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LUZIN GAPS ARE NOT COUNTABLY PARACOMPACT

MICHAEL HRUŠÁK, CHARLES J. G. MORGAN, AND SAMUEL G. DA SILVA

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Abstract. In this paper we show that if an almost disjoint family $\mathcal{A}$ is a Luzin gap then the corresponding space $\Psi(\mathcal{A})$ is not countably paracompact. This answers a Question of the third author, posed in Vol. 25 of this journal, about the value of the cardinal invariant $\mathfrak{ncp}$ ([7]). By going a little further, we also show that countably paracompact Ψ-spaces of size ω_1 violate the condition of being Luzin gaps in a very strong way.

1. Preliminaries and Introduction

In what follows, ω denotes the set of all natural numbers (and the least limit ordinal). $[\omega]^\omega$ and $[\omega]^{<\omega}$ denote, respectively, the family of all infinite subsets of ω and the family of all finite subsets of ω. ω_1 denotes the first uncountable cardinal. For a given set X, $|X|$ denotes the cardinality of X.

A set $\mathcal{A} \subseteq [\omega]^\omega$ is said to be an *almost disjoint* (or *a.d.*) *family* if every pair of distinct elements of $\mathcal{A}$ has finite intersection. For every a.d. family $\mathcal{A}$ one may construct a corresponding Ψ-*space*, $\Psi(\mathcal{A})$, whose underlying set is given by $\mathcal{A} \cup \omega$. The points in ω are declared isolated and the basic neighbourhoods of a point $A \in \mathcal{A}$ are given by the sets $\{A\} \cup (A \setminus F)$ for $F \in [\omega]^{<\omega}$. Clearly, ω is a dense set of isolated points, $\mathcal{A}$ is a closed and discrete subset of $\Psi(\mathcal{A})$ and it is easy to check that $\Psi(\mathcal{A})$ is a Hausdorff zero-dimensional (thus, completely regular) first-countable locally compact separable space. Moreover, it is well-known that any Hausdorff, first countable, locally compact separable space whose set of non-isolated points is non-empty and discrete turns out to be homeomorphic to a Ψ-space (see [1], p.154). As in [5], for a given a.d. family $\mathcal{A}$ and any topological property $\mathcal{P}$ we will take the expression "$\mathcal{A}$ satisfies $\mathcal{P}$" as synonymous with "$\Psi(\mathcal{A})$ satisfies $\mathcal{P}$".

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It is not surprising that topological properties of Ψ-spaces are often characterized by combinatorial properties of a.d. families. For instance, the combinatorial characterization of normality for Ψ-spaces is in terms of *separation of subfamilies.*

Definition 1.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be disjoint subfamilies of $[\omega]^\omega$. We say that $\mathcal{F}$ and $\mathcal{G}$ can be* **separated** *if there is a set $S \in [\omega]^\omega$ such that, for every $F \in \mathcal{F}$ and $G \in \mathcal{G}$, $F \setminus S$ and $G \cap S$ are both finite sets.*

Note that if S separates $\mathcal{F}$ from $\mathcal{G}$ then $\omega \setminus S$ separates $\mathcal{G}$ from $\mathcal{F}$, so the relation "can be separated" is symmetric. It is straightforward to check that, for a given a.d. family $\mathcal{A}$, disjoint subfamilies $\mathcal{F}$ and $\mathcal{G}$ can be separated if, and only if , $\mathcal{F}$ and $\mathcal{G}$ have disjoint neighbourhoods in $\Psi(\mathcal{A})$, and therefore a space $\Psi(\mathcal{A})$ is normal if, and only if, any two disjoint subfamilies of $\mathcal{A}$ can be separated.

By Jones's Lemma, there are no normal Ψ-spaces of size $2^\omega = \mathfrak{c}$, and therefore it is consistent that there are no uncountable normal Ψ-spaces. However, the statement "There is an a.d. family of size ω_1 which is not normal" is a theorem of ZFC. This follows from the existence of certain interesting, classical combinatorial objects, *Luzin gaps*, which are in some sense as far from being normal as possible.

Definition 1.2. *An almost disjoint family $\mathcal{A}$ is a* **Luzin gap** *if $|\mathcal{A}| = \omega_1$ and no pair of disjoint uncountable subfamilies of $\mathcal{A}$ can be separated.*

In some texts, Luzin gaps are called *Luzin families*(e.g. [2], p. 214). Here we follow the terminology from Scheepers' survey ([6], §1.6). For a construction of such families in ZFC see [1], pp.124–125.

In contrast, the statement "There is an a.d. family of size ω_1 which does not have property (a)" is not a theorem of ZFC; see [7], p.13.[1]

A topological property which is commonly compared with normality is *countable paracompactness.* A topological space is said to be countably paracompact if every countable open cover has a locally finite open refinement. For a.d. families, previous works due to the authors give the following combinatorial characterizations for this topological property.

Proposition 1.3 ([5],[7])**.** *Let $\mathcal{A} \subseteq [\omega]^\omega$ be an a.d. family and consider the corresponding space $\Psi(\mathcal{A})$. The following statements are equivalent:*
(i) $\Psi(\mathcal{A})$ is countably paracompact.
(ii) For every decreasing sequence $\langle \mathcal{F}_n : n < \omega \rangle$ of subsets of $\mathcal{A}$ such that $\bigcap_{n<\omega} \mathcal{F}_n = \emptyset$ there is a sequence $\langle E_n : n < \omega \rangle$ of subsets of ω satisfying the conditions:
(ii).1 $\forall n < \omega \ \forall A \in \mathcal{F}_n$ $(A \setminus E_n$ *is finite*$)$
(ii).2 $\forall A \in \mathcal{A} \ \exists n < \omega$ $(A \cap E_n$ *is finite*$)$

[1]For almost disjoint families, property (a) is characterized as follows: we say that an a.d. family $\mathcal{A}$ has property (a) if, and only if, for every function $f : \mathcal{A} \to \omega$ there is a set $P \subseteq \omega$ such that $0 < |P \cap (A \setminus f(A))| < \omega$ for every $A \in \mathcal{A}$ ([8]).

(*iii*) *For every partition* $\{\mathcal{A}_n : n < \omega\}$ *of* $\mathcal{A}$ *there is a sequence* $\langle E_n : n < \omega\rangle$ *of subsets of* ω *satisfying the conditions:*

(*iii*).1 $\forall n < \omega\ \forall m \geqslant n\ \forall A \in \mathcal{A}_m\ (A \setminus E_n$ *is finite*)

(*iii*).2 $\forall A \in \mathcal{A}\ \exists n < \omega\ (A \cap E_n$ *is finite*)

(*iv*) *For every partition* $\{\mathcal{A}_n : n < \omega\}$ *of* $\mathcal{A}$ *there is a* $\subseteq$*-decreasing sequence* $\langle E_n : n < \omega\rangle$ *of subsets of* ω *satisfying the conditions:*

(*iv*).1 $\forall n < \omega\ \forall A \in \mathcal{A}_n\ (A \setminus E_n$ *is finite*)

(*iv*).2 $\forall A \in \mathcal{A}\ \exists n < \omega\ (A \cap E_n$ *is finite*)

(*v*) *For every function* $g : \mathcal{A} \to \omega$ *there are a* $\subseteq$*-decreasing sequence* $\langle E_n : n < \omega\rangle$ *of subsets of* ω *and a function* $f : \mathcal{A} \to \omega$ *satisfying the conditions:*

(*v*).1 $\forall A \in \mathcal{A}\ (A \setminus E_{g(A)}$ *is finite*)

(*v*).2 $\forall A \in \mathcal{A}\ (A \cap E_{f(A)}$ *is finite*)

Notice that for any $\mathcal{A}$, g and f as in (*v*) of the preceding proposition, one must have $g(A) < f(A)$ for every $A \in \mathcal{A}$, i.e., f dominates g. As this easy remark suggests, there is a connection between uncountable countably paracompact a.d. families and continuum size dominating families, as we now explain.

Watson has shown that the existence of countably paracompact separable spaces with uncountable closed discrete subsets is equivalent to the existence of dominating families of size not larger than $\mathfrak{c}$ in the family of functions of ω_1 into ω ([9]), and the existence of such "small dominating families" is related to large cardinals, by previous results due to Jech and Prikry ([4]; see [5], p.435, for more comments on this subject). In the restricted case of Ψ-spaces, one can easily apply item (*iv*) of the preceding proposition and give a purely combinatorial proof for the following:

Proposition 1.4. *The existence of a countably paracompact a.d. family of size* ω_1 *implies the existence of a dominating family of size not larger than* 2^ω *in* ${}^{\omega_1}\omega$.

Proof. Suppose that $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is a countably paracompact a.d. family. Let $\langle E^\alpha : \alpha < \kappa\rangle$, $\kappa \leqslant \mathfrak{c}$, be an enumeration of all decreasing sequences of subsets of ω (say $E^\alpha = \langle E^\alpha_n : n > \omega\rangle$) which satisfy (*iv*).2 of Proposition 1.3. For every $\alpha < \kappa$ define a function $f_\alpha : \omega_1 \to \omega$ such that, for every $\beta < \omega_1$,

$$f_\alpha(\beta) = \begin{cases} \max\{n : A_\beta \setminus E^\alpha_n \text{ is finite}\} & \text{if the set is non-empty} \\ 0 & \text{otherwise.} \end{cases}$$

As each E^α is decreasing and satisfies (*iv*).2, every function of the family $\mathcal{F} = \{f_\alpha : \alpha < \kappa\}$ is well defined. We claim that $\mathcal{F}$ is a dominating family. For if $g : \omega_1 \to \omega$ is an arbitrary function and $\{\mathcal{A}_n : n < \omega\}$ is the partition of $\mathcal{A}$ given by $\mathcal{A}_n = \{A \in \mathcal{A} : g(A) = n\}$, then, by countable paracompactness, there is a decreasing sequence E which satisfies both conditions of item (*iv*) of Proposition 1.3, and therefore there is $\xi < \kappa$ such that $E = E^\xi$. Then g is dominated by f_ξ, because if $\beta < \omega_1$ and $m = g(\beta)$ then $A_\beta \setminus E^\xi_m$ is a finite set, and therefore $g(\beta) = m \leqslant \max\{n : A_\beta \setminus E^\xi_n \text{ is finite}\} = f_\xi(\beta)$. ∎

Fleissner has shown that there are no countably paracompact separable spaces containing closed and discrete subsets of size $\mathfrak{c}$ ([3]). Consequently there are no countably paracompact a.d. families of size $\mathfrak{c}$.

Inspired by natural comparisons with normality, the third author asked in Vol. 25 of this journal ([7]) whether the statement "There is an a.d. family of size ω_1 which is not countably paracompact" is a theorem of ZFC, or, equivalently, he asked if $\mathfrak{ncp} = \omega_1$, where $\mathfrak{ncp}$ is the uncountable cardinal invariant given by

$$\mathfrak{ncp} = min\{\kappa\colon \text{there is an almost disjoint family } \mathcal{A} \text{ with } |\mathcal{A}| = \kappa \text{ whose corresponding space } \Psi(\mathcal{A}) \text{ is not countably paracompact}\}$$

As remarked in [7], the existence of a Luzin gap which is not countably paracompact would give a positive answer to this question, and the paper referred to ends in consideration of this topic.

In this paper we answer this Question about the value of $\mathfrak{ncp}$ by showing that *no* Luzin gap is countably paracompact, and, moreover, countably paracompact a.d. families of size ω_1 violate the condition of being a Luzin gap in a very strong way.

2. Luzin gaps are not countably paracompact

We start this section by giving a direct proof of our main result.

Theorem 2.1. *If $\mathcal{A}$ is a Luzin gap, then $\Psi(\mathcal{A})$ is not countably paracompact.*

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be a Luzin gap and let $\{\mathcal{A}_n : n < \omega\}$ be any partition of $\mathcal{A}$ satisfying $|\mathcal{A}_n| = \omega_1$ for every $n < \omega$. We will check that there is no sequence $\langle E_n : n < \omega\rangle$ of subsets of ω satisfying both of clauses (iii).1 and (iii).2 of Proposition 1.3.

Suppose that $\langle E_n : n < \omega\rangle$ is a sequence of subsets of ω such that (iii).1 holds.

Claim. For all $n < \omega$, $\{\alpha < \omega_1 : A_\alpha \cap E_n \text{ is finite}\}$ is a countable set.

Proof of the claim. Suppose towards a contradiction that there is $n < \omega$ such that $X = \{\alpha : A_\alpha \cap E_n \text{ is finite}\}$ is uncountable. It follows (by (iii).1) that $\mathcal{A}' = \{A_\alpha : \alpha \in X\}$ and $\mathcal{A}_n$ are disjoint uncountable sets which are separated by E_n. This contradicts our hypothesis that $\mathcal{A}$ is Luzin. □

So, letting $Y_n = \{\alpha : A_\alpha \cap E_n \text{ is finite}\}$ for $n < \omega$, we have $|Y_n| \leqslant \omega$ for all n and therefore

$$Y = \bigcup_{n<\omega} Y_n = \{\alpha : \exists n \ (A_\alpha \cap E_n \text{ is finite})\}$$

is also a countable set. Pick $\zeta \in \omega_1 \setminus Y$. Then A_ζ is a counterexample to (iii).2. ■

We have already remarked that there are Luzin gaps in ZFC. Therefore, the following corollary holds.

Corollary 2.2. $\mathfrak{ncp} = \omega_1$.

Let us go a little further. Notice that, in order to guarantee that a given a.d. family (of size ω_1) is not a Luzin gap, one just has to exhibit a pair of disjoint uncountable subfamilies which can be separated. In the following result, we show that if an uncountable a.d. family $\mathcal{A}$ is countably paracompact then *every pair* of disjoint uncountable subfamilies of $\mathcal{A}$ is able to provide a pair of uncountable subsets which can be separated.

Theorem 2.3. *Let $\mathcal{A}$ be an uncountable countably paracompact a.d. family and let $\mathcal{B}$, $\mathcal{C}$ be any disjoint uncountable subsets of $\mathcal{A}$. Then, there are uncountable $\mathcal{B}' \subseteq \mathcal{B}$ and $\mathcal{C}' \subseteq \mathcal{C}$ such that $\mathcal{B}'$ and $\mathcal{C}'$ can be separated.*

Proof. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be as in the statement, and let $\{\mathcal{C}_n : n < \omega\}$ be a partition of $\mathcal{C}$ into countably many uncountable sets. Define $g : \mathcal{A} \to \omega$ such that, for every $A \in \mathcal{A}$,

$$g(A) = \begin{cases} 0 & \text{if } A \notin \mathcal{B} \cup \mathcal{C} \\ 1 & \text{if } A \in \mathcal{B} \\ n+2 & \text{if } A \in \mathcal{C}_n \end{cases}$$

By countable paracompactness, let $\langle E_n : n < \omega \rangle$ and $f : \mathcal{A} \to \omega$ be as in item (v) of Proposition 1.3. Recall that if $g(A) = k$ then $A \setminus E_k$ is a finite set, and if $f(A) = k$ then $A \cap E_k$ is a finite set.

For every $n < \omega$ let $\mathcal{B}_n = \{A \in \mathcal{B} : f(A) = n\}$. As $g(A) = 1$ for every $A \in \mathcal{B}$, $\mathcal{B}_0 = \mathcal{B}_1 = \emptyset$ and $\{\mathcal{B}_n : n \in im(f) \setminus 2\}$ is a partition of the uncountable set $\mathcal{B}$ into countably many sets. Let $k \geqslant 2$ be minimal such that $\mathcal{B}_k$ is uncountable. Then E_k separates the uncountable sets $\{A \in \mathcal{A} : g(A) = k\} = \mathcal{C}_{k-2} \subseteq \mathcal{C}$ and $\mathcal{B}' = \mathcal{B}_k \subseteq \mathcal{B}$. ■

In the following section, we show how to ensure an analogous separation property for countably many uncountable disjoint subsets of an uncountable countably paracompact a.d. family.

3. On countable families of disjoint uncountable subsets

We start this section by ensuring the existence of functions satisfying certain requirements.

Proposition 3.1. *Let $\{X_n : n < \omega\}$ be a partition of an uncountable set X into countably many uncountable sets. Then there is a function $g : X \to \omega$ satisfying the following requirements:*

(i) For every $m < \omega$, if $g(x) = g(y)$ and $x \in X_m$ then $y \in X_m$;
(ii) For every $m < \omega$, $g \restriction X_m$ has infinite image.
(iii) For every $i < \omega$, $\{x \in X : g(x) = i\}$ is uncountable.

Proof. Consider any bijection between $\omega \times \omega$ and ω and, using this bijection, consider a countable family $\{s_n : n < \omega\}$ of injective sequences of natural numbers, say $s_n = \langle s_{n,k} : k < \omega \rangle$, such that $\{im(s_n) : n < \omega\}$ is a partition of ω. Now partition each set X_m into countably many uncountable sets, say $X_{m,k}$ for $k < \omega$, and let g map all points of $X_{m,k}$ to the natural number $s_{m,k}$. ■

Functions as in the preceding proposition play a central role in the following:

Theorem 3.2. *Let $\mathcal{A}$ be an uncountable countably paracompact a.d. family and let $\{\mathcal{A}_n : n < \omega\}$ be a partition of $\mathcal{A}$ into countably many uncountable subsets. Suppose $g : \mathcal{A} \to \omega$ satisfies all requirements of the preceding proposition and, for such g, let $\langle E_n : n < \omega \rangle$ and $f : \mathcal{A} \to \omega$ be given as in item (v) of Proposition 1.3. Then, for any $m, n < \omega$ and for any $r < \omega$ such that $\{A \in \mathcal{A} : g(A) = r\} \subseteq \mathcal{A}_m$ there are $s = s(r)$ and $t = t(r)$ satisfying $r < s \leqslant t < \omega$ and such that*

$$\mathcal{A}'_m = \{A \in \mathcal{A} : g(A) = r \text{ and } f(A) = s\} \text{ and}$$
$$\mathcal{A}'_n = \{A \in \mathcal{A} : g(A) = t\}$$

are uncountable subsets of, respectively, $\mathcal{A}_m$ and $\mathcal{A}_n$, and, moreover, $\mathcal{A}'_n$ and $\mathcal{A}'_m$ are separated by E_t.

Proof. Let $\mathcal{A}$, $\{\mathcal{A}_n : n < \omega\}$, g, $\langle E_n : n < \omega \rangle$, f, m, n, r be as in statement. Let s be minimal such that $\{A : g(A) = r \text{ and } f(A) = s\}$ is uncountable; notice that $s > r$. Let $t \geqslant s$ be minimal such that $\{A \in \mathcal{A} : g(A) = t\}$ is an uncountable subset of $\mathcal{A}_n$ (using that g satisfies the requirements of the previous proposition). Taking

$$\mathcal{A}'_m = \{A \in \mathcal{A} : g(A) = r \text{ and } f(A) = s\} \subseteq \mathcal{A}_m, \text{ and}$$
$$\mathcal{A}'_n = \{A \in \mathcal{A} : g(A) = t\} \subseteq \mathcal{A}_n,$$

one easily checks that E_t separates the (uncountable) sets given by $\mathcal{A}'_n$ and $\mathcal{A}'_m$. ■

For a given g satisfying the desired requirements, the preceding theorem may be applied for every pair m, n of natural numbers. Therefore we have the following

Corollary 3.3. *Let $\mathcal{A}$ be an uncountable countably paracompact a.d. family and let $\{\mathcal{A}_n : n < \omega\}$ be a partition of $\mathcal{A}$ into countably many uncountable subsets. Then there is a $\subseteq$-decreasing sequence of sets of natural numbers $\mathcal{E} = \langle E_n : n < \omega \rangle$ such that, for each $m, n < \omega$, there are uncountable $\mathcal{A}^{m,n}_m \subseteq \mathcal{A}_m$ and $\mathcal{A}^{m,n}_n \subseteq \mathcal{A}_n$ and an infinite set of natural numbers $E^{m,n} \in im(\mathcal{E})$ such that $E^{m,n}$ separates $\mathcal{A}^{m,n}_n$ and $\mathcal{A}^{m,n}_m$.*

With an easy induction, we have also the following

Corollary 3.4. *Let $\mathcal{A}$ be an uncountable countably paracompact a.d. family and let $\{\mathcal{A}_n : n < \omega\}$ be a partition of $\mathcal{A}$ into countably many uncountable subsets. Then there are families $\{\mathcal{B}_n : n < \omega\}$ and $\{D_n : n < \omega\}$ such that, for all $n < \omega$, $\mathcal{B}_n$*

is an uncountable subset of $\mathcal{A}_n$ *and* D_n *is an infinite subset of* ω *which separates* $\mathcal{B}_{n+1}$ *and* $\mathcal{B}_n$.

Proof. We apply the preceding theorem only for pairs $n, n+1$. Indeed: for a given g as desired, let r_o be any natural number such that $\{A \in \mathcal{A} : g(A) = r_o\}$ is a subset of $\mathcal{A}_0$ and, applying the theorem for $m = 0$ and $n = 1$, let $s_0 = s(r_0)$ and $t_0 = t(r_0)$. Now we proceed inductively, taking, for all $n \geqslant 0$, $r_{n+1} = t_n$, $s_{n+1} = s(r_{n+1})$ and $t_{n+1} = t(r_{n+1})$. The result follows by taking the sets

$$\{A : g(A) = r_n \text{ and } f(A) = s_n\}$$

as the $\mathcal{B}_n$ and the sets E_{t_n} as the D_n. □

It is straightforward to check that analogous results may be obtained for countable families of disjoint uncountable subsets of a given countably paracompact a.d. family: for any such family $\{\mathcal{A}_n : n < \omega\}$, it suffices to consider the partition $\{\mathcal{B}_n : n < \omega\}$ given by $\mathcal{B}_0 = \mathcal{A} \setminus (\bigcup_{n<\omega} \mathcal{A}_n)$ and $\mathcal{B}_{n+1} = \mathcal{A}_n$ for every $n < \omega$ and slightly change all the preceding arguments. We may work with functions g with $g(0) = 0$ and such that requirements which are analogous to those in Proposition 3.1 are done only for positive values of g.

In this paper we have been interested in countably paracompact a.d. families. As we have seen, while being normal is equivalent to the property that any pair of disjoint subfamilies can be separated (Definition 1.1), and being a Luzin gap to that no pair of disjoint uncountable subfamilies can be separated, being countably paracompact implies that any pair of disjoint uncountable subfamilies contain uncountable subfamilies which can be separated – as well as the more complex separation properties of this section. Exploring further connections between topological and separation properties of a.d. families is an interesting topic for further research.

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(Michael Hrušák) Instituto de Matemáticas, Universidad Nacional Autónoma de México, Xangari, 58089, Morelia, Michoacán, México

E-mail address: michael@matmor.unam.mx

(Charles J.G. Morgan) Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK, and

Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa, Avenida Professor Gama Pinto, 2, 1649-003 Lisboa, Portugal

E-mail address: charles.morgan@ucl.ac.uk

(Samuel G. da Silva) Instituto de Matemática, Universidade Federal da Bahia, Campus de Ondina, Av. Adhemar de Barros, S/N, Ondina, CEP 40170-110, Salvador, BA, Brazil

E-mail address: samuel@ufba.br

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