

## Intersection numbers of families of ideals

M. Hrušák · C. A. Martínez-Ranero ·  
U. A. Ramos-García · O. A. Téllez-Nieto

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**Abstract** We study the intersection number of families of tall ideals. We show that the intersection number of the class of analytic  $P$ -ideals is equal to the bounding number  $\mathfrak{b}$ , the intersection number of the class of all meager ideals is equal to  $\mathfrak{h}$  and the intersection number of the class of all  $F_\sigma$  ideals is between  $\mathfrak{h}$  and  $\mathfrak{b}$ , consistently different from both.

**Keywords** Tall ideals · Distributivity number · Cardinal invariants of the continuum

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M. Hrušák · U. A. Ramos-García · O. A. Téllez-Nieto (✉)  
Centro de Ciencias Matemáticas, UNAM,  
Apartado Postal 61-3, Xangari,  
58089 Morelia, Michoacán, México  
e-mail: otellez@matmor.unam.mx

U. A. Ramos-García  
e-mail: ariet@matmor.unam.mx

M. Hrušák  
e-mail: michael@matmor.unam.mx

C. A. Martínez-Ranero  
Department of Mathematics, University of Toronto,  
Room 6290, 40 St. George Street, Toronto, ON, Canada  
e-mail: azarel@math.toronto.edu

## 1 Introduction

In Plewik [17] it is proved that the intersection of less than  $\mathfrak{h}$  non-meager ideals is a non-meager ideal and he showed that there exists a family of size  $\mathfrak{d}$  of non-meager ideals which has empty intersection. In Plewik [16] the same author proved that the intersection of less than  $\mathfrak{c}$  ultrafilters is a non-meager filter. In Talagrand [19] it is proved that the intersection of countably many non-measurable filters is a non-measurable filter and in Bartoszyński and Shelah [3] proved that it is consistent with **ZFC** that the intersection of a family of less than  $\mathfrak{c}$  ultrafilters has measure zero.

In this paper we investigate how many tall ideals from a given class  $\Gamma$  of ideals on  $\omega$  are needed so that their intersection is not tall.

The first result of this sort is essentially due to Balcar et al. [1], who showed that there is a base tree of height  $\mathfrak{h}$  in  $\mathcal{P}(\omega)/\text{fin}$  and, in effect, showed that  $\mathfrak{h}$  is the minimal size of a family of tall ideals on  $\omega$  whose intersection is **fin** (equivalently, not tall).

**Definition 1** Let  $\Gamma$  be a class of tall ideals on  $\omega$  such that  $\bigcap \Gamma = \text{fin}$  (that is, for all  $A \in [\omega]^\omega$  there is  $\mathcal{I} \in \Gamma$  such that  $A \notin \mathcal{I}$ ). The *intersection number* of  $\Gamma$  is defined as  $\mathfrak{h}_\Gamma = \min\{|\Omega| : \Omega \subseteq \Gamma \text{ } (\bigcap \Omega \text{ is not tall})\}$ .

We consider the intersection number for several naturally occurring classes of ideals. In particular, we show that the intersection number of the class of analytic  $P$ -ideals is equal to the bounding number  $\mathfrak{b}$ , the intersection number of the class of all meager ideals is equal to  $\mathfrak{h}$  and the intersection number of the class of all  $F_\sigma$  ideals is between  $\mathfrak{h}$  and  $\mathfrak{b}$  and is consistently different from both of them.

We assume knowledge of the method of forcing as well as the basic theory of cardinal invariants of the continuum as covered in [2]. Our notation is standard and follows [2] and [11]. In particular, for a cardinal  $\kappa$  and a set  $A$ ,  $[A]^\kappa$  denotes  $\{X \subseteq A : |X| = \kappa\}$ . For any given function  $\varphi$  we denote by  $\varphi''A$  and  $\varphi^{-1}[A]$  the sets  $\{\varphi(x) : x \in A\}$  and  $\{x : \varphi(x) \in A\}$ , respectively. For any two sets  $A$  and  $B$ , we say that  $A$  is *almost contained* in  $B$ , in symbols  $A \subseteq^* B$ , if  $A \setminus B$  is finite. For functions  $f, g \in \omega^\omega$  we write  $f \leq^* g$  to mean that there is  $m \in \omega$  such that  $f(n) \leq g(n)$  for all  $n \geq m$ . An *interval partition* is a partition of  $\omega$  into finite intervals  $\mathcal{I} = \{I_n = [i_n, i_{n+1}) : n \in \omega\}$ . We say that the interval partition  $\mathcal{I} = \{I_n : n \in \omega\}$  *dominates* another interval partition  $\mathcal{J} = \{J_n : n \in \omega\}$  if there exists  $m \in \omega$  such that for all  $n > m$  there is  $k \in \omega$  such that  $J_k \subseteq I_n$ . Recall that the *bounding number*  $\mathfrak{b}$  is the least cardinal of a  $\leq^*$ -unbounded family of functions in  $\omega^\omega$ . Equivalently, it is the least cardinality of a family  $\mathcal{F}$  of partitions of  $\omega$  into intervals, such that there is no partition that dominates every element of  $\mathcal{F}$  (see [4]). A family  $\mathcal{S} \subseteq \mathcal{P}(\omega)$  is a *splitting family* if for every infinite  $A \subseteq \omega$  there is an  $S \in \mathcal{S}$  such that both  $S \cap A$  and  $A \setminus S$  are infinite. The *splitting number*  $\mathfrak{s}$  is the minimal size of a splitting family in  $\mathcal{P}(\omega)$ . We say that a family  $\mathcal{D}$  of infinite subsets of  $\omega$  is *dense* in  $[\omega]^\omega$  if for all  $A \in [\omega]^\omega$  there is  $D \in \mathcal{D}$  almost contained in  $A$ .  $\mathcal{D}$  is *open* if it is downward closed under  $\subseteq^*$ . The *distributivity number*  $\mathfrak{h}$  of  $\mathcal{P}(\omega)/\text{fin}$  is the smallest size of a family of dense open sets with empty intersection.

An *ideal* on  $X$  is a family of subsets of  $X$  closed under finite unions and subsets. We assume throughout the paper that all ideals contain all singletons  $\{x\}$  for all  $x \in X$ . An ideal  $\mathcal{I}$  on  $\omega$  is *tall* if for all  $X \in [\omega]^\omega$  there is an  $I \in \mathcal{I}$  such that  $I \cap X$  is infinite. All the ideals that we consider are tall. A *filter*  $\mathcal{F}$  on  $\omega$  is a family of subsets of  $\omega$  such

that  $\{X \subseteq \omega : \omega \setminus X \in \mathcal{F}\}$  is an ideal on  $\omega$  and an *ultrafilter* is a maximal filter with respect to inclusion, equivalently, for all  $X \subseteq \omega$ , either  $X \in \mathcal{F}$  or  $\omega \setminus X \in \mathcal{F}$ .

Ideals and filters on  $\omega$ , as subsets of  $\mathcal{P}(\omega)$  can be seen as subsets of the Cantor’s set  $2^\omega$  (equipped with the product topology), by identifying each subset of  $\omega$  with its characteristic function. When we speak about analytic complexity or some topological property of a filter or an ideal we refer to this topology. In particular, recall that a set is *meager* if it is the countable union of nowhere dense sets. Thus, an ideal  $\mathcal{I}$  is meager if it is meager seen as a subset of the Cantor’s set.

The *uniformity of the null ideal*  $\text{non}(\mathcal{N})$  is the least cardinality of a subset of the real line which is not of Lebesgue measure zero. The *additivity of the meager ideal*  $\text{add}(\mathcal{M})$  is the least  $\kappa$  such that the meager ideal is not  $\kappa$ -additive. The *covering number of the meager ideal*  $\text{cov}(\mathcal{M})$  is the smallest size of a family of meager sets which cover the real line.

### 2 ZFC (in)equalities

Let  $\Gamma$  be a class of tall ideals. We say that  $\Gamma$  is *closed under restrictions and translations* if given  $\mathcal{I} \in \Gamma$ ,  $X \notin \mathcal{I}$  and  $f$  a bijection between  $X$  and  $\omega$ , the set  $\mathcal{I} \upharpoonright_f X = \{f[I \cap X] : I \in \mathcal{I}\}$  is an ideal in the class  $\Gamma$ . It is easy to see that all classes that we consider are closed under restrictions and translations.

Let  $\Gamma$  be a class of tall ideals closed under restrictions and translations. Suppose that  $\Omega \subseteq \Gamma$  satisfies  $\bigcap \Omega$  is not tall and  $X \in [\omega]^\omega$  is a witness of that, then  $\Omega' = \{\mathcal{I} \upharpoonright_f X : \mathcal{I} \in \Omega\}$  is a subclass of  $\Gamma$  and  $\bigcap \Omega' = \text{fin}$ . Therefore, the intersection number of  $\Gamma$  can be defined as  $\min\{|\Omega| : \Omega \subseteq \Gamma \wedge \bigcap \Omega = \text{fin}\}$ .

We will use the following simple fact several times in the paper.

**Lemma 1** *Let  $\Gamma, \Delta$  be classes of tall ideals on  $\omega$ . If for each  $\mathcal{I} \in \Gamma$  there is  $\mathcal{J} \in \Delta$  such that  $\mathcal{J} \subseteq \mathcal{I}$ , then  $\mathfrak{h}_\Delta \leq \mathfrak{h}_\Gamma$ . In particular, if  $\Gamma \subseteq \Delta$ , then  $\mathfrak{h}_\Delta \leq \mathfrak{h}_\Gamma$ .*

*Proof* Let  $\mathcal{H} = \{\mathcal{I}_\alpha : \alpha < \mathfrak{h}_\Gamma\} \subseteq \Gamma$  be a family such that  $\bigcap \mathcal{H} = \text{fin}$ . For each  $\alpha$ , let  $\mathcal{J}_\alpha \subseteq \mathcal{I}_\alpha$  such that  $\mathcal{J}_\alpha \in \Delta$ . Then  $\mathcal{H}' = \{\mathcal{J}_\alpha : \alpha < \mathfrak{h}_\Gamma\} \subseteq \Delta$  and  $\bigcap \mathcal{H}' = \text{fin}$ , therefore  $\mathfrak{h}_\Delta \leq \mathfrak{h}_\Gamma$ . □

A family  $\mathcal{A}$  of infinite subsets of  $\omega$  is an *almost disjoint* family if for any  $A, B \in \mathcal{A}$ ,  $A \cap B$  is a finite set. A *maximal almost disjoint* (MAD) family is an infinite almost disjoint family of subsets of  $\omega$ , maximal with respect to inclusion.

As it has already been mentioned,  $\mathfrak{h}$  is the smallest possible value of the intersection number. The following theorem shows that the families of MAD and meager ideals realize the same intersection number  $\mathfrak{h}$ .

**Proposition 1**  $\mathfrak{h}_{\text{MAD}} = \mathfrak{h}_{\text{meager}} = \mathfrak{h}$ .

*Proof* It is sufficient to prove that  $\mathfrak{h}_{\text{MAD}} \leq \mathfrak{h}$  and  $\mathfrak{h}_{\text{meager}} \leq \mathfrak{h}_{\text{MAD}}$ . For the first inequality, let  $\mathcal{I}$  be a tall ideal. Then  $\mathcal{I}$  is a dense open family in  $[\omega]^\omega$ . It follows from proposition 6.18 of [4], that there is a MAD family  $\mathcal{A}$  such that the ideal generated by  $\mathcal{A}$ ,  $\mathcal{I}(\mathcal{A}) = \{X \subseteq \omega : \exists \mathcal{B} \in [\mathcal{A}]^{<\omega} (X \subseteq^* \bigcup \mathcal{B})\}$  is a subset of  $\mathcal{I}$ . From Lemma 1 we have the inequality. For the second inequality, recall that in Mathias [13] it is

proved that the ideals based on MAD families are meager (that is,  $\text{MAD} \subseteq \text{meager}$ ). The result follows from Lemma 1 again.  $\square$

Let  $\text{max}$  denote the class of *maximal ideals*. Recall that an ideal  $\mathcal{I}$  is *maximal* if its dual filter is an ultrafilter. The following proposition shows that the intersection number of the class of maximal ideals is the greatest possible.

**Proposition 2**  $\mathfrak{h}_{\text{max}} = \mathfrak{c}$ .

*Proof* It suffices to show that the intersection of less than  $\mathfrak{c}$  maximal ideals is a tall ideal. Let  $\kappa < \mathfrak{c}$  be given and let  $\{\mathcal{I}_\alpha : \alpha < \kappa\}$  be a family of maximal ideals. Given an  $A \in [\omega]^\omega$ , let  $\{A_\xi : \xi < \mathfrak{c}\}$  be an almost disjoint family of infinite subsets of  $A$ . First observe that for a fixed  $\alpha < \kappa$ ,  $|\{A_\xi : \xi < \mathfrak{c}\} \setminus \mathcal{I}_\alpha| \leq 1$ . To see this, pick  $\xi < \mathfrak{c}$  such that  $A_\xi \notin \mathcal{I}_\alpha$ . If  $\chi \neq \xi$  then  $A_\xi \cap A_\chi =^* \emptyset$ ,  $A_\chi \subseteq^* \omega \setminus A_\xi$ . We have  $\omega \setminus A_\xi \in \mathcal{I}_\alpha$  since the ideal is maximal. Therefore,  $A_\chi \in \mathcal{I}_\alpha$ . It easily follows from the observation that there is a  $\xi_0 < \mathfrak{c}$  such that  $A_{\xi_0} \in \mathcal{I}_\alpha \cap [A]^\omega$  for all  $\alpha < \kappa$ .  $\square$

Recall that an ideal  $\mathcal{I}_f$  is *summable* if there is  $f : \omega \rightarrow (0, \infty)$  such that  $\lim_{n \rightarrow \infty} f(n) = 0$ ,  $\sum_{n \in \omega} f(n) = \infty$  and  $\mathcal{I} = \{A \subseteq \omega : \sum_{n \in A} f(n) < \infty\}$ . An ideal  $\mathcal{I}$  is a *P-ideal* if for any sequence  $\langle I_n : n \in \omega \rangle \subseteq \mathcal{I}$  there is  $I \in \mathcal{I}$  such that  $I_n \subseteq^* I$  for all  $n \in \omega$ . An ideal  $\mathcal{I}$  is  *$\omega$ -hitting* if for any sequence  $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$  there is  $I \in \mathcal{I}$  such that  $|A_n \cap I| = \aleph_0$  (equivalently,  $A_n \cap I \neq \emptyset$  for all  $n \in \omega$ ).

A *lower semicontinuous submeasure* on a set  $X$  is a function  $\varphi : \mathcal{P}(X) \rightarrow [0, \infty]$  satisfying  $\varphi(\emptyset) = 0$ ;  $\varphi(A) \leq \varphi(B)$  whenever  $A \subseteq B$ ;  $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$  and  $\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A \cap n)$  for all  $A, B \subseteq X$ . If  $\varphi$  is a lower semicontinuous submeasure on  $\omega$  then the ideals  $\text{Fin}(\varphi) = \{A \subseteq \omega : \varphi(A) < \infty\}$  and  $\text{Exh}(\varphi) = \{A \subseteq \omega : \lim_{n \rightarrow \infty} \varphi(A \setminus n) = 0\}$  are  $F_\sigma$  and  $F_{\sigma\delta}$  P-ideals, respectively.

In Solecki [18] it is shown that if  $\mathcal{I}$  is an analytic P-ideal, then there is a lower semicontinuous submeasure  $\varphi$  on  $\omega$  such that  $\mathcal{I} = \text{Exh}(\varphi)$  and in Mazur [14] it is proved that if an ideal  $\mathcal{I}$  is  $F_\sigma$  then there is a lower semicontinuous submeasure  $\varphi$  such that  $\mathcal{I} = \text{Fin}(\varphi)$ .

An  $F_\sigma$  ideal  $\mathcal{I}$  is *fragmented* (see [10]) if there is partition  $\langle I_n : n \in \omega \rangle$  of  $\omega$  in finite sets and submeasures  $\varphi_n : \mathcal{P}(I_n) \rightarrow [0, \infty)$  on  $I_n$  such that  $\mathcal{I} = \{A \subseteq \omega : \exists k \forall n (\varphi_n(A \cap I_n) \leq k)\}$

Since  $F_\sigma \subseteq \dots \subseteq \text{Borel} \subseteq \text{analytic} \text{ and } \text{summable} \subseteq \text{analytic P-ideal} \subseteq \text{Borel } \omega\text{-hitting} \subseteq \omega\text{-hitting}$ , we have from Lemma 1 the following inequalities.

**Proposition 3** 1.  $\mathfrak{h} \leq \mathfrak{h}_{\text{analytic}} \leq \mathfrak{h}_{\text{Borel}} \leq \dots \leq \mathfrak{h}_{F_\sigma}$ .

2.  $\mathfrak{h} \leq \mathfrak{h}_{\omega\text{-hitting}} \leq \mathfrak{h}_{\text{Borel } \omega\text{-hitting}} \leq \mathfrak{h}_{\text{analytic P-ideal}} \leq \mathfrak{h}_{\text{summable}}$ .

*Proof* (1) is obvious. For (2), if  $\mathcal{I}_f$  is a summable ideal, then the lower semicontinuous submeasure  $\varphi$  on  $\omega$  defined by  $\varphi(A) = \sum_{n \in A} f(n)$  shows that  $\mathcal{I}_f$  is an analytic P-ideal as  $\mathcal{I} = \text{Exh}(\varphi)$ .

Let  $\mathcal{I}$  be an analytic P-ideal. Let us see that  $\mathcal{I}$  is Borel  $\omega$ -hitting. From Solecki's theorem we know that  $\mathcal{I}$  is an  $F_{\sigma\delta}$  ideal. Let  $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$ . Since  $\mathcal{I}$  is tall, for every  $n \in \omega$  there is  $I_n \in \mathcal{I}$  such that  $I_n \cap A_n$  is infinite. Let  $I \in \mathcal{I}$  be such that  $I_n \subseteq^* I$  for all  $n \in \omega$ . Then  $I \cap A_n$  is infinite for all  $n \in \omega$ .  $\square$

The next class of ideals that we consider is the class of *eventually different* ideals. We consider this class for two reasons: the first one is because its intersection number admits a simple combinatorial characterization and the second one is, it allows us to relate the intersection number of the classes seen so far with the classical cardinal invariants  $\mathfrak{b}$ ,  $\mathfrak{s}$  and  $\text{non}(\mathcal{N})$ .

**Definition 2** Let  $f \in \omega^\omega$  be such that  $|\text{ran}(f)| = \aleph_0$  and  $\limsup_{n \rightarrow \infty} |f^{-1}(n)| = \infty$ . We define the (tall  $F_\sigma$ ) ideal  $\mathcal{ED}_f = \{A \subseteq \omega : \exists m \forall l \geq m |A \cap f^{-1}(l)| \leq m\}$ . The class of *ED-ideals* is defined as the class

$$\text{ED} = \{\mathcal{ED}_f : f \in \omega^\omega \wedge |\text{ran}(f)| = \aleph_0 \wedge \limsup_{n \rightarrow \infty} |f^{-1}(n)| = \infty\}.$$

The class of *ED<sub>fin</sub>-ideals* is defined by

$$\text{ED}_{\text{fin}} = \{\mathcal{ED}_f : f \in \omega^\omega \wedge f \text{ is finite-to-one} \wedge \limsup_{n \rightarrow \infty} |f^{-1}(n)| = \infty\}.$$

Note that

$$\begin{aligned} \mathfrak{h}_{\text{ED}} &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall A \in [\omega]^\omega \exists f \in \mathcal{F} (\forall k \exists^\infty n (|f^{-1}(n) \cap A| > k))\} \\ \mathfrak{h}_{\text{ED}_{\text{fin}}} &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall A \in [\omega]^\omega \exists f \in \mathcal{F} \text{ finite-to-one} \\ &\quad (\forall k \exists^\infty n (|f^{-1}(n) \cap A| > k))\}. \end{aligned}$$

From Lemma 1 and  $\text{ED}_{\text{fin}} \subseteq \text{ED} \subseteq F_\sigma$ , we obtain the following inequalities.

**Proposition 4**  $\mathfrak{h}_{F_\sigma} \leq \mathfrak{h}_{\text{ED}} \leq \mathfrak{h}_{\text{ED}_{\text{fin}}}$ . □

We can estimate the values of  $\mathfrak{h}_{\text{ED}}$  and  $\mathfrak{h}_{\text{ED}_{\text{fin}}}$ .

**Theorem 1**  $\mathfrak{h}_{\text{ED}_{\text{fin}}} = \mathfrak{b}$ .

*Proof* First we prove that  $\mathfrak{h}_{\text{ED}_{\text{fin}}} \leq \mathfrak{b}$ . Let  $\kappa < \mathfrak{h}_{\text{ED}_{\text{fin}}}$  and  $\langle P_\alpha : \alpha < \kappa \rangle$  be a family of partitions of  $\omega$  into intervals where  $P_\alpha = \langle I_n^\alpha : n \in \omega \rangle$ . Define  $f_\alpha : \omega \rightarrow \omega$  by  $f_\alpha(x) = n$  if  $x \in I_n^\alpha$  (that means that  $f_\alpha^{-1}(n) = I_n^\alpha$  for all  $n \in \omega$ ). Since  $f_\alpha$  is finite-to-one for all  $\alpha < \kappa$  and  $\kappa < \mathfrak{h}_{\text{ED}_{\text{fin}}}$ , there is an  $A \in [\omega]^\omega$  such that for each  $\alpha < \kappa$  there are  $k_\alpha, m_\alpha \in \omega$  such that  $|f_\alpha^{-1}(n) \cap A| \leq k_\alpha$  for all  $n > m_\alpha$ . Let  $e_A : \omega \rightarrow \omega$  be the enumerating function of  $A$  ( $e_A(n)$  is the  $n$ -th element of  $A$ ) and define the following partition of  $\omega$  into intervals:

$$\begin{aligned} J_0 &= [0, e_A(0)); \\ J_{n+1} &= [e_A(s_n), e_A(s_{n+1})), \end{aligned}$$

where  $s_n = \sum_{i=0}^n i$ , for  $n \geq 1$ . Note that  $|J_n \cap A| = n$ .

We claim that  $P = \langle J_n : n \in \omega \rangle$  dominates  $P_\alpha$  for all  $\alpha < \kappa$ . Fix  $\alpha < \kappa$ , let  $k_\alpha, m_\alpha \in \omega$  be such that  $|f_\alpha^{-1}(n) \cap A| \leq k_\alpha$  for all  $n > m_\alpha$ . Let  $N \in \omega$  such that  $N > \max\{3k_\alpha, m_\alpha\}$  and  $e_A(s_{N-1}) \in I_k^\alpha$  implies  $k \geq m_\alpha$ . Let us see that for each  $m \geq N$  there is an  $r \in \omega$  such that  $I_r^\alpha \subseteq J_m$ .

For  $m \geq N$ , let  $r_0 = \min\{n \in \omega : I_n^\alpha \cap J_m \neq \emptyset\}$ . By the second condition on  $N$ ,  $r_0 \geq m_\alpha$ . If  $I_{r_0} \subseteq J_m$  we are done. If not, we claim that  $I_{r_0+1} \subseteq J_m$ . Suppose not, then  $J_m \subseteq I_{r_0}^\alpha \cup I_{r_0+1}^\alpha$  and therefore  $A \cap J_m \subseteq A \cap (I_{r_0}^\alpha \cup I_{r_0+1}^\alpha)$  which implies that  $|A \cap J_m| \leq |A \cap (I_{r_0}^\alpha \cup I_{r_0+1}^\alpha)|$  but  $|A \cap J_m| = m \geq 3k_\alpha$  while  $|A \cap (I_{r_0}^\alpha \cup I_{r_0+1}^\alpha)| \leq 2k_\alpha$ . It is a contradiction.

On the other hand, let  $\kappa < \mathfrak{b}$  and  $\langle f_\alpha : \alpha < \kappa \rangle \subseteq \omega^\omega$  be a family of finite-to-one functions. For each  $\alpha < \kappa$  we define a partition of  $\omega$  into intervals as follows:

$$I_0^\alpha = [0, k_0),$$

where  $k_0 = \min\{m \in \omega : f_\alpha^{-1}(0) \subseteq [0, m)\}$ , and

$$I_{n+1}^\alpha = [k_n, k_{n+1}),$$

where  $k_{n+1} > k_n$  and satisfies  $\forall x \in I_n^\alpha (f_\alpha^{-1}[f_\alpha(x)] \subseteq k_{n+1})$ . Put  $P_\alpha = \langle I_n^\alpha : n \in \omega \rangle$ . Observe that  $f_\alpha^{-1}(m)$  is contained in at most two consecutive intervals of  $P_\alpha$ , for all  $m \in \text{ran}(f_\alpha)$ . Let  $P = \langle J_n : n \in \omega \rangle$  be a partition dominating the family  $\langle P_\alpha : \alpha < \kappa \rangle$  (that is, for each  $\alpha < \kappa$  there is  $N_\alpha \in \omega$  such that for all  $n \geq N_\alpha$  there exists  $r \in \omega$  such that  $I_r^\alpha \subseteq J_n$ ) and let  $A$  be a selector of  $P$ . For  $\alpha < \kappa$ , consider  $N_\alpha$  and  $r$  such that  $I_r^\alpha \subseteq J_{N_\alpha}$ . If  $m_\alpha = \max\{f_\alpha(x) : x \in I_r^\alpha\}$ , then for each  $n \geq m_\alpha$ ,  $|f_\alpha^{-1}(n) \cap A| < 3$ , because  $f_\alpha^{-1}(n)$  is contained in at most two intervals of  $P_\alpha$ . This proves that  $\langle f_\alpha : \alpha < \kappa \rangle$  is not a witness for  $\mathfrak{h}_{\text{ED}_{\text{fin}}}$ , and therefore  $\mathfrak{b} \leq \mathfrak{h}_{\text{ED}_{\text{fin}}}$ .  $\square$

Recall that an ideal  $\mathcal{I}$  is a  $Q$ -ideal if for every partition  $\langle F_n : n < \omega \rangle$  of  $\omega$  into finite sets, there is an  $\mathcal{I}$ -positive set  $X$  such that  $|X \cap F_n| \leq 1$  for all  $n < \omega$ .

We use the following Theorem from [9].

**Theorem 2** *For each Borel ideal  $\mathcal{I}$ , the following are equivalent:*

1.  $\mathcal{I}$  is not a  $Q$ -ideal,
2.  $\mathcal{I}$  is an  $\omega$ -hitting ideal.  $\square$

**Lemma 2** 1.  $\mathfrak{h}_{\text{summable}} \leq \mathfrak{b}$ .

2.  $\mathfrak{h}_{\text{ED}_{\text{fin}}} \leq \mathfrak{h}_{\text{Borel } \omega\text{-hitting}}$ .
3.  $\mathfrak{h}_{\text{Borel } \omega\text{-hitting}} \leq \mathfrak{h}_{\text{fragmented}}$ .
4.  $\mathfrak{h}_{\text{fragmented}} \leq \mathfrak{h}_{\text{ED}_{\text{fin}}}$ .

*Proof* For (1), we use the characterization of  $\mathfrak{b}$  from [7], that is,

$$\mathfrak{b} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq c_0 \wedge \forall X \in [\omega]^\omega \exists s \in \mathcal{S} (s \upharpoonright X \notin \ell_1)\},$$

where  $c_0$  and  $\ell_1$  denote the standard Banach spaces of sequences of reals.

Let  $\kappa < \mathfrak{h}_{\text{summable}}$  and  $\mathcal{S} = \{s_\alpha : \alpha < \kappa\} \subseteq c_0$ , without loss of generality we can suppose that  $\sum_{n \in \omega} s_\alpha(n) = \infty$ . For each  $\alpha < \kappa$ , we define the summable ideal  $\mathcal{I}_\alpha = \{A \subseteq \omega : \sum_{n \in A} s_\alpha(n) < \infty\}$ . Since  $\kappa < \mathfrak{h}_{\text{summable}}$ , there is an  $A \in \bigcap_{\alpha < \kappa} \mathcal{I}_\alpha \cap [\omega]^\omega$ . Then  $s_\alpha \upharpoonright A \in \ell_1$ , and therefore  $\kappa < \mathfrak{b}$ .

Let us see (2). By the Theorem 2, if  $\mathcal{I}$  is a Borel  $\omega$ -hitting ideal, then  $\mathcal{I}$  is not a  $Q$ -ideal, that means that there is a partition  $\langle F_n : n \in \omega \rangle$  of  $\omega$  into finite sets such that

every selector of the partition belongs to  $\mathcal{I}$ . Consider the function  $f : \omega \rightarrow \omega$  given by  $f(x) = n$ , where  $x \in F_n$ , it is easy to see that  $f$  is finite-to-one and  $\mathcal{ED}_f \subseteq \mathcal{I}$  (each element of  $\mathcal{ED}_f$  is a finite union of selectors of  $\langle F_n : n \in \omega \rangle$ ). Thus, Lemma 1 gives the desired conclusion.

In order to show (3), let  $\mathcal{I}$  be a fragmented ideal, with  $\langle I_n : n \in \omega \rangle$  and  $\langle \varphi_n : n \in \omega \rangle$  witnessing it. Let us see that  $\mathcal{I}$  is an  $\omega$ -hitting ideal. Suppose not. Then by Theorem 2,  $\mathcal{I}$  is a Q-ideal. That means that for  $\langle I_n : n \in \omega \rangle$  there is an  $\mathcal{I}$ -positive set  $X$  such that  $|X \cap I_n| \leq 1$  for all  $n \in \omega$ . If  $X \cap I_n \neq \emptyset$ , let  $x_n \in X \cap I_n$ . Since  $X$  is an  $\mathcal{I}$ -positive set, then  $\sup\{\varphi_n(X \cap I_n) : n \in \omega\} = \sup\{\varphi_n(x_n) : X \cap I_n \neq \emptyset\} = \infty$ . Then there is a  $Y \subseteq X$  infinite such that the  $n$ -th element of  $Y$  has submeasure at least  $n$ . Therefore,  $\mathcal{I}$  is not a tall ideal, a contradiction. Again Lemma 1 gives the conclusion.

For (4), note that each ideal  $\text{ED}_{fin}$  is fragmented via  $I_n = f^{-1}(n)$  and  $\varphi_n(a) = |a|$ . □

**Theorem 3**  $\mathfrak{b} = \mathfrak{h}_{\text{ED}_{fin}} = \mathfrak{h}_{\text{Borel } \omega\text{-hitting}} = \mathfrak{h}_{\text{analytic P-ideal}} = \mathfrak{h}_{\text{summable}} = \mathfrak{h}_{\text{fragmented}}$ .

*Proof* Follows directly from 2 of Proposition 3, Theorem 1 and Lemma 2. □

In order to simplify the calculations for  $\mathfrak{h}_{\text{ED}}$  we introduce the following cardinal

$$\nu = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall A \in [\omega]^\omega \exists f \in \mathcal{F} (\forall n \in \omega (|A \cap f^{-1}(n)| = \aleph_0))\}.$$

Obviously  $\mathfrak{h}_{\text{ED}} \leq \nu$ .

It is not known if  $\mathfrak{h}_{\text{ED}}$  is equal to any of the known cardinal invariants, as in the case of  $\mathfrak{h}_{\text{ED}_{fin}}$ . However, we can bound it from both sides.

**Theorem 4**  $\min\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{h}_{\text{ED}} \leq \min\{\mathfrak{b}, \text{non}(\mathcal{N})\}$ .

*Proof* Obviously  $\mathfrak{h}_{\text{ED}} \leq \mathfrak{h}_{\text{ED}_{fin}} = \mathfrak{b}$ . Now we will prove that  $\nu \leq \text{non}(\mathcal{N})$ . Consider the measure  $\mu_0$  on  $\omega$  given by  $\mu_0(n) = \frac{1}{2^{n+1}}$  and let  $\mu$  be the product measure on  $\omega^\omega$ . Let  $N_A = \{f \in \omega^\omega : \exists n \in \omega (|A \cap f^{-1}(n)| < \aleph_0)\}$ .

We show that  $\mu(N_A) = 0$ . Observe that

$$N_A = \bigcup_{n \in \omega, F \in [A]^{<\omega}} N_A(n, F),$$

where  $N_A(n, F) = \{f \in \omega^\omega : f^{-1}(n) \cap A = F\}$ .

Fix  $n \in \omega$  and  $F = \{a_0, \dots, a_r\} \in [A]^{<\omega}$  (in increasing order). Let us see that  $\mu(N_A(n, F)) = 0$ , and therefore  $\mu(N_A) = 0$ .

$$\begin{aligned} \mu(N_A(n, F)) &= \lim_{m \rightarrow \infty} (\mu_0(\omega \setminus \{n\}))^{a_0} (\mu_0(\{n\})) (\mu_0(\omega \setminus \{n\}))^{a_1 - a_0 - 1} (\mu_0(\{n\})) \\ &\quad \dots (\mu_0(\omega \setminus \{n\}))^{a_r - a_{r-1} - 1} (\mu_0(\{n\})) (\mu_0(\omega \setminus \{n\}))^m \\ &= \lim_{m \rightarrow \infty} (\mu_0(\omega \setminus \{n\}))^{a_r - r - 1} (\mu_0(\{n\}))^{r-1} (\mu_0(\omega \setminus \{n\}))^m \\ &= \lim_{m \rightarrow \infty} \left(1 - \frac{1}{2^{n+1}}\right)^{a_r - r - 1} \left(\frac{1}{2^{n+1}}\right)^{r-1} \left(1 - \frac{1}{2^{n+1}}\right)^m = 0 \end{aligned}$$

Take  $\kappa < \nu$  and  $\mathcal{F} \subseteq \omega^\omega$  where  $|\mathcal{F}| = \kappa$ . Then, there is  $A \in \omega^\omega$  such that for all  $f \in \mathcal{F}$  there is  $n \in \omega$  such that  $|f^{-1}(n) \cap A| < \aleph_0$ . Then  $\mathcal{F} \subseteq N_A$ , and therefore  $\mathcal{F}$  is a null set.

Recall that

$$\min\{\mathfrak{b}, \mathfrak{s}\} = \min\{|\mathcal{X}| : \forall \varphi \in \mathcal{X} (\varphi : [\omega]^2 \rightarrow 2) \wedge \forall A \in [\omega]^\omega \exists \varphi \in \mathcal{X} \forall n \in \omega (\varphi'' [A \setminus n]^2 = 2)\}$$

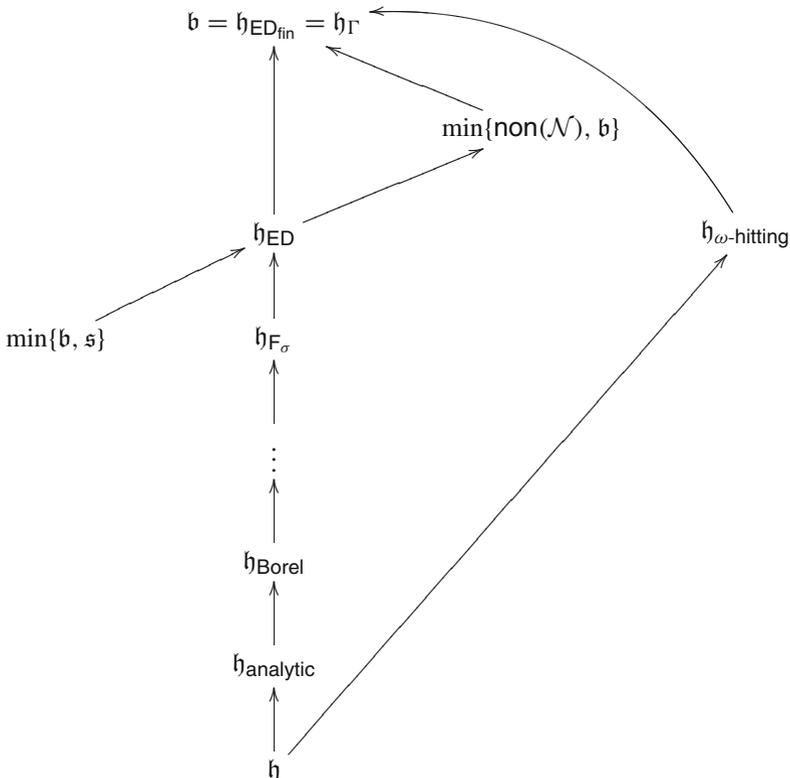
(see [4], Theorem 3.5).

Let  $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$  and  $\mathcal{F} = \{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$ . For each  $\alpha < \kappa$  we define  $\varphi_\alpha : [\omega]^2 \rightarrow 2$  as follows:  $\varphi_\alpha(\{m, n\}) = 0$  if and only if  $f_\alpha(n) = f_\alpha(m)$ . Since  $\kappa < \min\{\mathfrak{b}, \mathfrak{s}\}$ , there is  $A \in [\omega]^\omega$  such that for every  $\alpha < \kappa$  exists  $n_\alpha \in \omega$  such that  $|\varphi_\alpha'' [A \setminus n_\alpha]^2| = 1$ . Now the proof proceed by cases.

If  $\varphi'' [A \setminus n_\alpha]^2 = \{0\}$  and  $m_\alpha = \max\{f_\alpha(k) : k \leq n_\alpha + 1\}$ , then  $f^{-1}(n) \cap A = \emptyset$  for all  $n > m_\alpha$ .

If  $\varphi'' [A \setminus n_\alpha]^2 = \{1\}$  and  $m_\alpha = \max\{f_\alpha(k) : k \leq n_\alpha + 1\}$ , then  $|f^{-1}(n) \cap A| = 1$  for all  $n > m_\alpha$ . Therefore,  $\mathfrak{h}_{ED} \geq \min\{\mathfrak{b}, \mathfrak{s}\}$ . □

The relations that we have seen so far can be summarized in the following diagram ( $\Gamma$  is any of the following classes of tall ideals: Borel  $\omega$ -hitting, analytic P-ideal, summable or fragmented). The arrow goes from a smaller cardinal to the larger.



### 3 Consistency results

In this section we will show that each of the following statements is consistent with **ZFC**:

1.  $\mathfrak{h} < \mathfrak{h}_{\text{analytic}}$ .
2.  $\mathfrak{h}_{F_\sigma} = \mathfrak{h}_{\text{ED}} < \mathfrak{h}_{\omega\text{-hitting}}$ .
3.  $\mathfrak{h}_{\text{ED}} = \mathfrak{h}_{\omega\text{-hitting}} < \mathfrak{b}$
4.  $\mathfrak{h}_{\text{ED}} < \text{add}(\mathcal{M})$ .

#### 3.1 The consistency of $\mathfrak{h} < \mathfrak{h}_{\text{analytic}}$

We use the following forcing notions (see [2]).

*The Laver forcing*  $\mathbb{L}$ :  $T \in \mathbb{L}$  if and only if  $T \subseteq \omega^{<\omega}$  is a tree, there is  $s_T \in T$  (called *stem of T*) such that for all  $t \in T$  either  $t \subseteq s_T$  or  $s_T \subseteq t$  and for all  $t \in T$  if  $t \supseteq s_T$ , then  $\text{succ}_T(t) = \{n \in \omega : t \frown n \in T\}$  is infinite. For  $T, T' \in \mathbb{L}$ , define  $T' \leq T$  if  $T' \subseteq T$ .

*The Mathias forcing*  $\mathbb{M}$  :  $\langle s, A \rangle \in \mathbb{M}$  if and only if  $s \in [\omega]^{<\omega}$ ,  $A \in [\omega]^\omega$  and  $\max(s) < \min(A)$ . If  $\langle s, A \rangle, \langle s', A' \rangle \in \mathbb{M}$  define  $\langle s, A \rangle \leq \langle s', A' \rangle \in \mathbb{M}$  if and only if  $s' \subseteq s$ ,  $A \subseteq A'$  and  $s \setminus s' \subseteq A'$ .

*The Mathias forcing associated to an ultrafilter*  $\mathcal{U}$ ,  $\mathbb{M}_{\mathcal{U}}$  :  $\langle s, A \rangle \in \mathbb{M}$  if and only if  $s \in [\omega]^{<\omega}$ ,  $A \in \mathcal{U}$  and  $\max(s) < \min(A)$ . The same order as  $\mathbb{M}$ .

$\mathbb{LM}$  denotes the two step iteration  $\mathbb{L} * \mathbb{M}$ , and for a forcing notion  $\mathbb{P}$ ,  $\mathbb{P}_{\omega_2}$  denotes the countable support iteration of  $\mathbb{P}$  of length  $\omega_2$ .

We recall the following theorem due to Mathias [13].

**Theorem 5** (Mathias, [13]) *Let  $\mathcal{U}$  be an ultrafilter on  $\omega$ . Then  $\mathcal{U}$  is selective if and only if  $\mathcal{U} \cap \mathcal{I} \neq \emptyset$  for each tall analytic ideal  $\mathcal{I}$ .* □

**Lemma 3** (Folklore)  $\mathbb{M} \simeq \mathcal{P}(\omega)/\text{fin} * \mathbb{M}_{\dot{\mathcal{U}}}$ , where  $\dot{\mathcal{U}}$  is the selective ultrafilter added by  $\mathcal{P}(\omega)/\text{fin}$ .

*Proof* Consider the mapping  $\iota : \mathbb{M} \rightarrow \mathcal{P}(\omega)/\text{fin} * \mathbb{M}_{\mathcal{U}}$  given by  $\iota(\langle a, A \rangle) = \langle A, (a, A) \rangle$ . It is easy to see that  $\iota$  is a dense embedding. □

As  $\mathcal{P}(\omega)/\text{fin}$  adds a selective ultrafilter  $\mathcal{U}$  and  $\mathbb{M}_{\mathcal{U}}$  adds a pseudo-intersection of  $\mathcal{U}$ , if  $G$  is  $\mathbb{M}$ -generic over  $V$ , then  $V[G] = V[\mathcal{U}][A]$  where  $A$  is the pseudo-intersection added by  $\mathbb{M}_{\mathcal{U}}$ . If  $G$  is a  $\mathbb{L}$ -generic over  $V$  and  $f_G$  is the Laver real added by  $G$ , then we write  $V[f_G]$  instead of  $V[G]$ .

**Theorem 6** *It is consistent with **ZFC** that  $\mathfrak{h} = \omega_1$  and  $\mathfrak{h}_{\text{analytic}} = \omega_2$ .*

*Proof* It is shown in [6] that if  $V \models \mathbf{CH}$  and  $G$  a  $\mathbb{LM}_{\omega_2}$ -generic over  $V$ , then  $V[G] \models \mathfrak{h} = \omega_1$ .

Let us show that  $V[G] \models \mathfrak{h}_{\text{analytic}} = \omega_2$ . Let  $\langle \mathcal{I}_\alpha : \alpha < \omega_1 \rangle \in V[G]$  be a family of analytic ideals on  $\omega$ .

*Claim* There exists  $\beta < \omega_2$  such that  $\mathcal{I}_\alpha \in V[G_\beta]$  for all  $\alpha < \omega_1$ .

Fix  $\alpha < \omega_1$ , since the ideal  $\mathcal{I}_\alpha$  is analytic, it is the continuous image of a Polish space. As a continuous function from a Polish space is determined by the values in a countable dense subset there is  $\beta_\alpha < \omega_2$  such that  $\mathcal{I}_\alpha \in V[G_{\beta_\alpha}]$ . Let  $\beta = \sup\{\beta_\alpha : \alpha < \omega_1\}$ . Then  $\beta < \omega_2$  and  $\mathcal{I}_\alpha \in V[G_\beta]$  for all  $\alpha < \omega_1$ .

Having fixed such  $\beta$  note that, by Schoenfield’s absoluteness,  $V[G_\beta]$  as well as any larger model thinks that  $\mathcal{I}_\alpha$  is tall for every  $\alpha < \omega_1$ . Now, from the previous remark,  $V[G_{\beta+1}] \simeq V[G_\beta][f][\mathcal{U}][A]$ , where  $f$  is the Laver real,  $\mathcal{U}$  is the selective ultrafilter added by  $\mathcal{P}(\omega)/\text{fin}$  and  $A$  is the pseudo-intersection of  $\mathcal{U}$ . By the theorem of Mathias, there is  $I_\alpha \in \mathcal{U} \cap \mathcal{I}_\alpha$  for each  $\alpha < \omega_1$ . Since  $A \subseteq^* I_\alpha$ , we have that  $A \in \mathcal{I}_\alpha$  for all  $\alpha < \omega_1$ . Thus,  $A \in \bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha$ .  $\square$

### 3.2 The consistency of $\mathfrak{h}_{F_\sigma} = \mathfrak{h}_{\text{ED}} < \mathfrak{h}_{\omega\text{-hitting}}$

The forcing notion that we use is the Laver forcing. First we show that the range of a Laver real belongs to any  $\omega$ -hitting ideal.

**Lemma 4** *Let  $\mathcal{I} \in V$  be an  $\omega$ -hitting ideal. If  $G$  is  $\mathbb{L}$ -generic over  $V$ ,  $f_G$  is the Laver real added by  $G$  and  $A = \text{ran}(f_G)$ , then  $V[G] \models A \in \mathcal{I}$ .*

*Proof* We show that the set  $\{S \in \mathbb{L} : S \Vdash “A \in \mathcal{I}”\}$  is dense. Let  $T \in \mathbb{L}$  be a Laver condition. Since  $\mathcal{I}$  is  $\omega$ -hitting, there is  $I \in \mathcal{I}$  such that  $|I \cap \text{succ}_T(t)| = \omega$  for all  $t \in T$  with  $s_T \subseteq t$ . Define  $T' \leq T$  by recursion as follows:  $s_{T'} = s_T$ ,  $\text{succ}_{T'}(s_{T'}) = \text{succ}_T(s_T) \cap I$ . Suppose defined  $\text{succ}_{T'}(t)$  for  $|t| = |s_{T'}| + n$ . For  $t \in T'$  with  $|t| = |s_{T'}| + n + 1$ , let  $\text{succ}_{T'}(t) = \text{succ}_T(t) \cap I$ . Hence,  $T'$  is a Laver condition such that for each  $t \in T'$ ,  $\text{succ}_{T'}(t) \subseteq I$ . That means that  $T' \Vdash “f_G(n) \in I”$  for all  $n \geq |s_{T'}|$ , then  $T' \Vdash “\text{ran}(f_G) = A \subseteq^* I”$  and therefore  $T' \Vdash “A \in \mathcal{I}”$ .  $\square$

Now we prove the consistency of the statement.

**Theorem 7** *It is consistent with ZFC that  $\mathfrak{h}_{\text{ED}} = \omega_1$  and  $\mathfrak{h}_{\omega\text{-hitting}} = \omega_2$ .*

*Proof* Let  $V \models \text{CH}$  and  $G$  be a  $\mathbb{L}_{\omega_2}$ -generic over  $V$ . In [15] it is shown that  $V[G] \models \text{non}(\mathcal{N}) = \omega_1$ , and from Theorem 4, we have that  $V[G] \models \mathfrak{h}_{\text{ED}} = \omega_1$ .

It remains to verify that  $V[G] \models \mathfrak{h}_{\omega\text{-hitting}} > \omega_1$ . Let  $\langle \mathcal{I}_\alpha : \alpha < \omega_1 \rangle \in V[G]$  be a family of  $\omega$ -hitting ideals. The following claim will be necessary to finish the proof.

*Claim* There is  $\gamma < \omega_2$  such that  $V[G_\gamma] \models \mathcal{I}_\alpha \cap V[G_\gamma]$  is  $\omega$ -hitting for all  $\alpha < \omega_1$ .

*Proof of Claim* Start with  $\alpha_0 < \omega_2$ . In  $V[G_{\alpha_0}]$ , enumerate all sequences of infinite subsets of  $\omega$ ,  $\langle \langle A_n^\xi : n \in \omega \rangle : \xi < \omega_1 \rangle$  (by CH, there are only  $\omega_1$ ). For each  $\xi < \omega_1$ , let  $I_\alpha^\xi \in \mathcal{I}_\alpha$  be such that  $(\text{in } V[G]) I_\alpha^\xi \cap A_n^\xi \neq \emptyset$  for all  $n \in \omega$ . The set  $\{I_\alpha^\xi : \alpha, \xi < \omega_1\}$  has cardinality  $\omega_1$ , then there is  $\alpha_1 < \omega_2$  such that  $I_\alpha^\xi \in V[G_{\alpha_1}]$  for all  $\alpha < \omega_1$  and all  $\xi < \omega_1$ . Iterating this process  $\omega_1$  times, we find  $\alpha_{\omega_1}$ . Then  $\gamma = \alpha_{\omega_1}$  works. If  $\langle A_n : n \in \omega \rangle \in V[G_\gamma]$  there is  $\xi < \omega_1$  such that  $\langle A_n : n \in \omega \rangle \in V[G_{\alpha_\xi}]$ . In  $V[G_{\alpha_{\xi+1}}]$ , we know that for each  $\alpha < \omega_1$ , there is  $I_\alpha \in \mathcal{I}_\alpha$  such that  $I_\alpha \cap A_n \neq \emptyset$ . But  $V[G_{\alpha_{\xi+1}}] \subseteq V[G_\gamma]$ .  $\square$

Let  $\gamma < \omega_2$  obtained from the claim. Now, in  $V[G_{\gamma+1}]$  we have by the Lemma 4 that the range  $A$  of the  $(\gamma + 1)$ -st Laver real is an infinite set that belongs to each  $\mathcal{I}_\alpha$ , that is,  $A \in \bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha$ .  $\square$

### 3.3 The consistency of $\mathfrak{h}_{\text{ED}} = \mathfrak{h}_{\omega\text{-hitting}} < \mathfrak{b}$

For this consistency proof we use the random forcing  $\mathbb{B}(\omega_1)$ . Let  $\mu$  be the standard product measure on  $2^{\omega_1}$  and  $\mathcal{N}_{\omega_1} = \{X \subseteq 2^{\omega_1} : \mu(X) = 0\}$ . For  $A, B \in \mathbf{Borel}(2^{\omega_1})$  let  $A \simeq B$  if and only if  $A \Delta B \in \mathcal{N}_{\omega_1}$  and denote  $[A]_{\mathcal{N}}$  the equivalence class of the set  $A$  with respect this equivalence relation. Define  $\mathbb{B}(\omega_1) = \{[A]_{\mathcal{N}} : A \in \mathbf{Borel}(2^{\omega_1})\}$  with the order  $[A]_{\mathcal{N}} \leq [B]_{\mathcal{N}}$  if  $A \setminus B \in \mathcal{N}_{\omega_1}$ .

The random forcing  $\mathbb{B}(\omega_1)$  preserves  $\mathfrak{b}$  (see [2]) and adds  $\omega_1$  reals in the following way: if  $G$  is  $\mathbb{B}(\omega_1)$ -generic filter and  $r_G \in 2^{\omega_1}$  is the generic function, then the  $\alpha$ -th real is defined by  $r_\alpha(n) = r_G(\alpha \cdot \omega + n)$  for  $\alpha < \omega_1$ .

We can see  $V[G]$  as  $V[r_\alpha : \alpha < \omega_1]$ , where  $r_\alpha : \omega \rightarrow 2$  is the  $\alpha$ -th random real added by  $G$ .

**Theorem 8** *It is consistent with ZFC that  $\mathfrak{h}_{\text{ED}} = \mathfrak{h}_{\omega\text{-hitting}} = \omega_1$  and  $\mathfrak{b} = \omega_2$ .*

*Proof* Start with a model  $V$  such that  $\mathfrak{b} = \omega_2 = \mathfrak{c}$  (for example, the model obtained in Theorem 7 works) and let  $G$  be a  $\mathbb{B}(\omega_1)$ -generic over  $V$ . Then,  $V[G] \models \mathfrak{non}(\mathcal{N}) = \omega_1 < \mathfrak{b} = \omega_2$ , because, as already mentioned, random forcing preserve  $\mathfrak{b}$  and  $\{r_\alpha : \alpha < \omega_1\}$  is a witness for  $\mathfrak{non}(\mathcal{N}) = \omega_1$ .

Let us show that  $V[G] \models \mathfrak{h}_{\omega\text{-hitting}} = \omega_1$ . Let  $\{r_\alpha : \alpha < \omega_1\}$  be the  $\omega_1$  random reals added by  $G$ . For  $\alpha, \beta < \omega_1$ , let  $J_\beta = r_\beta^{-1}(1)$  and  $\mathcal{I}_\alpha = \langle J_\beta : \beta > \alpha \rangle$ .

We claim that  $\mathcal{I}_\alpha$  is an  $\omega$ -hitting ideal (and therefore, tall) for each  $\alpha < \omega_1$ , and  $\bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha = \mathbf{fin}$ .

Observe that, by the product lemma for random forcing (see [12]), it is sufficient to show that if  $\langle A_n : n \in \omega \rangle \in V[r_\gamma : \gamma < \alpha]$ , then  $J_\beta \cap A_n \neq \emptyset$  for all  $\beta > \alpha$  and for all  $n \in \omega$ . For this, note that  $\mu[\bigcap_{\beta > \alpha} J_\beta \cap A_n = \emptyset] = \mu(\{f \in 2^{\omega_1} : \forall k \in A_n (f(\beta + k) = 0)\}) = 0$ , for each  $n \in \omega$ .

To check that  $\bigcap_{\alpha < \omega_1} \mathcal{I}_\alpha = \mathbf{fin}$ , note that if  $V[G] \models A \in [\omega]^\omega$ , then there is  $\alpha < \omega_1$  such that  $A \in V[G_\alpha]$ . Now,  $\mu[A \subseteq J_\beta] = \mu(\{f \in 2^{\omega_1} : \forall k \in A (f(\beta + k) = 1)\}) = 0$ , which implies  $A \not\subseteq J_\beta$  for all  $\beta > \alpha$ . □

*Remark 1* It follows that the cardinal invariant  $\mathfrak{h}_{\omega\text{-hitting}}$  is not *tame*.<sup>1</sup> As a consequence of theorem 6.1.11 of [20] (under an appropriate large cardinal assumption), for every tame cardinal invariant  $\mathfrak{j}$ , if  $\mathfrak{j} < \mathfrak{b}$  holds in some forcing extension, then it holds in  $V^{\mathbb{L}_{\omega_2}}$ . Theorem 7 shows that  $V^{\mathbb{L}_{\omega_2}} \models \mathfrak{h}_{\omega\text{-hitting}} = \mathfrak{b}$ . On the other hand, Theorem 8 shows that  $V^{\mathbb{L}_{\omega_2} * \mathbb{B}(\omega_1)} \models \mathfrak{h}_{\omega\text{-hitting}} = \omega_1 < \mathfrak{b} = \omega_2$ .

### 3.4 The consistency of $\mathfrak{h}_{\text{ED}} < \mathfrak{add}(\mathcal{M})$

We consider  $\mathbb{L}_{\text{Fr}}$ , *Laver forcing* associated with the Fréchet filter  $\text{Fr}$  (the filter of co-finite sets of  $\omega$ ). It is defined as the set of those trees  $T \subseteq \omega^{<\omega}$  for which there is  $s_T \in T$  (the *stem* of  $T$ ) such that for all  $t \in T$ ,  $t \subseteq s_T$  or  $s_T \subseteq t$  and such that for

<sup>1</sup> Recall that a cardinal invariant  $\mathfrak{j}$  is *tame* (see [20]) if it is the minimum size of a set  $A \subseteq X$ , where  $X$  is a Polish space, with properties  $\phi(A)$ , and  $\forall x \in X \exists y \in A \theta(x, y)$  where  $\phi$  quantifies over natural numbers and elements of  $A$  only and  $\theta$  is a projective formula not mentioning the set  $A$ .

all  $t \in T$ , with  $t \supseteq s_T$  the set  $\text{succ}_T(t) = \{n \in \omega : t \cap n \in T\} \in \text{Fr}$ . It is ordered by inclusion. It is well known that the forcing  $\mathbb{L}_{\text{Fr}}$  is  $\sigma$ -centered.

Similar to the definition of an  $\omega$ -hitting family of sets, we say that a family  $\mathcal{F} \subseteq \omega^\omega$  is  $\omega$ -hitting if given  $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$  there is a  $f \in \mathcal{F}$  such that  $f^{-1}(m) \cap A_n$  is infinite for all  $m$  and  $n$ . An important property of  $\omega$ -hitting families of functions, which will be used several times in what follows, is that if an  $\omega$ -hitting family is partitioned into countably many pieces, then at least one of the pieces is  $\omega$ -hitting.

We now turn to the preservation of  $\omega$ -hitting for functions in iterations. The argument is based on [5]. In order to do that, we introduce a stronger property: We say that a forcing notion  $\mathbb{P}$  *strongly preserves  $\omega$ -hitting for functions* if for every  $\mathbb{P}$ -name  $\dot{A}$  for an infinite subset of  $\omega$  there is a  $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$  such that for any  $f \in \omega^\omega$ ,  $f^{-1}(m) \cap A_n$  is infinite for all  $m$  and  $n$  then  $\Vdash_{\mathbb{P}} \text{“} f^{-1}(m) \cap \dot{A} \text{ is infinite for all } m\text{”}$ . Clearly, every forcing notion that strongly preserves  $\omega$ -hitting for functions preserves  $\omega$ -hitting for functions.

**Lemma 5**  $\mathbb{L}_{\text{Fr}}$  *strongly preserves  $\omega$ -hitting for functions.*

*Proof* Let  $\dot{A}$  be an  $\mathbb{L}_{\text{Fr}}$ -name for a countable subset of  $\omega$ . Aiming for a contradiction, assume that for each  $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$  there is  $f \in \omega^\omega$  such that  $f^{-1}(m) \cap A_n$  is infinite for all  $m$  and  $n$ , yet there are a condition  $T_f$  and natural numbers  $n_f, m_f$  such that

$$T_f \Vdash \text{“} f^{-1}(m_f) \cap \dot{A} \subseteq n_f\text{”}. \quad (\star)$$

Let  $\mathcal{F}$  be the family of all such  $f \in \omega^\omega$ , that is, the family of all  $f \in \omega^\omega$  such that there are a condition  $T_f$  and natural numbers  $n_f, m_f$  such that  $T_f \Vdash \text{“} f^{-1}(m_f) \cap \dot{A} \subseteq n_f\text{”}$ . By our assumption  $\mathcal{F}$  is  $\omega$ -hitting.

Recall the standard rank analysis for Laver forcing. For  $s \in \omega^{<\omega}$ , say  $s$  favors  $k \in \dot{A}$  if there is no condition  $T \in \mathbb{L}_{\text{Fr}}$  with stem  $s$  such that  $T \Vdash \text{“} k \notin \dot{A}\text{”}$ , or equivalently, every condition  $T \in \mathbb{L}_{\text{Fr}}$  with stem  $s$  has an extension  $T'$  such that  $T' \Vdash \text{“} k \in \dot{A}\text{”}$ . Define the rank  $\text{rk}(s)$  by recursion on the ordinals by

$$\text{rk}(s) = 0 \Leftrightarrow \begin{cases} \text{either } \exists K \in [\omega]^\omega \forall k \in K (s \text{ favors } k \in \dot{A}) \\ \text{or } \exists X \in [\omega]^\omega, \exists f : X \rightarrow \omega \text{ finite-to-one} \\ \forall l \in X (s \frown l \text{ favors } f(l) \in \dot{A}) \end{cases}$$

and  $\text{rk}(s) \leq \alpha$  if and only if there is a  $X \in [\omega]^\omega$  such that  $\text{rk}(s \frown l) < \alpha$  for all  $l \in X$ , when  $\alpha > 0$ .

*Claim*  $\text{rk}(s) < \infty$  for all  $s$ .

*Proof of Claim* Assume  $\text{rk}(s) = \infty$ . So  $K = \{k : s \text{ favors } k \in \dot{A}\}$  is finite. Recursively build  $T \in \mathbb{L}_{\text{Fr}}$  with stem  $s$  such that for all  $t \in T$  extending  $s$ ,

- $\text{rk}(t) = \infty$ , and
- $\{k : t \text{ favors } k \in \dot{A}\} \subseteq K$ .

Let such  $t$  be given. First, there is  $X_0 \in \text{Fr}$  such that  $\text{rk}(t \smallfrown l) = \infty$  for all  $l \in X_0$ . Let  $X_1 = \{l \in X_0 : \exists k \notin K (t \smallfrown l \text{ favors } k \in \dot{A})\}$ . If  $X_1$  is infinite, then we can define a function as in the definition of  $\text{rk}$ , and so  $\text{rk}(t) = 0$ , a contradiction. Thus  $X_1$  is finite and  $X_0 \setminus X_1 \in \text{Fr}$ . For  $t \smallfrown l$  with  $l \in X_0 \setminus X_1$ , both clauses above are satisfied, and the construction proceeds.

Now find  $T' \leq T$  and  $k \notin K$  such that  $T' \Vdash "k \in \dot{A}"$ . Then the stem of  $T'$  in particular favors  $k \in \dot{A}$ , a contradiction.  $\square$

Let  $s_f$  be the stem of  $T_f$ . By strengthening  $T_f$ , if necessary, we may assume that  $\text{rk}(s_f) = 0$  for all  $f \in \mathcal{F}$ . Since  $\mathcal{F}$  is  $\omega$ -hitting, there are  $s$  and natural numbers  $n, m$  such that the family  $\mathcal{F}_{s,n,m} = \{f \in \mathcal{F} : s = s_f, n = n_f \text{ and } m = m_f\}$  is  $\omega$ -hitting. Fix such  $s, n$  and  $m$ .

We consider two cases, according to the definition of  $\text{rk}$ .

*Case 1*  $\exists K \in [\omega]^\omega \forall k \in K (s \text{ favors } k \in \dot{A})$

Let  $f \in \mathcal{F}_{s,n,m}$  be such that  $f^{-1}(m) \cap K$  is infinite. So there is  $k > n$  such that  $k \in f^{-1}(m) \cap K$ . Thus there is  $T' \leq T_f$  with  $T' \Vdash "k \in \dot{A}"$ , a contradiction to the initial assumption  $(\star)$ .

*Case 2*  $\exists X \in [\omega]^\omega, f : X \rightarrow \omega$  finite-to-one  $\forall l \in X (s \smallfrown l \text{ favors } f(l) \in \dot{A})$ .

Let  $g \in \mathcal{F}_{s,n,m}$  be such that  $g^{-1}(m) \cap \text{ran}(f)$  is infinite. Since  $X \subseteq^* \text{succ}_{T_g}(s)$ , there is a  $k \in g^{-1}(m) \cap \text{ran}(f)$  with  $k > n$  such that  $f^{-1}(k) \cap \text{succ}_{T_g}(s) \neq \emptyset$ . Let  $l \in f^{-1}(k) \cap \text{succ}_{T_g}(s)$ . Thus  $s \smallfrown l$  favors  $k \in \dot{A}$ . Hence there is  $T \leq T_g$  whose stem extends  $s \smallfrown l$  such that  $T \Vdash "k \in \dot{A}"$ , again a contradiction.  $\square$

**Lemma 6** *Finite support iteration of forcings strongly preserving  $\omega$ -hitting for functions strongly preserves  $\omega$ -hitting for functions.*

*Proof* This is a standard argument. We provide the details for the sake of completeness. Obviously, it suffices to consider limit stages of cofinality  $\omega$ .

Let  $\langle \mathbb{P}_k, \dot{Q}_k : k \in \omega \rangle$  be a finite support iteration of ccc forcing such that

$$\Vdash_{\mathbb{P}_k} \text{"}\dot{Q}_k \text{ strongly preserves } \omega\text{-hitting for functions"}$$

for each  $k \in \omega$ .

Let  $\dot{A}$  be a  $\mathbb{P}_\omega$ -name for an infinite subset of  $\omega$ . In the intermediate extension  $V[G_k]$  find a decreasing sequence of conditions  $\langle p_{n,k} : n \in \omega \rangle \subset \mathbb{P}_{[k,\omega]}$  and infinite subsets  $A_{n,k}$  of  $\omega$  such that

$$p_{n,k} \Vdash_{\mathbb{P}_{[k,\omega]}} \text{"the first } n \text{ elements of } A_{m,k} \text{ and } \dot{A} \text{ agree for } m \leq n"$$

where  $\mathbb{P}_\omega = \mathbb{P}_k * \mathbb{P}_{[k,\omega]}$ . The  $A_{n,k}$  are approximations to  $\dot{A}$ .

Now, as each  $\mathbb{P}_k$  strongly preserves  $\omega$ -hitting for functions, there is a  $\langle A_{n,k}^m : m \in \omega \rangle \subseteq [\omega]^\omega$  such that for every  $f \in \omega^\omega$ , if  $f^{-1}(i) \cap A_{n,k}^m$  is infinite for all  $i$  and  $m$  then

$$\Vdash_{\mathbb{P}_k} \text{"}f^{-1}(i) \cap \dot{A}_{n,k} \text{ is infinte for all } i"$$

Consider  $\langle A_{n,k}^m : n, k, m \in \omega \rangle$  and let  $f \in \omega^\omega$  be such that  $f^{-1}(i) \cap A_{n,k}^m$  is infinite for all  $n, k$  and  $m$ . To finish the proof, it suffices to show that

$$\Vdash_{\mathbb{P}_\omega} \text{“} f^{-1}(i) \cap \dot{A} \text{ is infinite for all } i\text{”}.$$

If not, then there are a  $q \in \mathbb{P}_\omega$ ,  $i \in \omega$  and  $m \in \omega$  such that  $q \Vdash_{\mathbb{P}_\omega} \text{“} f^{-1}(i) \cap \dot{A} \subseteq m\text{”}$ . Let  $k$  be such that  $q \in \mathbb{P}_k$ .

Let  $G_k$  be a  $\mathbb{P}_k$ -generic such that  $q \in G_k$ . As  $f^{-1}(i) \cap A_{m,k}$  is infinite, let  $l \geq m$  with  $l \in f^{-1}(i) \cap A_{m,k}$ . For large enough  $n$ ,

$$p_{n,k} \Vdash_{\mathbb{P}_{(k,\omega)}} \text{“} l \in \dot{A}\text{”}.$$

Since  $q \in G_k$ , this contradicts the initial assumption about  $q$ . □

Combining the previous two lemmas, we obtain the following consistency result.

**Theorem 9** *It is consistent with ZFC that  $\mathfrak{h}_{\text{ED}} = \omega_1$  and  $\text{add}(\mathcal{M}) = \omega_2$ .*

*Proof* Start with a model of **CH** and iterate the forcing  $\mathbb{L}_{\text{Fr}}$  with finite support  $\omega_2$  times. To establish the first assertion, let  $\kappa = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \text{ is } \omega\text{-hitting}\}$ . Then  $\mathfrak{h}_{\text{ED}} \leq \kappa$ . By Lemmas 5 and 6, the cardinal  $\kappa$  is preserved along the iteration, and hence  $V^{\mathbb{L}_{\text{Fr}}^{\omega_2}} \models \mathfrak{h}_{\text{ED}} = \omega_1$ . On the other hand, it is well known that  $\mathbb{L}_{\text{Fr}}$  adds a Cohen real and also adds a dominating real, and since  $\text{add}(\mathcal{M}) = \min\{\text{cov}(\mathcal{M}), \mathfrak{b}\}$  (see [2]), it follows that  $V^{\mathbb{L}_{\text{Fr}}^{\omega_2}} \models \text{add}(\mathcal{M}) = \omega_2$ . □

### 4 Final remarks and questions

In [8] the author asked which of the following inequalities can be consistently strict:  $\mathfrak{h} \leq \mathfrak{h}_{\text{analytic}} \leq \mathfrak{h}_{\text{Borel}} \leq \dots \leq \mathfrak{h}_{F_\sigma} \leq \mathfrak{b}$ . There is no known consistency result that distinguishes between the intersection numbers of analytic ideals and that of Borel (or even  $F_\sigma$ ) ideals. Note that a positive answer to the following question provides an answer to the previous question.

*Question 1* [8] Let  $\mathcal{I}$  be a tall Borel (analytic) ideal. Is there a tall  $F_\sigma$  ideal  $\mathcal{J}$  such that  $\mathcal{J} \subseteq \mathcal{I}$ ?

The author of [8] (the first listed author of this note) also claimed that “obviously  $\mathfrak{h}_{F_\sigma} \leq \min\{\mathfrak{b}, \mathfrak{s}\}$ ”. We do not know whether this is true, but definitely, it does not seem obvious.

*Question 2* Is  $\mathfrak{h}_{F_\sigma} \leq \mathfrak{s}$ ?

*Question 3* Is  $\mathfrak{h}_{\text{ED}} = \min\{\mathfrak{b}, \mathfrak{s}\}$ ?

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