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Precompact Fréchet topologies on Abelian groups

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ABSTRACT

We study precompact Fréchet topologies on countable Abelian groups. For every countable Abelian group G we introduce the notion of a γ_G-set and show that there is a precompact Fréchet non-metrizable topology on G if and only if there is an uncountable γ_G-set that separates points of G. We show that, assuming the existence of an uncountable γ-set, there is a non-metrizable precompact Fréchet topology on every countable Abelian group, and assuming $\mathfrak{p} > \omega_1$, there is a non-metrizable Fréchet topology on every countable group which admits a non-discrete topology at all. We further study the notion of a γ_G-set and show that the minimal size of a subset of the dual group G^* which is not a γ_G-set is the pseudointersection number $\mathfrak{p}$ for any countable Abelian group G.

1. Introduction

A Hausdorff topological space X is *Fréchet–Urysohn* (or just Fréchet) if whenever a point $x \in X$ is in the closure of a set $A \subset X$, there is a sequence of elements of A converging to x. The classical metrization theorem of G. Birkhoff and S. Kakutani states that a T_1 topological group is metrizable if and only if it is first countable. There are non-separable Fréchet topological groups which are not metrizable, e.g., the Σ-product of uncountably many copies of $\mathbb{Z}_2$. V.I. Malykhin in 1978 (see [2,7,12]) asked:

Problem 1.1 *(Malykhin).* Is there a separable Fréchet topological group that is not metrizable?

Since a group with a dense metrizable subgroup is metrizable, the problem can be reformulated as asking for the existence of a countable Fréchet topological group that is not metrizable.

It is well known that the answer to Malykhin's problem is consistently positive. For instance, under either of the following assumptions: $\mathfrak{p} > \omega_1$, $\mathfrak{p} = \mathfrak{b}$ and the existence of an uncountable γ-set, there is a non-metrizable Fréchet group topology on the Boolean group $([\omega]^{<\omega}, \Delta)$ of finite subsets of ω with the symmetric difference as the group operation (see [14] and [15]). It is well known that the existence of an uncountable γ-set is the weakest of the three previous assumptions (see [6] and [16]). Another example of a separable non-metrizable Fréchet topological group can be obtained also from

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an uncountable γ-set using results from $C_p(X)$ theory. In [6] it is shown that $C_p(X)$ is separable Fréchet non-metrizable if and only if X is an uncountable γ-set.

It is *a priori* not clear what role the group structure plays in Malykhin's problem. A.Ju. Ol'šanskiĭ showed that there is a countable group G which admits only discrete topology (see [1, (13.4)]). In particular, there is no non-metrizable Fréchet topology on G. As far as we know non-metrizable Fréchet topologies were considered only on the Boolean group (see [14, 15,18,7,4]) or on additive subgroups of vector spaces, e.g., $C_p(X)$ (see [6,15,18]). Recall that a group is *topologizable* if it admits a non-discrete Hausdorff group topology. Here we show that, consistently, there is a non-metrizable Fréchet group topology on every countable topologizable group.

M. Ismail in [8] (see also [13]) proved that every locally compact group of countable tightness is metrizable, and so every locally compact Fréchet group is metrizable. Recall that a topological group is *precompact* (or equivalently *totally bounded* [22]) if it is a subgroup of some compact group (e.g., if finitely many translates of every neighborhood of the identity element cover the entire group). The Σ-product of uncountably many copies of $\mathbb{Z}_2$ is also an example of a non-separable precompact Fréchet group that is not metrizable. We consider the following variation on Problem 1.1:

Question 1.2. Is there a separable precompact Fréchet topological group that is not metrizable?

This question can also be reformulated as asking for the existence of a countable precompact Fréchet topological group that is not metrizable. The study of precompact Fréchet topologies was suggested by Shakhmatov in [20], but as far as we know, no actual work in the area has been done.

We concentrate on the study of precompact Fréchet topologies on countable Abelian groups. For every countable Abelian group G we introduce the notion of a γ_G-set and show that there is a precompact Fréchet non-metrizable topology on G if and only if there is an uncountable γ_G-set (subset of the dual group G^*) that separates points of G. There is a close relationship between the notions of γ-set and γ_G-set, e.g., we show that, assuming the existence of an uncountable γ-set, there is an uncountable γ_G-set that separates points of G for every countable Abelian group G. Thus, the answer to Question 1.2 is consistently positive. We further study the notion of a γ_G-set and show that the minimal size of a subset of the dual group G^* which is not a γ_G-set is the pseudointersection number $\mathfrak{p}$ for any countable Abelian group G.

2. Notation and terminology

Our set-theoretic notation is mostly standard and follows [10]. In particular, ω stands for the set of all natural numbers (finite ordinals) and $[\omega]^\omega$ the set of all infinite subsets of ω. $A \subseteq^* B$ denotes that A is *almost contained in* B, i.e. $A \setminus B$ is finite. Recall also that a family of subsets of ω is *centered* if any finite subfamily has infinite intersection. The *pseudointersection number* $\mathfrak{p}$ is the minimal size of a centered family of subsets of ω without an infinite pseudointersection (i.e., without a common lower bound in the $\subseteq^*$ order). For functions $f, g \in \omega^\omega$ we write $f <^* g$ to mean that there is $m \in \omega$ such that $f(n) < g(n)$ for all $n \geqslant m$. Recall that the *bounding number* $\mathfrak{b}$ is the least cardinal of a $<^*$-unbounded family of functions in ω^ω.

Recall that an open cover $\mathcal{U}$ of a topological space X is an *ω-cover* if for every finite $F \subseteq X$ there exists $U \in \mathcal{U}$ with $F \subseteq U$. An open cover $\mathcal{U}$ of a topological space X is a *γ-cover* if $\mathcal{U}$ is infinite and every $x \in X$ is in all but finitely many $U \in \mathcal{U}$. In particular, every γ-cover is an ω-cover. A space X is a *γ-space* if every ω-cover of X contains a countable γ-subcover. A γ-space which is separable metric is called *γ-set*. It is not difficult to see that every countable space is a γ-space. The cardinal $\mathsf{non}(\gamma\text{-set})$ is defined as the least cardinality of a set which is not a γ-set. It is well known that $\mathfrak{p} = \mathsf{non}(\gamma\text{-set})$ (see [6]) so, in particular, every separable metric space of size $< \mathfrak{p}$ is a γ-set. The γ-space notion has the following diagonal sequence property.

Lemma 2.1. *([6]) X is a γ-space if and only if for every sequence $\langle \mathcal{U}_n \colon n \in \omega \rangle$ of open ω-covers of X there exists $U_n \in \mathcal{U}_n$ such that $\{U_n \colon n \in \omega\}$ forms a γ-subcover of X.* □

As every γ-set has strong measure zero [6], there are no uncountable γ-sets in the Laver model for consistency of Borel conjecture [11], and hence, the existence of an uncountable γ-set is independent of **ZFC**.

The circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ and will be written additively. We denote by $\mathbb{Z}(p^\infty)$ the quasicyclic p-group, for a prime p. The neutral element of an Abelian group G will be denoted by 0_G or simply 0 and in the case of $\mathbb{T}$ by $\mathbf{0}$. The norm $\|x\|$ on $\mathbb{T}$ is defined as the length of the shortest arc between $\mathbf{0}$ and x. The open symmetric arc $T_m = \{x \in \mathbb{T} \colon \|x\| < \frac{1}{m}\}$ will often be used.

Given an Abelian group G and a prime number p, the *p-torsion part* or, equivalently, *p-primary component* of G is

$$G_p = \{g \in G \colon p^n g = 0_G, \text{ for some } n \in \omega\}.$$

If $G = G_p$ for some prime p, then G is called a *torsion p-group*. It is well known that every torsion Abelian group G is isomorphic to the direct sum $\bigoplus_p G_p$.

A non-empty subset A of an Abelian group G not containing the neutral element is *independent* if for every finite set $a_1, \dots, a_n$ of distinct elements of A and integers $m_1, \dots, m_n$, the equality $m_1 a_1 + \dots + m_n a_n = 0_G$ implies that $m_i a_i = 0_G$

for all $i = 1, \dots, n$. Kuratowski–Zorn lemma implies that every independent subset of a group G is contained in a maximal independent subset. The *torsion-free rank* of an Abelian group G or, in symbols, $r_0(G)$ is the cardinality of a maximal independent subset of elements of infinite order in G. Similarly, for a prime p, the *p-rank* $r_p(G)$ of the group G is the cardinality of a maximal independent subset of elements of p-power orders in G. The *Prüfer rank*, or just *the rank of* G is defined as:

$$r(G) = r_0(G) + \sum_p r_p(G).$$

It is clear that $r(G) = r_0(G)$ if the group G is torsion-free, and $r(G) = r_p(G)$ if G is a torsion p-group (see [19, Section 4.2]).

The topological groups G considered here will be Abelian and Hausdorff. The symbol G_d will denote the group G endowed with the discrete topology.

We will need some facts from the literature concerning the Pontryagin–van Kampen duality. Recall that given an Abelian topological group G its (dual) *group of characters* is

$$G^* = \{x : G \to \mathbb{T}\colon\ x \text{ is a continuous homomorphism}\}$$

with the *compact-open* topology. The *evaluation mapping* $e : G \to \mathbb{T}^{G^*}$ is defined by the formula $e(g)(x) = x(g)$ for all $g \in G$ and $x \in G^*$.

The next theorem summarizes certain known facts from the literature.

Theorem 2.2. *([17] and [9]) Let G be an Abelian locally compact group. Then*:

(i) *G^* is Abelian locally compact.*
(ii) *G^{**} is naturally isomorphic to G, via the evaluation mapping e.*
(iii) *G^* is compact if and only if G is discrete.*
(iv) *A compact Abelian group G is metrizable if and only if G^* is countable.* □

Note that $|(G_d)^*| \geqslant \mathfrak{c}$ whenever G is infinite.

3. Precompact Fréchet topologies

Recall that $X \subseteq (G_d)^*$ *separates points* of G if for every $g \in G$ with $g \neq 0_G$ there is an $x \in X$ such that $x(g) \neq \mathbf{0}$.

Definition 3.1. Given G an Abelian group and $X \subseteq (G_d)^*$ that separates points of G let τ_X be the weakest topology on G which makes all $x \in X$ continuous. The symbol G_X will denote the group G endowed with the topology τ_X.

For $X \subseteq (G_d)^*$, the *diagonal product* of the family X, denoted by r_X, is the mapping from G into $\mathbb{T}^X$ defined by $r_X(g)(x) = x(g)$ for all $g \in G$ and $x \in X$. Note that if X separates points of G, then r_X is an embedding of G to the product group $\mathbb{T}^X$. It is easily seen that the topology τ_X is just the subspace topology induced by r_X.

Proposition 3.2. *([5]) Let G be an Abelian group and let $X \subseteq (G_d)^*$ separate points of G. Then G_X is a precompact Hausdorff group. Moreover, every precompact Hausdorff group topology on G is of the form τ_X.* □

Given an Abelian group G, $g \in G$ and $m > 0$ let

$$U_g^m = \left\{x \in (G_d)^*\colon\ x(g) \in T_m\right\}$$

and given $A \subseteq G$ let

$$\mathcal{U}_A^m = \left\{U_a^m\colon\ a \in A\right\}.$$

Note that the sets of the form U_g^m are open neighborhoods of the identity element in the (compact separable metric) topology on $(G_d)^*$.

Lemma 3.3. *Let G be an Abelian group, let $X \subseteq (G_d)^*$ separate points of G and let A be an infinite subset of G. Then*:

(i) *$\mathcal{U}_A^m$ is an ω-cover of X for every $m > 0$ if and only if 0_G belongs to the τ_X-closure of A in G;*
(ii) *$\mathcal{U}_A^m$ is a γ-cover of X for every $m > 0$ if and only if A τ_X-converges to 0_G in G (i.e., every neighborhood of 0_G contains all but finitely many elements of A).*

Proof. Using the diagonal product r_X, we can identify G_X with $r_X[G]$ in $\mathbb{T}^X$.

(i) $\mathcal{U}^m_A$ is an ω-cover of X for every $m > 0$ if and only if for every $m > 0$ and for all $F \in [X]^{<\omega}$ there is $a \in A$ such that $x(a) \in T_m$ for each $x \in F$ ($F \subseteq U^m_a$). This is equivalent to the fact that 0_G belongs to τ_X-closure of A in G, because $U(F, m) = \{g \in G\colon\ x(g) \in T_m$ for every $x \in F\}$ is a basic neighborhood of 0_G in $\mathbb{T}^X$.

(ii) Suppose that $\mathcal{U}^m_A$ is a γ-cover of X for every $m > 0$, and let $U(F, m)$ be a basic neighborhood of 0_G in $\mathbb{T}^X$. Then $x \in U^m_a$ for every $x \in F$ and for all but finitely many $a \in A$, or equivalently, $A \subseteq^* U(\{x\}, m)$ for every $x \in F$. By the finiteness of F, it follows that $A \subseteq^* U(F, m)$.

Conversely, suppose that A τ_X-converges to 0_G in G. Fix $m > 0$ and $x \in X$. By convergence, it follows that $A \subseteq^* U(\{x\}, m)$, or equivalently, $x \in U^m_a$ for all but finitely many $a \in A$. □

The following definition will play an important role in this paper.

Definition 3.4. An infinite set $X \subseteq (G_d)^*$ is a *γ_G-space*, if for every infinite $A \subseteq G$ if $\mathcal{U}^m_A$ is an ω-cover of X for every $m > 0$, then there is a countable $B \subseteq A$ such that $\mathcal{U}^m_B$ is a γ-cover of X for every $m > 0$. In the special case when G is a countable group, we will say that X is a *γ_G-set* if it is a γ_G-space.

Combining the previous definition and Lemma 3.3, we obtain the first main result of this paper.

Theorem 3.5. *Let G be an Abelian group and let $X \subseteq (G_d)^*$ separate points of G. Then G_X is Fréchet if and only if X is a γ_G-space.* □

The following result is well known [5].

Theorem 3.6. *([5]) Let G be an Abelian group and let $X \subseteq (G_d)^*$ separate points of G. Then G_X is metrizable if and only if X is countable.* □

Combining the last two theorems, we obtain the following conclusion.

Corollary 3.7. *The existence of a non-metrizable precompact Fréchet group topology on a countable Abelian group G is equivalent to the existence of an uncountable γ_G-set that separates points of G.* □

There is a close relationship between the notions of γ-space and γ_G-space.

Proposition 3.8. *If $X \subseteq (G_d)^*$ is a γ-space, then X is a γ_G-space.*

Proof. Suppose that A is an infinite subset of G such that $\mathcal{U}^m_A$ is an ω-cover of X for every $m > 0$. Since X is a γ-space, we can apply Lemma 2.1 to find $U^{n+1}_{a_n} \in \mathcal{U}^{n+1}_A$ such that $\mathcal{U} = \{U^{n+1}_{a_n}\colon\ n \in \omega\}$ forms a γ-subcover of X. Let $B = \{a_n\colon\ n \in \omega\}$. Then $\mathcal{U}^m_B$ is a γ-cover of X for every $m > 0$. Indeed, let $m > 0$ and $x \in X$, since $\mathcal{U}$ is a γ-subcover of X, there is a $k \in \omega$ with $k \geqslant m$ such that $x \in U^{i+1}_{a_i} \subseteq U^m_{a_i}$ for every $i \geqslant k$. □

As every countable set is a γ-set, by the previous proposition every countable set $X \subseteq (G_d)^*$ is a γ_G-set.

With the help of Proposition 3.8, we obtain another interesting conclusion.

Corollary 3.9. *Assuming the existence of an uncountable γ-set, every countable Abelian group admits a non-metrizable precompact Fréchet group topology.*

Proof. Let X be an uncountable γ-set and let G be a countable Abelian group. Since γ-sets are zero-dimensional and $(G_d)^*$ is a perfect Polish space we can assume without loss of generality that $X \subseteq (G_d)^*$.

On the other hand, since G is countable, there is a countable $Y \subset (G_d)^*$ that separates points of G (see e.g. [3, (1.1.8)]). Let $Z = X \cup Y$. Using Lemma 2.1, it is easy to see that Z is also a γ-set. Then Z is a γ_G-set by Proposition 3.8. Therefore, by Corollary 3.7, τ_Z is a non-metrizable precompact Fréchet group topology on G. □

4. γ_G-Sets

In this section we further study the notion of a γ_G-set.

Lemma 4.1 *(Preservation lemma). Let $f : H \to G$ be a homomorphism and let $f^* : (G_d)^* \to (H_d)^*$ be the induced homomorphism given by $x \mapsto x \circ f$ for every $x \in (G_d)^*$. If $X \subseteq (G_d)^*$ is a γ_G-space then $f^*[X]$ is a γ_H-space.*

Proof. Let $X \subseteq (G_d)^*$ be a γ_G-space and let $Y = f^*[X]$. Suppose that $C \subseteq H$ is an infinite set such that $\mathcal{V}^m_C$ is an ω-cover of Y for every $m > 0$, where $\mathcal{V}^m_C = \{V^m_c\colon c \in C\}$ and $V^m_c = \{y \in (H_d)^*\colon y(c) \in T_m\}$. We may also assume that $\mathcal{V}^m_C$ is infinite for every $m > 0$. Put $A = f[C]$.

Claim 4.2. *A is an infinite set such that $\mathcal{U}^m_A$ is an ω-cover of X for every $m > 0$, where $\mathcal{U}^m_A = \{U^m_a\colon a \in A\}$ and $U^m_a = \{x \in (G_d)^*\colon x(a) \in T_m\}$.*

Proof of Claim 4.2. Fix $m > 0$. Note that, if $x \in (G_d)^*$ and $c \in C$, then $x(f(c)) = f^*(x)(c)$ and, hence, $f^{*-1}[V^m_c] = U^m_{f(c)}$. Since $\mathcal{V}^m_C$ is infinite, it follows that A is an infinite set. Assume now that $E \subset X$ is a finite set. Put $F = f^*[E]$. Then, there is $c \in C$ with $F \subseteq V^m_c$. It follows that $E \subseteq f^{*-1}[F] \subseteq f^{*-1}[V^m_c] = U^m_{f(c)}$. Thus, $\mathcal{U}^m_A$ is an ω-cover of X. □

Now, since X is γ_G-space, there is a countable $B \subseteq A$ such that $\mathcal{U}^m_B$ is a γ-cover of X for every $m > 0$. Let $\varphi : B \to \bigcup_{b \in B} f^{-1}[b]$ be a choice function, and put $D = \varphi[B]$. Then, D is a countable subset of C such that $\mathcal{V}^m_D$ is a γ-cover of Y for every $m > 0$. □

Now, note that f^* is a surjection if and only if f is an injection and f^* is an injection if and only if f is a surjection. Therefore, we obtain the following conclusion.

Theorem 4.3. *The existence of an uncountable γ_G-set for some countable Abelian group G, implies the existence of an uncountable $\gamma_{\mathbb{Z}_\omega}$-set, where $\mathbb{Z}_\omega = \bigoplus_\omega \mathbb{Z}$ is the free Abelian group on countably many generators.*

Proof. It is well known from the theory of free Abelian groups that there is a surjective homomorphism $f : \mathbb{Z}_\omega \to G$ for any countable Abelian group G. So f^* is an injection and therefore the theorem follows directly from Lemma 4.1. □

We let, $\mathrm{non}(\gamma_G\text{-set}) = \min\{|X|\colon X \subseteq (G_d)^* \text{ is not a } \gamma_G\text{-set}\}$ and establish the second main result of the paper.

Theorem 4.4. *Let G be a countable Abelian group. Then* $\mathrm{non}(\gamma_G\text{-set}) = \mathfrak{p}$.

Proof. The inequality $\mathrm{non}(\gamma_G\text{-set}) \geqslant \mathfrak{p}$ follows directly from Proposition 3.8 and the fact that $\mathfrak{p} = \mathrm{non}(\gamma\text{-set})$.

To establish the other inequality, we need the next lemma.

Lemma 4.5. *Let H be a subgroup of G, then* $\mathrm{non}(\gamma_G\text{-set}) \leqslant \mathrm{non}(\gamma_H\text{-set})$.

Proof. Let $f : H \to G$ be a monomorphism, and suppose that $Y \subseteq (H_d)^*$ is not a γ_H-set. Since f is an injection, f^* is a surjection and hence there is an $X \subseteq (G_d)^*$ such that $|X| = |Y|$ and $Y = f^*[X]$. By Lemma 4.1 it follows that X is not a γ_G-set. □

Thus, to show that $\mathrm{non}(\gamma_G\text{-set}) \leqslant \mathfrak{p}$, it is enough to verify that $\mathrm{non}(\gamma_H\text{-set}) = \mathfrak{p}$, for some subgroup H of G.

We need a fact concerning structural theory of Abelian groups.

Fact 4.6. *Any countable Abelian group G contains an isomorphic copy of one of the following: $\mathbb{Z}$, K, $\mathbb{Z}(p^\infty)$ for a prime p, where K is a group generated by an infinite independent set.*

Proof. Suppose that G does not contain an isomorphic copy of $\mathbb{Z}$ or K. Then $r(G)$ must be finite and $r_0(G) = 0$. Therefore, there is a prime p such that G_p is infinite with $r(G_p)$ finite. By 4.3.13 in [19], it follows that G_p is a direct sum of finitely many cyclic and quasicyclic groups. The fact now follows. □

Claim 4.7. $\mathrm{non}(\gamma_K\text{-set}) = \mathfrak{p}$.

Proof of Claim 4.7. Let $A = \{a_n\colon n \in \omega\}$ be an independent set of non-zero elements of K such that $K = \langle A\rangle$. So $K = \bigoplus_{n\in\omega}\langle a_n\rangle$. Let $\mathcal{F} \subseteq [\omega]^\omega$ be a centered family without an infinite pseudointersection. Note that for every $n \in \omega$, we can find $b_n \in \mathbb{T}$ with $b_n \notin T_4$ such that $\langle b_n\rangle$ is isomorphic to a subgroup of $\langle a_n\rangle$. Now, for every $F \in \mathcal{F}$ and for each $n \in \omega$ consider the function $x_{F,n} : \langle a_n\rangle \to \mathbb{T}$ defined by

$$x_{F,n}(ma_n) = \begin{cases} \mathbf{0}, & \text{if } n \in F; \\ mb_n, & \text{otherwise.} \end{cases}$$

Then $x_{F,n}$ is a group homomorphism. By universal property of the direct sum, there exists a unique homomorphism $x_F : K \to \mathbb{T}$ making the following diagrams commute ($x_{F,n} = x_F \circ \iota_n$):

$$\begin{tikzcd}
\langle a_n\rangle \arrow[r, "\iota_n"] \arrow[dr, "x_{F,n}"'] \arrow[dr, phantom, "\curvearrowright", shift left=3] & K \arrow[d, dashed, "x_F"] \\
& \mathbb{T}
\end{tikzcd}$$

where ι_n is the inclusion. Let $X = \{x_F\colon\ F \in \mathcal{F}\}$ and we show that X is not a γ_K-set. Clearly $|X| = |\mathcal{F}|$.

Fix $m > 0$. Let $\{F_i\colon\ i < k\}$ be a finite subset of $\mathcal{F}$. Since $\mathcal{F}$ is a centered family, there is $n \in \bigcap_{i<k} F_i$. Then, $x_{F_i}(a_n) = \mathbf{0}$ for every $i < k$, and this implies that $\{x_{F_i}\colon\ i < k\} \subseteq U^m_{a_n}$. Therefore, $\mathcal{U}^m_A$ is an ω-cover of X for every $m > 0$. Assume now that B is an infinite subset of A and verify that $\mathcal{U}^4_B$ is not a γ-cover of X. Suppose to the contrary that for every $F \in \mathcal{F}$ we have $x_F \in U^4_b$ (or equivalently, $x_F(b) = \mathbf{0}$) for all but finitely many $b \in B$. Put $E = \{n \in \omega\colon\ a_n \in B\}$. Notice that, for every $F \in \mathcal{F}$, $x_F(a_n) = \mathbf{0}$ is equivalent to $n \in F$. Therefore, E is an infinite pseudointersection of $\mathcal{F}$, which contradicts our assumption about $\mathcal{F}$. □

Claim 4.8. $\mathrm{non}(\gamma_{\mathbb{Z}(p^\infty)}$*-set*$) = \mathfrak{p}$.

Proof of Claim 4.8. Consider $\mathbb{Z}(p^\infty)$ as a subgroup of $\mathbb{T}$, i.e. $\mathbb{Z}(p^\infty) = \langle a_n\colon\ n \geqslant 1\rangle$, where $a_n = \frac{1}{p^n} + \mathbb{Z}$ for $n \geqslant 1$.

To prove the claim, we need the next technical lemma.

Lemma 4.9. *Let $N = \{2^n\colon\ n \geqslant 1\}$. Then, for each $F \subseteq N$ there is $\bar{x}_F \in (\mathbb{Z}(p^\infty)_d)^*$ with the following properties*:

(i) *$\bar{x}_F(a_n) \in T_4$ if and only if $n \in F$; and*
(ii) *$\bar{x}_F(a_n) \in T_n$ if $n \in F$.*

Proof. Let F be a subset of N. Put

$$k = \begin{cases} 1, & \text{if } p = 2;\\ \frac{p-1}{2}, & \text{when } p > 2,\end{cases}$$

and $A = \{a_n\colon\ n \geqslant 1\}$. Define a mapping $x_F : A \to \mathbb{T}$ in the following way:

(1) $x_F(a_1) = \mathbf{0}$.
(2) $x_F(a_{2^n}) = \begin{cases} \frac{x_F(a_{2^{n-1}})}{p^{2^{n-1}}}, & \text{if } 2^n \in F;\\ \frac{x_F(a_{2^{n-1}})}{p^{2^{n-1}}} + (\frac{k}{p} + \mathbb{Z}), & \text{otherwise,}\end{cases}$ when $n \geqslant 1$.
(3) $x_F(a_m) = p^{2^n - m} x_F(a_{2^n})$, when $2^{n-1} \leqslant m \leqslant 2^n$.

It is easy to check that $p x_F(a_1) = \mathbf{0}$ and $p x_F(a_{n+1}) = x_F(a_n)$ for $n \geqslant 1$. This guarantees that x_F can be extended to a homomorphism $\bar{x}_F : \mathbb{Z}(p^\infty) \to \mathbb{T}$.

It remains to verify that $\bar{x}_F$ satisfies (i) and (ii). For this, we have to distinguish two cases.

Case 1. $p = 2$. First, note that $\|\frac{1}{m}(r + \mathbb{Z})\| = \frac{1}{m}\|r + \mathbb{Z}\|$, for all $m > 0$ and $0 \leqslant r \leqslant \frac{1}{2}$. So, by items (1) and (2), it follows that

$$\bigl\|x_F(a_{2^n})\bigr\| = \begin{cases} \frac{\|x_F(a_{2^{n-1}})\|}{2^{2^{n-1}}}, & \text{if } 2^n \in F;\\ \frac{1}{2} - \frac{\|x_F(a_{2^{n-1}})\|}{2^{2^{n-1}}}, & \text{otherwise,}\end{cases} \tag{$\star$}$$

when $n \geqslant 1$. Clearly $\frac{\|x_F(a_{2^{n-1}})\|}{2^{2^{n-1}}} < \min\{\frac{1}{2^n}, \frac{1}{4}\}$, for every $n \geqslant 1$. Therefore, it follows from ($\star$) that $\bar{x}_F$ satisfies (i) and (ii).

Case 2. $p > 2$. Similarly to the previous case, it is easy to see that

$$\bigl\|x_F(a_{2^n})\bigr\| = \begin{cases} \frac{\|x_F(a_{2^{n-1}})\|}{p^{2^{n-1}}}, & \text{if } 2^n \in F;\\ \frac{k}{p} + \frac{\|x_F(a_{2^{n-1}})\|}{p^{2^{n-1}}}, & \text{otherwise,}\end{cases} \tag{$\star\star$}$$

when $n \geqslant 1$. Also $\frac{\|x_F(a_{2^{n-1}})\|}{p^{2^{n-1}}} < \min\{\frac{1}{2^n}, \frac{1}{4}\}$, for every $n \geqslant 1$. Therefore, it follows from ($\star\star$) that also $\bar{x}_F$ satisfies (i) and (ii) for this case. □

Let N be as in Lemma 4.9 and let $\mathcal{F} \subseteq [N]^\omega$ be a centered family without an infinite pseudointersection. Then, by Lemma 4.9, for every $F \in \mathcal{F}$ there is $\bar{x}_F \in (\mathbb{Z}(p^\infty)_d)^*$ such that $\bar{x}_F$ satisfies (i) and (ii) of this lemma. Let $X = \{\bar{x}_F\colon\ F \in \mathcal{F}\}$ and we show that X is not $\gamma_{\mathbb{Z}(p^\infty)}$-set. Clearly $|X| = |\mathcal{F}|$.

Fix $m > 0$. Let $\{F_i\colon\ i<k\}$ be a finite subset of $\mathcal{F}$. Since $\mathcal{F}$ is a centered family, there is $n \in \bigcap_{i<k} F_i$ such that $n > m$. By Lemma 4.9(ii), it follows that $\|\bar{x}_{F_i}(a_n)\| < \frac{1}{n} < \frac{1}{m}$ for every $i < k$, and thus $\{\bar{x}_{F_i}\colon\ i<k\} \subseteq U^m_{a_n}$. Therefore, $\mathcal{U}^m_{A_N}$ is an ω-cover of X for every $m > 0$, where $A_N = \{a_n\colon\ n \in N\}$. Assume now that B is an infinite subset of A_N and verify that $\mathcal{U}^4_B$ is not a γ-cover of X. Aiming at a contradiction, suppose that for every $F \in \mathcal{F}$, $\bar{x}_F \in U^4_b$ for all but finitely many $b \in B$. Put $E = \{n\colon\ a_n \in B\}$. By Lemma 4.9(i), it follows that $E \subseteq^* F$ for every $F \in \mathcal{F}$, which contradicts our assumption about $\mathcal{F}$. □

Claim 4.10. $\mathsf{non}(\gamma_{\mathbb{Z}}\text{-}set) = \mathfrak{p}$.

Proof of Claim 4.10. It is well known that the dual group $(\mathbb{Z}_d)^*$ is isomorphic to $\mathbb{T}$. Thus, each $x \in \mathbb{T}$ can be identified with the homomorphism of $\mathbb{Z}$ to $\mathbb{T}$ defined by $x(n) := nx$ for every $n \in \mathbb{Z}$.

To establish the claim, we prove the following lemma.

Lemma 4.11. *Let $A = \{2^{4^n}\colon\ n \in \omega\}$. Then, for every $F \subseteq A$ there is $x_F \in \mathbb{T}$ with the following properties*:

(i) *$nx_F \in T_4$ if and only if $n \in F$; and*
(ii) *$nx_F \in T_n$ if $n \in F$.*

Proof. Let F be a subset of A and put $E = \{n\colon\ 2^{4^n} \in F\}$. For each $n \in \omega$ consider the interval $J_n = [-\frac{1}{2^{4^n}}, \frac{1}{2^{4^n}}]$, and for every $s \in 2^{<\omega}$ put

$$I_s = \frac{1}{2} + \sum_{i<|s|} \frac{1-s(i)}{2^{4^i+1}} + J_{|s|}.$$

These intervals satisfy the following properties:

(a) if $s \subseteq t$, then $I_t \subseteq I_s$;
(b) $I_s + \mathbb{Z} \subseteq U^4_{2^{4^i}}$ if and only if $s(i) = 1$;
(c) if $s(i) = 1$, then $I_s + \mathbb{Z} \subseteq U^{2^{4^i}}_{2^{4^i}}$.

Indeed, to see (a), first note that

$$I_t = \frac{1}{2} + \sum_{i<|s|} \frac{1-s(i)}{2^{4^i+1}} + \overbrace{\sum_{|s|\leqslant i<|t|} \frac{1-t(i)}{2^{4^i+1}} + J_{|t|}}^{(\star)},$$

and to verify that $(\star) \subseteq J_{|s|}$, it suffices to show that

$$\frac{1}{2^{4^{|t|}}} + \sum_{|s|\leqslant i<|t|} \frac{1-t(i)}{2^{4^i+1}} \leqslant \frac{1}{2^{4^{|s|}}}.$$

Without loss of generality we can assume that $|s| < |t|$. Thus $|t| \geqslant |s|+1$ and since $4^{|s|+i} - 1 \geqslant 4^{|s|} + i$ for all $i \geqslant 1$, it follows that

$$\begin{aligned} \frac{1}{2^{4^{|t|}}} + \sum_{|s|\leqslant i<|t|} \frac{1-t(i)}{2^{4^i+1}} &\leqslant \frac{1}{2^{4^{|s|+1}}} + \frac{1}{2^{4^{|s|}+1}} + \frac{1}{2^2}\sum_{i\geqslant 1} \frac{1}{2^{4^{|s|+i}-1}} \\ &\leqslant \frac{1}{2^2}\cdot\frac{1}{2^{4^{|s|}}} + \frac{1}{2}\cdot\frac{1}{2^{4^{|s|}}} + \frac{1}{2^2}\cdot\frac{1}{2^{4^{|s|}}}\sum_{i\geqslant 1}\frac{1}{2^i} = \frac{1}{2^{4^{|s|}}}. \end{aligned}$$

To prove (b) and (c), note that

$$U^m_n = \{x \in \mathbb{T}\colon\ nx \in T_m\} = \sum_{k\in n}\left(\frac{k}{n} + I_{n,m}\right) + \mathbb{Z},$$

where $I_{n,m} = (-\frac{1}{nm}, \frac{1}{nm})$. Suppose that $s(i) = 1$. Then

$$I_{s\restriction_{i+1}} = \frac{1}{2} + \sum_{j<i} \frac{1-s(j)}{2^{4^j+1}} + J_{i+1}.$$

Clearly, $\frac{1}{2} + \sum_{j<i} \frac{1-s(j)}{2^{4^j+1}} = \frac{k}{2^{4^i}}$, for some $k \in 2^{4^i}$. Since $\frac{1}{2^{4^{i+1}}} < \min\{\frac{1}{4} \cdot \frac{1}{2^{4^i}}, \frac{1}{2^{4^i}} \cdot \frac{1}{2^{4^i}}\}$, it follows that $I_s + \mathbb{Z} \subseteq I_{s\restriction_{i+1}} + \mathbb{Z} \subseteq U^4_{2^{4^i}} \cap U^{2^{4^i}}_{2^{4^i}}$. This prove (c) and the first part of (b).

For the second part of (b), suppose that $s(i) = 0$. Let $k \in 2^{4^i}$ such that $\frac{k}{2^{4^i}} = \frac{1}{2} + \sum_{j<i} \frac{1-s(j)}{2^{4^j+1}}$. Then

$$I_{s\restriction_{i+1}} = \frac{k}{2^{4^i}} + \frac{1}{2} \cdot \frac{1}{2^{4^i}} + J_{i+1}.$$

Since $\frac{1}{2^{4^{i+1}}} < \frac{1}{4} \cdot \frac{1}{2^{4^i}}$, it follows that $\frac{1}{2} \cdot \frac{1}{2^{4^i}} + J_{i+1} \subseteq [\frac{1}{4} \cdot \frac{1}{2^{4^i}}, \frac{3}{4} \cdot \frac{1}{2^{4^i}}]$. Then $(I_{s\restriction_{i+1}} + \mathbb{Z}) \cap U^4_{2^{4^i}} = \emptyset$, and hence also $(I_s + \mathbb{Z}) \cap U^4_{2^{4^i}} = \emptyset$.

By item (a), take an $x_F \in \bigcap_{n\in\omega} I_{\chi_E\restriction_n} + \mathbb{Z}$, where χ_E is the characteristic function of E. Therefore, it follows from (b) and (c) that x_F satisfies (i) and (ii). □

Let A be as in Lemma 4.11 and let $\mathcal{F} \subseteq [A]^\omega$ be a centered family without an infinite pseudointersection. Then, by Lemma 4.11, for every $F \in \mathcal{F}$ there is $x_F \in \mathbb{T}$ such that x_F satisfies (i) and (ii) of this lemma. Let $X = \{x_F\colon F \in \mathcal{F}\}$ and we show that X is not $\gamma_{\mathbb{Z}}$-set. Clearly $|X| = |\mathcal{F}|$.

Fix $m > 0$. Let $\{F_i\colon i < k\}$ be a finite subset of $\mathcal{F}$. Since $\mathcal{F}$ is a centered family, there is $n \in \bigcap_{i<k} F_i$ such that $n > m$. By Lemma 4.11(ii), it follows that $\|nx_{F_i}\| < \frac{1}{n} < \frac{1}{m}$ for every $i < k$, and thus $\{x_{F_i}\colon i < k\} \subseteq U^m_n$. Therefore, $\mathcal{U}^m_A$ is an ω-cover of X for every $m > 0$. Assume now that B is an infinite subset of A and verify that $\mathcal{U}^4_B$ is not a γ-cover of X. Aiming at a contradiction, suppose that for every $F \in \mathcal{F}$, $x_F \in U^4_b$ for all but finitely many $b \in B$. By Lemma 4.11(i), it follows that $B \subseteq^* F$ for every $F \in \mathcal{F}$, which contradicts our assumption about $\mathcal{F}$. □

The theorem is proved. □

5. Final remarks and questions

It has been suggested to us by the anonymous referee to consider also the question of existence of non-metrizable Fréchet group topologies on non-Abelian groups.

Definition 5.1. ([23]) For every countable topologizable group G, let $\mathfrak{p}_G$ denote the minimum character of a non-discrete Hausdorff group topology on G which cannot be refined to a non-discrete metrizable group topology.

Theorem 5.2. *([23]) For every countable topologizable group G, $\mathfrak{p}_G = \mathfrak{p}$.* □

In particular, every countable group admitting a non-discrete Hausdorff group topology admits a non-metrizable one.

Theorem 5.3 ($\mathfrak{p} > \omega_1$). *Every countable topologizable group admits a non-metrizable group topology which is Fréchet.*

Proof. It is a theorem of **ZFC** that every countable group which admits a group topology which is not metrizable (of uncountable weight) also admits a group topology of weight ω_1. To see this fix an arbitrary group topology τ on a countable group G of uncountable weight. Recursively choose countably generated filters $\mathcal{F}_\alpha$, $\alpha < \omega_1$ contained in the filter of τ-neighborhoods of e_G so that

(i) each $\mathcal{F}_\alpha$ satisfies the axioms of a neighborhood filter of e_G in a Hausdorff group topology on G, and
(ii) for $\alpha < \beta$ the filter $\mathcal{F}_\alpha$ is properly contained in the filter $\mathcal{F}_\beta$.

Then $\mathcal{F} = \bigcup_{\alpha<\omega_1} \mathcal{F}_\alpha$ is a filter of neighborhoods of e_G in a Hausdorff group topology by (i) and the topology has weight ω_1 by (ii). On the other hand, it is well known that every countable space of weight less than $\mathfrak{p}$ is Fréchet [15]. □

Note that if the original group topology is precompact then so is the resulting Fréchet group topology. In particular, assuming $\mathfrak{p} > \omega_1$ every countable group which admits a non-metrizable precompact group topology also admits a non-metrizable precompact Fréchet group topology. Thus, the results of Ol'šanskiĭ and his school [21] are the only hindrance to a possible existence of non-metrizable Fréchet group topologies.

Question 5.4. Is it consistent with **ZFC** that every $\gamma_{\mathbb{Z}_\omega}$-set is countable, or equivalently, is it consistent with **ZFC** that every countable Abelian precompact Fréchet group is metrizable?[3]

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[3] The authors have recently answered the question in the affirmative.

We believe that methods developed in [4] could be of use here.

Question 5.5. Is there an uncountable $\gamma_{\mathbb{Z}_\omega}$-set in the Laver model?

It seems unlikely, but at the moment we do not know whether the existence of a countable non-metrizable Fréchet topological group implies the existence of an uncountable γ-set.

Question 5.6. Is it consistent with **ZFC** that every γ-set is countable but there is a countable non-metrizable Fréchet topological group?

We do not even know, whether the existence of an uncountable γ_G-set implies the existence of an uncountable γ-set.

Question 5.7. Is it consistent with **ZFC** that every γ-set is countable but there is an uncountable γ_G-set for some group G?

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