Since the poset P_I is c.c.c., the set $B \setminus \bigcup_n C_n$ is in the ideal I and there must be a number $n \in \omega$ such that the Borel set C_n is I-positive. Set $C = C_n \subset B$; we will be done if we show that the function $f \restriction C$ is continuous.

Suppose $r \in C$ and $O \subset \omega^\omega$ is a basic open set such that $f(r) \in O$. We must produce a basic open set $P \subset 2^\omega$ such that $r \in P$ and for every real $s \in P \cap C$, $f(r) \in O$. Look at the M-generic filter $G \subset M \cap P(J)$ associated with the real r: $G = \{\langle t, b\rangle \in P(J) \cap M : \forall m \in \omega\, m \in t \to r(m) = 1 \wedge m \in b \to r(m) = 0\}$. By the forcing theorem, there must be a condition $\langle t, b\rangle \in G$ which forces $\dot{f}(\chi(\dot{a}_{gen})) \in O$. Let $m \in \omega$ be a natural number larger than n, larger than all elements of the finite set t, and larger than all elements of the finite set $b \setminus a$. It is enough to show that whenever $s \in C$ is a real such that $s \restriction m = r \restriction m$ then $f(s) \in O$. A brief inspection reveals that the condition $\langle t, b\rangle$ belongs to the M-generic filter associated with the real s, and by the forcing theorem applied in the model M, it must be the case that $f(s) \in O$ as desired.

A similar proof can be used to show that the Hechler forcing in the natural presentation has the continuous reading of names, while the eventually different real forcing does not have the continuous reading of names.

Example 2.10 The eventually different real forcing does not have the continuous reading of names in any presentation. Here, if I is a σ-ideal on a Polish space X such that the forcing P_I is proper, a different presentation is just a Borel bijection $\pi : X \to Y$ of X and another Polish space and the ideal J on Y defined by $A \in J \leftrightarrow \pi^{-1}A \in I$. Since Borel injective images of Borel sets are Borel, it follows that P_J and P_I are isomorphic partial orders. The eventually different real forcing P is the set of all pairs $p = \langle t_p, f_p \rangle$ where t_p is a finite sequence of natural numbers and f_p is a finite set of functions in ω^ω. The ordering is defined by $q \le p$ if $t_p \subset t_q$, $f_p \subset f_q$ and $(t_q \setminus t_p) \cap \bigcup f_p = 0$. The forcing P adds an element $\dot{x}_{\text{gen}}$ of the Baire space as the union of the first coordinates of the conditions in the generic filter. The function $\dot{x}_{\text{gen}}$ has finite intersection with every function in the ground model. The forcing P is clearly σ-centered since any two conditions with the same first coordinate are compatible. Let I be the σ-ideal of all Borel sets $B \subset \omega^\omega$ such that $P \Vdash$"$\dot{x}_{\text{gen}} \notin \dot{B}$" so P is in the forcing sense equivalent to the poset P_I.

First we claim that it is enough to show that for no Polish topology τ on the Baire space extending the standard Baire space topology the forcing P_I has the τ-continuous reading of names. To see this, note that if $\pi : \omega^\omega \to Y$ and J is a presentation of the eventually different forcing, then there is a Polish topology τ on the Baire space which gives the same Borel structure as the original one and makes all the π-preimages of open subsets of Y open [11, 13.A]. It is easy to see that if the forcing P_J had the continuous reading of names, then so would P_I in the topology τ.

Let $B_n : n \in \omega$ enumerate a basis for the topology τ. These are all Borel subsets of the Baire space ω^ω and so there are countable antichains $A_n : n \in \omega$ in the forcing P such that every condition in A_n forces $\dot{x}_{\text{gen}} \in \dot{B}_n$ and the antichains are maximal with respect to this property.

A piece of notation and an easy construction: for a finite set $f \subset \omega^\omega$ of functions and a number $l \in \omega$ write $f(l) = \{x(l) : x \in f\}$. For every number $m \in \omega$ choose a set f_m of $m + 1$ many functions in the Baire space which return mutually distinct values at every input and moreover such that for all numbers $k, n \in \omega$ and every condition $q \in A_n$ there is a number $l > k$ such that $f_m(l) \cap f_q(l) = 0$. Let $h : \omega^\omega \to \omega^\omega$ be the partial Borel function defined by $h(x)(m) =$the least number k such that $x(l) \notin f_m(l)$ for all numbers $l > k$. Note that the function h is defined on all but I-many points in the Baire space. We claim that there is no Borel I-positive set $C \subset \omega^\omega$ such that $h \restriction C$ is a τ-continuous function.

Suppose there in fact is such a set $C \subset \omega^\omega$. Find a condition $p \in P$ such that $p \Vdash \dot{x}_{\text{gen}} \in \dot{C}$ and let $m = |f_p|$. The sets $C_k = \{x \in C : h(x)(m) = k\} : k \in \omega$ exhaust all of C and so for one of them, p must have an extension forcing $\dot{x}_{\text{gen}}$ into it. This set C_k is relatively τ-open in the set C, and there must be a set $a \subset \omega$ such that $C_k = C \cap \bigcup_{n \in a} B_n$. Since the set C_k is I-positive, there must be a number $n \in a$ and a condition $q \in A_n$ such that p, q are compatible conditions. Now use the property of the finite set $f_m \subset \omega^\omega$ to find a number $l > k, |s_p|, |s_q|$ such that $f_m(l) \cap f_q(l) = 0$. Since there are $m + 1$ many functions in the finite set $f_m \subset \omega^\omega$ and only m many functions in the set f_p, there is a function $y \in f_m$ such that $y(l) \notin f_p(l)$. It is now easy to find a finite sequence s extending both s_p and s_q such that the condition $r = \langle s, f_p \cup f_q \rangle$ is a lower bound of p, q and $s(l) = y(l)$. Since the condition r forces both $\dot{x}_{\text{gen}} \in \dot{C}$ and $\dot{x}_{\text{gen}} \in \dot{B}_n$, any sufficiently generic point $x \in \omega^\omega$ below the condition r will belong to the intersection $B_n \cap C$. However, for every such a point it is the case that $h(x)(m) > l > k$, contradicting the assumption that $B_n \cap C \subset C_k$!

The continuous reading of names is a rather slippery property of ideals. It is not preserved under Borel isomorphism of ideals. This is to say that there are σ-ideals I and J on Polish spaces X and Y and a Borel bijection $f : X \to Y$ such that a set $A \subset Y$ is in the ideal J iff its f-preimage is in the ideal I, but the poset P_I does have the continuous reading of names while P_J does not. An instructive example is that of the Steprāns forcing, 2.7. Note that since Borel injective images of Borel sets are Borel, in this case the function f can be naturally extended to an isomorphism of the posets P_I and P_J. This means that the continuous reading of names is, in fact, a property of a presentation of forcing as opposed to a property of the forcing itself. Even so, the continuous reading of names is perceived as a natural and useful property. We state two of its important features.

Claim 2.11 [20, 2.2.2(2)] Suppose that I is a σ-ideal on a Polish space X. If P_I is a proper forcing notion with the continuous reading of names then every Borel I-positive set has a G_δ I-positive subset.

Proof Suppose that $B \subset X$ is a Borel I-positive set, a projection of a closed set $E \subset X \times \omega^\omega$. Since the poset P_I is proper, there must be an I-positive Borel set $D \subset B$ and a Borel function f defined on the set D whose graph is a subset of the set E. Use the continuous reading of names and thin out the set D if necessary so as to make the function $f \restriction D$ continuous. Every partial continuous function can be extended to a continuous function with a G_δ domain. Let $D \subset C$, $f \subset g$ be such a G_δ set and a continuous extension, with D still dense in C. It is immediate that $g : C \to \omega^\omega$ is a function whose graph is a subset of the set E. Then $C \subset B$ is an I-positive G_δ-subset of the set B.

The opposite implication does not hold: compact sets are dense in the natural presentation of Steprāns forcing [20, 2.3.46], while the continuous reading of names fails.

Fact 2.12 (LC) *The continuous reading of names is preserved under the countable support iteration of universally Baire proper forcings of the form P_I.*

This is proved in the forthcoming [21]. In conjunction with the previous claim, this means for example that G_δ sets are dense in the countable support iteration of Laver forcing.

3 Proof of Theorem 1.3

The proof is the same for both the Cantor and Baire space, and we will treat the Baire space case.

Definition 3.1 The function $\pi : \mathcal{P}(\omega^{<\omega}) \to \mathcal{P}(\omega^\omega)$ is defined by $\pi(a) = \{r \in \omega^\omega : \exists^\infty n\ r \restriction n \in a\}$.

Clearly, the range of the function π is exactly the collection of all G_δ-subsets of ω^ω, and the function π preserves inclusion. Moreover, if I is a σ-ideal on ω^ω then $a \in tr(I)$ if and only if $\pi(a) \in I$, and the map $\pi \restriction Q_{tr(I)} : Q_{tr(I)} \to P_I$ preserves compatibility. For the remainder of the section fix a σ-ideal I on ω^ω such that the poset P_I is proper and has the continuous reading of names, and write $J = tr(I)$.

Claim 3.2 1. For every set $a \notin J$ and for every I-positive G_δ subset $B \subset \pi(a)$ there is a set $b \subset a$ such that $\pi(b) = B$.
2. Q_J forces $\pi''\dot{G}$ to be a P_I-generic filter, where $\dot{G}$ is the name for the Q_J generic filter.

Proof The second item immediately follows from the first and Claim 2.11. For the first one, suppose $B \subset \pi(a)$ is an I-positive G_δ set, $B = \bigcap_n O_n$ for some open sets O_n. By induction on $n \in \omega$ build sets $a_n \subset a$ in the following way:

1. Each $a_n \subset \omega^{<\omega}$ is an antichain and it refines a_{n-1}. For notational convenience let a_{-1} be the singleton containing the empty sequence.
2. $B \subset \bigcup_{t \in a_n} [t] \subset O_n$.

After the construction is complete, writing $b = \bigcup_n a_n$ we will have $\pi(b) = B$ as required. Suppose the antichain a_n has been obtained. For each $t \in a_n$ let $c(t)$ be the collection of all proper extensions $u \in \omega^{<\omega}$ of t such that $[u] \subset O_{n+1}$, and no proper initial segment of u longer than t has this property. Note that $c(t) \subset \omega^{<\omega}$ is an antichain. For each $u \in c(t)$ let $d(u)$ be the collection of all proper extensions v of u which are in the set a, such that no proper initial segment of v longer than u has this property. Note that each $d(u)$ is an antichain. It is not difficult to verify that the set $a_{n+1} = \bigcup\{d(u) : u \in c(t), t \in a_n\}$ has the desired properties.

Claim 3.3 The poset Q_J is proper.

Proof Let M be a countable elementary submodel of a large structure with $I \in M$ and let $a \in Q_J \cap M$ be a condition. Let $\langle D_n : n \in \omega\rangle$ be an enumeration of all open dense subsets of the poset Q_J in the model M. We will find sets $a_n \subset a$ and functions $g_n : a_n \to D_n \cap M$ with the following properties.

1. Each set $a_n \subset \omega^{<\omega}$ is an antichain and it refines a_{n-1}.
2. The set $b = \bigcup_n a_n \subset a$ is J-positive.
3. For each sequence $t \in a_n$ the set $\{u \in b : t \subset u\}$ is a subset of $g_n(t)$.

It follows that the set $b \subset a$ is the required M-master condition in the poset Q_J. To see this, choose a J-positive set $c \subset b$ and an open dense set $D = D_n \in M$ for some number n. For each sequence $t \in a_n$ write $b_t = \{u \in b : t \subset u\}$. Since the set $a_n \subset \omega^{<\omega}$ is an antichain, it is the case that $\pi(c) = \bigcup_{t \in a_n} \pi(c \cap b_t)$ and therefore one of the sets $c \cap b_t : t \in a_n$ must be J-positive. Such a set $c \cap b_t \subset c$ has the condition $g_n(t) \in D_n \cap M$ above it as required.

To perform the construction, find an M-master condition $B \subset \pi(a)$ for the poset P_I. Thinning out the condition B we may assume that for every dense set $E \in M$ of the poset P_I, $B \subset \bigcup(E \cap M)$. Thinning out the condition B even further, by Claim 2.1, we may assume that for every set $C \in P_I \cap M$ the intersection $C \cap B$ is relatively open in B. Thinning out the condition B further still we may assume that it is a G_δ set such that for every basic open set O, $B \cap O \notin I \leftrightarrow B \cap O \neq 0$. Fix a representation $B = \bigcap_n O_n$, for some open sets O_n.

The induction hypotheses on the construction of the sets a_n are the following.

1. Each $a_n \subset \omega^{<\omega}$ is an antichain and it refines a_{n-1}.
2. $B \subset \bigcup_{t \in a_n}[t] \subset O_n$.
3. For every n, $g_n(t) \subset \{u \in a : t \subset u\}$ is a condition in the open dense set D_n. For every $n \in m$, $t \in a_n$ and $u \in a_m$ such that $t \subset u$, $u \in g_n(t)$ and $g_m(u) \subset g_n(t)$. For notational convenience put $g_{-1}(0) = a$.
4. For each $t \in a_n$, $B \cap [t]$ is a nonempty subset of $\pi(g_n(t))$.

Now suppose that a_n, g_n have been constructed. Fix a node $t \in a_n$. We will show how the part of the antichain a_{n+1} below t will be constructed. Let E be the part of the open dense set $D_{n+1} \subset Q_J$ below the condition $g_n(t) \in Q_J$. Claim 3.2 shows that the set $\pi''(E)$ is dense below the condition $\pi(g_n(t))$. Then $B \cap [t] \subset \bigcup(\pi'' E \cap M) = \bigcup \pi''(E \cap M)$ by the choice of the M-master condition B. Note that for every condition $p \in E \cap M$ the set $\pi(p) \cap B \subset B \cap [t]$ is relatively open by the choice of the condition $B \in P_I$ again. It is now easy to build an antichain $d \subset \omega^{<\omega}$ below the node t so that for every $u \in d$ it is the case that $[u] \subset O_{n+1}$ and there is a condition $p(u) \in E \cap M$ such that $B \cap [u]$ is a nonempty subset of $\pi(p(u))$, and $B \cap [t] \subset \bigcup_{u \in d}[u]$. Let c be then the collection of all nodes $v \in \omega^{<\omega}$ such that there is some $u \in d$ such that $u \subset v$, $v \in p(u)$, $B \cap [v] \neq 0$ and no proper initial segment of v is an extension of u in $p(u)$. The set c is an antichain below the node t, and it is the part of the antichain a_{n+1} below t. For every node $v \in c$ let $g_{n+1}(v) = p(u) \cap \{w \in \omega^{<\omega} : v \subset w\}$. The induction hypotheses are easily seen to be satisfied.

Claim 3.4 The remainder poset $R = Q_J / P_I$ preserves stationary subsets of ω_1 and it is $\aleph_0$-distributive.

Note that the proof below leaves open the possibility that the remainder collapses the stationarity of the set $[\omega_2]^{\aleph_0}_0 \setminus V$ and therefore fails to be proper.

Proof Here the remainder poset R is computed via the Q_J-name for the P_I-generic filter obtained in Claim 3.2. Note that writing $\dot{r}_{\mathrm{gen}}$ for the canonical P_I-generic real we have $t \subset \dot{r}_{\mathrm{gen}} \leftrightarrow$ the set $\{u \in \omega^{<\omega} : t \subset u\}$ is in the Q_J-generic filter, this for every sequence $t \in \omega^{<\omega}$.

The fact that $P_I \Vdash$"$\dot{R}$ is stationary preserving" follows abstractly from the proof of the previous claim: if M is a countable elementary submodel containing the ideal I and $B \in P_I$ is any M-master condition for the poset P_I then there is an M-master condition $b \in Q_J$ such that $\pi(b) \subset B$. Namely, suppose that $\dot{S}$ is a P_I-name for a stationary subset of ω_1 and $\dot{C}$ is a Q_J-name for a club in ω_1. We must find a condition $b \in Q_J$ and an ordinal $\alpha \in \omega_1$ such that $b \Vdash \check{\alpha} \in \dot{S} \cap \dot{C}$. Note that as $\dot{S}$ is forced to be stationary, there must be a model M and a M-master condition B forcing $\check{M} \cap \omega_1 \in \dot{S}$.

Writing $\alpha = M \cap \omega_1$ and finding an M-master condition $b \in Q_J$ such that $\pi(b) \subset B$ we see that b, α work as required.

For the distributivity, suppose that $\dot{f}$ is a Q_J-name for an ω-sequence of ordinals. We must prove that $\dot{f} \in V[\dot{r}_{\text{gen}}]$. To this end, revisit the proof of the previous claim again. Assume that $\dot{f} \in M$ and for each number $k \in \omega$ find a number $n_k \in \omega$ such that the conditions in the open dense set $D_{n_k} \subset Q_J$ decide the value of $\dot{f}(\check{k})$. Look again at the master condition $b = \bigcup_n a_n$. It is not difficult to see that b forces that for each $n \in \omega$ there is exactly one initial segment of the real $\dot{r}_{\text{gen}}$ in the set a_n; call it t_n. Consequently, the sequence $\dot{f}$ can be recovered in the model $V[\dot{r}_{\text{gen}}]$ by the following formula: $\dot{f}(\check{k})$ is that ordinal which is forced by the condition $g_{n_k}(t_{n_k})$ to be the value of $\dot{f}(\check{k})$.

This completes the proof of Theorem 1.3, the rest of this section is devoted to speculations about the surrounding issues.

It is interesting to see what the $\aleph_0$-distributive tail Q_J/P_I can be. From the definitions it is equal to the collection of all ground model sets $a \subset \omega^{<\omega}$ such that the P_I-generic real has infinitely many initial segments in a, ordered by inclusion. In many cases it is, in the forcing sense, equivalent to $\mathcal{P}(\omega)/\text{fin}$ of the P_I extension. To prove this it is enough to show that P_I forces every infinite subset of the generic real $\dot{r}_{\text{gen}}$ (understood now as a path through $\omega^{<\omega}$) to have an infinite subset of the form $a \cap \dot{r}_{\text{gen}}$ for some set a in the ground model. We can verify this property in a great number of cases and disprove in others, but we do not have a suitable general criterion.

Proposition 3.5 *Let I be a σ-ideal on 2^ω σ-generated by a σ-compact family of closed sets. The forcing P_I is proper and bounding, and writing $J = tr(I)$, $Q_J = P_I * \mathcal{P}(\omega)/\text{fin}$.*

Here, the hyperspace of closed subsets of 2^ω is equipped with the usual Hausdorff topology, and a family of closed sets is σ-compact if it is a countable union of compact sets.

Proof The ideals I considered in this proposition form a class considered in [7]. There it is proved that the poset P_I is proper and bounding; it has the continuous reading of names simply because the ideal is generated by closed sets—Example 2.5. In fact a standard determinacy argument [7] Corollary 3.21 shows the following: fix a σ-ideal I σ-generated by a collection $F = \bigcup_n F_n$ in which the sets $F_n \subset K(2^\omega)$ are closed. Call a tree $T \subset 2^{<\omega}$ *I-fat* if for every node $t \in T$ and every number n there is a number m such that no set in F_n meets all the open sets determined by the extensions of the node t in the tree T of length m. Then a Borel set $B \subset 2^\omega$ is I-positive if and only if it contains all branches of some I-fat tree. Therefore the poset of I-fat trees is naturally isomorphic with a dense subset of the poset P_I and below we will identify it with P_I.

We will show that $P_I \Vdash$"every infinite subset of $\dot{r}_{\text{gen}}$ has an infinite subset of the form $a \cap \dot{r}_{\text{gen}}$ for some set a in the ground model". Suppose $T \in P_I$ is an I-fat tree, $T \Vdash$"$\dot{x} \subset \dot{r}_{\text{gen}}$ is an infinite set". A standard fusion argument will give an I-fat tree $S \subset T$ such that for every number n there is $m > n$ such that every sequence $s \in S$ of length m has an initial segment of length $\geq n$ in the set $a = \{t \in S : S \restriction t \Vdash \check{t} \in \dot{x}\} \subset S$. Clearly $S \Vdash$"$\check{a} \cap \dot{r}_{\text{gen}}$ is an infinite subset of $\dot{x}$" as desired.

The following definition is not standard. It is an attempt to restate a commonly used combinatorial forcing property in topological terms.

Definition 3.6 Let I be a σ-ideal on some Polish space X with a fixed metric d. We say that the poset P_I has the *pure decision property* (with respect to the metric d) if for every I-positive Borel set $B \subset X$ and every Borel map $f : (B, d) \to (Y, e)$ into a compact metric space there is a Borel I-positive set $C \subset B$ on which the map f is a contraction.

Example 3.7 The Laver forcing has the pure decision property in the standard representation, with respect to the metric of least difference on ω^ω: $d(x, y) = 2^{-n}$ where n is the smallest number where the functions $x, y \in \omega^\omega$ differ.

Proof Let B be Borel I-positive set and $f : (B, d) \to (Y, e)$ be a Borel map into a compact metric space. Thinning out the set B if necessary we may assume that $B = [T]$ for some Laver tree $T \subset \omega^{<\omega}$. To simplify the notation assume that T has an empty trunk.

Before we proceed recall the well known fact that for every Laver tree S and Borel partition $[S] = \bigcup_{i \in n} A_i$ into finitely many pieces there is a Laver tree $U \subset S$ with the same trunk such that the set $[U]$ is included in one of the pieces of the partition.

Now for every n find a finite 2^{-n-1}-network $y_n \subset Y$, that is, a set such that every point of the space Y is 2^{-n-1}-close to one of its elements. By induction on $n \in \omega$ build a fusion sequence of Laver trees T_n so that $T_0 = T$, T_{n+1} agrees with T_n on sequences of length $n + 1$ and for every such a sequence $t \in T_n$ there is an element $x_t \in y_n$ such that for every path r through T_{n+1} extending the sequence t, the element $f(r) \in Y$ is 2^{-n-1}-close to x_t. This is possible by the observation in the previous paragraph. Note that by the triangle inequality this means that for two such paths r_0, r_1 the elements $f(r_0), f(r_1) \in Y$ will have e-distance $\leq 2^{-n}$. Let S be the fusion of the sequence of trees T_n. It is not difficult to see that the set $C = [S]$ has the required properties.

Proposition 3.8 *If I is a σ-ideal on ω^ω such that the poset P_I is proper and has the pure decision property with respect to the metric of least difference on ω^ω, then $Q_{tr(I)} = P_I * \mathcal{P}(\omega)/\mathrm{fin}$.*

Proof Note that the pure decision property implies the continuous reading of names.

Suppose that $B \in P_I$ forces $\dot{x} \subset \dot{r}_{\mathrm{gen}}$ to be an infinite set. Since the poset P_I is proper, thinning out the set B if necessary we can find a Borel map $f : B \to 2^\omega$ such that $B \Vdash$“$\dot{x} = \{\dot{r}_{\mathrm{gen}} \restriction n : n \in \dot{f}(\dot{r}_{\mathrm{gen}})\}$”. Consider the metric e of least difference on 2^ω and use the pure decision property to find an I-positive set $C \subset B$ such that $f : C \to 2^\omega$ is a contraction. This means that for every sequence $t \in \omega^{<\omega}$, all reals $r \in C$ extending the sequence t return the same value $b(t) \in 2$ for $f(r)(|t|)$. Let $a = \{t \in \omega^{<\omega} : b(t) = 1\}$. It follows from the definitions that $C \Vdash \dot{x} = a \cap \dot{r}_{\mathrm{gen}}$, and the proposition follows.

Example 3.9 The Cohen poset forces that there is an infinite set $x \subset \dot{r}_{\mathrm{gen}}$ without an infinite subset of the form $a \cap \dot{r}_{\mathrm{gen}}$, $a \in V$. Just let an initial segment t of $\dot{r}_{\mathrm{gen}}$ into $\dot{x}$ if and only if $\dot{r}_{\mathrm{gen}}(|t|) = 0$.

As a final remark in this section, once we produced so many ideals J for which the factor forcing Q_J is proper, we should also produce some for which it is not proper. The following proposition of independent interest shows how to do exactly that in several ways. First, an instrumental definition.

Definition 3.10 Let β be a limit ordinal. We say that an inclusion-decreasing sequence $\langle I_\alpha : \alpha \in \beta\rangle$ of σ-ideals on a Polish space *does not stabilize* if for every ordinal $\alpha \in \beta$ and every I_α-positive Borel set B there is an ordinal $\alpha \in \gamma \in \beta$ and a Borel set $C \subset B$ which is I_α-small but I_γ-positive. This is equivalent to saying that, writing $I = \bigcap_\alpha I_\alpha$, the sets $I_\alpha \cap P_I$ are all dense in P_I. Restated again, this is equivalent to saying that for every I-positive Borel set B, $I \restriction B \neq I_\alpha \restriction B$—hence the terminology.

Similarly, we say that an inclusion-decreasing sequence $\langle J_\alpha : \alpha \in \beta\rangle$ of ideals on some countable set X does not stabilize if for every ordinal $\alpha \in \beta$ and J_α-positive set $a \subset X$ there is an ordinal $\alpha \in \gamma \in \beta$ and a set $b \subset a$ which is J_α-small but J_γ-positive. This is the same as to say, writing $J = \bigcap_\alpha J_\alpha$, that the sets $J_\alpha \cap Q_J$ are dense in the factor forcing Q_J.

Proposition 3.11 *Assume the Continuum Hypothesis. If I is a σ-ideal on a Polish space, then*

1. *P_I collapses $\aleph_1$ if and only if $I = \bigcap_{n\in\omega} I_n$ for an inclusion-decreasing sequence of σ-ideals which does not stabilize.*
2. *Suppose P_I preserves $\aleph_1$. P_I is nowhere c.c.c. if and only if $I = \bigcap_{\alpha\in\omega_1} I_\alpha$ for an inclusion-decreasing sequence of σ-ideals which does not stabilize.*

If J is an ideal on a countable set, then

3. *Q_J adds an unbounded real if and only if $J = \bigcap_{n\in\omega} J_n$ for an inclusion-decreasing sequence of ideals which does not stabilize.*
4. *If $J = \bigcap_{n\in\omega} J_n$ for an inclusion-decreasing sequence of P-ideals which does not stabilize, then Q_J collapses $\aleph_1$.*

Proof For the first equivalence, assume that P_I collapses $\aleph_1$. Let $\dot f : \check\omega \to \check\omega_1$ be a name for a function with cofinal range. For every number $n \in \omega$ let I_n be the ideal generated by sets $B \in P_I$ which force the first n values of the function $\dot f$ to be bounded by some fixed countable ordinal, together with all sets in the ideal I. It is clear that $\langle I_n : n \in \omega\rangle$ is an inclusion-decreasing sequence of σ-ideals which does not stabilize, and $I = \bigcap_n I_n$. On the other hand, suppose that $I = \bigcap_n I_n$ for some inclusion-decreasing sequence of σ-ideals which does not stabilize. Since the ideals I_n are dense in the poset P_I, we can pick a maximal antichain $A_n \subset I_n$ from each, and by CH it will be enough to show that every condition in P_I is compatible with uncountably many elements of one of these antichains. Indeed, if $B \in P_I$ is a condition, then $B \notin I_n$ for some number n, and B must be compatible with uncountably many elements of the antichain A_n, because if $X \subset A_n$ is a countable set, then $C = \bigcup X \in I_n$ and the condition $B\setminus C \notin I_n$ is a condition incompatible with all elements of the set X.

For the second equivalence, assume that the poset P_I preserves $\aleph_1$ and is nowhere c.c.c. Then there is a name $\dot f$ for a function from ω_1 to itself which is not bounded by any ground model such function. To see this, let $\langle M_\alpha : \alpha \in \omega_1\rangle$ be a tower of countable elementary submodels of some large structure, and define $\dot{f}(\alpha) = \min\{\beta \in \omega_1$: for every maximal antichain $A \in M_\alpha$ the unique element in it which belongs to the generic filter is in the model $M_\beta\}$. Since the forcing P_I preserves $\aleph_1$, and by CH $P_I \subset \bigcup_\alpha M_\alpha$, this is well-defined. If $p \in P_I$ is a condition and $g : \omega_1 \to \omega_1$ is a function, find an ordinal $\alpha \in \omega_1$ such that $p \in M_\alpha$, a maximal antichain $A \in M_\alpha$ which has uncountably many elements below the condition p, and an element $q \in A \setminus M_{g(\alpha)}$ below the condition p. Then $q \Vdash$"$g(\alpha) \in \dot{f}(\alpha)$", and it follows that the function $\dot{f}$ is unbounded.

Now, given an ordinal α let I_α be the ideal generated by the sets $B \in P_I$ for which there is a countable ordinal β such that B forces all values $\{\dot{f}(\gamma) : \gamma \in \alpha\}$ to be smaller than β, together with all sets in the ideal I. It is clear that $\langle I_\alpha : \alpha \in \omega_1\rangle$ is an inclusion decreasing sequence of σ-ideals. Since the function $\dot{f}$ is not dominated by any ground model function, it is the case that $I = \bigcap_\alpha I_\alpha$, and since the forcing P_I preserves $\aleph_1$, the sequence of ideals does not stabilize.

For the other direction, let $I = \bigcap_\alpha I_\alpha$. Suppose $B \in P_I$ is a Borel set; we must find an uncountable antichain below it. It must be the case that $B \notin I_\alpha$ for some countable ordinal α. Now since the sequence of ideals does not stabilize, the ideal I_α is dense in the poset P_I, and therefore there must be a maximal antichain A below B which consists solely of I_α-small sets. This antichain must be uncountable, because otherwise $\bigcup A \in I_\alpha$ and $B \setminus \bigcup A \notin I_\alpha$ is a condition in P_I which avoids all elements of the maximal antichain, a contradiction.

For the third equivalence, first suppose that $Q_J \Vdash$"$\dot{f} \in \omega^\omega$ is an unbounded function." Let $J_n = \{a \subset \omega$: there is a number m such that a forces the first n values of the function $\dot{f}$ to be smaller than $m\}$. It is immediate that $\langle J_n : n \in \omega\rangle$ is an inclusion-decreasing sequence of ideals which does not stabilize. Since $\dot{f}$ is forced unbounded, $J = \bigcap J_n$. On the other hand, suppose that $J = \bigcap_n J_n$ is the intersection of an inclusion decreasing sequence of ideals which does not stabilize. Each ideal J_n is dense in Q_J, so we can find a maximal antichain $A_n \subset J_n$. Now, suppose $a \in Q_J$. There are two cases. In the first case, there is no condition $b \subset a$ which is compatible with at most countably many elements of $\bigcup_n A_n$. Then $a \Vdash$"$\aleph_1$ is collapsed and by the CH an unbounded real is added." In the second case, there is such a condition b, compatible only with elements $\{a_n^k : k \in \omega\}$ of the antichain A_n. Let $\dot{f} \in \omega^\omega$ be defined by $f(n) =$ the unique k such that a_n^k is in the generic filter. The condition b forces this function to be well-defined, and we will be done if we prove that it forces it not to be bounded by any ground model function. Indeed, if $c \subset b$ is a condition and $g \in \omega^\omega$ is a function, it must be the case that $c \notin J_n$ for some number n, $d = \bigcup_{k \in g(n)} a_n^k \in J_n$, $c \setminus d \notin J_n$ and clearly $c \setminus d \subset c$ is a condition forcing $g(n) \leq \dot{f}(n)$.

Finally, for the fourth item, suppose that $J = \bigcap_n J_n$ is an intersection of an inclusion decreasing sequence of P-ideals which does not stabilize. Every ideal J_n is dense in the factor Q_J, therefore we can find a maximal antichain $A_n \subset J_n$. We will be done if we show that every condition $a \in Q_J$ is compatible with uncountably many elements of one of these antichains. Indeed, if $a \in Q_J$, then $a \notin J_n$ for some number n, and a must be compatible with uncountably many elements of the antichain A_n. This is true because if $X \subset A_n$ is a countable set then there is a set $b \in J_n$ containing all elements of X modulo finite, and then $a \setminus b \notin J_n$ is a condition which is incompatible with all elements of the set X!

The following example answers a question of Ilijas Farah—Question 4.3 of [6].

Example 3.12 An analytic P-ideal J such that the factor Q_J collapses $\aleph_1$. Let $\langle \alpha_n : n \in \omega \rangle$ be a decreasing sequence of positive real numbers smaller than 1. Let J_n be the summable P-ideal associated with the weight function $k \mapsto k^{-\alpha_n}$. We claim that $\langle J_n : n \in \omega \rangle$ is an inclusion-decreasing sequence of ideals which does not stabilize. The inclusions are clear. To see that stabilization is impossible, choose a number n and a set $a \notin J_n$. We will produce a set $b \subset a$, $b \in J_n \setminus J_{n+1}$. By induction on $m \in \omega$ find mutually disjoint finite sets $b_m \subset a$ such that $\Sigma_{k \in b_m} k^{-\alpha_n} \leq 2^{-m}$ while $\Sigma_{k \in b_m} k^{-\alpha_{n+1}} \geq 1$. Then $b = \bigcup_m b_m$ will be as desired. To find the set b_m, first find a number $k_m \in \omega$ such that for every $k > k_m$ it is the case that $k^{-\alpha_n} \leq 2^{-m-1} k^{-\alpha_{n+1}} \leq 2^{-m-1}$ and then find a finite set b_m consisting of numbers larger than k_m such that the sum $\Sigma_{k \in b_m} k^{-\alpha_n}$ is between 2^{-m-1} and 2^{-m}.

Let $J = \bigcap_n J_n$. This is an $F_{\sigma\delta}$ ideal, and a simple diagonalization argument shows that it is a tall P-ideal. The proposition shows that the factor Q_J collapses $\aleph_1$ in the presence of CH. If CH fails, the argument only shows that Q_J is not proper, and we do not know if it has to collapse $\mathfrak{c}$ to $\aleph_0$.

Note that this ideal is of minimal possible complexity for the factor Q_J to be improper. All quotients of F_σ ideals are σ-closed by a theorem of Just and Krawczyk [10].

Example 3.13 Let $\langle K_n : n \in \omega \rangle$ be a decreasing sequence of ideals on ω which does not stabilize. Consider the forcings L_n of all trees $T \subset \omega^{<\omega}$ such that every node $s \in T$ longer than some fixed $t \in T$ splits into K_n-positively many immediate successors. It is not difficult to show that the posets L_n are proper and have the continuous reading of names–the arguments closely follow those for Laver forcing. Let $I_n : n \in \omega$ be the σ-ideals on ω^ω associated with these forcings; a Borel set $B \subset \omega^\omega$ is I_n-positive if and only if $[T] \subset B$ for some tree $T \in L_n$. It is not difficult to see that the ideals form an inclusion-decreasing sequence which does not stabilize. Let $J_n = tr(I_n)$, let $I = \bigcap_n I_n$, and let $J = \bigcap_n J_n = tr(I)$. Now P_I is not proper by the proposition. Since each of the forcings P_{I_n} has the continuous reading of names, a review of the proof of Claim 3.2 shows that P_I naturally regularly embeds into Q_J. Ergo, the forcing Q_J cannot be proper either.

4 The proof of Theorem 1.6

As in the previous section, fix a σ-ideal I on 2^ω (the ω^ω case is identical) such that the poset P_I is proper and has the continuous reading of names, and let $J = tr(I)$.

Suppose that some condition $B \in P_I$ forces some ideal K on ω to be destroyed, by an infinite set $\dot{x} \subset \omega$ with finite intersection with every ground model element of the ideal K. Let $\dot{f}(n)$ be defined to be the nth element of the set $\dot{x}$. By the continuous reading of names, there is a set $a \subset 2^{<\omega}$ such that $a = \bigcup_n a_n$, $a_n \subset 2^{<\omega}$ is an antichain, a_{n+1} refines a_n, $\pi(a) \subset B$ is an I-positive G_δ set, and for every number n and every sequence $t \in a_n$ the condition $\{r \in \pi(a) : t \subset r\}$ decides the value of $\dot{f}(n)$ to be some definite number $g(t) \in \omega$. We claim that the function $g : a \to \omega$ is a Katětov reduction of the ideal K to $J \restriction a$. And indeed, if $c \in K$ were a set such that the preimage $b = g^{-1}c$ is J-positive, then clearly $\pi(b) \subset \pi(a) \subset B$ is a condition forcing the set c to have an infinite intersection with the set $\dot{x}$, contrary to the choice of B and $\dot{x}$.

On the other hand, if an ideal K has Katětov reduction $g : a \to \omega$ to the ideal $J \restriction a$ for some J-positive set $a \subset 2^{<\omega}$, then the condition $\pi(a) \in P_I$ forces that the generic path $\dot{r}_{\text{gen}}$ destroys the ideal $J \restriction a$ and the set $g''(\dot{r}_{\text{gen}} \cap a) \subset \omega$ destroys the ideal K. The first statement is immediate from the definition of the function π, and for the second statement note that if some set $c \in K$ had infinite intersection with the set $g''(\dot{r}_{\text{gen}} \cap a)$, then its preimage $b = g^{-1}c \subset a$ would have to have infinite intersection with $\dot{r}_{\text{gen}}$, contradicting the fact that $b \in J$. Theorem 1.6 follows.

Given a particular forcing P_I, Theorem 1.6 gives a satisfactory characterization of the collection of the ideals which it destroys. The opposite question also makes sense: given an ideal on ω, is it easy to recognize those forcings which destroy it? The following observation plays an important role in answering this question. Recall that for a tall ideal J on ω, the cardinal $\mathrm{cov}^*(J)$ is defined as $\min\{|A| : A \subset J \wedge \forall a \in [\omega]^\omega\ \exists b \in A\ |a \cap b| = \aleph_0\}$ [8].

Proposition 4.1 *Suppose that I is a σ-ideal on ω^ω generated by analytic sets such that P_I is a proper forcing with the continuous reading of names. Let $J = tr(I)$. Then*

$$\mathrm{cov}(I) \leq \mathrm{cov}^*(J) \leq \max\{\mathrm{cov}(I), \mathfrak{d}\}.$$

Proof The first inequality is easy. If $A \subset J$ is a family such that $\pi''A$ does not cover the whole space, any path through $\omega^{<\omega}$ converging to a point in $\omega^\omega \setminus \bigcup \pi''A$ is an infinite subset of $\omega^{<\omega}$ which has finite intersection with all elements of the family A.

The second inequality requires more care. First fix several auxiliary objects. Let $\kappa = \max\{\mathrm{cov}(I), \mathfrak{d}\}$. Let F be the collection of all functions $f : \omega^{<\omega} \to \mathcal{P}(\omega^{<\omega})$ such that for every sequence $t \in \omega^{<\omega}$, the value $f(t)$ is a finite set of extensions of the sequence t including t itself. Since $\kappa \geq \mathfrak{d}$, there is a collection $\{f_\alpha : \alpha \in \kappa\} \subset F$ such that for every function $f \in F$ there is an ordinal $\alpha \in \kappa$ such that $f(t) \subset f_\alpha(t)$ holds for every sequence $t \in \omega^{<\omega}$. Fix also a collection $\{C_\alpha : \alpha \in \kappa\}$ of analytic sets in the ideal I covering the whole space. Since every analytic set is a union of $\leq \mathfrak{d}$ many compact sets, we may assume that all sets C_α are in fact compact, $C_\alpha = [T_\alpha]$ for some finitely branching tree $T_\alpha \subset \omega^{<\omega}$.

To construct the family witnessing $\mathrm{cov}^*(J) \leq \kappa$, for ordinals $\alpha, \beta \in \kappa$ define a set $a_{\alpha,\beta} \subset \omega^{<\omega}$ in the following way: by induction on $n \in \omega$ find numbers m_n such that $m_0 = 0$ and m_{n+1} is longer than all sequences in the set $a_{\alpha,\beta}(n) = \bigcup\{f_\alpha(t) : t \in T_\beta \cap \omega^{m_n}\}$. In the end, let $a_{\alpha,\beta} = \bigcup_n a_{\alpha,\beta}(n)$. Also, for each sequence $t \in \omega^{<\omega}$ and every ordinal $\alpha \in \kappa$, let $b_{\alpha,t} = \bigcup_{n\in\omega} f_\alpha(t^\frown n)$. We claim that the collection $A = \{a_{\alpha,\beta} : \alpha, \beta \in \kappa\} \cup \{b_{\alpha,t} : \alpha \in \kappa, t \in \omega^{<\omega}\}$ is a subset of the trace ideal J and every infinite subset of $\omega^{<\omega}$ has an infinite intersection with one of its elements.

It is not difficult to see that $a_{\alpha,\beta} \in J$, since every path meeting infinitely many elements of the set $a_{\alpha,\beta}$ must meet infinitely many elements of the tree T_β, and therefore $\pi(a_{\alpha,\beta}) = [T_\beta] \in I$. Also, trivially, $\pi(b_{\alpha,t}) = 0 \in I$, and so $A \subset J$. Now suppose $x \subset \omega^{<\omega}$ is an infinite set. Let $S \subset \omega^{<\omega}$ be the tree of all sequences $s \in \omega^{<\omega}$ with infinitely many extensions in the set x. There are two cases. Either the tree S has some terminal node t. This means that there are infinitely many elements of the set x below the node t but only finitely many below all of its immediate extensions. It is immediately clear that then some set $b_{\alpha,t} \in A$ covers the infinite set of all extensions of the sequence t which are in x. If the tree S has no terminal nodes then it has to have a cofinal branch, which then is an element of some set $[T_\beta]$. Define a function $f \in F$ by setting $f(t) = \{t$ and some extension of the sequence t which is in the set x if there is one$\}$. If $\alpha \in \kappa$ is then an ordinal such that $\forall t \in \omega^{<\omega}\ f(t) \subset f_\alpha(t)$, it is clear that the set $a_{\alpha,\beta} \in A$ has an infinite intersection with the set x as desired.

Example 4.2 A forcing destroys the trace of the K_σ-ideal I on ω^ω if and only if it adds an unbounded real. For every ideal K on ω, if there is some forcing which adds an unbounded real and does not kill K then Miller forcing is such.

Proof Note that $\mathtt{cov}(I) = \mathfrak{d}$. The argument in the proposition then exactly proves the first equivalence. For the other sentence, note that if some forcing P adds an unbounded real and preserves the ideal K, then $K \not\leq_K tr(I)$ because P destroys $tr(I)$. But Theorem 1.6 then says that P_I, the Miller forcing, preserves K as well.

Example 4.3 The dominating number $\mathfrak{d}$ cannot be omitted in the statement of the proposition. There is a forcing which does not add random reals even in iteration but destroys $tr(\mathtt{null})$. However, there is no such forcing which is bounding.

Proof Every ideal K on ω can be destroyed by a σ-centered forcing, namely the forcing $P(K)$. This is in particular true when $K = tr(\mathtt{null})$. Centered forcings do not add random reals even when they are iterated with finite support.

Example 4.4 In some particular cases the upper bound in the proposition is not optimal. If I is the Laver ideal then $\mathtt{cov}(I) = \mathtt{cov}(tr(I)) = \mathfrak{b}$ and a forcing adds a dominating real if and only if it destroys $tr(I)$.

5 The complexity of the trace ideals

The complexity of the trace ideals is closely tied to the complexity of the σ-ideals generating them. Recall that a σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ [11, 29.E, 35.9] if for every analytic set $A \subset 2^\omega \times X$ the collection $\{y \in 2^\omega :$the vertical section A_y is in $I\}$ is co-analytic.

Proposition 5.1 *Suppose that I is a σ-ideal on ω^ω such that the factor forcing P_I is proper and has the continuous reading of names, and every analytic I-positive set has a Borel I-positive subset. The following are equivalent*:

- *I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$*
- *the trace ideal $tr(I)$ is co-analytic.*

Proof The top to bottom direction follows immediately from the definitions. If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then consider the set $A \subset \mathcal{P}(\omega^{<\omega}) \times \omega^\omega$ given by $\langle a, x\rangle \in A$ if $x \in \pi(a)$. This is clearly a Borel set, and $a \in tr(I) \leftrightarrow A_a \in I$ by the definitions. The latter is a coanalytic condition.

For the bottom to top direction suppose that $tr(I)$ is co-analytic. Let $A \subset 2^\omega \times \omega^\omega$ be an analytic set, with a tree $T \subset (2 \times \omega \times \omega)^{<\omega}$ such that $A = p[T]$. The proof will be complete if we show that for every $y \in 2^\omega$, $A_y \notin I$ if and only if there is a set $b \subset \omega^{<\omega}$ decomposed into antichains $b = \bigcup_n a_n$ and a function $g : b \to T$ for $n \in \omega$ such that

- the antichain a_{n+1} refines a_n
- g preserves extension and whenever $u \in a_n$ then $g(u)$ is a sequence of length n whose first coordinates form an initial segment of u and y
- $b \notin tr(I)$.

Note that this is an analytic condition. To prove this equivalence, if there are such objects b and g, it is clear that the I-positive set $\pi(b)$ is a subset of A_y and therefore $A_y \notin I$. On the other hand, if $A_y \notin I$ then A_y has a Borel I-positive subset C by the assumptions, and by the properness and the continuous reading of names of the poset P_I there is even a Borel I-positive G_δ-set $D \subset C$ and a continuous function $f : D \to [T]$ such that for every real $r \in D$, the second coordinate of the value $f(r)$ is just r itself. It is then easy to construct b and g as above in such a way that $\pi(b) = D$ and for every real $r \in D$, $f(r) = \bigcup_{u \subset r} g(u)$.

This lemma gives us a rather good criterion for checking whether a given trace ideal is co-analytic or not. If the poset P_I adds a dominating real then the ideal I is not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, [20, C.0.16]. A quick review of forcings used in practice shows that many of them which do not add dominating reals (such as the Cohen or Solovay real) are associated with $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideals. However, Arnold Miller [14] constructed a definable c.c.c. ideal I such that the poset P_I does not add a dominating real while the ideal I is still not $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Therefore a careful check for this property is frequently necessary.

In many cases ocurring in practice, the trace ideal is in fact co-analytic and not Borel. Typically, if I is the ideal of countable sets then the trace ideal is a complete co-analytic set since the collection of uncountable closed sets is. In other cases, the trace ideal is Borel, such as when I is the meager ideal or the Lebesgue null ideal. We have no good criterion as to when that happens. We have just a conjecture:

Conjecture 5.2 Suppose that I is a σ-ideal on 2^ω or ω^ω such that the factor poset P_I is proper with continuous reading of names. If the trace ideal is analytic then it is in fact Borel.

This conjecture can be viewed as a variation on the Kechris–Louveau–Woodin theorem [12], which says that analytic σ-ideals of closed sets are G_δ. We can verify it in a good number of cases:

Lemma 5.3 *Suppose that I is a σ-ideal on ω^ω such that P_I is proper and bounding. If the trace ideal is analytic then it is Borel.*

Proof Since the poset P_I is bounding, compact sets are dense in it and it has the continuous reading of names, Example 2.4. If a set $a \subset \omega^{<\omega}$ is not in the trace ideal, apply these two properties below the condition $\pi(a)$ to the name $\dot{f}(n)$ =the nth initial segment of the generic real in the set $\check{a}$. It follows that a set $a \subset \omega^{<\omega}$ is not in the trace ideal if and only if there is a tree T and disjoint finite subsets $\{b_n : n \in \omega\}$ of T such that $[T] \notin I$ and each b_n is a maximal antichain in T consisting only of elements of the set a. What is the complexity of the latter statement? The trace ideal restricted to trees is analytic, therefore Borel by the Kechris–Louveau–Woodin theorem [12], and so this is an analytic statement. Thus the trace ideal is both analytic and co-analytic, therefore Borel.

The trace ideals can be Borel in a number of other situations. If I is a c.c.c. ergodic Π^1_1 on Σ^1_1 σ-ideal on the Baire space then the trace ideal $tr(I)$ is Borel. Recall that the ideal I is ergodic [20, 5.4.1], if there is a countable Borel equivalence relation E on ω^ω such that every Borel E-invariant set is in I or its complement is in I. Then a set $a \subset \omega^{<\omega}$ is in the trace ideal iff $\pi(a) \in I$ iff the complement of the E-saturation of the set $\pi(a)$ is not in I. These are a coanalytic and an analytic statement, respectively, showing that $tr(I)$ is a Borel ideal.

Specific examples of c.c.c. ergodic Π^1_1 on Σ^1_1 σ-ideals with the continuous reading of names include the Cohen and random forcing as well as their finite iterations. Other examples are hard to come by, and the following example identifies a large class.

Example 5.4 Suppose that J is an analytic P-ideal, and I is the σ-ideal associated with the Prikry poset $P(J)$. Then the following are equivalent:

1. J is F_σ
2. I is Π^1_1 on Σ^1_1
3. $P(J)$ does not add a dominating real
4. the trace ideal is Borel.

Proof For the implication (1)→(2) write $J^{<\omega} = \{a \subset [\omega]^{<\aleph_0} \setminus \{0\} : \exists b \in J \forall x \in a\ x \cap b \neq 0\}$. It is clear that this is an ideal on the set $[\omega]^{<\aleph_0} \setminus \{0\}$. A useful observation:

Claim 5.5 If J is an F_σ ideal then $J^{<\omega}$ is F_σ again.

Proof By a theorem of Mazur [13] there is a lower semicontinuous submeasure μ on $\mathcal{P}(\omega)$ such that $J = \{a \subset \omega : \mu(a) < \infty\}$. (A submeasure on $\mathcal{P}(\omega)$ is lower semicontinuous if its value on a given set is just the supremum of its values on the finite subsets of the set.) Let $\mu^{<\omega}$ be a function on $\mathcal{P}([\omega]^{<\aleph_0})$ defined by $\mu^{<\omega}(b) = \inf\{\mu(a) : \forall x \in b\ x \cap a \neq 0\}$. It is not difficult to verify that this is a lower semicontinuous submeasure such that $J^{<\omega} = \{b \subset [\omega]^{<\aleph_0} : \mu^{<\omega}(b) < \infty\}$. The claim follows.

By [20, C.0.14], to prove the (1)→(2) implication of the Example it is just necessary to show that the collection of countable subsets of $P(J)$ which are maximal antichains is a Borel set. In order to do this, let $A \subset P(J)$ be a countable set. Then A is a maximal antichain if and only if it is an antichain and for every finite set $t \subset \omega$, every condition of the form $\langle t, a \rangle$ is compatible with some element of A. The latter condition is equivalent to: either there is some condition $\langle u, b \rangle \in A$ such that $u \subset t$ and $b \cap t \setminus u = 0$, or the set $a_t = \{x \subset \omega : \exists b\ \langle t \cup x, b \rangle \in A\}$ is not in the ideal $J^{<\omega}$. By the Claim, this is a Borel statement.

(2) implies (3) by [20, C.0.16]. (3) implies (1) by a result of Solecki: if an analytic P-ideal J is not F_σ then the ideal $0\times$Fin is Rudin–Blass reducible to J [15, 3.3].

Let $f : \omega \to \omega \times \omega$ be such a finite-to-one reduction. Since the $P(J)$ generic set $\dot{a}_{gen} \subset \omega$ has a finite intersection with a ground model set $b \subset \omega$ if and only if $b \in J$, it immediately follows that $f''\dot{a}_{gen}$ has a finite intersection with a ground model set $b \subset \omega \times \omega$ if and only if $b \in 0\times\text{Fin}$. Let $g \in \omega^\omega$ be defined by $g(n) = \min\{m \in \omega : \langle n, m\rangle \in f''\dot{a}_{gen}\}$. A brief inspection reveals that this is a well-defined function modulo finite dominating all ground model functions.

This leaves us with the equivalence of (2) and (4). Note that the forcing P_I has the continuous reading of names by Example 2.9 and so (4) implies (2) by Proposition 5.1. For the opposite direction note that (2) implies the trace ideal is co-analytic by that same Proposition, so it is enough to show from (2) that the trace ideal is analytic. To this end, use the ergodicity of the ideal I again. For a set $a \subset 2^\omega$, $a \in tr(I)$ if and only if the complement of the closure of the set $\pi(a)$ under finite changes is I-positive, which is an analytic condition by (2).

6 Open questions

Question 6.1 Let I be the meager ideal on 2^ω, let $J = tr(I)$. What is the remainder forcing Q_J/P_I? Similarly for the Lebesgue measure zero ideal.

Question 6.2 Is there a simple preservation criterion on the forcing P_I which is equivalent to the remainder forcing being equal to $\mathcal{P}(\omega)/\text{fin}$?

Question 6.3 Is every proper forcing of the form P_I regularly embeddable into a proper forcing of the form Q_J?

Question 6.4 Assume CH. Is it true that for every ideal J on a countable set, the factor forcing Q_J collapses $\aleph_1$ if and only if it is $\aleph_0$-generated?

Question 6.5 The various definable improper forcings produced in the paper should collapse $\mathfrak{c}$ to $\aleph_0$ in ZFC. Is this really true?

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