Pseudocompactness of hyperspaces✩

Michael Hrušák a,*, Fernando Hernández-Hernández b, Iván Martínez-Ruiz c

a Instituto de Matemáticas, UNAM Morelia, Mich., Mexico
b Facultad de Ciencias Físico Matemáticas, UMSNH Morelia, Mich., Mexico
c Instituto de Física y Matemáticas, UMSNH Morelia, Mich., Mexico

Received 12 November 2006; received in revised form 1 July 2007; accepted 4 July 2007

Abstract

We consider the following question of Ginsburg: Is there any relationship between the pseudocompactness of $X^\omega$ and that of the hyperspace $2^X$? We do that first in the context of Mrówka–Isbell spaces $\Psi(A)$ associated with a maximal almost disjoint (MAD) family $A$ on $\omega$ answering a question of J. Cao and T. Nogura. The space $\Psi(A)^\omega$ is pseudocompact for every MAD family $A$. We show that

1. $(p = c) 2^\Psi(A)$ is pseudocompact for every MAD family $A$.
2. $(h < c)$ There is a MAD family $A$ such that $2^\Psi(A)$ is not pseudocompact.

We also construct a ZFC example of a space $X$ such that $X^\omega$ is pseudocompact, yet $2^X$ is not.

© 2007 Elsevier B.V. All rights reserved.

MSC: 54C65; 54B20; 03E0

Keywords: Hyperspace; Vietoris topology; $\Psi$-space; Mrówka–Isbell space; MAD family; Pseudocompact

1. Introduction

The hyperspace of a space $X$ (denoted by $2^X$) consists of all closed nonempty subsets of $X$. We consider $2^X$ equipped with the *Vietoris topology*, i.e., the topology generated by sets of the form:

$\langle U; V_0, \ldots, V_n \rangle = \{ F \in 2^X : F \subseteq U \text{ and } F \cap V_i \neq \emptyset \text{ for every } i \leq n \},$

where $U, V_0, \ldots, V_n$ are nonempty open subsets of $X$. The equivalence between compactness of a space and its hyperspace was established by Michael and Vietoris. It is natural to ask if there is a similar relationship with respect

✩ Authors gratefully acknowledge support from PAPIIT grant IN106705 and CONACyT grant 46337. The first author’s research was also partially supported by GA ČR grant 201-03-0933. Third author is supported by the Consejo Nacional de Ciencia y Tecnología CONACyT, Mexico. Scholarship 189295.

* Corresponding author.

E-mail addresses: michael@matmor.unam.mx (M. Hrušák), fernando@matmor.unam.mx (F. Hernández-Hernández), ivan@matmor.unam.mx (I. Martínez-Ruiz).

0166-8641/$ – see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2007.07.001
to weaker compactness type properties, in particular, with respect to countable compactness and pseudocompactness. Recall that a space \( X \) is pseudocompact if every continuous real-valued function \( f \) on \( X \) is bounded.\(^1\) J. Ginsburg obtained some partial results:

**Theorem 1.1.** ([6])

(a) If every power of a space \( X \) is countably compact then so is \( 2^X \).
(b) If \( 2^X \) is countably compact (pseudocompact) then so is every finite power of \( X \).

Ginsburg also presented an example of a completely regular space such that every finite power is countably compact, hence pseudocompact, but whose hyperspace is not pseudocompact. He asked: *Is there any relationship between the pseudocompactness of \( X^\omega \) and that of the hyperspace \( 2^X \)?* His question was brought to our attention by J. Cao, T. Nogura and A. Tomita and their article [4] in which they considered Ginsburg’s question and gave the following partial answer:

**Theorem 1.2.** ([4]) If \( X \) is a homogeneous Tychonoff space such that \( 2^X \) is pseudocompact then \( X^\omega \) is pseudocompact.

J. Cao and T. Nogura, in a private conversation, explicitly asked whether \( 2^X \) is pseudocompact for some/every Mrówka–Isbell space \( X \).

In this note we answer Cao and Nogura’s question by showing that \( p = c \) implies that \( 2^\Psi(A) \) is a pseudocompact space for every MAD family \( A \), while \( h < c \) implies that there is a MAD family \( A \) for which \( 2^\Psi(A) \) is not pseudocompact. We also construct a ZFC example of a space \( X \) such that \( X^\omega \) is pseudocompact, yet \( 2^X \) is not pseudocompact. This answers Ginsburg question in one direction and shows that the equivalent of (a) of Theorem 1.1 does not hold for pseudocompactness.

1 Many authors (e.g., [5]) consider pseudocompactness only for Tychonoff spaces. However, most of our hyperspaces are not Tychonoff. In fact, a hyperspace \( 2^X \) is a Tychonoff space if and only if \( X \) is normal (see [5, p. 121]). For Tychonoff spaces, pseudocompactness is equivalent to the fact that there is no infinite discrete family of open subsets of \( X \). This is in general not true for Hausdorff spaces. Nevertheless, in the context of hyperspaces all the “standard” definitions of pseudocompactness coincide (see [6]).

2. Preliminaries

An infinite family \( A \subseteq [\omega]^\omega \) is almost disjoint (AD) if every two distinct elements of \( A \) have finite intersection. A family \( A \) is maximal almost disjoint (MAD) if it is almost disjoint and maximal with this property.

**Definition 2.1.** Let \( A \) be an AD family. The *Mrówka–Isbell space* \( \Psi(A) \) associated to \( A \) is defined as follows: The underlying set is \( \omega \cup A \), all elements of \( \omega \) are isolated, and basic neighborhoods of \( A \in A \) are of the form \( \{A\} \cup (A \setminus F) \) for some finite set \( F \).

It follows immediately from the definition that \( \Psi(A) \) is a first countable, locally compact space. If \( A \) is infinite then \( \Psi(A) \) is not countably compact and \( \Psi(A) \) is pseudocompact if and only if \( A \) is a MAD family, see, e.g., [9].

Recall that a subset \( A \) of topological space \( X \) is relatively countably compact in \( X \) if every \( E \in [A]^\omega \) has an accumulation point in \( X \). One way to show that a space is pseudocompact is to show that it has a dense relatively countably compact subspace. The following lemma is easy to prove.

**Lemma 2.2.** Let \( X \) have a dense set \( D \) of isolated points. Then the following are equivalent:

1. \( X \) is pseudocompact.
2. \( D \) is relatively countably compact in \( X \).

---

\(^1\) Many authors (e.g., [5]) consider pseudocompactness only for Tychonoff spaces. However, most of our hyperspaces are not Tychonoff. In fact, a hyperspace \( 2^X \) is a Tychonoff space if and only if \( X \) is normal (see [5, p. 121]). For Tychonoff spaces, pseudocompactness is equivalent to the fact that there is no infinite discrete family of open subsets of \( X \). This is in general not true for Hausdorff spaces. Nevertheless, in the context of hyperspaces all the “standard” definitions of pseudocompactness coincide (see [6]).
A subset $A$ of a topological space $X$ is relatively sequentially compact in $X$ if every sequence of elements of $A$ has a subsequence which is convergent in $X$. If $X$ has a dense set which is even relatively sequentially compact then all powers of $X$ are pseudocompact.

**Proposition 2.3.** If $X$ has a dense subset which is relatively sequentially compact in $X$, then $X^\omega$ is pseudocompact.

**Proof.** Assume $D \subseteq X$ is relatively sequentially compact and dense. The same proof (see, e.g., [9, Theorem 6.9]) that shows that countable product of sequentially compact spaces is sequentially compact, shows that $D^\omega$ is a dense relatively sequentially compact subspace of $X^\omega$, which directly implies pseudocompactness of $X^\omega$. □

As a direct consequence we get:

**Lemma 2.4.** $(\Psi (A))^\omega$ is pseudocompact for every MAD family $A$.

Let $\text{Fin}$ denote the set of all nonempty finite subsets of $\omega$. The following lemma, which is easy to prove, will be used frequently.

**Lemma 2.5.** If $X$ is a topological space such that $\omega$ is the dense set of isolated points of $X$, then $\text{Fin}$ is a dense set of isolated points in $2^X$.

### 3. Hyperspaces of Mrówka–Isbell spaces under MA$_\sigma$-centered

In this section we show that it is consistent with ZFC that $2^{\Psi (A)}$ is pseudocompact for every MAD family $A$.

We say that a set $A$ is almost contained in $B$, $A \subseteq^* B$, if $A \setminus B$ is finite. $A =^* B$ means $A \subseteq^* B$ and $B \subseteq^* A$. Recall that a family $\mathcal{F} \subseteq [\omega]^\omega$ is centered if the intersection of any finite subset of $\mathcal{F}$ is infinite. The pseudo-intersection number $\mathfrak{p}$ is defined as the minimal size of a centered family $\mathcal{F} \subseteq [\omega]^\omega$ without a pseudo-intersection, i.e., the minimal size of a centered family $\mathcal{F} \subseteq [\omega]^\omega$ such that for every $A \in [\omega]^\omega$ there is an $F \in \mathcal{F}$ such that $A \setminus F$ is infinite. By a theorem of M.G. Bell [1], the assumption $\mathfrak{p} = \mathfrak{c}$ is equivalent to Martin’s Axiom for $\sigma$-centered partial orders.

We introduce the following notation: Let $A \subseteq \mathcal{P}(\omega)$ be an almost disjoint family. Given a one-to-one sequence $Y = \langle F_n : n \in \omega \rangle \subseteq \text{Fin}$ and $A \subseteq \omega$, let

- $I_A = \{n \in \omega : A \cap F_n \neq \emptyset\}$,
- $M_A = \{n \in \omega : F_n \subseteq A\}$.

If, moreover, $F \in 2^{\Psi (A)}$ let

- $\mathcal{F}_F = \{I_{A \setminus k} : A \in F \cap A, \ k \in \omega\} \cup \{I_{[n]} : n \in F \cap \omega\}$.

The next lemma is the main technical tool for this section.

**Lemma 3.1.** Given a one-to-one sequence $Y = \langle F_n : n \in \omega \rangle \subseteq \text{Fin}$ and $F \subseteq \Psi (A)$, the following are equivalent:

1. $F$ is an accumulation point of $Y$ in the hyperspace $2^{\Psi (A)}$.
2. For every $P \subseteq \omega$ that satisfies $F \cap \omega \subseteq P$ and $(\forall A \in F \cap A) (A \subseteq^* P)$, the family $\mathcal{F}_F \cup \{M_P\}$ is centered.

Note that, in particular, (2) implies that $\mathcal{F}_F$ is centered, as $P = \omega$ satisfies the hypothesis for any $Y$ and $F$.

**Proof.** Suppose that $F$ is an accumulation point of $Y$. Let $P \subseteq \omega$ be such that $F \cap \omega \subseteq P$ and $A \subseteq^* P$ for all $A \in F \cap A$. Then $V = P \cup (F \cap A)$ is an open subset of $\Psi (A)$ which contains $F$. To see that $\mathcal{F}_F \cup \{M_P\}$ is centered let $Q = \{I_{A_0 \setminus k_0}, \ldots, I_{A_m \setminus k_m}\} \cup \{I_{[a_0]}, \ldots, I_{[a_1]}\} \cup \{M_P\} \subseteq \mathcal{F}_F \cup \{M_P\},$
where \( A_i \in F \cap \mathcal{A}, \ k_i \in \omega \) for all \( i \leq m \) and \( a_i \in F \cap \omega \) for all \( i \leq l \). Then
\[
U = \{ V; \{ A_0 \} \cup A_0 \setminus k_0, \ldots, \{ A_m \} \cup A_m \setminus k_m, \{ a_0 \}, \ldots, \{ a_l \} \}
\]
is a neighborhood of \( F \) in \( 2^{\Psi(A)} \) and therefore \( Y \cap U \) is infinite. There is an \( I \in [\omega]^\omega \) such that \( (A_j \setminus k_j) \cap F_i \neq \emptyset \) and \( \{ \alpha_i \} \cap F_i \neq \emptyset \) for all \( i \in I \) and all \( j \leq m, \ k \leq l \). By the definition of \( I_F \) and \( M_P \), it follows that \( I \subseteq \bigcap Q \) and so \( \mathcal{F}_F \cup \{ M_P \} \) is centered.

Conversely, assume that \( \mathcal{F}_F \cup \{ M_P \} \) is centered and let
\[
U = \{ V; \{ A_0 \} \cup A_0 \setminus k_0, \ldots, \{ A_m \} \cup A_m \setminus k_m, \{ a_0 \}, \ldots, \{ a_l \} \}
\]
be a basic neighborhood of \( F \) in \( 2^{\Psi(A)} \). Let \( P = V \cap \omega \). Since \( F \subseteq V, F \cap \omega \subseteq P \) and \( A \subseteq^* P \) for all \( A \in F \cap \mathcal{A} \). As \( \mathcal{F}_F \cup \{ M_P \} \) is centered,
\[
\bigcap \{ I_{A \setminus k_i}; \ i \leq m \} \cap \bigcap \{ I_{\alpha_i}; \ i \leq l \} \cap \{ M_P \}
\]
is infinite. Therefore so is \( U \cap Y \). This shows that \( F \) is an accumulation point of \( Y \). \( \square \)

**Theorem 3.2.** \((p = c) \ 2^{\Psi(A)} \) is pseudocompact for every MAD family \( A \).

**Proof.** By Lemmas 2.5 and 2.2, it suffices to show that given a MAD family \( A \), every one-to-one sequence \( Y = \langle F_n; n \in \omega \rangle \subseteq \text{Fin} \) has an accumulation point in \( 2^{\Psi(A)} \). Consider such a \( Y \) and let \( \{ P_\alpha; \ \alpha < c \} \) be an enumeration of \( [\omega]^\omega \), where each element is listed \( c \)-many times and \( P_0 = \omega \). Recursively construct a family \( \{ E_\alpha; \ \alpha < c \} \) such that, for every \( \alpha < c \):

1. \( E_\alpha \subseteq \Psi(A) \),
2. \( |E_\alpha| \leq |\alpha| + \omega \),
3. \( \alpha \subseteq \beta \) implies \( E_\alpha \subseteq E_\beta \),
4. \( \mathcal{F}_\alpha = \{ I_{A \setminus k_i}; A \in E_\alpha \cap \mathcal{A}, k \in \omega \} \cup \{ I_{\alpha_i}; n \in E_\alpha \cap \omega \} \) is centered, and
5. one of the following occurs:
   
   a. \( \{ E_\alpha \cap \omega \} \setminus P_\alpha \neq \emptyset \),
   b. there is an \( A \in E_\alpha \cap \mathcal{A} \) such that \( A \nsubseteq^* P_\alpha \), or
   c. \( \mathcal{F}_\alpha \cup \{ M_{P_\alpha} \} \) is centered.

The accumulation point of \( Y \) is going to be the closure of \( \bigcup_{\alpha < c} E_\alpha \). Next we will carry out the recursive construction.

There is an \( A \in \mathcal{A} \), such that for every \( k \in \omega \) there is an \( n \in \omega \) such that \( (A \setminus k) \cap F_n \neq \emptyset \). Let \( E_0 = \{ A \} \). As \( P_0 = \omega \), (1)–(5) hold.

Assume that \( 0 < \alpha < c \) and that \( E_\beta \) has been constructed for all \( \beta < \alpha \). Since \( \mathcal{F}_\beta \) is centered for all \( \beta < \alpha \), so is \( \mathcal{F}_\beta = \bigcup_{\beta < \alpha} \mathcal{F}_\beta \). Now, if \( \mathcal{F}_\alpha \cup \{ M_{P_\alpha} \} \) is centered, then letting \( E_\alpha = \bigcup_{\beta < \alpha} E_\beta \) all properties (1)–(5) are satisfied. If \( \mathcal{F}_\alpha \cup \{ M_{P_\alpha} \} \) is not centered, then as \( \mathcal{F} \) is centered and \( |\mathcal{F}| < p \), there is a \( J \in [\omega]^\omega \) almost contained in all elements of \( \mathcal{F} \) such that \( |J \cap M_{P_\alpha}| = \emptyset \). Consider the following two cases:

**Case 1:** There is an \( m \in \omega \setminus P_\alpha \) such that \( \{ n \in J; m \in F_n \} \) is infinite.

Let \( E_\alpha = \bigcup_{\beta < \alpha} E_\beta \cup \{ m \} \). All clauses but (4) are evidently true. To see that the fourth clause also holds for \( E_\alpha \), take
\[
G = \{ I_{A_0 \setminus k_0}, \ldots, I_{A_i \setminus k_i}, I_{\alpha_0}, \ldots, I_{\alpha_i}, I_{[m]} \} \subseteq \mathcal{F} \cup \{ I_{[m]} \}.
\]
Since \( \{ n \in J; m \in F_n \} \subseteq^* F \) for all \( F \in \mathcal{F} \), then
\[
\{ n \in J; m \in F_n \} \subseteq^* \bigcup_{i \leq s} \{ I_{A_i \setminus k_i} \cap I_{\alpha_i} \cap I_{[m]} \},
\]
thus \( \bigcup G \) is infinite and therefore \( \mathcal{F}_\alpha \) is centered.

**Case 2:** \( \{ n \in \omega; m \in F_n \} \) is finite for all \( m \in \omega \setminus P_\alpha \).
It follows that \( \bigcup_{n<\omega} F_n \setminus P_\alpha \) is infinite and hence there is \( A \in \mathcal{A} \) such that \( |A \cap \bigcup_{n<\omega} F_n| = \omega \). In this case let \( E_\alpha = \bigcup_{\beta<\alpha} E_\beta \cup \{A\} \). Again, only clause (4) requires verification. Take

\[ G = \{I_{A_0}\mid k_0, \ldots, I_{A_j}\mid k_j, I_{[a_0]}, \ldots, I_{[a_l]}, I_{A_j}\mid k\} \subseteq \mathcal{F}. \]

As the set \( \bigcup_{n<\omega} F_n \setminus P_\alpha \) is infinite, so is the set \( \{n \in J : A \cap F_n \neq \emptyset\} \). Moreover,

\[ \{n \in J : A \cap F_n \neq \emptyset\} \subseteq J \subseteq \forall F \]

for all \( F \in \mathcal{F} \). So, \( \{n \in J : A \cap F_n \neq \emptyset\} \subseteq \bigcap G \) and therefore \( \mathcal{F}_\alpha \) is centered. This completes the construction of the sets \( E_\alpha \).

Let \( E \) be the closure (in \( \Psi(\mathcal{A}) \)) of \( \bigcup_{\alpha<\omega} E_\alpha \). We claim that \( E \) is an accumulation point of \( Y \) in \( 2^{\Psi(\mathcal{A})} \). By Lemma 3.1, it suffices to show that for every \( P \subseteq \omega \) one of the following holds:

1. \((E \cap \omega) \setminus P \neq \emptyset.
2. \text{there is } A \in E \cap \mathcal{A} \text{ such that } |A \setminus P| = \aleph_0, \text{ or}
3. \( \mathcal{F}_E \cup \{MP\} \) is centered.

First we show that \( \mathcal{F}_E \) is centered. Clearly \( \mathcal{F} = \bigcup_{\alpha<\omega} \mathcal{F}_\alpha \) is centered and \( E \setminus \bigcup_{\alpha<\omega} E_\alpha \subseteq \mathcal{A} \), since all elements of \( \omega \) are isolated. Moreover, \( (A \setminus k) \cap \bigcup_{\alpha<\omega} E_\alpha \neq \emptyset \) for each \( A \in E \setminus \bigcup_{\alpha<\omega} E_\alpha \) and each \( k \in \omega \). Thus, given any \( A \in E \setminus \bigcup_{\alpha<\omega} E_\alpha \) and \( k \in \omega \), there is an \( m \in (A \setminus k) \cap \bigcup_{\alpha<\omega} E_\alpha \). Note that for this \( m \) we have that \( I_{[m]} \subseteq I_{A \setminus k} \). This implies that for all \( F \in \mathcal{F}_E \) there is an \( G \in \mathcal{F} \) such that \( G \subseteq F \). As \( \mathcal{F} \) is centered, so is \( \mathcal{F}_E \).

Finally, consider \( P \subseteq \omega \) and assume that \( \mathcal{F}_E \cup \{MP\} \) is not centered. There are \( A_0, \ldots, A_n \in E \cap \mathcal{A}, k_0, \ldots, k_n \in \omega \) and \( m_0, \ldots, m_k \in E \cap \omega \) such that \( \bigcap_{i \leq n} I_{A_i \setminus k_i} \cap \bigcap_{i \leq k} I_{[m_i]} \cap MP = \emptyset \).

For each \( i \leq n \) such that \( A_i \in E \setminus \bigcup_{\alpha<\omega} E_\alpha \) there is an ordinal \( \alpha_i < \omega \) and there is an \( \alpha_i \in E_{\alpha_i} \) such that \( I_{[m_i]} \subseteq I_{A_i \setminus k_i} \), as we saw in the previous paragraph. Choose \( \beta < \omega \) greater than all the \( \alpha_i \)’s and such that \( A_j \in E_\beta \) for all elements \( A_j, j \leq n \), that are members of \( \bigcup_{\alpha<\omega} E_\alpha \). Let \( \alpha < \omega \) be such that \( P = P_\alpha \) and \( \alpha > \beta \). It follows that \( \mathcal{F}_\alpha \cup \{MP\} \) is not centered either. Therefore, \( \bigcup_{\alpha<\omega} E_\alpha \setminus P \neq \emptyset \) or there is \( A \in E_\alpha \subseteq E \) such that \( |A \setminus P| = \aleph_0 \). This completes the proof.

4. Non-pseudocompact \( 2^{\Psi(\mathcal{A})} \)

In this section we will provide a consistent example of a MAD family \( \mathcal{A} \) such that \( 2^{\Psi(\mathcal{A})} \) is not pseudocompact.

Recall that \( D \subseteq [\omega]^{\omega} \) is dense if for every \( B \in [\omega]^{\omega} \) there is \( D \in D \) such that \( D \subseteq^* B \). The distributivity number \( \mathfrak{h} \) of \( [\omega]^{\omega} \) is defined as the minimal size of a collection of dense downward closed subsets of \( [\omega]^{\omega} \) whose intersection is empty. It is well known that \( p \leq \mathfrak{h} \leq \omega \) and that both inequalities are consistently strict.

**Theorem 4.1.** ([3]) There is a family \( \mathcal{T} \subseteq [\omega]^{\omega} \) such that

1. \( \mathcal{T} \) is a tree (ordered by \( \supseteq^* \)) of height \( \mathfrak{h} \).
2. Each level of \( \mathcal{T} \) is a maximal antichain in \( [\omega]^{\omega} \) (a MAD family).
3. Each \( D \in \mathcal{T} \) has \( \mathfrak{h} \)-many immediate successors.
4. \( \mathcal{T} \) is a dense subset of \( [\omega]^{\omega} \).

This is the base tree theorem of B. Balcar, J. Pelant and P. Simon.

**Theorem 4.2.** (\( \mathfrak{h} < \omega \)) There is a MAD family \( \mathcal{A} \) such that \( 2^{\Psi(\mathcal{A})} \) is not pseudocompact.

**Proof.** Fix a base tree \( \mathcal{T} \) of height \( \mathfrak{h} \) as in Theorem 4.1. For \( A \subseteq 2^{<\omega} \) let \( \pi_A = \{n \in \omega : A \cap 2^{<\omega} \neq \emptyset\} \). Let \( \mathcal{A} \subseteq [2^{<\mathfrak{h}}]^{\omega} \) be such that
Problem 4.3. Is there, in ZFC, a MAD family $\mathcal{A}$ such that $2^{\Psi(\mathcal{A})}$ is pseudocompact?

5. ZFC example

Ginsburg’s Theorem 1.1(a) says that $2^X$ is countably compact whenever every power of the space $X$ is countably compact. It is a known and easy to prove fact that, for a topological space $X$, the space $X^\omega$ is pseudocompact if and only if $X^\kappa$ is pseudocompact for every cardinal $\kappa$. In this section we construct a space $X$ (subspace of $\beta\omega$—the Čech–Stone compactification of $\omega$) such that $X^\omega$ is pseudocompact, yet $2^X$ is not. Therefore, a result analogous to Ginsburg’s does not hold for pseudocompactness.

Given an ordinal number $\alpha$, we denote the set of limit ordinals below $\alpha$ by $\operatorname{lim}(\alpha)$. As usual, we identify $\beta\omega$ with the set all ultrafilters on $\omega$ and, in particular, $\omega^* = \beta\omega \setminus \omega$ with the set of all free ultrafilters on $\omega$. For a set $A \subseteq \omega$, let $A^* = \{p \in \omega^* : A \in p\}$ and $\mathcal{A} = A \cup A^*$. Recall that given a topological space $X$, an ultrafilter $q \in \omega^*$, a point $x \in X$ and a sequence of points $\langle x_n : n \in \omega \rangle \subseteq X$, we say that $x$ is a $q$-limit of the sequence $\langle x_n : n \in \omega \rangle$, $x = q\text{-}\lim(x_n : n \in \omega)$, if for every neighborhood $U$ of $x$ the set $\{m \in \omega : x_m \in U\}$ is an element of $q$.

Theorem 5.1. There is a subspace $X$ of $\beta\omega$ such that $X^\omega$ is pseudocompact yet $2^X$ is not.

Proof. Enumerate all sequences of elements of $\omega^\omega$ by $\langle f_\alpha : \alpha \in \operatorname{lim}(\omega) \rangle$, where each $f_\alpha = \langle f_{\alpha,n} : n \in \omega \rangle$. Let $Y = \langle F_n : n \in \omega \rangle$, where $F_n = [2^n, 2^{n+1})$. Given $U \subseteq \omega$, let $\pi_U = \{n \in \omega : U \cap F_n \neq \emptyset\}$, and for an ultrafilter $p$, let $\pi(p) = \{\pi_U : U \in p\}$, and observe that $\pi(p)$ is an ultrafilter as well.

To carry out the construction we will choose, for $\alpha \in \operatorname{lim}(\omega)$, an ultrafilter $q_\alpha \in \omega^*$ and a set $X_\alpha = \{p_\alpha + m : m \in \omega\} \subseteq \beta\omega$ so that, for every $\alpha \in \operatorname{lim}(\omega)$ and $m \in \omega$:

1. $p_\alpha + m = q_\alpha\text{-}\lim(f_{\alpha,n}(m) : n \in \omega)$,
2. there is $U \in p_\alpha + m$ such that $U$ is a partial selector of $\{F_k : k \in \omega\}$, i.e., $|F_k \cap U| \leq 1$ for each $k \in \omega$,
3. for every $\beta < \alpha$, there is $U \in p_\alpha + m$ and $V \in p_\beta$ such that $\pi_U \cap \pi_V = \emptyset$.

Assume that this can be accomplished and let $X = \omega \cup \bigcup\{X_\alpha : \alpha < \operatorname{lim}(\omega)\}$. To show that $2^X$ is not pseudocompact it suffices to prove that $Y$ has no accumulation point in $2^X$. Aiming towards a contradiction, assume that $F \in 2^X$ is an accumulation point of $Y$.
Note that \( F \cap \omega = \emptyset \). If \( F \cap \omega \neq \emptyset \) let \( m \in F \cap \omega \). There is a unique \( k \in \omega \) such that \( m \in F_k \). Put \( W = \langle X; \{m\} \rangle \). Then \( W \) is a neighborhood of \( F \) and \( W \cap Y = \{F_k\} \). Thus \( F \) cannot be an accumulation point.

**Case 1:** \( F \) is countable.

By property (2) above, for every \( p \in F \) there is \( U_p \in p \) such that \( U_p \) is a partial selector of \( Y \). Choose \( K = \{x_m: m \in \omega\} \) such that \( x_m \in F_m \) for every \( m \in \omega \) and \( |U_p \cap K| < \omega \) for every \( p \in F \). Put

\[
W = \{H \in 2^X: H \cap \overline{K} = \emptyset\} = \langle X \setminus \overline{K} \rangle.
\]

Notice that \( |U_p \cap K| < \omega \) implies \( p \notin \overline{K} \) and thus \( W \) is a neighborhood of \( F \). Moreover, \( F_m \notin W \) as \( x_m \in \overline{K} \cap F_m \) for every \( m \in \omega \). Hence \( W \cap Y = \emptyset \), contradiction.

**Case 2:** \( F \) is uncountable.

Property (3) above ensures the existence of \( p, q \in F, U \subseteq p \) and \( V \subseteq q \) such that \( \pi_U \cap \pi_V = \emptyset \). Let \( W = \langle X; \overline{U} \cap X, \overline{V} \cap X \rangle \). Then \( W \) is a neighborhood of \( F \). But, \( F_k \notin W \) for every \( k \in \omega \), contradiction.

To show that \( X^\omega \) is pseudocompact, let \( \langle h_n: n \in \omega \rangle \) be a sequence of elements of \( \omega^\omega \). There exists \( \alpha \in \omega \) such that \( f_\alpha = \langle h_n: n \in \omega \rangle \). Define \( h \in X^\omega \) by \( h(m) = \text{q}_{\alpha}(m) \lim h(n) \). Clause (1) assures that \( h \) is the \( \text{q}_{\alpha} \)-limit of \( \langle h_n: n \in \omega \rangle \). Thus \( X^\omega \) is pseudocompact by Proposition 2.3.

To conclude the proof of the theorem, we show how to carry out the recursive construction satisfying the properties (1)–(3). Suppose we are at stage \( \alpha \) and that the sets \( X_\beta \) and the ultrafilters \( q_\beta \) have been chosen for \( \beta \in \lim(\alpha) \). Let \( g_m(n) = f_{\alpha,n}(m) \), for every \( m, n \in \omega \).

**Claim 1.** There exists \( C \in [\omega]^\omega \) such that:

(a) For every \( m \in \omega \) there is a \( k \in \omega \) such that \( g_m \upharpoonright (C \setminus k) \) is constant or \( g_m \upharpoonright (C \setminus k) \) is injective, and \( g_m[C \setminus k] \) is a partial selector of \( Y \),
(b) for every \( m \neq n \in \omega \), \( g_m \upharpoonright C = \ast g_n \upharpoonright C \) or \( g_m[C] \cap g_n[C] = \ast \emptyset \),
(c) for every \( \beta < \alpha \) and every \( m \in \omega \) such that \( g_m \upharpoonright C \) is not eventually constant, \( \pi_{g_m[C]} \cap \pi_V = \emptyset \) for some \( V \in p_\beta \).

**Proof.** In order to prove the claim, let

\[
N = \{\pi(p) \in \omega^\omega: (\exists \beta < \alpha)(p \in X_\beta)\}
\]

and recursively construct a decreasing sequence \( \{A_m: m \in \omega\} \subseteq [\omega]^\omega \) such that for every \( m \in \omega \):

1. \( g_m \upharpoonright A_m \) is constant or \( g_m \upharpoonright A_m \) is injective, and \( g_m[A_m] \) is a partial selector of \( Y \),
2. for every \( k < m \in \omega \), \( g_m \upharpoonright A_m = \ast g_k \upharpoonright A_m \) or \( g_m[A_m] \cap g_k[A_m] = \ast \emptyset \), and
3. \( (\pi_{g_m[A_m]})^\ast \cap N = \emptyset \).

Put \( A_{-1} = \omega \). To carry out the construction assume that \( \{A_k: k < m\} \) have been defined and consider the function \( g_m \). It is easy to find \( B \) an infinite subset of \( A_{m-1} \) such that (1) \( g_m \upharpoonright B \) is constant or \( g_m \upharpoonright B \) is injective, \( g_m[B] \) is a partial selector of \( Y \) and (2) for every \( k < m \in \omega \), \( g_m \upharpoonright B = \ast g_k \upharpoonright B \) or \( g_m[B] \cap g_k[B] = \ast \emptyset \). Since \( |N| < c \), \( N \) is nowhere dense in \( \beta_\omega \) (this follows from the existence of AD families of size \( c \), see [10]). Thus there is an infinite \( D \subseteq g_m[B] \) such that \( (\pi_D)^\ast \cap N = \emptyset \). Let \( A_m = g_m^{-1}[D] \). Conditions (1)–(3) are obviously satisfied.

To conclude the proof of the claim, choose \( C \in [\omega]^\omega \) such that \( C \subseteq \ast A_m \) for every \( m \in \omega \). The set \( C \) clearly satisfies the first two clauses. To see that it also satisfies the third, let \( \beta < \alpha \) and let \( m \in \omega \). Since \( C \subseteq \ast A_m \), then

\[
\pi(p_\beta) \notin (\pi_{g_m[C]})^\ast
\]

and hence there is \( V \in p_\beta \) such that \( \pi_V \cap \pi_{g_m[C]} = \emptyset \).

Choose \( q_\alpha \in (C)^\ast \) and let \( p_{\alpha + m} = q_\alpha \text{-lim}(g_m(n): n \in \omega) \in (g_m[C])^\ast \) for every \( m \in \omega \). Then \( q_\alpha \) and \( X_\alpha = \{p_{\alpha + m}: m \in \omega\} \) satisfy the properties (1)–(3). Indeed, the fact that (1) and (2) hold follows directly from the construction. To check (3), let \( \beta < \alpha \) and \( m \in \omega \). By (c) there is a \( V \in p_\beta \) such that \( \pi_U \cap \pi_V = \emptyset \), where

\[
U = g_m[C] \in p_{\alpha + m}. \quad \square
\]

**References**


