# ALMOST DISJOINT FAMILIES AND THE GEOMETRY OF NONSEPARABLE SPHERES 

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#### Abstract

We consider uncountable almost disjoint families of subsets of $\mathbb{N}$, the Johnson-Lindenstrauss Banach spaces $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ induced by them, and their natural equivalent renormings $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$. We introduce a partial order $\mathbb{P}_{\mathcal{A}}$ and characterize some geometric properties of the spheres of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ and of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ in terms of combinatorial properties of $\mathbb{P}_{\mathcal{A}}$. This provides a range of independence and absolute results concerning the above mentioned geometric properties via combinatorial properties of almost disjoint families. Exploiting the extreme behavior of some known and some new almost disjoint families we show the existence of Banach spaces where the unit spheres display surprising geometry: (1) There is a Banach space of density continuum whose unit sphere is the union of countably many sets of diameters strictly less than 1. (2) It is consistent that for every $\rho>0$ there is a nonseparable Banach space, where for every $\delta>0$ there is $\varepsilon>0$ such that every uncountable $(1-\varepsilon)$-separated set of elements of the unit sphere contains two elements distant by less than 1 and two elements distant at least by $2-\rho-\delta$. It should be noted that for every $\varepsilon>0$ every nonseparable Banach space has a plenty of uncountable $(1-\varepsilon)$-separated sets by the Riesz Lemma.

We also obtain a consistent dichotomy for the spaces of the form $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ : The Open Coloring Axiom implies that the unit sphere of every Banach space of the form $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ either is the union of countably many sets of diameter strictly less than 1 or it contains an uncountable $(2-\varepsilon)$-separated set for every $\varepsilon>0$.


## 1. INTRODUCTION

A family $\mathcal{A}$ of subsets of $\mathbb{N}$ will be called an a.d.-family if it is uncountable and consists of infinite sets which are pairwise almost disjoint, i.e., $A \cap B$ is finite for any two distinct $A, B \in \mathcal{A}$. Such a.d.-families induce diverse topological and abstract analytic structures which constitute interesting (counter)examples (e.g., [1, 2, 3, 9, 14, 22]) in various parts of mathematics. In this paper we provide applications in the geometry of the spheres of Banach spaces.

It was W. Johnson and N. Lindenstrauss who first considered a natural Banach space induced by an a.d.-family $\mathcal{A}$. We denote it as $\mathcal{X}_{\mathcal{A}}$. It is the closure of the linear span in $\ell_{\infty}$ with the supremum norm (denoted $\left\|\|_{\infty}\right.$ ) of the set

$$
c_{0} \cup\left\{1_{A}: A \in \mathcal{A}\right\},
$$

[^0]where $1_{A}$ stands for the characteristic function of $A \subseteq \mathbb{N}$. It is easy to see that $\mathcal{X}_{\mathcal{A}}$ is linearly isometric to the Banach space $C_{0}\left(\Psi_{\mathcal{A}}\right)$ of all real-valued continuous functions on $\Psi_{\mathcal{A}}$ vanishing at infinity, where $\Psi_{\mathcal{A}}$ is the locally compact space induced by $\mathcal{A}$, first considered by Alexandroff and Urysohn, and later often called Mrówka-Isbell space ( $[12]$ ). In this paper we also consider $\mathcal{X}_{\mathcal{A}}$ with a different, but equivalent norm $\left\|\|_{\infty, 2}\right.$ defined for $f \in \mathcal{X}_{\mathcal{A}}$ by
$$
\|f\|_{\infty, 2}=\|f\|_{\infty}+\sqrt{\sum_{n \in \mathbb{N}} \frac{f(n)^{2}}{2^{n+1}}} .
$$

Analyzing the dependence of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ on the combinatorial properties of the a.d.-family $\mathcal{A}$ we obtain new examples of nonseparable Banach spaces where the geometry of the unit sphere features completely unknown until now character.

Theorem 1. There is a Banach space $(\mathcal{X},\| \|)$ of density continuum (of the form $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ for some a.d.-family $\left.\mathcal{A}\right)$, where the unit sphere is the union of countably many sets, each of diameter strictly less than 1.

Proof. It is well known that there are a.d.-families $\mathcal{A}$ of cardinality $\mathfrak{c}$ which are $\mathbb{R}$-embeddable (see definition [25). By Proposition 27 the splitting partial order $\mathbb{P}_{\mathcal{A}}$ (Definition (5) for $\mathcal{A}$ which is $\mathbb{R}$-embeddable is $\sigma$-centered. Now Proposition 63 yields the Theorem.

This is a strengthening of the result of [24] where first nonseparable Banach spaces with no uncountable ( $1+$ )-separated sets in the sphere (i.e., such that the distances between distinct points are strictly bigger than 1) were constructed. Clearly the above space has this property as well. As in [24] one can prove that such spaces do not admit uncountable Auerbach systems or uncountable equilateral sets (sets where distances between any two points are the same) or the unit ball cannot be packed with uncountably many balls of diameter $1 / 3$. Also in 24 it was shown that there are a.d.-families (known as Luzin families) such that the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ does contain an uncountable (1+)-separated set. In fact, here we obtain the following consistent dichotomy (here $(2-\varepsilon)$-separated means that any two distinct points are distant at least by $2-\varepsilon$ ):

Theorem 2. Assume the Open Coloring Axiom OCA. Suppose that $\mathcal{A}$ is an a.d.family. Then the unit sphere of the Banach space $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ either is the union of countably many sets, each of diameter strictly less than 1 or it contains an uncountable $(2-\varepsilon)$-separated set for every $\varepsilon>0$.

Proof. Use proposition 67
It follows from Proposition 68 that the above dichotomy is not provable in ZFC for any $\varepsilon>0$. Most of the results of this paper should be seen in the context of the Riesz lemma of 1916 ([30]) which says that given a Banach space $\mathcal{X}$, and its closed proper subspace $\mathcal{Y} \subseteq \mathcal{X}$ and $\varepsilon>0$ there is $x$ in the unit sphere of $\mathcal{X}$ such that $\|x-y\|>1-\varepsilon$ for every $y \in \mathcal{Y}$. This allows to construct ( $1-\varepsilon$ )-separated sets of cardinality $\kappa$ in any Banach space of density equal to $\kappa$, where $\kappa$ is an infinite cardinal. It was Kottman who showed that the unit sphere of any separable Banach space contains an infinite (1+)-separated set ([25]) and Elton and Odell who showed that it always contains an infinite $(1+\varepsilon)$-separated set for some $\varepsilon>0$ ([6]). The search for a nonseparable version of Kottman's result produced many deep
postitive partial results which culminated in [10 but ended in the negative result of 24]. Theorem 1 provides even stronger failure of any nonseparable version of Kottman's result. Theorem 2 shows that we can uniformize, at least consistently, the behavior of $(1-\varepsilon)$-separated sets of the spheres of the spaces $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ for some a.d.-family $\mathcal{A}$ obtaining uniform behavior on an uncountable set. We consistently obtain Banach spaces where the behavior of the norm on the unit sphere is far from uniform even on each uncountable $(1-\varepsilon)$-separated set:

Theorem 3. It is consistent (e.g., follows from CH) that for every $\rho>0$ there exist a nonseparable Banach space $(\mathcal{X},\| \|)$ (a subspace of a space of the form $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ for some a.d.-family $\mathcal{A}$ ) such that for every $\delta>0$ there is $\varepsilon>0$ such that whenever and $\mathcal{Y} \subseteq \mathcal{X}$ is a $(1-\varepsilon)$-separated subset of the unit sphere of $\mathcal{X}$, then

- there are distinct $x, y \in \mathcal{Y}$ such that $\|x-y\|<1$, and
- there are $x, y \in \mathcal{Y}$ such that $\|x-y\|>2-\delta-\rho$.

Proof. By either of the Propositions [32, 33, 34, it is consistent that antiramsey a.d.-families exist. Now Proposition 69 yields the theorem.

Note that analogous phenomena cannot occur for infinite instead of uncountable $(1-\varepsilon)$-subsets of the unit spheres since by Theorem 1 of [28] every Banach space admits an equivalent renorming arbitrarily close to the original one which does admit 1-equilateral subset of the sphere (i.e., such a set where distances between any two distinct points are equal to 1 ). Consequently every infinite dimensional unit sphere in a Banach spaces admits an infinite subset $\mathcal{Y}$ such that $1-\varepsilon \leq\left\|y-y^{\prime}\right\| \leq$ $1+\varepsilon$ for any distinct $y, y^{\prime} \in \mathcal{Y}$ (while even nonseparable Banach spaces may not admit infinite equilateral sets - see [23]).

By choosing yet another a.d.-family we can obtain the geometry of the sphere completely opposite to the one from Theorem 1

Theorem 4. There exist a nonseparable Banach space ( $\mathcal{X},\| \|$ ) (a subspace of a space of the form $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ for some a.d.-family $\left.\mathcal{A}\right)$ and its separable subspace $\mathcal{Y}$ such that

- for every $\delta>0$ there is $\varepsilon>0$ such that whenever $\mathcal{Z} \subseteq \mathcal{X}$ is a subset of the unit sphere of $\mathcal{X}$ such that $\{[z] \mathcal{Y}: z \in \mathcal{Z}\}$ is a $(1-\varepsilon)$-separated subset of $\mathcal{X} / \mathcal{Y}$, then $\mathcal{Z}$ is the union of countably many sets which are $(2-\delta)$ separated.
- $\mathcal{X} / \mathcal{Y}$ does not admit an uncountable $(1+\varepsilon)$-separated set for any $\varepsilon>0$.

Proof. It is well known that Luzin families exist ([27, 14]). They are $L$-families by Proposition 22. Now Proposition 58 yields the first part of the theorem. For the second part note that $\mathcal{Y}$ obtained from Proposition 58 is $c_{0}$ and $\mathcal{X}_{\mathcal{A}} / c_{0}$ is isometric to $c_{0}\left(\omega_{1}\right)$ where it is well known that there are no uncountable $(1+\varepsilon)$-separated sets (6) for any $\varepsilon>0$.

As in [24] our main method to obtain the above Banach spaces is a combinatorial analysis of underlying a.d.-families which is greatly refined here. It turned out that the right tool for this analysis is the following partial order associated with an a.d.-family:

Definition 5. Suppose that $\mathcal{A}$ is an a.d.-family. The splitting partial order $\mathbb{P}_{\mathcal{A}}$ consists of all pairs $p=\left(A_{p}, B_{p}\right)$ such that $A_{p}, B_{p} \subseteq \mathbb{N}, A_{p} \cap B_{p}=\emptyset$ and there are finite subsets $a_{p}, b_{p}$ of $\mathcal{A}$ such that $A_{p}=^{*} \bigcup a_{p}$ and $B_{p}={ }^{*} \bigcup b_{p}$.

We say that $p \leq q$ if $A_{p} \supseteq A_{q}$ and $B_{p} \supseteq B_{q}$.
We call $p, q \in \mathbb{P}_{\mathcal{A}}$ essentially distinct if $a_{p} \neq a_{q}$ and $b_{p} \neq b_{q}$.
Although we do not force with the order $\mathbb{P}_{\mathcal{A}}$, we use forcing terminology, i.e., $p, q \in \mathbb{P}_{\mathcal{A}}$ are compatible if there is $r \in \mathbb{P}_{\mathcal{A}}$ such that $r \leq p, q$ and otherwise they are incompatible. Elements of $\mathbb{P}_{\mathcal{A}}$ will be called conditions. $\mathbb{P}_{\mathcal{A}}$ satisfies the c.c.c. if it does not admit an uncountable pairwise incompatible subset. A subset $\mathbb{P} \subseteq \mathbb{P}_{\mathcal{A}}$ is centered if for any $p_{1}, \ldots p_{k} \in \mathbb{P}$ there is $q \in \mathbb{P}_{\mathcal{A}}$ such that $q \leq p_{1}, \ldots p_{k} . \mathbb{P}_{\mathcal{A}}$ has precaliber $\kappa$, for $\kappa$ a cardinal, if every subset of $\mathbb{P}_{\mathcal{A}}$ of cardinality $\kappa$ contains a further subset of cardinality $\kappa$ which is centered. $\mathbb{P}_{\mathcal{A}}$ is said to be $\sigma$-centered if $\mathbb{P}=\bigcup_{n \in \mathbb{N}} \mathbb{P}_{n}$ and each $\mathbb{P}_{n}$ is centered. The core of our arguments leading to Theorems 1 and 2 is based on the following:

Theorem 6. Suppose that $\mathcal{A}$ is an a.d.-family.
(1) The following are equivalent:
(a) $\mathbb{P}_{\mathcal{A}}$ satisfies the c.c.c.
(b) The unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ does not admit an uncountable $(1+\varepsilon)$ separated set for some (equivalently for $\varepsilon=1$ ) $\varepsilon>0$.
(c) The unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ does not admit an uncountable 1separated set (equivalently $(2-\varepsilon)$-separated for some $\varepsilon>0$ ).
(2) The following are equivalent for a cardinal $\kappa$ of uncountable cofinality:
(a) $\mathbb{P}_{\mathcal{A}}$ has precaliber $\kappa$.
(b) For every $\varepsilon>0$ and every subset $\mathcal{Y}$ of the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ of cardinality $\kappa$ there is $\mathcal{Z} \subseteq \mathcal{Y}$ of cardinality $\kappa$ which has diameter less than $1+\varepsilon$.
(c) For every $\varepsilon>0$ and every subset $\mathcal{Y}$ of the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ of cardinality $\kappa$ there is $\mathcal{Z} \subseteq \mathcal{Y}$ of cardinality $\kappa$ which has diameter less than 1.
(3) The following are equivalent
(a) $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.
(b) For every $\varepsilon>0$ the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ is the union of countably many sets of diameters less than $1+\varepsilon$.
(c) The unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ is the union of countably many sets of diameters less than 1 .

Proof. For (1) use Propositions 46, 65 and 66, For (2) use Propositions 47 , 64 and 66. For (3) use Propositions 48, 63 and 66

To capture the combinatorics of $\mathbb{P}_{\mathcal{A}}$ behind Theorems 3 and 4 we introduce the following:

Definition 7. An a.d.-family $\mathcal{A}$ is called antiramsey if whenever an uncountable $\mathbb{P} \subseteq \mathbb{P}_{\mathcal{A}}$ consist of essentially distinct conditions, then there are distinct $p, q \in \mathbb{P}$ which are compatible and there $p, q \in \mathbb{P}$ which are incompatible.

Definition 8. An a.d.-family $\mathcal{A}$ is called L-family if whenever $\mathbb{P} \subseteq \mathbb{P}_{\mathcal{A}}$ consist of essentially distinct conditions, then $\mathbb{P}=\bigcup_{n \in \mathbb{N}} \mathbb{P}_{n}$ and each $\mathbb{P}_{n}$ consists of pairwise incompatible conditions.

To be able to take advantage of the links between the spaces $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ and $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ and the splitting partial order $\mathbb{P}_{\mathcal{A}}$ we need to analyze the structure
of the order $\mathbb{P}_{\mathcal{A}}$ for various a.d.-families $\mathcal{A}$. This is the most important ingredient of this paper allowing to prove the first four theorems. Besided antiramsey a.d.-families and $L$-families we consider in the context of the above properties of the partial order $\mathbb{P}_{\mathcal{A}}$ known types of almost disjoint families such Luzin families, $\mathbb{R}$-embedddable families and families admitting an $n$-Luzin gap and obtain the following:
Theorem 9. Suppose that $\mathcal{A}$ is an a.d.-family.
(a) $\mathcal{A}$ does not contain an $n$-Luzin gap for any $n \in \mathbb{N}$ implies that $\mathbb{P}_{\mathcal{A}}$ satisfies the c.c.c., which implies that $\mathcal{A}$ does not contain a 2 -Luzin gap.
(b) None of the above implications can be reversed in ZFC.
(c) The conditions of (a) are equivalent under OCA.
(2) (a) If $\mathcal{A}$ is $\mathbb{R}$-embeddable, then $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered. So such a.d.-families exist in ZFC. The implication cannot be reversed.
(b) If $\mathcal{A}$ is a maximal a.d.-family, then $\mathbb{P}_{\mathcal{A}}$ is not $\sigma$-centered.
(3) (a) (MA) Suppose that $\mathcal{A}$ is an a.d.-family of cardinality less than $\mathfrak{c}$. Then either $\mathbb{P}_{\mathcal{A}}$ does not satisfy c.c.c. or $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.
(b) (OCA) Suppose that $\mathcal{A}$ is an a.d.-family. Then either $\mathbb{P}_{\mathcal{A}}$ does not satisfy c.c.c. or $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.
(4) If $\mathcal{A}$ is a Luzin family, then it is an L-family. In particular L-families exist.
(5) It is consistent (and independent) that antiramsey a.d.-families exist, for example under CH and in many models of the negation of CH . They may be maximal.

Proof. 1 (a): Propositions 17 181 1(b): Propositions 19, 16, 1 (c): Propositions 20 21] 2 (a) Propositions 27, 28, 2 (b): 29, 3 (a): Proposition 30, 3 (b) Proposition 31) 4: Proposition 22, 5: Propositions 32, 33, 34,

Theorems 6 an 9 joined together yield a myriad of corollaries. For example under OCA the unit sphere of the space $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ admits a 1 -separated set if and only if $\mathcal{A}$ admits a 2 -Luzin gap, or for no maximal a.d.-family $\mathcal{A}$ the unit sphere of the space $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ is the union of countably many sets of diameters strictly less than 1.

The structure of the paper is the following: Section 2 contains some preliminaries. In Section 3 we investigate combinatorial properties of the forcing $\mathbb{P}_{\mathcal{A}}$ in the context of combinatorial properties of almost disjoint families $\mathcal{A}$, in particular we obtain all elements needed for Theorem [9, Section 4 is devoted to a graph which can code all the information of the compatibility graph of $\mathbb{P}_{\mathcal{A}}$ but the vertices of it are pairs of finite subsets of the a.d.-family $\mathcal{A}$. This section is not used in other sections but provides an alternative expression of the results of the other sections. In Section 5 we link the geometric properties of the sphere of spaces $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ with the combinatorial properties of the splitting partial order $\mathbb{P}_{\mathcal{A}}$. Equivalences like in Theorem 6 are possible for antiramsey and $L$-families and the spaces ( $\mathcal{X}_{\mathcal{A}},\| \|_{\infty}$ ) which are presented in Propositions 53 and 54 and for spaces $\left(\mathcal{X},\| \|_{\infty, 2}\right)$ but are less natural as one needs to include additional technical conditions since the spheres of any infinite dimensional Banach spaces contain uncountable sets of small diameters. To avoid these technical conditions in Section 6 we consider certain subspaces $\mathcal{X}_{\mathcal{A}, \phi}$ of the spaces $\mathcal{X}_{\mathcal{A}}$ for $\mathcal{A}$ being antiramsey and being an $L$-family and obtain Theorem 4 for $\mathcal{A}$ being an $L$-family. In Section 7 we investigate the spaces $\mathcal{X}_{\mathcal{A}}$ with the norm
$\left\|\|_{\infty, 2}\right.$, in particular we provide the space needed for Theorem In Section 8 we produce the space of Theorem 3,

## 2. Preliminaries

2.1. Notation and terminology. The a.d.-families are defined in the introduction. Antiramsey families and $L$-families are defined in Defintions 7 and 8 .

For $A, B \subseteq \mathbb{N}$ we write $A \subseteq^{*} B$ if $A \backslash B$ is finite; $A=^{*} B$ if $A \subseteq^{*} B$ and $B \subseteq^{*} A$. By $1_{A}$ we mean the characteristic function of $A \subseteq \mathbb{N}$. $[X]^{<\omega}$ denotes the family of all finite subsets of a set $X . f \mid X$ stands for the restriction of a function $f$ to $X$. By $2^{<\omega}$ we mean all finite 0 -1-sequences. If $t \in 2^{<\omega}$, then $t^{\subset} i \in 2^{<\omega}$ for $i=0,1$ is the extension of $t$ by $i$. The symbol $\mathfrak{c}$ stands for the cardinality of the continuum. Martin's axiom is denoted MA, the continuum hypothesis is denoted CH and the Zermelo-Fraenkel theory is denoted by ZFC. For more information on MA, CH, ZFC see [15.

The Open Coloring Axiom (OCA) is the following statement: Given a metric separable $X$ and a partition $K_{0} \cup K_{1}$ such that $\left\{(x, y):\{x, y\} \in K_{0}\right\}$ is open either there is an uncountable 0 -homogeneous set $Y \subseteq X$ or $X$ is the union of countably many 1-homogeneous sets. For more information on OCA see [15].

All Banach spaces considered in this paper are infinite dimensional and over the reals. $S_{\mathcal{X}}$ denotes the unit sphere of a Banach space $\mathcal{X}$. Sometimes we consider two norms $\left\|\|_{1}\right.$ and $\| \|_{2}$ on the same Banach spaces $\mathcal{X}$, then the corresponding spheres are denoted by $S_{\mathcal{X},\| \|_{1}}$ and $S_{\mathcal{X},\| \|_{2}}$ respectively. $\left\|\|_{\infty}\right.$ stands for the supremum norm while $\left\|\|_{2}\right.$ is the standard norm in the Hilbert space $\ell_{2}$. A subset $\mathcal{Y}$ of a Banach space $(\mathcal{X},\| \| \mathcal{X})$ is called $\delta$-separated $((\delta+)$-separated, $\delta$-equilateral) for $\delta>0$ if $\left\|x-x^{\prime}\right\|_{\mathcal{X}} \geq \delta\left(\left\|x-x^{\prime}\right\|_{\mathcal{X}}>\delta,\left\|x-x^{\prime}\right\|_{\mathcal{X}}=\delta\right)$ for any two distinct $x, x^{\prime} \in \mathcal{Y}$. A subset $\mathcal{Y}$ of a Banach space $(\mathcal{X},\| \| \mathcal{X})$ is called equilateral if it is $\delta$-equilateral for some $\delta>0$. Note that a Banach space $\mathcal{X}$ admits an uncountable (infinite) equilateral set if and only if it $S_{\mathcal{X}}$ admits an uncountable (infinite) 1-equilateral set (scale the set and translate one of its elements to 0 ).

An antichain in $\mathbb{P}_{\mathcal{A}}$ is a set of pairwise incompatible conditions. Some other forcing terminology is recalled below Definition 5

All the undefined terminology should be standard and can be find in the following books: 7] for topology, [8 for Banach spaces and [15] for set theory.
2.2. The partial order $\mathbb{P}_{\mathcal{A}}$. Essentially distinct conditions are defined together with the order $\mathbb{P}_{\mathcal{A}}$ in Definition 5.

Lemma 10. Suppose that $\mathcal{A}=\left\{A_{\alpha}: \alpha<\lambda\right\}$ is an a.d.-family, $\lambda$ is a cardinal, $\kappa$ is a regular uncountable cardinal and $\left\{p_{\xi}: \xi<\kappa\right\} \subseteq \mathbb{P}_{\mathcal{A}}$. Then there is $\Gamma \subseteq \kappa$ of cardinality $\kappa$ and $\left\{p_{\xi}^{\prime}: \xi \in \Gamma\right\} \subseteq \mathbb{P}_{\mathcal{A}}$ such that for every $\xi, \eta \in \Gamma$ we have that $p_{\xi}$ and $p_{\eta}$ are compatible in $\mathbb{P}_{\mathcal{A}}$ if and only if $p_{\xi}^{\prime}$ and $p_{\eta}^{\prime}$ are compatible in $\mathbb{P}_{\mathcal{A}}$ and moreover there are $k, l, m \in \mathbb{N}$ and $a_{\xi}^{\prime} \in[\mathcal{A}]^{k}$, $b_{\xi}^{\prime} \in[\mathcal{A}]^{l}$ for $\xi \in \Gamma$ and disjoint $E, F \subseteq m$ such that for all distinct $\xi, \eta \in \Gamma$ the following hold:

- $A_{p_{\xi}^{\prime}}=\left(\bigcup a_{\xi}^{\prime} \backslash m\right) \cup E$,
- $B_{p_{\xi}^{\prime}}=\left(\bigcup b_{\xi}^{\prime} \backslash m\right) \cup F$,
- $A_{\alpha} \cap A_{\alpha^{\prime}} \subseteq m$ for any distinct $\alpha, \alpha^{\prime} \in a_{\xi}^{\prime} \cup b_{\xi}^{\prime}$,
- $\left(a_{\xi}^{\prime} \cup b_{\xi}^{\prime}\right) \cap\left(a_{\eta}^{\prime} \cup b_{\eta}^{\prime}\right)=\emptyset=a_{\xi}^{\prime} \cap b_{\xi}^{\prime}$.

Proof. By Definition 5 there are $a_{\xi}, b_{\xi} \in[\mathcal{A}]^{<\omega}$ and $m_{\xi} \in \mathbb{N}$ and disjoint $E_{\xi}, F_{\xi} \subseteq$ $m_{\xi}$ such that

$$
p_{\xi}=\left(\left(\bigcup a_{\xi} \backslash m_{\xi}\right) \cup E_{\xi},\left(\bigcup b_{\xi} \backslash m_{\xi}\right) \cup F_{\xi}\right)
$$

Moreover, necessarily $a_{\xi} \cap b_{\xi}=\emptyset$. By increasing $m_{\xi}$ and modifying $E_{\xi}$ s and $F_{\xi}$ s we may assume that $A_{\alpha} \cap A_{\alpha^{\prime}} \subseteq m_{\xi}$ for all distinct $\alpha, \alpha^{\prime} \in a_{\xi} \cup b_{\xi}$.

By passing to a subset of cardinality $\kappa$ we may assume that there are $k^{\prime}, l^{\prime}, m \in \mathbb{N}$ and $E, F, G, H \in[\mathbb{N}]^{<\omega}$ such that $m=m_{\xi}, E=E_{\xi}, F=F_{\xi}$ for all $\xi<\kappa$ and $\left|a_{\xi}\right|=k^{\prime}$ and $\left|b_{\xi}\right|=l^{\prime}$ for any $\xi \in \kappa$.

If either of the sets $\left\{a_{\xi}: \xi<\kappa\right\}$ or $\left\{b_{\xi}: \xi<\kappa\right\}$ is of cardinality less than $\kappa$ we may find $\Gamma \subseteq \kappa$ of cardinality $\kappa$ such that the set has only one element which means that the entire $\left\{p_{\xi}: \xi \in \Gamma\right\}$ is pairwise compatible. So $p_{\xi}^{\prime}=(\emptyset, \emptyset)$ works with $E=F=a_{\xi}^{\prime}=b_{\xi}^{\prime}=\emptyset$ and $k, l, m=0$.

Otherwise we may assume that all $a_{\xi}$ 's and $b_{\xi}$ 's are distinct. Using the Delta System Lemma for families of finite sets of cardinality $\kappa$ by passing to an uncountable set we may assume that $\left\{a_{\xi}: \xi<\omega_{1}\right\}$ and $\left\{b_{\xi}: \xi<\omega_{1}\right\}$ form $\Delta$-systems with roots $a, b \in[\mathcal{A}]^{<\omega}$ respectively.

Let $a_{\xi}^{\prime}=a_{\xi} \backslash a, b_{\xi}^{\prime}=b_{\xi} \backslash b$ and $k=\left|a_{\xi}^{\prime}\right|, l=\left|b_{\xi}^{\prime}\right|$. Using the Delta System Lemma again for $\left\{\left(a_{\xi}^{\prime} \cup b_{\xi}^{\prime}\right): \xi<\kappa\right\}$ we find $\Gamma \subseteq \kappa$ of cardinality $\kappa$ such that additionally $\left(a_{\xi}^{\prime} \cup b_{\xi}^{\prime}\right) \cap\left(a_{\eta}^{\prime} \cup b_{\eta}^{\prime}\right)=\emptyset$ for every distinct $\xi, \eta \in \Gamma$. For $\xi \in \Gamma$ we put

$$
p_{\xi}^{\prime}=\left(\left(\bigcup a_{\xi}^{\prime} \backslash m\right) \cup E,\left(\bigcup b_{\xi}^{\prime} \backslash m\right) \cup F\right)
$$

Now $p_{\xi}$ is incompatible with $p_{\eta}$ if and only if

$$
\left(\left(\bigcup a_{\xi} \backslash m\right) \cap\left(\bigcup b_{\eta} \backslash m\right)\right) \cap\left(\left(\bigcup a_{\eta} \backslash m\right) \cap\left(\bigcup b_{\xi} \backslash m\right)\right) \neq \emptyset
$$

And $p_{\xi}^{\prime}$ is incompatible with $p_{\eta}^{\prime}$ if and only if

$$
\left(\left(\bigcup a_{\xi}^{\prime} \backslash m\right) \cap\left(\bigcup b_{\eta}^{\prime} \backslash m\right)\right) \cap\left(\left(\bigcup a_{\eta}^{\prime} \backslash m\right) \cap\left(\bigcup b_{\xi}^{\prime} \backslash m\right)\right) \neq \emptyset
$$

But these two conditions are equivalent since $\bigcup a \cap \bigcup b \backslash m=\emptyset$ and $((\bigcup a \cup \bigcup b) \backslash$ $m) \cap\left(\left(\bigcup a_{\xi}^{\prime} \cup \bigcup b_{\xi}^{\prime}\right) \backslash m\right)=\emptyset$ and $((\bigcup a \cup \bigcup b) \backslash m) \cap\left(\left(\bigcup a_{\eta}^{\prime} \cup \bigcup b_{\eta}^{\prime}\right) \backslash m\right)=\emptyset$ for any $\xi, \eta \in \Gamma$.

## 3. The splitting partial order $\mathbb{P}_{\mathcal{A}}$ and the combinatorial properties of $\mathcal{A}$

3.1. Luzin properties of $\mathcal{A}$. In this subsection we will use two "Luzin properties" of a.d.-families expressed in the following two definitions.

Definition 11. An a.d.-family $\mathcal{A}$ is called a Luzin family if $\mathcal{A}=\left\{A_{\xi}: \xi<\omega_{1}\right\}$ and $f_{\eta}: \eta \rightarrow \mathbb{N}$ is finite-to-one for each $\eta<\omega_{1}$, where $f_{\eta}(\xi)=\max \left(A_{\xi} \cap A_{\eta}\right)$ for each $\xi<\eta$.

Definition 12 ([13]). Let $n \in \mathbb{N}$ and $\mathcal{A}$ be an a.d.-family. Let $\mathcal{B}_{i}=\left\{B_{\alpha}^{i}: \alpha<\omega_{1}\right\}$ be disjoint subfamilies of $\mathcal{A}$ for $i<n$. We say that $\left(\mathcal{B}_{i}: i<n\right)$ is an n-Luzin gap if there is $m \in \mathbb{N}$ such that
(1) $B_{\alpha}^{i} \cap B_{\alpha}^{j} \subseteq m$ for all $i<j<n$ and $\alpha<\omega_{1}$ and
(2) $\bigcup_{i \neq j}\left(B_{\alpha}^{i} \cap B_{\beta}^{j}\right) \nsubseteq m$ for all distinct $\alpha, \beta<\omega_{1}$.

We say that $\mathcal{A}$ contains an n-Luzin gap if there is an $n$-Luzin gap $\left(\mathcal{B}_{i}: i<n\right)$ where each $\mathcal{B}_{i}$ is a subfamily of $\mathcal{A}$.

The results of this subsection clarify the relations among the above Luzin properties. The existence of an $n$-Luzin gap in an a.d.-family $\mathcal{A}$ implies the existence of a $k$-Luzin gap in $\mathcal{A}$ for $k>n>1$ (Proposition (15). Being a Luzin family is much stronger, as one can choose $n$-Luzin gaps in such a family within any $n$ many uncountable subfamiles (Proposition [13) and there are in ZFC families admitting a 2-Luzin gap, while not containing any Luzin subfamilies (Proposition 14). It is shown in 13 (Proposition 2.14) that it is consistent that there are a.d.-families which contain 3-Luzin gaps but no 2-Luzin gaps. This will also follow from several our results in the following sections. However OCA implies that an a.d.-family contains $n$-Luzin gap if and only if it contains a $k$-Luzin gap for any $n, k \in \mathbb{N} \backslash\{0,1\}$ (proposition 16 which is implicitly included in [13: Lemma 2.2 and Proposition 2.9)

Proposition 13. Let $\mathcal{A}$ be a Luzin a.d.-family and let $\mathcal{C}_{i} \subseteq \mathcal{A}$ be uncountable for each $i<n \in \mathbb{N} \backslash\{0,1\}$. Then for each $i<n$ there are uncountable $\mathcal{B}_{i} \subseteq \mathcal{C}_{i}$ such that $\left(\mathcal{B}_{i}: i<n\right)$ an $n$-Luzin gap.

Proof. By passing to uncountable subsets we may assume that the $\mathcal{C}_{i}$ s are pairwise disjoint. Thinning out further for each $i<n$ we may find an uncountable $\mathcal{B}_{i} \subseteq \mathcal{C}_{i}$ such that (1) of Definition 12 is satisfied for some $m$. Let $\left(\xi_{\alpha}^{i}: \alpha<\omega_{1}\right)$ be such that $\mathcal{B}_{i}=\left\{A_{\xi_{\alpha}^{i}}: \alpha<\omega_{1}\right\}$ and $\xi_{\alpha}^{i}<\xi_{\beta}^{j}$ for each $i, j<n$ and $\alpha<\beta<\omega_{1}$. Using Definition 11 we obtain a regressive function $f: \omega_{1} \rightarrow \omega_{1}$ such that for every limit $\beta \in \omega_{1}$ we have $\max \left(A_{\xi_{\alpha}}^{i} \cap A_{\xi_{\beta}}^{j}\right)>m$ for every $\alpha \in(f(\beta), \beta)$ and every $i, j<n$. By the Pressing Down Lemma we obtain an uncountable $\Gamma \subseteq \omega_{1}$ such that $A_{\xi_{\alpha}}^{i} \cap A_{\xi_{\beta}}^{j} \nsubseteq m$ for every $i, j<n$ and every distinct $\alpha, \beta \in \Gamma$ as required in (2) of Definition 12 .

Proposition 14. There is an almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ of cardinality $\mathfrak{c}$ such that $\mathcal{A}$ contains a 2-Luzin gap but does not contain a Luzin subfamily.

Proof. It is enough to construct such a family for $\kappa=\mathfrak{c}$. This construction is due to Juris Steprans. For every $x \in 2^{\mathbb{N}}$ consider $A_{x}^{0}=\left\{t \in 2^{<\omega}: t \frown 0 \subseteq x\right\}$ and $A_{x}^{1}=$ $\left\{t \in 2^{<\omega}: t^{\frown} 1 \subseteq x\right\}$. Note that $A_{x}^{0} \cap A_{x}^{1}=\emptyset$ and $\mathcal{A}=\left\{A_{x}^{0}, A_{x}^{1}: x \in 2^{\mathbb{N}}\right\} \subseteq \wp\left(2^{<\omega}\right)$ is almost disjoint. Of course we can identify $2^{<\omega}$ with $\mathbb{N}$. We claim that $\left(\mathcal{A}_{0}, \mathcal{A}_{1}\right)$ is a 2-Luzin gap as witnessed by $m=0$, where $\mathcal{A}_{i}=\left\{A_{x}^{i}: x \in 2^{\mathbb{N}}\right\}$ for $i \in\{0,1\}$.

Given distinct $x, y \in 2^{\mathbb{N}}$ there is $t \in 2^{<\omega}$ such that $t \subseteq x, y$ and $t^{\frown} i \subseteq x \backslash y$ and $t^{\frown}(1-i) \subseteq y \backslash x$ for some $i \in\{0,1\}$. Then $t \in A_{x}^{i} \cap A_{y}^{(1-i)}$, so (2) of Definition 12 is satisfied as well.

To see that $\mathcal{A}$ contains no Luzin subfamily, consider any uncountable $\mathcal{B} \subseteq \mathcal{A}$. By passing to an uncountable subfamily we may assume that $\mathcal{B} \subseteq \mathcal{A}_{i}$ for some $i \in$ $\{0,1\}$. Now, note that given an uncountable set of reals, there are two uncountable subsets of it included in disjoint open sets. This proves that the condition of Definition 11 fails for such $\mathcal{B}$.

Proposition 15. Suppose that $\mathcal{A}$ is an a.d.-family and $n<k$ are elements of $\mathbb{N} \backslash\{0,1\}$. If $\mathcal{A}$ contains an $n$-Luzin gap, then it contains a $k$-Luzin gap.
Proof. Let $\left(\mathcal{B}_{i}: i<n\right)$ for $\mathcal{B}_{i}=\left\{B_{\alpha}^{i}: \alpha<\omega_{1}\right\}$ be an $n$-Luzin gap as witnessed by $m \in \mathbb{N}$. First let us make an observation that actually any $m^{\prime} \geq m$ witnesses it for $\left(\mathcal{B}_{i}^{\prime}: i<n\right)$ where $\mathcal{B}_{i}^{\prime}=\left\{B_{\alpha}^{i}: \alpha \in \Gamma\right\}$ for some uncountable $\Gamma \subseteq \omega_{1}$. Indeed,
the clause (1) of Definition 12 is clear for such $m^{\prime}$ and any uncountable $\Gamma \subseteq \omega_{1}$. To obtain (2) find an uncountable $\Gamma \subseteq \omega_{1}$ such that $B_{\alpha}^{i} \cap m^{\prime}=B_{\beta}^{i} \cap m^{\prime}$ for all $\alpha, \beta \in \Gamma$ and any $i<n$. Then whenever $\alpha, \beta$ are distinct elements of $\Gamma$ and $i, j<n$ are distinct, by (1) we have $B_{\alpha}^{i} \cap B_{\beta}^{j} \cap m^{\prime}=B_{\alpha}^{i} \cap B_{\alpha}^{j} \cap m^{\prime} \subseteq m$, hence by (2) for ( $\mathcal{B}_{i}: i<n$ ) we have $B_{\alpha}^{i} \cap B_{\beta}^{j} \nsubseteq m^{\prime}$ for some distinct $i, j<n$.

Now to prove the proposition, by passing to a subset we may assume that $\mathcal{A} \backslash$ $\bigcup\left\{\mathcal{B}_{i}: i<n\right\}$ is uncountable, so we can find $\mathcal{B}_{j} \subseteq \mathcal{A} \backslash \bigcup\left\{\mathcal{B}_{i}: i<n\right\}$ for $n \leq j<k$ such that $\mathcal{B}_{i} \cap \mathcal{B}_{j}=\emptyset$ for $i<j<k$. Thinning out further we may assume that there is $m^{\prime}>m$ such that $B_{\alpha}^{i} \cap B_{\alpha}^{j} \subseteq m^{\prime}$ for all $i<j<n$ and $\alpha<\omega_{1}$. Passing to the uncountable $\Gamma \subseteq \omega_{1}$ from our initial observation we obtain a $k$-Luzin gap.

Proposition 16. Assume OCA. Suppose that $\mathcal{A}$ is an a.d.-family and $n<k$ are elements of $\mathbb{N} \backslash\{0,1\}$. $\mathcal{A}$ contains an $n$-Luzin gap if and only if $\mathcal{A}$ contains a $k$-Luzin gap.

Proof. First let us note that the following version of OCA follows from the standard one: Given $l \in \mathbb{N} \backslash\{0\}$ and a cover $K_{0} \cup \cdots \cup K_{l}$ of $X \subseteq 2^{\mathbb{N}}$ such that $\{(x, y)$ : $\left.\{x, y\} \in K_{l^{\prime}}\right\}$ is open in $X^{2}$ for any $l^{\prime} \leq l$ there is $l^{\prime} \leq l$ and an uncountable $Y \subseteq X$ which is $l^{\prime}$-homogeneous. This should be clear, one needs to apply iterating the standard OCA obtaining either an uncountable homogeneous $Y \subseteq X$ in some part $K_{l^{\prime}}$ or obtaining an uncountable set $Y \subseteq X$ whose pairs avoid $K_{l^{\prime}}$. In the first case we are done, in the second case we work with $Y$ instead of $X$. The remaining colors cover the pairs of $Y$ and $\left\{(x, y):\{x, y\} \in K_{l^{\prime \prime}} \cap Y\right\}$ are open in $Y^{2}$ or $l^{\prime \prime} \leq l$ distinct than $l^{\prime}$. This procedure can be continued until the desired set is found.

Let $\left(\mathcal{B}_{i}: i<n\right)$ for $\mathcal{B}_{i}=\left\{B_{\alpha}^{i}: \alpha<\omega_{1}\right\}$ be a $k$-Luzin gap as witnessed by $m \in \mathbb{N}$. For

$$
X=\left\{x_{\alpha}=\left(1_{B_{\alpha}^{0}}, \ldots, 1_{B_{\alpha}^{k-1}}\right) \in\left(2^{\mathbb{N}}\right)^{k}: \alpha<\omega_{1}\right\}
$$

define a cover of $[X]^{2}$ by parts $\left\{K_{\{i, j\}}:\{i, j\} \in[k]^{2}\right\}$ by declaring that if $\left\{x_{\alpha}, x_{\beta}\right\}$ belongs to a $K_{\{i, j\}}$ then $\left(B_{\alpha}^{i} \cap B_{\beta}^{j}\right) \cup\left(B_{\alpha}^{i} \cap B_{\beta}^{j}\right) \nsubseteq m$. Since $\left(\mathcal{B}_{i}: i<n\right)$ is assumed to be $k$-Luzin, there is always such an $\{i, j\}$. This proves that this is really a cover. Moreover $\left\{\left(x_{\alpha}, x_{\beta}\right):\left\{x_{\alpha}, x_{\beta}\right\} \in K_{\{i, j\}}\right\}$ is open for each $\{i, j\} \in[k]^{2}$. So by our version of OCA we obtain an uncountable $\Gamma \subseteq \omega_{1}$ and $i^{\prime}<j^{\prime}<k$ such that $\left(B_{\alpha}^{i^{\prime}} \cap B_{\beta}^{j^{\prime}}\right) \cup\left(B_{\alpha}^{i^{\prime}} \cap B_{\beta}^{j^{\prime}}\right) \nsubseteq m$ for every distinct $\alpha, \beta \in \Gamma$ and consequently $\left(\mathcal{B}_{i^{\prime}}, \mathcal{B}_{j^{\prime}}\right)$ is a 2 -Luzin gap.

It follows from Proposition 15 that $\mathcal{A}$ contains an $n$-Luzin gap for any $n \geq 2$.
3.2. The c.c.c. of $\mathbb{P}_{\mathcal{A}}$ and $n$-Luzin gaps in $\mathcal{A}$. In this subsection we analyze the relations between the existence of $n$-Luzin gaps in an a.d.-family $\mathcal{A}$ and the countable chain condition of the partial order $\mathbb{P}_{\mathcal{A}}$. A 2 -Luzin gap in $\mathcal{A}$ yields an uncountable antichain in $\mathbb{P}_{\mathcal{A}}$ which yields an $n$-Luzin gap in $\mathcal{A}$ (Propositions 17 18). But none of the implications can be reversed in ZFC (Proposition 20, 21) and 2-Luzin cannot be weakened in ZFC to 3-Luzin in Proposition 17 (Proposition 20). However OCA implies that the c.c.c. of $\mathbb{P}_{\mathcal{A}}$ is equivalent to the existence of a $n$-Luzin gap in $\mathcal{A}$ for some $n \in \mathbb{N}$ (Proposition 19).

Proposition 17. Suppose that $\mathcal{A}$ is an a.d.-family. If $\mathcal{A}$ contains a 2 -Luzin gap, then $\mathbb{P}_{\mathcal{A}}$ fails to satisfy c.c.c. In particular, $\mathbb{P}_{\mathcal{A}}$ fails to satisfy c.c.c. if $\mathcal{A}$ is a Luzin family.

Proof. Let $\left\{\mathcal{B}_{1}, \mathcal{B}_{2}\right\}$ be a 2-Luzin gap contained by $\mathcal{A}$. Then $\mathbb{C}=\left\{\left(B_{\alpha}^{0} \backslash m, B_{\alpha}^{1} \backslash m\right)\right.$ : $\left.\alpha<\omega_{1}\right\} \subseteq \mathbb{P}_{\mathcal{A}}$ by (1) of Definition 12, By (2) of Definition 12 either $\left(B_{\alpha}^{0} \backslash m\right) \cap$ $\left(B_{\alpha}^{1} \backslash m\right) \neq \emptyset$ or $\left(B_{\alpha}^{0} \backslash m\right) \cap\left(B_{\alpha}^{1} \backslash m\right) \neq \emptyset$ which means that $\mathbb{C}$ is an uncountable antichain of $\mathbb{P}_{\mathcal{A}}$.

For the proof of the last part of the proposition apply Proposition 13
Proposition 18. Suppose that $\mathcal{A}$ is an a.d.-family. If $\mathbb{P}_{\mathcal{A}}$ fails to be c.c.c., then $\mathcal{A}$ contains an $n$-Luzin gap for some $n \in \mathbb{N}$.
Proof. Let $\left\{p_{\xi}: \xi<\omega_{1}\right\}$ be an uncountable antichain in $\mathbb{P}_{\mathcal{A}}$. By Lemma 10 we may assume that there are $k, l, m \in \mathbb{N}$ and $a_{\xi} \in[\mathcal{A}]^{k}, b_{\xi} \in[\mathcal{A}]^{l}$ and $E, F \subseteq m$ such that $p_{\xi} \mathrm{s}$ are as $p_{\xi}^{\prime} \mathrm{S}$ in Lemma 10, Let $a_{\xi}=\left\{A_{\xi}^{i}: i<k\right\}$ and $b_{\xi}=\left\{A_{\xi}^{i}: k \leq i<k+l\right\}$ for $\xi<\kappa$. By passing to an uncountable subset we may assume that $A_{\xi}^{i} \cap m=A_{\eta}^{i} \cap m$ for all $\xi<\eta<\omega_{1}$.

Put $n=k+l$ and $\mathcal{A}_{i}=\left\{A_{\xi}^{i}: \xi<\omega\right\}$ and note that $\left(\mathcal{A}_{i}: i<n\right)$ is an $n$-Luzin gap as witnessed by $m$ : condition (1) of Definition 12 is clear from the choice of $m$ based on Lemma 10 and condition (2) follows from the fact that by the incompatibility of $p_{\xi}$ and $p_{\eta}$ for each $\xi<\eta<\omega_{1}$ we have $i<k \leq j<n$ such that either $\left(\left(A_{\xi}^{i} \backslash m\right) \cup E\right) \cap\left(\left(A_{\eta}^{j} \backslash m\right) \cup F\right) \neq \emptyset$ or $\left(\left(A_{\eta}^{i} \backslash m\right) \backslash E\right) \cap\left(\left(A_{\xi}^{j} \backslash m\right) \cup F\right) \neq \emptyset$; however the fact that $E, F \subseteq m$ are disjoint implies that the incompatibility of $p_{\xi}$ and $p_{\eta}$ is equivalent to the alternative of $A_{\xi}^{i} \cap A_{\eta}^{j} \nsubseteq m$ or $A_{\eta}^{i} \cap A_{\xi}^{j} \nsubseteq m$ respectively.
Proposition 19. Assume OCA. Suppose that $\mathcal{A}$ is an a.d.-family. $\mathbb{P}_{\mathcal{A}}$ is c.c.c. if and only if $\mathcal{A}$ contains an n-Luzin gap for every (some) $n \in \mathbb{N} \backslash\{0,1\}$.
Proof. Apply Propositions 16, 17, 18.
Proposition 20. There is a $\sigma$-centered forcing notion $\mathbb{P}$ such that $\mathbb{P}$ forces that there is an a.d.-family $\mathcal{A}$ such that $\mathbb{P}_{\mathcal{A}}$ satisfies c.c.c. and $\mathcal{A}$ contains a 3 -Luzin gap.
Proof. We consider a forcing notion $\mathbb{P}$ consisting of conditions

$$
p=\left(n_{p}, a_{p},\left(A_{p}^{0}(\xi), A_{p}^{1}(\xi), A_{p}^{2}(\xi): \xi \in a_{p}\right)\right)
$$

satisfying
(1) $n_{p} \in \mathbb{N}, a_{p} \in\left[\omega_{1}\right]^{<\omega}$,
(2) $A_{p}^{i}(\xi) \subseteq n_{p}$ for $i<3$ and $\xi \in a_{p}$,
(3) $A_{p}^{i}(\xi) \cap A_{p}^{j}(\xi)=\emptyset$ for all $\xi \in a_{p}$ and $i<j<3$,
(4) for any distinct $\xi, \eta \in a_{p}$ there are distinct $i, j<3$ such that

$$
A_{p}^{i}(\xi) \cap A_{p}^{j}(\eta) \neq \emptyset
$$

For $p, q \in \mathbb{P}$ we say that $p \leq q$ if
(5) $n_{p} \geq n_{q}, a_{p} \supseteq a_{q}$,
(6) $A_{p}^{i}(\xi) \cap n_{q}=A_{q}^{i}(\xi) \cap n_{q}$ for all $i<3$ and $\xi \in a_{q}$,
(7) $\left(\bigcup_{i<3} A_{p}^{i}(\xi)\right) \cap\left(\bigcup_{i<3} A_{p}^{i}(\eta)\right) \subseteq n_{q}$ for any distinct $\xi, \eta \in a_{q}$.

First we will prove that if $p, q \in \mathbb{P}$ and $n_{p}=n_{q}$ and $A_{p}^{i}(\xi)=A_{q}^{i}(\xi)$ for all $i<3$ and $\xi \in a_{p} \cap a_{q}$, then $p$ and $q$ are compatible in $\mathbb{P}$. This and the fact that $X^{\omega_{1}}$ with the product topology is separable for any finite $X$ imply that $\mathbb{P}$ is $\sigma$-centered.

For this first note that if $a_{q} \subseteq a_{p}$ (and the above hypotheses hold), then $p \leq q$. So we may assume that $a_{p} \backslash a_{q} \neq \emptyset \neq a_{q} \backslash a_{p}$. To construct $r \in \mathbb{P}$ which is stronger
than $p$ and $q$ define $n=n_{p}=n_{q}$ and $a_{r}=a_{p} \cup a_{q}$. Let $k=\left|a_{p} \backslash q_{q}\right| \cdot\left|a_{q} \backslash a_{p}\right|$ and put $n_{r}=n+k$. Let $k \backslash n=\left\{k_{\xi, \eta}: \xi \in a_{p} \backslash a_{q}, \eta \in a_{q} \backslash a_{p}\right\}$. Define

$$
A_{r}^{i}(\xi)= \begin{cases}A_{p}^{i}(\xi)=A_{q}^{i}(\xi) & \text { if } \xi \in a_{p} \cap a_{q} \text { and } i=0,1,2 \\ A_{p}^{i}(\xi) \cup\left\{k_{\xi, \eta}: \eta \in a_{q} \backslash a_{p}\right\} & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i=0 \\ A_{q}^{i}(\xi) \cup\left\{k_{\xi, \eta}: \xi \in a_{p} \backslash a_{q}\right\} & \text { if } \xi \in a_{q} \backslash a_{q} \text { and } i=1 \\ A_{p}^{i}(\xi) & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i=1,2 \\ A_{q}^{i}(\xi) & \text { if } \xi \in a_{q} \backslash a_{p} \text { and } i=0,2\end{cases}
$$

Then it is clear that $r \in \mathbb{P}$ because $k_{\xi, \eta} \in A_{r}^{0}(\xi) \cap A_{r}^{1}(\eta)$ for $\xi \in a_{p} \backslash a_{q}$ and $\eta \in a_{q} \backslash a_{p}$. Moreover $r \leq p, q$ because all $k_{\xi, \eta} \mathrm{s}$ are distinct. This completes the proof that $\mathbb{P}$ is $\sigma$-centered.

Let $\dot{\mathbb{G}}$ be a $\mathbb{P}$-name for a generic filter in $\mathbb{P}$. For $i<3$ and $\xi<\omega_{1}$ let $\dot{A}_{\xi}^{i}$ be a $\mathbb{P}$-name for a subset of $\mathbb{N}$ such that $\mathbb{P}$ forces that $\dot{A}_{\xi}^{i}=\bigcup\left\{A_{p}^{i}(\xi): p \in \dot{\mathbb{G}}\right\}$. Standard density arguments imply that $\mathbb{P}$ forces that $\dot{A}_{\xi}^{i}$ is an infinite subset of $\mathbb{N}$ and that

$$
\mathcal{A}_{\dot{\mathbb{G}}}=\left\{\dot{A}_{\xi}^{i}: i<3, \xi<\omega_{1}\right\}
$$

is an a.d.-family. We will prove that $\mathcal{A}_{\dot{G}}$ is the reqiuired family. Let $\mathcal{A}_{\dot{\mathbb{G}}}^{i}=\left\{\dot{A}_{\xi}^{i}\right.$ : $\left.\xi<\omega_{1}\right\}$ for $i<3$. By a standard density argument and conditions (3) and (4) above $\mathbb{P}$ forces that $\left(\mathcal{A}_{\overparen{\mathbb{G}}}^{0}, \mathcal{A}_{\overparen{\mathbb{G}}}^{1}, \mathcal{A}_{\dot{\mathbb{G}}}^{2}\right)$ is a 3 -Luzin gap as witnessed by $m=0$. The rest of the proof is devoted to proving that $\mathbb{P}_{\mathcal{A}_{\dot{\mathrm{G}}}}$ satisfies the c.c.c.

Let $\left\{\dot{\rho}_{\alpha}: \alpha<\omega_{1}\right\}$ be $\mathbb{P}$-names for elements of an uncountable subset of $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. By Lemma 10 and by passing to a further uncountable subset we may assume that there are conditions $p_{\alpha} \in \mathbb{P}_{\mathcal{A}}$ for $\alpha<\omega_{1}$ and $k, l, m \in \mathbb{N}$ and disjoint $a_{\alpha} \in\left[\omega_{1} \times 3\right]^{k}$, $b_{\alpha} \in\left[\omega_{1} \times 3\right]^{l}$ and $E, F \subseteq m$ such that

$$
\begin{equation*}
p_{\alpha} \Vdash \dot{\rho}_{\alpha}=\left(\left(\bigcup_{(\xi, i) \in \check{a}_{\alpha}} \dot{A}_{\xi}^{i} \backslash \check{m}\right) \cup \check{E},\left(\bigcup_{(\xi, i) \in \check{b}_{\alpha}} \dot{A}_{\xi}^{i} \backslash \check{m}\right) \cup \check{F}\right) . \tag{*}
\end{equation*}
$$

By a standard density arguments and passing to a further uncountable subset we may assume that for each $\alpha<\omega_{1}$ we have $n=n_{p_{\alpha}} \geq m$ and $a_{\alpha} \cup b_{\alpha} \subseteq a_{p_{\alpha}}$. Moreover by another thinning out we may assume that there are bijections $\phi_{\beta, \alpha}: a_{p_{\alpha}} \rightarrow a_{p_{\beta}}$ such that for every $\alpha<\beta<\omega_{1}$ and $i \in 3$ we have

- $\phi_{\beta, \alpha}(\xi)=\xi$ for $\xi \in a_{p_{\alpha}} \cap a_{p_{\beta}}$,
- $A_{p_{\alpha}}^{i}(\xi)=A_{p_{\beta}}^{i}\left(\phi_{\beta, \alpha}(\xi)\right)$ for $\xi \in a_{p_{\alpha}}$,
- $(\xi, i) \in a_{\alpha}$ if and only if $\left(\phi_{\beta, \alpha}(\xi), i\right) \in a_{\beta}$ for $\xi \in a_{p_{\alpha}}$ and for $i<3$,
- $(\xi, i) \in b_{\alpha}$ if and only if $\left(\phi_{\beta, \alpha}(\xi), i\right) \in b_{\beta}$ for $\xi \in a_{p_{\alpha}}$ and for $i<3$.

Pick any two $\alpha<\beta<\omega_{1}$. Put $p=p_{\alpha}$ and $q=p_{\beta}$. We will construct $r \leq p, q$ such that $r$ forces that $\rho_{\alpha}$ and $\rho_{\beta}$ are compatible in $\mathbb{P}_{\mathcal{A}_{G}}$. Put $a_{r}=a_{p} \cup a_{q}$ and $n_{r}=n+n^{\prime}$ where $n^{\prime}=\left|a_{p} \backslash a_{q}\right|=\left|a_{q} \backslash a_{p}\right|$. To define $A_{r}^{i}(\xi)$ s for $\xi \in a_{r}$ and $i<3$ we need to fix $\psi: a_{p} \backslash a_{q} \rightarrow[3]^{2}$ such that if $\psi(\xi)=\{i, j\}$, then either $a_{\alpha} \cap\{(\xi, i),(\xi, j)\}=\emptyset$ or $b_{\alpha} \cap\{(\xi, i),(\xi, j)\}=\emptyset$. Such $\psi(\xi)$ can be found because the complements of $a_{\alpha}$ and of $b_{\alpha}$ cover $a_{p_{\alpha}} \times 3$ as $a_{\alpha} \cap b_{\alpha}=\emptyset$ and so out of three elements $(\xi, 0),(\xi, 1),(\xi, 2)$ two must be in the same complement. Note that by the choice of $\alpha$ and $\beta$ we also have $a_{\alpha} \cap\{(\xi, i),(\xi, j)\}=\emptyset$ if and only if $a_{\beta} \cap\{(\phi(\xi), i),(\phi(\xi), j)\}=\emptyset$ and $b_{\alpha} \cap\{(\xi, i),(\xi, j)\}=\emptyset$ if and only if $b_{\beta} \cap\{(\phi(\xi), i),(\phi(\xi), j)\}=\emptyset$. Let $a_{p} \backslash a_{q}=$
$\left\{\xi_{k^{\prime}}: k^{\prime}<n^{\prime}\right\}$ and $\phi\left(\xi_{k^{\prime}}\right)=\eta_{k^{\prime}}$ for $k^{\prime}<n^{\prime}$. We are ready to define

$$
A_{r}^{i}(\xi)= \begin{cases}A_{p}^{i}(\xi)=A_{q}^{i}(\xi) & \text { if } \xi \in a_{p} \cap a_{q} \text { and } i=0,1,2 \\ A_{p}^{i}(\xi) \cup\left\{n+k^{\prime}\right\} & \text { if } \xi=\xi_{k^{\prime}} \text { and } i=\min (\psi(\xi)) \\ A_{p}^{i}(\xi) & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i \neq \min (\psi(\xi)), \\ A_{q}^{i}(\xi) \cup\left\{n+k^{\prime}\right\} & \text { if } \xi=\eta_{k^{\prime}} \text { and } i=\max (\psi(\xi)) \\ A_{q}^{i}(\xi) & \text { if } \xi \in a_{q} \backslash a_{p} \text { and } i \neq \min \left(\psi\left(\phi^{-1}(\xi)\right)\right)\end{cases}
$$

It should be clear that $r \in \mathbb{P}$ and $r \leq p, q$. The remaining part is devoted to proving that $r$ forces that $\rho_{\alpha}$ and $\rho_{\beta}$ are compatible in $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. By $\left(^{*}\right)$ it is enough to prove that for any $(\xi, i) \in a_{\alpha}$ and $(\eta, j) \in b_{\beta}$ we have $r \Vdash \dot{A}_{\xi}^{i} \cap \dot{A}_{\eta}^{j} \backslash \check{m}=\emptyset$ and for any $(\xi, i) \in b_{\alpha}$ and $(\eta, j) \in a_{\beta}$ we have $r \Vdash \dot{A}_{\xi}^{i} \cap \dot{A}_{\eta}^{j} \backslash \check{m}=\emptyset$. For this by (3) and (7) it is enough to prove that for any $(\xi, i) \in a_{\alpha}$ and $(\eta, j) \in b_{\beta}$ we have we have $A_{r}^{i}(\xi) \cap A_{r}^{j}(\eta) \backslash m=\emptyset$ and for any $(\xi, i) \in b_{\alpha}$ and $(\eta, j) \in a_{\beta}$ we have $A_{r}^{i}(\xi) \cap A_{r}^{j}(\eta) \backslash m=\emptyset$. We have

$$
\left[A_{r}^{i}(\xi) \cap A_{r}^{j}(\eta) \backslash m\right] \cap n=\left[A_{q}^{i}\left(\phi_{\beta, \alpha}(\xi)\right) \cap A_{q}^{j}(\eta) \backslash m\right] \cap n
$$

which must be empty if $\left(\phi_{\alpha, \beta}(\xi), i\right) \in a_{\beta}$ and $(\eta, j) \in b_{\beta}$ or if $\left(\phi_{\alpha, \beta}(\xi), i\right) \in b_{\beta}$ and $(\eta, j) \in a_{\beta}$ because $\dot{\rho}$ is forced to be a condition of $\mathbb{P}_{\mathcal{A}_{\vec{G}}}$. This however happens if and only if $(\xi, i) \in a_{\alpha}$ and $(\eta, j) \in b_{\beta}$ and $(\xi, i) \in b_{\alpha}$ and $(\eta, j) \in a_{\beta}$ respectively. So we are left with proving $\left[A_{r}^{i}(\xi) \cap A_{r}^{j}(\eta) \backslash m\right] \cap\left(n_{r} \backslash n\right)=\emptyset$ under appropriate hypotheses. This follows from the construction, more specifically from the choice of $\psi$ which guarantees that $A_{r}^{i}(\xi) \cap A_{r}^{j}(\eta)$ intersect above $n$ only if $\eta=\phi_{\beta, \alpha}(\xi)$ and $i=\min \left(\psi(\xi)\right.$ and $j=\max (\psi(\xi))$. As for $\eta=\psi_{\beta, \alpha}(\xi)$ we have that $(\eta, j) \in a_{\beta}$ $\left((\eta, j) \in b_{\beta}\right)$ if and only if $(\xi, j) \in a_{\alpha}\left((\xi, j) \in b_{\alpha}\right)$, the choice of $\psi$ guarantees that the intersection is empty.

Proposition 21. There is a $\sigma$-centered forcing notion $\mathbb{P}$ such that $\mathbb{P}$ forces that there is an a.d.-family $\mathcal{A}$ such that $\mathbb{P}_{\mathcal{A}}$ admits an uncountable antichain and $\mathcal{A}$ does not contain a 2-Luzin gap.

Proof. We consider a forcing notion $\mathbb{P}$ consisting of conditions

$$
p=\left(n_{p}, a_{p},\left(A_{p}^{0}(\xi), A_{p}^{1}(\xi), A_{p}^{2}(\xi), A_{p}^{3}(\xi): \xi \in a_{p}\right)\right)
$$

satisfying
(1) $n_{p} \in \mathbb{N}, a_{p} \in\left[\omega_{1}\right]^{<\omega}$,
(2) $A_{p}^{i}(\xi) \subseteq n_{p}$ for $i<4$ and $\xi \in a_{p}$,
(3) $A_{p}^{i}(\xi) \cap A_{p}^{j}(\xi)=\emptyset$ for all $\xi \in a_{p}$ and $i<j<4$,
(4) for any distinct $\xi, \eta \in a_{p}$ there is $i<3$ such that

$$
\left[A_{p}^{i}(\xi) \cap A_{p}^{3}(\eta)\right] \cup\left[A_{p}^{3}(\xi) \cap A_{p}^{i}(\eta)\right] \neq \emptyset
$$

For $p, q \in \mathbb{P}$ we say that $p \leq q$ if
(5) $n_{p} \geq n_{q}, a_{p} \supseteq a_{q}$,
(6) $A_{p}^{i}(\xi) \cap n_{q}=A_{q}^{i}(\xi) \cap n_{q}$ for all $i<4$ and $\xi \in a_{q}$,
(7) $\left(\bigcup_{i<4} A_{p}^{i}(\xi)\right) \cap\left(\bigcup_{i<4} A_{p}^{i}(\eta)\right) \subseteq n_{q}$ for any distinct $\xi, \eta \in a_{q}$.

First we will prove that if $p, q \in \mathbb{P}$ and $n_{p}=n_{q}$ and $A_{p}^{i}(\xi)=A_{q}^{i}(\xi)$ for all $i<4$ and $\xi \in a_{p} \cap a_{q}$, then $p$ and $q$ are compatible in $\mathbb{P}$. This and the fact that $X^{\omega_{1}}$ with the product topology is separable for any finite $X$ imply that $\mathbb{P}$ is $\sigma$-centered.

For this first note that if $a_{q} \subseteq a_{p}$ (and the above hypotheses hold), then $p \leq q$. So we may assume that $a_{p} \backslash a_{q} \neq \emptyset \neq a_{q} \backslash a_{p}$. To construct $r \in \mathbb{P}$ which is stronger than $p$ and $q$ define $n=n_{p}=n_{q}$ and $a_{r}=a_{p} \cup a_{q}$. Let $k=\left|a_{p} \backslash q_{q}\right| \cdot\left|a_{q} \backslash a_{p}\right|$ and put $n_{r}=n+k$. Let $k \backslash n=\left\{k_{\xi, \eta}: \xi \in a_{p} \backslash a_{q}, \eta \in a_{q} \backslash a_{p}\right\}$. Define

$$
A_{r}^{i}(\xi)= \begin{cases}A_{p}^{i}(\xi)=A_{q}^{i}(\xi) & \text { if } \xi \in a_{p} \cap a_{q} \text { and } i<4 \\ A_{p}^{i}(\xi) \cup\left\{k_{\xi, \eta}: \eta \in a_{q} \backslash a_{p}\right\} & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i=0 \\ A_{q}^{i}(\xi) \cup\left\{k_{\xi, \eta}: \xi \in a_{p} \backslash a_{q}\right\} & \text { if } \xi \in a_{q} \backslash a_{p} \text { and } i=3 \\ A_{p}^{i}(\xi) & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i=1,2,3 \\ A_{q}^{i}(\xi) & \text { if } \xi \in a_{q} \backslash a_{p} \text { and } i=0,1,2\end{cases}
$$

Then it is clear that $r \in \mathbb{P}$ because $k_{\xi, \eta} \in A_{r}^{0}(\xi) \cap A_{r}^{3}(\eta)$ for $\xi \in a_{p} \backslash a_{q}$ and $\eta \in a_{q} \backslash a_{p}$. Moreover $r \leq p, q$ because all $k_{\xi, \eta} \mathrm{s}$ are distinct. This completes the proof that $\mathbb{P}$ is $\sigma$-centered.

Let $\dot{G}$ be a $\mathbb{P}$-name for a generic filter in $\mathbb{P}$. For $i<4$ and $\xi<\omega_{1}$ let $\dot{A}_{\xi}^{i}$ be a $\mathbb{P}$-name for a subset of $\mathbb{N}$ such that $\mathbb{P}$ forces that $\dot{A}_{\xi}^{i}=\bigcup\left\{A_{p}^{i}(\xi): p \in \dot{\mathbb{G}}\right\}$. Standard density arguments imply that $\mathbb{P}$ forces that $\dot{A}_{\xi}^{i}$ is an infinite subset of $\mathbb{N}$ and that

$$
\mathcal{A}_{\dot{G}}=\left\{\dot{A}_{\xi}^{i}: i<4, \xi<\omega_{1}\right\}
$$

is an a.d.-family. We will prove that $\mathcal{A}_{\dot{G}}$ is the required family. Let $\dot{\rho}_{\xi}=\left(\bigcup_{i<3} \dot{A}_{\xi}^{i}, \dot{A}_{\xi}^{3}\right)$ for $\xi<\omega_{1}$. By a standard density argument and conditions (3) and (4) above $\mathbb{P}$ forces that $\left\{\dot{\rho}_{\xi}: \xi<\omega_{1}\right)$ is an antichain in $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. The rest of the proof is devoted to showing that $\mathcal{A}_{\dot{G}}$ does not admit a 2 -Luzin gap.

Let $\dot{\xi}_{\alpha}, \dot{\eta}_{\alpha}$ for $\alpha<\omega_{1}$ be $\mathbb{P}$-names for elements of $\omega_{1}$ let $\dot{k}_{\alpha}, i_{\alpha}$ for $\alpha<\omega_{1}$ be $\mathbb{P}$-names for elements of 4 and let $\dot{m}$ be a $\mathbb{P}$-name for an element of $\mathbb{N}$ such that $\mathbb{P}$ forces that

- $\left\langle\dot{\xi}_{\alpha}, \dot{k}_{\alpha}\right\rangle \neq\left\langle\dot{\eta}_{\alpha}, i_{\alpha}\right\rangle$
- $\left\{\left\langle\dot{\xi}_{\alpha}, \dot{k}_{\alpha}\right\rangle,\left\langle\dot{\eta}_{\alpha}, \dot{i}_{\alpha}\right\rangle\right\} \cap\left\{\left\langle\dot{\xi}_{\beta}, \dot{k}_{\beta}\right\rangle,\left\langle\dot{\eta}_{\beta}, \dot{l}_{\beta}\right\rangle\right\}=\emptyset$
- $\dot{A}_{\xi_{\alpha}}^{k_{\alpha}} \cap \dot{A}_{\eta_{\alpha}}^{i_{\alpha}} \subseteq \dot{m}$
for every $\alpha<\beta<\omega_{1}$. It is enough to show that for every $p \in \mathbb{P}$ there is $\alpha<\beta<\omega_{1}$ and $q \leq p$ which forces that

$$
\begin{equation*}
A_{\dot{\xi}_{\alpha}}^{\dot{k}_{\alpha}} \cap A_{\dot{\eta}_{\beta}}^{i_{\beta}}, A_{\dot{\eta}_{\alpha}}^{i_{\alpha}} \cap A_{\dot{\xi}_{\beta}}^{\dot{k}_{\beta}} \subseteq \dot{m} \tag{*}
\end{equation*}
$$

Below any condition of $\mathbb{P}$ there are conditions $p_{\alpha}$ for $\alpha<\omega_{1}$ and $\xi_{\alpha}, \eta_{\alpha}<\omega_{1}$ and $k_{\alpha}, l_{\alpha}, m \in \mathbb{N}$ such that for each $\alpha$ the condition $p_{\alpha}$ forces that $\dot{\xi}_{\alpha}=\check{\xi}_{\alpha}, \dot{\eta}_{\alpha}=$ $\check{\eta}_{\alpha}, \dot{k}_{\alpha}=\check{k}_{\alpha}, \dot{l}_{\alpha}=\check{l}_{\alpha}, \dot{m}=\check{m}$ (note that $\xi_{\alpha}$ may be equal to $\eta_{\alpha}$ or $k_{\alpha}$ may be equal to $l_{\alpha}$ ).

By a standard density arguments and passing to an uncountable subset we may assume that for each $\alpha<\omega_{1}$ we have $n=n_{p_{\alpha}}, k_{\alpha}=k$ and $l_{\alpha}=l$. Moreover by another thinning out we may assume that $\left(a_{p_{\alpha}}: \alpha<\omega_{1}\right)$ forms a $\Delta$-system and that there are bijections $\phi_{\beta, \alpha}: a_{p_{\alpha}} \rightarrow a_{p_{\beta}}$ such that for every $\alpha<\beta<\omega_{1}$ and $i<4$ we have

- $\phi_{\beta, \alpha}(\xi)=\xi$ for $\xi \in a_{p_{\alpha}} \cap a_{p_{\beta}}$,
- $A_{p_{\alpha}}^{i}(\xi)=A_{p_{\beta}}^{i}(\phi(\xi))$ for $\xi \in a_{p_{\alpha}}$,
- $\phi_{\beta, \alpha}\left(\xi_{\alpha}\right)=\xi_{\beta}$
- $\phi_{\beta, \alpha}\left(\eta_{\alpha}\right)=\eta_{\beta}$

Since the forcing is c.c.c. and forces that $\left\{\left\langle\dot{\xi}_{\alpha}, \dot{k}_{\alpha}\right\rangle,\left\langle\dot{\eta}_{\alpha}, \dot{l}_{\alpha}\right\rangle\right\} \cap\left\{\left\langle\dot{\xi}_{\beta}, \dot{k}_{\beta}\right\rangle,\left\langle\dot{\eta}_{\beta}, \dot{l}_{\beta}\right\rangle\right\}=\emptyset$ we can conclude that $\xi_{\alpha}, \eta_{\alpha} \in a_{p_{\alpha}} \backslash a_{p_{\beta}}$ for any distinct $\alpha, \beta<\omega_{1}$.

Pick any two $\alpha<\beta<\omega_{1}$. Put $p=p_{\alpha}$ and $q=p_{\beta}$. We will construct $r \leq p, q$ such that $r$ forces $\left(^{*}\right)$. Let $j<3$ be different than $k, l$. To construct $r \in \mathbb{P}$ which is stronger than $p$ and $q$ define $a_{r}=a_{p} \cup a_{q}$. Let $k=\left|a_{p} \backslash q_{q}\right| \cdot\left|a_{q} \backslash a_{p}\right|$ and put $n_{r}=n+k$. Let $k \backslash n=\left\{k_{\xi, \eta}: \xi \in a_{p} \backslash a_{q}, \eta \in a_{q} \backslash a_{p}\right\}$. Define

$$
A_{r}^{i}(\xi)= \begin{cases}A_{p}^{i}(\xi)=A_{q}^{i}(\xi) & \text { if } \xi \in a_{p} \cap a_{q} \text { and } i<4 \\ A_{p}^{i}(\xi) \cup\left\{k_{\xi, \eta}: \eta \in a_{q} \backslash a_{p}\right\} & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i=j \\ A_{q}^{i}(\xi) \cup\left\{k_{\xi, \eta}: \xi \in a_{p} \backslash a_{q}\right\} & \text { if } \xi \in a_{q} \backslash a_{q} \text { and } i=3 \\ A_{p}^{i}(\xi) & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } i \neq j \\ A_{q}^{i}(\xi) & \text { if } \xi \in a_{q} \backslash a_{p} \text { and } i<3\end{cases}
$$

Then it is clear that $r \in \mathbb{P}$ because $k_{\xi, \eta} \in A_{r}^{j}(\xi) \cap A_{r}^{3}(\eta)$ for $\xi \in a_{p} \backslash a_{q}$ and $\eta \in a_{q} \backslash a_{p}$. Moreover $r \leq p, q$ because all $k_{\xi, \eta} \mathrm{S}$ are distinct. The remaining part is devoted to proving $\left(^{*}\right)$. By (3) and (7) it is enough to prove that $A_{r}^{k}\left(\xi_{\alpha}\right) \cap A_{r}^{l}\left(\eta_{\beta}\right), A_{r}^{l}\left(\eta_{\alpha}\right) \cap$ $A_{r}^{k}\left(\xi_{\beta}\right) \subseteq m$. As $A_{r}^{k}\left(\xi_{\alpha}\right) \cap n=A_{p}^{k}\left(\xi_{\alpha}\right) \cap n$ and $A_{r}^{l}\left(\eta_{\beta}\right) \cap n=A_{q}^{l}\left(\eta_{\beta}\right) \cap n=A_{p}^{l}\left(\eta_{\alpha}\right) \cap n$ we conclude that $A_{r}^{k}\left(\xi_{\alpha}\right) \cap A_{r}^{l}\left(\eta_{\beta}\right) \cap \subseteq A_{p}^{k}\left(\xi_{\alpha}\right) \cap A_{p}^{l}\left(\eta_{\alpha}\right)$ which is included in $m$ since $\mathbb{P}$ forces that $\dot{A}_{\xi_{\alpha}}^{\dot{k}_{\alpha}} \cap \dot{A}_{\eta_{\alpha}}^{i_{\alpha}} \subseteq \dot{m}$. Similarly $A_{r}^{l}\left(\eta_{\alpha}\right) \cap A_{r}^{k}\left(\xi_{\beta}\right) \cap n \subseteq m$.

Now no $k_{\xi, \eta}$ can belong to $A_{r}^{k}\left(\xi_{\alpha}\right) \cap A_{r}^{l}\left(\eta_{\beta}\right)$ or $A_{r}^{l}\left(\eta_{\alpha}\right) \cap A_{r}^{k}\left(\xi_{\beta}\right)$ because $j \notin$ $\{k, l\}$.
3.3. Big antichains in $\mathbb{P}_{\mathcal{A}}$. In this subsection we show that any Luzin family $\mathcal{A}$ exemplifies a spectacular failure of the c.c.c. of $\mathbb{P}_{\mathcal{A}}$, that is they are $L$-families (Proposition 22. Recall Definition 8 of an $L$-family). Luzin families are just of size $\omega_{1}$, but in ZFC there are a.d.-families $\mathcal{A}$ admitting antichains of size $\mathfrak{c}$ in $\mathbb{P}_{\mathcal{A}}$ (Proposition 23). Nevertheless consistently all a.d.-families of size $\mathfrak{c}$ have plenty of big sets of pairwise compatible elements (Proposition 24).
Proposition 22. If $\mathcal{A}$ is a Luzin family, it is an L-family.
Proof. Let $\mathcal{A}=\left\{A_{\alpha}: \alpha<\omega_{1}\right\}$. Let $f_{\eta}: \eta \rightarrow \mathbb{N}$ be the finite-to-one function which witnesses that $\mathcal{A}$ is a Luzin family as in Definition 11 ,

Let $\left\{p_{\xi}: \xi<\omega_{1}\right\}$ be essentially distinct and let and $A_{p_{\xi}}=\left(\bigcup a_{\xi} \backslash m_{\xi}\right) \cup F_{\xi}$ and $B_{p_{\xi}}=\left(\bigcup b_{\xi} \backslash m_{\xi}\right) \cup G_{\xi}$ for $\xi<\omega_{1}$, where $a_{\xi}, b_{\xi} \in[\mathcal{A}]^{<\omega}$ are such that $a_{\xi} \neq a_{\xi^{\prime}}$ and $b_{\xi} \neq b_{\xi^{\prime}}$ and $F_{\xi}, G_{\xi} \subseteq m_{\xi}$ are disjoint and $m_{\xi} \in \mathbb{N}$ for $\xi<\xi^{\prime}<\omega_{1}$. We may assume that $m_{\xi}=m, F_{\xi}=F$ and $G_{\xi}=G$ and $a_{\xi} \neq \emptyset \neq b_{\xi}$ for all $\xi<\omega_{1}$. Let

$$
\alpha_{\xi}=\max \left(\left\{\alpha: A_{\alpha} \in a_{\xi}\right\}\right), \beta_{\xi}=\max \left(\left\{\alpha: A_{\alpha} \in b_{\xi}\right\}\right)
$$

for $\xi<\omega_{1}$.
The fact that $p_{\xi}$ s are essentially distinct $\left(a_{\xi} \neq a_{\xi^{\prime}}\right.$ and $b_{\xi} \neq b_{\xi^{\prime}}$ for $\left.\xi<\xi^{\prime}<\omega_{1}\right)$ implies that for each $\xi<\omega_{1}$ there may be at most countably many $\xi^{\prime}<\omega_{1}$ such that $\alpha_{\xi}=\alpha_{\xi^{\prime}}$ or $\beta_{\xi}=\beta_{\xi^{\prime}}$. It follows that there is a function $g: \omega_{1} \rightarrow \omega_{1}$ such that $\beta_{\xi}<g\left(\alpha_{\xi}\right)$ and $\alpha_{\xi}<g\left(\beta_{\xi}\right)$ for any $\xi<\omega_{1}$.

Let $\left(\eta_{\alpha}: \alpha<\omega_{1}\right)$ be an enumeration of a club set $C \subseteq \omega_{1}$ such that $\beta<\alpha \in C$ implies $g(\beta)<\alpha$. Let $\mathbb{P}_{\alpha}=\left\{p_{\xi}: \alpha_{\xi}, \beta_{\xi} \in\left[\eta_{\alpha}, \eta_{\alpha+1}\right)\right\}$. $\mathbb{P}_{\alpha}$ s are pairwise disjoint. The choice of $C$ implies that $\left\{p_{\xi}: \xi<\omega_{1}\right\}=\bigcup_{\alpha<\omega_{1}} \mathbb{P}_{\alpha}$ (take $\alpha<\omega_{1}$ such that $\min \left(\alpha_{\xi}, \beta_{\xi}\right) \in\left[\eta_{\alpha}, \eta_{\alpha+1}\right)$ and note that $\max \left(\alpha_{\xi}, \beta_{\xi}\right) \in\left[\eta_{\alpha}, \eta_{\alpha+1}\right)$ as well $) . \mathbb{P}_{\alpha} \mathrm{s}$ are also countable. For $n \in \mathbb{N}$ let $\mathbb{Q}_{n} \subseteq\left\{p_{\xi}: \xi<\omega_{1}\right\}$ be such that $\left|\mathbb{Q}_{n} \cap \mathbb{P}_{\alpha}\right| \leq 1$ for every $\alpha<\omega_{1}$ and $\bigcup_{n \in \mathbb{N}} \mathbb{Q}_{n}=\left\{p_{\xi}: \xi<\omega_{1}\right\}$. It is enough to decompose each
uncountable $\mathbb{Q}_{n}$ into countably many subsets which are pairwise incompatible in $\mathbb{P}_{\mathcal{A}}$.

Fix $n \in \mathbb{N}$ such that $\mathbb{Q}_{n}$ is uncountable and define $h_{n}\left(p_{\xi}\right) \in \mathbb{N}$ for $p_{\xi} \in \mathbb{Q}_{n}$ by induction on $\alpha<\omega_{1}$ such that $p_{\xi} \in \mathbb{Q}_{n} \cap \mathbb{P}_{\alpha}$. If $h_{n} \mid\left(\mathbb{Q}_{n} \cap \bigcup_{\beta<\alpha} \mathbb{P}_{\alpha}\right)$ is already defined and $\mathbb{Q}_{n} \cap \mathbb{P}_{\alpha} \neq \emptyset$ is witnessed by $p_{\xi} \in \mathbb{Q}_{n} \cap \mathbb{P}_{\alpha}$, then define

$$
X_{\alpha}=\left\{p_{\xi^{\prime}}: p_{\xi^{\prime}} \in \mathbb{P}_{\beta}, \beta<\alpha, \min \left(f_{\alpha \xi}\left[\left[\eta_{\beta}, \eta_{\beta+1}\right)\right]\right) \leq m\right\}
$$

Since $f_{\alpha_{\xi}}$ is finite-to-one, the set $X_{\alpha}$ is a finite subset of $\left\{p_{\xi^{\prime}}: \xi^{\prime}<\xi\right\}$. Now choose $h_{n}\left(p_{\xi}\right)$ to be any element of $\mathbb{N}$ not belonging to $h_{n}\left[X_{\alpha}\right]$.

It follows from the construction of $h_{n}$ that $h_{n}\left(p_{\xi}\right)=h_{n}\left(p_{\xi}^{\prime}\right)$ for $\xi^{\prime}<\xi$ implies that $f_{\alpha_{\xi}}\left(\beta_{\xi^{\prime}}\right)>m$ and consequently that $\left(A_{\alpha_{\xi}} \cap B_{\beta_{\xi^{\prime}}}\right) \backslash m \neq \emptyset$ and so $p_{\xi}$ and $p_{\xi^{\prime}}$ are incompatible in $\mathbb{P}_{\mathcal{A}}$. This completes the proof as the fibers of $h_{n}$ yields the required decomposition of $\mathbb{Q}_{n}$.

Proposition 23. There is an almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ of cardinality $\mathfrak{c}$ such that $\mathbb{P}_{\mathcal{A}}$ contains an antichain of cardinality $\mathfrak{c}$.

Proof. This construction is due to Juris Steprans. For every $x \in 2^{\mathbb{N}}$ consider $A_{x}^{0}=$ $\left\{t \in 2^{<\omega}: t \frown 0 \subseteq x\right\}$ and $A_{x}^{1}=\left\{t \in 2^{<\omega}: t \frown 1 \subseteq x\right\}$. Note that $A_{x}^{0} \cap A_{x}^{1}=\emptyset$ and $\left\{A_{x}^{0}, A_{x}^{1}: x \in 2^{\mathbb{N}}\right\} \subseteq \wp\left(2^{<\omega}\right)$ is almost disjoint. Of course we can identify $2^{<\omega}$ with $\mathbb{N}$. Given distinct $x, y \in 2^{\mathbb{N}}$ there is $t \in 2^{<\omega}$ such that $t \subseteq x, y$ and $t^{\frown} i \subseteq x$ and $t^{\frown}(1-i) \subseteq y$ for some $i \in\{0,1\}$. Then $t \in A_{x}^{i} \cap A_{y}^{(1-i)}$ and so $\left(A_{x}^{0}, A_{x}^{1}\right)$ is incompatible with $\left(A_{y}^{0}, A_{y}^{1}\right)$. So $\left\{\left(A_{x}^{0}, A_{x}^{1}\right): x \in 2^{\mathbb{N}}\right\}$ is the required antichain.

Proposition 24. It is consistent that for every almost disjoint family of subsets of $\mathbb{N}$ of cardinality $\mathfrak{c}$ the forcing $\mathbb{P}_{\mathcal{A}}$ contains a set of cardinality $\mathfrak{c}$ of pairwise compatible essentially distinct conditions.

Proof. It is proved in [9] that in the iterated Sacks model every almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ of subsets of $\mathbb{N}$ contains a subfamily $\mathcal{B}$ of cardinality $\mathfrak{c}$ which is $\mathbb{R}$-embeddable. By Proposition 27 the forcing $\mathbb{P}_{\mathcal{B}}$ contains a pairwise compatible set of cardinality $\mathfrak{c}$ which is pairwise compatible set in $\mathbb{P}_{\mathcal{A}}$.

### 3.4. The $\sigma$-centeredness of $\mathbb{P}_{\mathcal{A}}$ and the $\mathbb{R}$-embeddability of $\mathcal{A}$.

Definition 25. An a.d.-family $\mathcal{A}$ is called $\mathbb{R}$-embeddable if and only if there is an injection $f: \mathbb{N} \rightarrow \mathbb{Q}$ such that for each $A \in \mathcal{A}$ the set $f[A]$ is the set of all terms of a convergent sequence to $r_{A} \in \mathbb{R} \backslash \mathbb{Q}$ and $r_{A} \neq r_{A^{\prime}}$ for all distinct $A, A^{\prime} \in \mathcal{A}$.
$\mathbb{R}$-embeddability was first formally defined in [11] using a slightly different condition. The equivalence of these definitions and several other conditions is proved in Lemma 2 of [9] (using a slightly different language of $\Psi$-spaces).

In this subsection we investigate the relation between the $\mathbb{R}$-embeddability of an a.d.-family $\mathcal{A}$ and the property of $\mathbb{P}_{\mathcal{A}}$ being $\sigma$-centered. If $\mathcal{A}$ id $\mathbb{R}$-embeddable, then $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered (Proposition 27) but the reverse implication fails in ZFC (Proposition 28). We also prove that no maximal a.d.-family $\mathcal{A}$ yields $\mathbb{P}_{\mathcal{A}} \sigma$-centered (Proposition 29).

Proposition 26. Suppose that $\mathcal{A}$ is an a.d.-family. $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered if and only if there are $A_{n}, B_{n} \subseteq \mathbb{N}$ for $n \in \mathbb{N}$ such that $A_{n} \cap B_{n}=\emptyset$ for all $n \in \mathbb{N}$ and for every $p \in \mathbb{P}_{\mathcal{A}}$ there is $n \in \mathbb{N}$ satisfying $A_{p} \subseteq A_{n}$ and $B_{p} \subseteq B_{n}$.

Proof. The sufficiency is clear. For the necessity, let $\mathbb{P}_{\mathcal{A}}=\bigcup_{n \in \mathbb{N}} \mathbb{P}_{n}$, where each $\mathbb{P}_{n}$ is pairwise compatible. Then $A_{n}=\bigcup\left\{A_{p}: p \in \mathbb{P}_{n}\right\}$ and $B_{n}=\bigcup\left\{B_{p}: p \in \mathbb{P}_{n}\right\}$ work.

Proposition 27. Suppose that $\mathcal{A}$ is an a.d.-family. If $\mathcal{A}$ is $\mathbb{R}$-embeddable, then $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.

Proof. Let $f: \mathbb{N} \rightarrow \mathbb{Q}$ witnesses the $\mathbb{R}$-embeddability of $\mathcal{A}$. Given $p \in \mathbb{P}_{\mathcal{A}}$ since $f$ is an injection, $f\left[A_{p}\right]$ and $f\left[B_{p}\right]$ are disjoint. Moreover both $f\left[A_{p}\right]$ and $f\left[B_{p}\right]$ are subsets of $\mathbb{Q}$ almost equal to finite unions of sequences converging to irrationals and the irrationals for $f\left[A_{p}\right]$ must be all distinct from the irrationals for $f\left[A_{p}\right]$. It follows that the closures of $f\left[A_{p}\right]$ and $f\left[B_{p}\right]$ consists of $f\left[A_{p}\right]$ and $f\left[B_{p}\right]$ and the irrational limits respectively, and so the closures of $f\left[A_{p}\right]$ and $f\left[B_{p}\right]$ are disjoint. It follows that there is $(U, V) \in \mathcal{B}^{2}$ such that $f\left[A_{p}\right] \subseteq U$ and $f\left[B_{p}\right] \subseteq V$, where $\mathcal{B}^{2}=\left\{\left(U_{n}, V_{n}\right): n \in \mathbb{N}\right\}$ is the set of all pairs $(U, V)$ such that $U \cap V=\emptyset$ and $U$ and $V$ are finite unions of intervals with rational end-points. It follows that $A_{n}=f^{-1}\left[U_{n}\right]$ and $B_{n}=f^{-1}\left[V_{n}\right]$ satisfy the condition from Proposition 26 and so $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.

Proposition 28. There is an a.d.-family $\mathcal{A}$ which is not $\mathbb{R}$-embeddable but $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.

Proof. Let $\mathcal{B}=\left\{B_{\alpha}: \alpha<\mathfrak{c}\right\}$ be any a.d.-family which is $\mathbb{R}$-embeddable. We will construct an a.d.-family $\mathcal{A}=\left\{A_{\xi}^{1}, A_{\xi}^{2}: \xi<\mathfrak{c}\right\}$ as in the proposition such that for each $\xi<\mathfrak{c}$ and each $i=0,1$ either there is $\alpha_{i}(\xi)<\mathfrak{c}$ such each $A_{\xi}^{i}=B_{\alpha_{i}(\xi)}$ or there are $\beta_{i}(\xi)<\gamma_{i}(\xi)<\mathfrak{c}$ such that $A_{\xi}^{i}=B_{\beta_{i}(\xi)} \cup B_{\gamma_{i}(\xi)}$. Note that the above property of elements of $\mathcal{A}$ implies that $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered as long as $\mathcal{A}$ is an a.d.-family. This follows from Proposition 27 is because $\mathbb{P}_{\mathcal{A}} \subseteq \mathbb{P}_{\mathcal{B}}$ in this case.

To construct recursively $\mathcal{A}$ let $\left\{f_{\xi}: \xi<\mathfrak{c}\right\}$ be an enumeration of all injective functions from $\mathbb{N}$ into $\mathbb{Q}$. Suppose that $\left\{A_{\xi}^{i}: \xi<\eta, i=0,1\right\}$ has been constructed for $\eta<\mathfrak{c}$. Let $X=\mathfrak{c} \backslash\left\{\alpha_{i}(\xi), \beta_{i}(\xi), \gamma_{i}(\xi): \xi<\eta\right\}$. We will pick the ordinals $\alpha_{i}(\xi), \beta_{i}(\xi), \gamma_{i}(\xi)$ from $X$. This will guarantee that $\mathcal{A}$ is an a.d.-family. At stage $\eta$ consider three cases. First case is when for each $\alpha \in X$ the set $f_{\xi}\left[B_{\alpha}\right]$ is the set of all terms of a convergent sequence and the limits $f_{\xi}\left[B_{\alpha}\right]$ and $f_{\xi}\left[B_{\alpha^{\prime}}\right]$ are distinct for distinct $\alpha, \alpha^{\prime} \in X$. In such a case define $A_{\xi}^{0}=B_{\alpha} \cup B_{\alpha^{\prime}}$ and $A_{\xi}^{1}=B_{\alpha^{\prime \prime}}$ for any distinct $\alpha, \alpha^{\prime} \alpha^{\prime \prime} \in X$. The second case is when for each $\alpha \in X$ the set $f_{\xi}\left[B_{\alpha}\right]$ is the set of all terms of a convergent sequence but the limits $f_{\xi}\left[B_{\alpha}\right]$ and $f_{\xi}\left[B_{\alpha^{\prime}}\right]$ are equal for some distinct $\alpha, \alpha^{\prime} \in X$. In such a case define $A_{\xi}^{0}=B_{\alpha}$ and $A_{\xi}^{1}=B_{\alpha^{\prime}}$. The third case is when there is $\alpha \in X$ such that $f_{\xi}\left[B_{\alpha}\right]$ is not the set of all terms of a convergent sequence. In such a case define $A_{\xi}^{0}=B_{\alpha}$ and $A_{\xi}^{1}=B_{\alpha^{\prime}}$ for any $\alpha^{\prime} \in X \backslash\{\alpha\}$.

It should be clear from the construction that no injection $f_{\xi}: \mathbb{N} \rightarrow \mathbb{Q}$ witnesses the fact that $\mathcal{A}$ is $\mathbb{R}$-embeddable, and hence $\mathcal{A}$ is as required.

Proposition 29. Suppose that $\mathcal{A}$ is a maximal a.d.-family. Then $\mathbb{P}_{\mathcal{A}}$ is not $\sigma$ centered.

Proof. Let $A_{n}, B_{n} \subseteq \mathbb{N}$ be as in Proposition[26. Assume that $\mathcal{A}$ is maximal. We will aim at arriving at a contradiction. By extending we may assume that $A_{n} \cup B_{n}=\mathbb{N}$ for every $n \in \mathbb{N}$. Define $F: \mathbb{N} \rightarrow\{0,1\}^{\mathbb{N}}$ by $F(k)(n)=0$ if and only if $k \in A_{n}$. For $A \in \mathcal{A}$ let $X_{A} \subseteq\{0,1\}^{\mathbb{N}}$ be the set of all accumulation points of $F[A]$. Clearly each
$X_{A}$ is a closed subset of $\{0,1\}^{\mathbb{N}}$ for $A \in \mathcal{A}$. Since $\mathcal{A}$ is an a.d.-family, if $A, B \in \mathcal{A}$ are distinct, then there is $m \in \mathbb{N}$ such that $A \cap B \subseteq m$ and so $(\{A \backslash m\},\{B \backslash m\}) \in \mathbb{P}_{\mathcal{A}}$. It follows from the choice of $A_{n}, B_{n}$ s that there is $n \in \mathbb{N}$ such that $F(k)(n) \neq F\left(k^{\prime}\right)(n)$ for $k \in A \backslash m$ and $k^{\prime} \in B \backslash m$ and consequently $X_{A} \cap X_{B}=\emptyset$ for any two distinct $A, B \in \mathcal{A}$. Similarly by using $(\{A \backslash m\},\{\{m\}\})$ for $m \in \mathbb{N}$ we conclude that $F[\mathbb{N}] \cap X_{A}=\emptyset$ for all $\mathcal{A} \in \mathcal{A}$; and taking $\left(\{\{m\}\},\left\{\left\{m^{\prime}\right\}\right\}\right)$ for distinct $m, m^{\prime} \in \mathbb{N}$ we conclude that $F$ is injective, in particular each $X_{A}$ for $A \in \mathcal{A}$ is nonempty by the compactness of $\{0,1\}^{\mathbb{N}}$.

By the metrizability of $\{0,1\}^{\mathbb{N}}$, if there $x \in F_{A}$ for $A \in \mathcal{A}$ which is in the closure of $F[\mathbb{N} \backslash A]$, there is a nontrivial subsequence of $F[\mathbb{N} \backslash A]$ convergent to $x$ (as $F[\mathbb{N} \backslash A]$ is disjoint from $\left.X_{A}\right)$. Then by the maximality of $\mathcal{A}$ there is $B \in \mathcal{A}$ such that $x \in F_{B}$ which contradicts our finding that $X_{A}$ s are pairwise disjoint for $A \in \mathcal{A}$. It follows that every point of $F_{A}$ for $A \in \mathcal{A}$ has an open basic neighborhood $U$ in $\{0,1\}$ such that $U \cap F_{B}=\emptyset$ for all $B \in \mathcal{A} \backslash\{A\}$. But this contradicts the fact that there are only countably many basic open sets of $\{0,1\}^{\mathbb{N}}$ and $\mathcal{A}$ must be uncountable as a maximal a.d.-family.
3.5. Antiramsey families and dichotomies. In this subsection we prove a consistent dichotomy (Proposition 31): $\mathbb{P}_{\mathcal{A}}$ is either $\sigma$-centered or fails to be c.c.c. which will find application in the following sections. This dichotomy is only consistent as consistently there exist strong counterexamples to it which we call antiramsey a.d.-families (Propositions 32, 33, 344). They also produce new interesting results concerning Banach spaces described in the following sections.
Proposition 30. Assume MA. Suppose that $\mathcal{A}$ is an a.d.-family of cardinality less than $\mathfrak{c}$. Then either $\mathbb{P}_{\mathcal{A}}$ does not satisfy c.c.c. or $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.
Proof. If $|\mathcal{A}|<\mathfrak{c}$, then $\left|\mathbb{P}_{\mathcal{A}}\right|<\mathfrak{c}$. The proposition follows from the well-known fact that under MA such c.c.c. forcings are $\sigma$-centered.

Proposition 31 (OCA). Suppose that $\mathcal{A}$ is a.d.-family. Then either $\mathbb{P}_{\mathcal{A}}$ does not satisfy c.c.c. or it $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.
Proof. Let

$$
X=\left\{\left(1_{A_{p}}, 1_{B_{p}}\right): p \in \mathbb{P}_{A}\right\} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}
$$

Following [13] define a coloring $c:[X]^{2} \rightarrow\{0,1\}$ as $c(x, y)=0$ if and only if $x=$ $\left(1_{A_{p}}, 1_{B_{p}}\right)$ for $p \in \mathbb{P}_{\mathcal{A}}$ and $y=\left(1_{A_{q}}, 1_{B_{q}}\right)$ for $q \in \mathbb{P}_{\mathcal{A}}$ and $p$ and $q$ are incompatible in $\mathbb{P}_{\mathcal{A}}$. Note that $c^{-1}[\{0\}]$ is open in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ because for distinct $x, y \in X$ we have $c(x, y)=1$ if and only if there is $n \in \mathbb{N}$ such that $x_{1}(n)=1=y_{2}(n)$ or $y_{1}(n)=1=x_{2}(n)$, where $x=\left(x_{1}, x_{2}\right)$ and $y=\left(y_{1}, y_{2}\right)$. So OCA can be applied to the coloring $c$ to conclude that either $X$ can be covered by countably many 1 -monochromatic sets, that is $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered or $X$ contains an uncountable 0 monochromatic set, that is $\mathbb{P}_{\mathcal{A}}$ does not satisfy the c.c.c.
Proposition $32(\mathrm{CH})$. For any uncountable regular cardinal $\kappa$ there is a c.c.c forcing $\mathbb{P}$ which forces that there is an antiramsey a.d.-family $\mathcal{A}$ of cardinality $\kappa=\mathfrak{c}$ which is a maximal a.d.-family.

Proof. Assume CH and choose a regular uncountable $\kappa$. We consider a forcing notion $\mathbb{P}$ consisting of conditions

$$
p=\left(n_{p}, a_{p},\left(A_{p}(\xi): \xi \in a_{p}\right)\right)
$$

satisfying
(1) $n_{p} \in \mathbb{N}, a_{p} \in[\kappa]^{<\omega}$,
(2) $A_{p}(\xi) \subseteq n_{p}$ for $\xi \in a_{p}$,

For $p, q \in \mathbb{P}$ we say that $p \leq q$ if
(5) $n_{p} \geq n_{q}, a_{p} \supseteq a_{q}$,
(6) $A_{p}(\xi) \cap n_{q}=A_{q}(\xi) \cap n_{q}$ for all $\xi \in a_{q}$,
(7) $\left.A_{p}(\xi)\right) \cap A_{p}(\eta) \subseteq n_{q}$ for any distinct $\xi, \eta \in a_{q}$.

First note that if $p, q \in \mathbb{P}$ and $n_{p}=n_{q}$ and $A_{p}(\xi)=A_{q}(\xi)$ for all $\xi \in a_{p} \cap a_{q}$, then $p$ and $q$ are compatible in $\mathbb{P}$. This shows that $\mathbb{P}$ satisfies the c.c.c.

Let $\dot{\mathbb{G}}$ be a $\mathbb{P}$-name for a generic filter in $\mathbb{P}$. For $\xi<\kappa$ let $\dot{A}_{\xi}$ be a $\mathbb{P}$-name for a subset of $\mathbb{N}$ such that $\mathbb{P}$ forces that $\dot{A}_{\xi}=\bigcup\left\{A_{p}(\xi): p \in \dot{\mathbb{G}}\right\}$. Standard density arguments imply that $\mathbb{P}$ forces that $\dot{A}_{\xi}$ is an infinite subset of $\mathbb{N}$ and that

$$
\mathcal{A}_{\dot{\mathbb{G}}}=\left\{\dot{A}_{\xi}: \xi<\kappa\right\}
$$

is an a.d.-family. We will prove that $\mathcal{A}_{\dot{\mathrm{G}}}$ is the required family.
Let $\left\{\dot{\rho}_{\alpha}: \alpha<\omega_{1}\right\}$ be $\mathbb{P}$-names for elements of an uncountable subset of $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. By Lemma 10 and by passing to a further uncountable subset we may assume that there are conditions $p_{\alpha} \in \mathbb{P}$ for $\alpha<\omega_{1}$ and $k, l, m \in \mathbb{N}$ and disjoint $a_{\alpha} \in\left[\omega_{1}\right]^{k}$, $b_{\alpha} \in\left[\omega_{1}\right]^{l}$ and $E, F \subseteq m$ such that

$$
\begin{equation*}
p_{\alpha} \Vdash \dot{\rho}_{\alpha}=\left(\left(\bigcup_{\xi \in \tilde{a}_{\alpha}} \dot{A}_{\xi} \backslash \check{m}\right) \cup \check{E},\left(\bigcup_{\xi \in \breve{b}_{\alpha}} \dot{A}_{\xi} \backslash \check{m}\right) \cup \check{F}\right) . \tag{*}
\end{equation*}
$$

By standard density arguments and passing to an uncountable subset we may assume that for each $\alpha<\omega_{1}$ we have $n=n_{p_{\alpha}} \geq m$ and $a_{\alpha}, b_{\alpha} \subseteq a_{p_{\alpha}}$. Moreover by another thinning out we may assume that there are bijections $\phi_{\beta, \alpha}: a_{p_{\alpha}} \rightarrow a_{p_{\beta}}$ such that for every $\alpha<\beta<\omega_{1}$ we have

- $\phi(\xi)=\xi$ for $\xi \in a_{p_{\alpha}} \cap a_{p_{\beta}}$,
- $A_{p_{\alpha}}(\xi)=A_{p_{\beta}}(\phi(\xi))$ for $\xi \in a_{p_{\alpha}}$,
- $\xi \in a_{\alpha}$ if and only if $\phi(\xi) \in a_{\beta}$ for $\xi \in a_{p_{\alpha}}$,
- $\xi \in b_{\alpha}$ if and only if $\phi(\xi) \in b_{\beta}$ for $\xi \in a_{p_{\alpha}}$.

Using the same argument as in the proof of the c.c.c. of $\mathbb{P}$ we see that any such conditions $p_{\alpha}$ and $p_{\beta}$ are compatible, so the fact that $\dot{a}_{\alpha} \cup \dot{b}_{\alpha}$ must be forced to be disjoint with $\dot{a}_{\beta} \cup \dot{b}_{\beta}$ implies that actually $a_{\alpha} \cup b_{\alpha} \subseteq a_{p_{\alpha}} \backslash a_{p_{\beta}}$ and $a_{\beta} \cup b_{\beta} \subseteq a_{p_{\beta}} \backslash a_{p_{\alpha}}$.

Pick any two $\alpha<\beta<\omega_{1}$. Put $p=p_{\alpha}$ and $q=p_{\beta}$. First we will construct $r \leq p, q$ such that $r$ forces that $\dot{\rho}_{\alpha}$ and $\dot{\rho}_{\beta}$ are compatible in $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. For this it is enough to put $a_{r}=a_{p} \cup a_{q}$ and $n_{r}=n$ and $A_{r}(\xi)=A_{p}(\xi)$ for $\xi \in a_{p}$ and $A_{r}(\xi)=A_{q}(\xi)$ for $\xi \in a_{q}$. For $\xi \in a_{\alpha}$ and $\eta \in b_{\beta}$ we have

$$
\left(A_{r}(\xi) \cap A_{r}(\eta)\right) \backslash m=\left(A_{q}(\phi(\xi)) \cap A_{q}(\eta)\right) \backslash m=\emptyset
$$

since $\phi(\xi) \in a_{\beta}$ and $q$ forces that $\dot{\rho}_{\beta}$ is a condition of $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. So by (7) we have that $r$ forces that $\left(A_{r}(\xi) \cap A_{r}(\eta)\right) \backslash m=\emptyset$ and so by the choice of $\xi, \eta$ we have that $r$ forces $\rho_{\alpha}$ to be compatible with $\rho_{\beta}$.

Now we will construct $s \leq p, q$ such that $s$ forces that $\rho_{\alpha}$ and $\rho_{\beta}$ are incompatible in $\mathbb{P}_{\mathcal{A}_{\dot{G}}}$. Note that the hypothesis of the proposition implies that $a_{\alpha}, a_{\beta}, b_{\alpha}, b_{\beta} \neq \emptyset$.

So pick $\xi_{0} \in a_{\alpha}$ and $\eta_{0} \in b_{\beta}$. Now put $a_{r}=a_{p} \cup a_{q}$ and $n_{r}=n+1$ and

$$
A_{r}(\xi)= \begin{cases}A_{p}(\xi)=A_{q}(\xi) & \text { if } \xi \in a_{p} \cap a_{q} \\ A_{p}(\xi) \cup\{n\} & \text { if } \xi=\xi_{0} \\ A_{p}(\xi) & \text { if } \xi \in a_{p} \backslash a_{q} \text { and } \xi \neq \xi_{0} \\ A_{q}(\xi) \cup\{n\} & \text { if } \xi=\eta_{0} \\ A_{q}(\xi) & \text { if } \xi \in a_{q} \backslash a_{p} \text { and } \xi \neq \eta_{0}\end{cases}
$$

It should be clear that $r \in \mathbb{P}$ and $r \leq p, q$ moreover $r$ forces that

$$
\left.\check{n} \in \dot{( } A\left(\xi_{0}\right) \backslash \check{m}\right) \cap\left(\dot{A}\left(\eta_{0}\right) \backslash \check{m}\right)
$$

so $r$ forces that $\dot{\rho}_{\alpha}$ is incompatible with $\dot{\rho}_{\beta}$.
For the maximality of $\mathcal{A}_{\dot{G}}$ suppose that there is a $\mathbb{P}$-name $\dot{A}$ for an infinite subset of $\mathbb{N}$ and $p \in \mathbb{P}$ which forces that $\dot{A}$ is almost disjoint from all $\dot{A}_{\xi}$ for $\xi<\kappa$. It follows that there is a name $\dot{X}$ for an uncountable subset of $\omega_{1}$ and $k \in \mathbb{N}$ and $q \leq p$ such that

$$
\begin{equation*}
q \Vdash \dot{A}_{\alpha} \cap \dot{A} \subseteq \check{k} \text { for all } \alpha \in \dot{X} \tag{*}
\end{equation*}
$$

Let $p_{\xi} \Vdash \check{\alpha}_{\xi} \in \dot{X}$ for $\alpha_{\xi} \in a_{p_{\xi}}$ and $\alpha_{\xi} \neq \alpha_{\eta}$ for $\xi \neq \eta$. By passing to an uncountable subset of $\left\{p_{\xi}: \xi<\omega_{1}\right\}$ we may assume that $n_{p_{\xi}}=n_{p_{\eta}}=n \in \mathbb{N}$ for all $\xi<\eta<\omega_{1}$ and $\left\{a_{p_{\xi}}: \xi<\omega_{1}\right\}$ forms a $\Delta$-system with root $\Delta$ and $A_{p_{\xi}}(\alpha)=A_{p_{\eta}}(\alpha)$ for every $\alpha \in \Delta$ and $\xi<\eta<\omega_{1}$. Let $q=\left(n, \Delta,\left(A_{p_{0}}(\alpha): \alpha \in \Delta\right)\right)$. There is $r \leq q$ which decides an element $m$ of $\dot{A}$ above $n$. Now it is easy to see that there is $\xi<\omega_{1}$ such that $\left(a_{p_{\xi}} \backslash \Delta\right) \cap a_{r}$ and one can construct $s \leq r, p_{\xi}$ such that $s$ forces that $m \in \dot{A}_{\alpha_{\xi}}$ which contradicts $(*)$.

The argument that $\mathbb{P}$ forces that $\mathfrak{c}$ is equal to $\kappa$ is standard (see e.g. [15]).
Proposition $33(\mathrm{CH})$. There is an antiramsey a.d.-family $\mathcal{A}$ which is a maximal a.d.-family.

Proof. Use CH to construct two enumerations. First enumeration is $\left(\mathbb{P}_{\xi}: \xi<\omega_{1}\right)$ of all countably infinite families of elements $p$ of the form $p=\left(A_{p}, B_{p}\right)$, where $A_{p}, B_{p} \subseteq \mathbb{N}$ and $A_{p} \cap B_{p}=\emptyset$. The second one is $\left\{X_{\xi}: \xi<\omega\right\}$ of all infinite subsets of $\mathbb{N}$ such that $X_{0}=\mathbb{N}$. We construct an antiramsey a.d.-family $\mathcal{C}=\left\{C_{\alpha}: \alpha<\omega_{1}\right\}$ by induction on $\alpha<\omega_{1}$. Let $\left\{C_{n}: n \in \omega\right\}$ be any family of pairwise disjoint infinite subsets of $\mathbb{N}$. Having constructed $\mathcal{C}_{\alpha}=\left\{C_{\beta}: \beta<\alpha\right\}$ for $\omega \leq \alpha<\omega_{1}$ we will describe the construction of $C_{\alpha}$. Let $\mathcal{I}_{\alpha}$ be the ideal generated by $\mathcal{C}_{\alpha}$ and finite subsets of $\mathbb{N}$. We will consider a partial order $\mathbb{Q}_{\alpha}$ consisting of conditions of the form $q=\left(n_{q}, s_{q}, I_{q}\right)$, where

- $n_{q} \in \mathbb{N}$
- $s_{q} \in 2^{<\omega}$
- $n_{q}=\left|s_{q}\right|$
- $I_{q} \in \mathcal{I}_{\alpha}$
- $s^{-1}[\{1\}] \cap I_{q}=\emptyset$.
and $q \leq r$ if $s_{q} \supseteq s_{r}$ and $I_{q} \supseteq I_{r}$ and $n_{q} \geq n_{r}$.

$$
C_{\alpha}=\bigcup\left\{s^{-1}[\{1\}]: s \in \mathbb{G}_{\alpha}\right\},
$$

where $\mathbb{G}_{\alpha}$ is a filter in $\mathbb{Q}_{\alpha}$ which meets certain countable family of dense subsets of $\mathbb{Q}_{\alpha}$. Below we will specify these dense sets. First consider sets

- $D_{\xi, i}^{1}=\left\{q \in \mathbb{Q}_{\alpha}: \exists j>i q(j)=1, \& j \in X_{\xi}\right\}$ for $i \in \mathbb{N}$ and $\xi<\omega_{1}$ such that $X_{\xi} \notin \mathcal{I}_{\alpha}$.
- $D_{\beta}^{2}=\left\{q \in \mathbb{Q}_{\alpha}: C_{\beta} \subseteq^{*} I_{q}\right\}$ for $\beta<\alpha$.

It is clear that since $\mathcal{C}_{\alpha}$ is almost disjoint, infinite and consiting of infinite sets, the above sets are dense in $\mathbb{Q}_{\alpha}$. It is clear that if $\mathbb{G}$ meets each $D_{X_{0}, i}^{1}$ and $D_{\beta}^{2}$ for all $i \in \mathbb{N}$ and all $\beta<\alpha$, then $C_{\alpha}$ is infinite and almost disjoint from all $C_{\beta}$ for $\beta<\alpha$.

For $m \in \mathbb{N}$ and $\xi<\omega_{1}$ such that $\mathbb{P}_{\xi}$ is a family of essentially distinct elements of $\mathbb{P}_{\mathcal{C}_{\alpha}}$ define further sets to be proved to be dense:

- $D_{\xi, m}^{3}=\left\{q \in \mathbb{Q}_{\alpha}: \exists j>m \exists p \in \mathbb{P}_{\xi} j \in A_{p} \cap s_{q}^{-1}[\{1\}]\right\}$.
- $D_{\xi, m}^{4}=\left\{q \in \mathbb{Q}_{\alpha}: \exists j>m \exists p \in \mathbb{P}_{\xi} j \in B_{p} \cap s_{q}^{-1}[\{1\}]\right\}$.

Let us check that the above sets are dense in $\mathbb{Q}_{\alpha}$ for $m \in \mathbb{N}$ and $\xi<\omega_{1}$ as above. Fix $r \in \mathbb{Q}_{\alpha}$. By the density of $D_{m}^{1}$ we may assume that $n_{r} \geq m$. As $\mathbb{P}_{\xi}$ is infinite and consists of essentially distinct conditions of $\mathbb{P}_{\mathcal{C}_{\alpha}}$, there is $p \in \mathbb{P}_{\xi}$ such that $A_{p} \not \mathbb{E}^{*} I_{r}$. Let $j \in A_{p} \backslash\left(I_{r} \cup n_{r}\right)$. Define $q \leq r$ by putting

$$
s_{q}=s_{r} \cup\left(0 \mid\left[n_{r}, j\right)\right) \cup\{\langle j, 1\rangle\}, n_{q}=j+1, I_{q}=I_{r} .
$$

It is clear that $q \in D_{\xi, m}^{3}$. An analogous argument works for $D_{\xi, m}^{4}$.
Now for $m \in \mathbb{N}, p^{\prime} \in \mathbb{P}_{\mathcal{C}_{\alpha}}$ and $\xi<\omega_{1}$ such that $\mathbb{P}_{\xi} \subseteq \mathbb{P}_{\mathcal{C}_{\alpha}}$ define auxiliary sets

- $E_{\xi, p^{\prime}, m}^{1}=\left\{q \in \mathbb{Q}_{\alpha}: \forall p \in \mathbb{P}_{\xi} p \| p^{\prime} \Rightarrow s_{q}^{-1}[\{1\}] \backslash m \nsubseteq A_{p}\right\}$
- $E_{\xi, p^{\prime}, m}^{2}=\left\{q \in \mathbb{Q}_{\alpha}: \forall p \in \mathbb{P}_{\xi} p \| p^{\prime} \Rightarrow s_{q}^{-1}[\{1\}] \backslash m \nsubseteq B_{p}\right\}$
- $E_{\xi, p^{\prime}, m}^{3}=\left\{q \in \mathbb{Q}_{\alpha}: \exists p \in \mathbb{P}_{\xi}, p \| p^{\prime} \& B_{p} \backslash m \subseteq I_{q}\right\}$.
- $E_{\xi, p^{\prime}, m}^{4}=\left\{q \in \mathbb{Q}_{\alpha}: \exists p \in \mathbb{P}_{\xi}, p \| p^{\prime} \& A_{p} \backslash m \subseteq I_{q}\right\}$.

Finally define the last batch of sets to be proven dense:
$\begin{aligned} \text { - } D_{\xi, p^{\prime}, m}^{5} & =E_{\xi, p^{\prime}, m}^{1} \cup E_{\xi, p^{\prime}, m}^{3} . \\ \text { - } D_{\xi, p^{\prime}, m}^{6} & =E_{\xi, p^{\prime}, m}^{2} \cup E_{\xi, p^{\prime}, m}^{4} .\end{aligned}$
We use this opportunity to note that
(*) if there is $q \in \mathbb{G}_{\alpha} \cap E_{\xi, p^{\prime}, m}^{3}$, then there is $p \in \mathbb{P}_{\xi}$ which is compatible with $\left(A_{p^{\prime}} \cup\left(C_{\alpha} \backslash m\right), B_{p^{\prime}}\right)$ as long as $\left(C_{\alpha} \backslash m\right) \cap B_{p^{\prime}}=\emptyset$,
since $C_{\alpha} \cap\left(B_{p} \backslash m\right) \subseteq\left(\mathbb{N} \backslash I_{q}\right) \cap I_{q}=\emptyset$. Likewise
${ }^{(* *)}$ if there is $q \in \mathbb{G}_{\alpha} \cap E_{\xi, p^{\prime}, m}^{4}$, then there is $p \in \mathbb{P}_{\xi}$ which is compatible with $\left(A_{p^{\prime}}, B_{p^{\prime}} \cup\left(C_{\alpha} \backslash m\right)\right)$ as long as $\left(C_{\alpha} \backslash m\right) \cap A_{p^{\prime}}=\emptyset$.
Let us prove that $D_{\xi, p^{\prime}, m}^{5}$ is dense if $\mathbb{P}_{\xi}$ is a collection of essentially distinct elements of $\mathbb{P}_{\mathcal{C}_{\alpha}}, p^{\prime} \in \mathbb{P}_{\mathcal{C}_{\alpha}}$ and $m \in \mathbb{N}$. Consider $r \in \mathbb{Q}_{\alpha}$ such that $n_{r} \geq m$. If for all $p \in \mathbb{P}_{\xi}$ such that $p$ and $p^{\prime}$ are compatible we have $s_{r}^{-1}[\{1\}] \backslash m \nsubseteq A_{p}$ then $r \in E_{\xi, p^{\prime}, m}^{1} \subseteq D_{\xi, p^{\prime}, m}^{5}$. Otherwise find $p \in \mathbb{P}_{\xi}$ compatible with $p^{\prime}$ such that $s_{r}^{-1}[\{1\}] \backslash m \subseteq A_{p}$. Consider

$$
s_{q}=\left(n_{r}, s_{r}, I_{r} \cup B_{p} \backslash m\right)
$$

It is a condition of $\mathbb{Q}_{\alpha}$ because $A_{p} \cap B_{p}=\emptyset$. It is clear that $q \in E_{\xi, p^{\prime}, m}^{3} \subseteq D_{\xi, p^{\prime}, m}^{5}$ and $q \leq r$ which completes the proof of the density of $D_{\xi, p^{\prime}, m}^{5}$. An analogous argument works for $D_{\xi, p^{\prime}, m}^{6}$.

Now we specify $\mathbb{G}_{\alpha}$ as a filter in $\mathbb{Q}_{\alpha}$ which meets $D_{\xi, i}^{1}$ s for all $i \in \mathbb{N}$ and $\xi<\alpha$ such that $X_{\xi} \notin \mathcal{I}_{\alpha}, D_{\beta}^{2} \mathrm{~s}$ for all $\beta<\alpha, D_{\xi, m^{3}}^{3} \mathrm{~s}$ and $D_{\xi, m}^{4} \mathrm{~s}$ for all $m \in \mathbb{N}$ and $\xi<\alpha$ such that $\mathbb{P}_{\xi} \subseteq \mathbb{P}_{\mathcal{C}_{\alpha}}$ is a collection of essentially distinct conditions, $D_{\xi, p^{\prime}, m^{5}}^{5}$ and $D_{\xi, p^{\prime}, m^{\prime}}^{6}$ for all $m \in \mathbb{N}$, all $p^{\prime} \in \mathbb{P}_{\mathcal{C}_{\alpha}}$ and all $\xi<\alpha$ such that $\mathbb{P}_{\xi} \subseteq \mathbb{P}_{\mathcal{C}_{\alpha}}$ is a
collection of essentially distinct conditions. This completes the inductive step of the construction of the a.d. family $\mathcal{C}=\left\{C_{\alpha}: \alpha<\omega_{1}\right\}$.

Now we prove that $\mathcal{C}$ is antiramsey. Consider an uncountable family $\mathbb{P} \subseteq \mathbb{P}_{\mathcal{C}}$ of essentially distinct conditions. By passing to an uncountable subfamily we may assume that there are $m \in \mathbb{N}$ and pairwise disjoint $F, G \subseteq m$ such that each $p \in \mathbb{P}$ is of the form

$$
p=\left(\left[\left(\bigcup a_{p}\right) \backslash m\right] \cup F,\left[\left(\bigcup b_{p}\right) \backslash m\right) \cup G\right)
$$

for some finite $a_{p}, b_{p} \subseteq \mathcal{C}$ such that $a_{p} \cap b_{p}=\emptyset$ and $\left(a_{p} \cup b_{p}\right) \cap\left(a_{q} \cup b_{q}\right)=\emptyset$ for any distinct $p, q \in \mathbb{P}$. There is an infinite and countable $\overline{\mathbb{P}} \subseteq \mathbb{P}$ such that for every $p^{\prime} \in \mathbb{P}$ and every $n \in \mathbb{N}$ there is $p \in \overline{\mathbb{P}}$ such that $A_{p} \cap n=A_{p^{\prime}} \cap m$ and $B_{p} \cap m=B_{p^{\prime}} \cap m$. There is $\xi<\omega_{1}$ such that $\overline{\mathbb{P}}=\mathbb{P}_{\xi}$ and there is $\xi<\eta<\omega_{1}$ such that $\mathbb{P}_{\xi} \subseteq \mathbb{P}_{\mathcal{C}_{\eta}}$. Pick $p^{\prime} \in \mathbb{P}$ such that $a_{p^{\prime}} \cap(\eta+1)=\emptyset=b_{p^{\prime}} \cap(\eta+1)$. We will show that there is $p \in \mathbb{P}_{\xi}$ such that $p$ and $p^{\prime}$ are compatible and that there is $p \in \mathbb{P}^{\prime}$ such that $p$ and $p^{\prime}$ are incompatible.

Let $a_{p^{\prime}} \cup b_{p^{\prime}}=\left\{\alpha_{1}, \ldots, \alpha_{k}\right\}$ in the increasing order. For finite disjoint $F^{\prime}, G^{\prime} \subseteq \mathbb{N}$ and $0 \leq l \leq k$ define as $p_{0}^{\prime}\left(F^{\prime}, G^{\prime}\right)=\left(F \cup F^{\prime}, G \cup G^{\prime}\right)$ and
$p_{l}^{\prime}\left(F^{\prime}, G^{\prime}\right)=\left(\left[\bigcup\left(a_{p^{\prime}} \cap\left\{C_{\alpha_{1}} \ldots, C_{\alpha_{l}}\right\}\right) \backslash m\right] \cup F \cup F^{\prime},\left[\bigcup\left(b_{p^{\prime}} \cap\left\{C_{\alpha_{1}} \ldots, C_{\alpha_{l}}\right\}\right) \backslash m\right] \cup G \cup G^{\prime}\right)$
for $2 \leq l \leq k$. By induction on $1 \leq l \leq k$ we will prove that for every finite $F^{\prime} \subseteq A_{p^{\prime}}$ and finite $G^{\prime} \subseteq B_{p^{\prime}}$ there is there is $p \in \mathbb{P}_{\xi}$ compatible with $p_{l}\left(F^{\prime}, G^{\prime}\right)$ and stronger than $\left(F^{\prime}, G^{\prime}\right)$. For $l=k$ and $F^{\prime}=G^{\prime}=\emptyset$ this will give the desired $p \in \mathbb{P}_{\xi}$ compatible with $p^{\prime}$.

For $l=0$ this follows from the choice of $\mathbb{P}_{\xi}$ since $p$ is compatible with $\left(F^{\prime}, G^{\prime}\right)$ if $F^{\prime} \subseteq A_{p^{\prime}}$ and $G^{\prime} \subseteq B_{p^{\prime}}$.

To prove it for $l>0$, fix $F^{\prime} \subseteq A_{p^{\prime}}, G^{\prime} \subseteq B_{p^{\prime}}$ and first assume that $\alpha_{l} \in a_{p^{\prime}}$ and consider $q \in G_{\alpha_{l}} \cap D_{\xi, p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right), m}^{5}$. By the inductive assumption there is $p \in \mathbb{P}_{\xi}$ compatible with $p_{l-1}^{\prime}\left(F^{\prime} \cup\left(s_{q}^{-1}[\{1\}] \backslash m\right), G^{\prime}\right)$ and stronger than $\left(F^{\prime} \cup\right.$ $\left.\left(s_{q}^{-1}[\{1\}] \backslash m\right), G^{\prime}\right)$, so $\left(s_{q}^{-1}[\{1\}] \backslash m\right) \subseteq A_{p}$ and $p$ is compatible with $p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right)$. It follows that $q \notin E_{\xi, p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right), m}^{1}$ and hence $q \in E_{\xi, p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right), m}^{3}$. By $\left(^{*}\right)$ it follows that $p_{l}^{\prime}\left(F^{\prime}, G^{\prime}\right)=\left(A_{p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right)} \cup\left(C_{\alpha_{l}} \backslash m\right), B_{p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right)}\right)$ is compatible with $p$ which completes the inductive step and allows to conclude the existence of the desired $p \in \mathbb{P}_{\xi}$ compatible with $p^{\prime}$. The case of $\alpha_{l} \in b_{p^{\prime}}$ is analogous and uses $\left(^{* *}\right)$ instead of $\left({ }^{*}\right)$ and $D_{\xi, p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right), m}^{6}$ instead of $D_{\xi, p_{l-1}^{\prime}\left(F^{\prime}, G^{\prime}\right), m}^{5}$.

To obtain $p \in \mathbb{P}_{\xi}$ incompatible with $p^{\prime}$ pick any $\alpha \in a_{p^{\prime}}$ and use the fact that $\mathbb{G}_{\alpha}$ intersects $D_{\xi, m}^{4}$ to find $j>m$ in $C_{\alpha} \cap B_{p}$. This guarantees that $p$ and $p^{\prime}$ are incompatible. This completes the proof of the fact that $\mathcal{C}$ is antiramsey.

To see that $\mathcal{C}$ is a maximal a.d.-family, it is enough to prove that whenever $X \subseteq \mathbb{N}$ is infinite, then there is $\alpha<\omega_{1}$ such that $X \cap C_{\alpha}$ is infinite. If $X \in \mathcal{I}=\bigcup_{\alpha<\omega_{1}} \mathcal{I}_{\alpha}$, there are finitely many generators $C_{\alpha}$ of $\mathcal{I}$ which cover $X$, so at least one of them must have infinite intersection with $X$. If $X \notin \mathcal{I}$, then consider $\xi<\omega_{1}$ such that $X=X_{\xi}$ and $\alpha>\xi$. Since $\mathbb{G}_{\alpha}$ intersects $D_{i, \xi}^{1}$ for all $i \in \mathbb{N}$ we conclude that $C_{\alpha} \cap X_{\xi}$ is infinite as required.

Proposition 34. In any model of ZFC obtained by adding a Cohen real there is an antiramsey a.d.-family $\mathcal{A}$ of cardinality $\omega_{1}$.

Proof. Let $V$ be the ground model, $\mathbb{Q}$ be the Cohen forcing i.e., $\{0,1\}^{<\omega}$ with the end-extension as the order, and let $\dot{c} \in\{0,1\}$ be the name for a Cohen real over $V$. Let $\mathcal{B}=\left\{B_{\xi}: \xi<\omega_{1}\right\}$ be a Luzin family (in the sense of Definition 11) in $V$. We claim that $\mathcal{A}=\left\{A_{\xi}: \xi<\omega_{1}\right\}$ satisfies the proposition in $V[c]$, where $\mathbb{Q}$ forces that

$$
\dot{A}_{\xi}=\check{B}_{\xi} \cap \dot{c}^{-1}[\{1\}]
$$

for every $\xi \in \omega_{1}$.
It is clear that $\mathcal{A}$ is an a.d.-family since its is a refinement of $\mathcal{B}$ and it is wellknown that a Cohen real intersects any ground model set on an infinite set. Work in $V$ and suppose that $\left\{\dot{p}_{\xi}: \xi<\omega_{1}\right\}$ are $\mathbb{Q}$-names for conditions of $\mathbb{P}_{\mathcal{A}}$ such that $A_{\dot{p}_{\xi}} \neq A_{\dot{p}_{\eta}}$ and $B_{\dot{p}_{\xi}} \neq B_{\dot{p}_{\eta}}$ for $\xi<\eta$. Let $p \in \mathbb{Q}$.

Deciding and using Lemma 10 and using the countability of $\mathbb{Q}$ we may assume that there are $a_{\xi}, b_{\xi} \in\left[\omega_{1}\right]^{<\omega}$ and $m \in \mathbb{N}$ and $E, F \subseteq m$ and a condition $q \in \mathbb{Q}$ such that $q \leq p$ forces that

$$
\dot{p}_{\xi}=\left(\left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{a}_{\xi}\right\} \backslash \check{m}\right) \cup \check{F},\left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{b}_{\xi}\right\} \backslash \check{m}\right) \cup \check{E}\right)
$$

and $E \cap F=\emptyset, a_{\xi} \cap b_{\xi}=\emptyset=\left(a_{\xi} \cup b_{\xi}\right) \cap\left(a_{\eta} \cup b_{\eta}\right)$ for any $\xi<\eta<\omega_{1}$. Let $k=|q|$. We may assume that $k \geq m$. Find uncountable $\Gamma \subseteq \omega_{1}$ such that

$$
\begin{align*}
& {\left[\left(\bigcup\left\{B_{\alpha}: \alpha \in a_{\xi}\right\} \backslash m\right) \cup F\right] \cap k=\left[\left(\bigcup\left\{B_{\alpha}: \alpha \in a_{\eta}\right\} \backslash m\right) \cup F\right] \cap k}  \tag{1}\\
& {\left[\left(\bigcup\left\{B_{\alpha}: \alpha \in b_{\xi}\right\} \backslash m\right) \cup F\right] \cap k=\left[\left(\bigcup\left\{B_{\alpha}: \alpha \in b_{\eta}\right\} \backslash m\right) \cup F\right] \cap k} \tag{2}
\end{align*}
$$

for every distinct $\xi, \eta \in \Gamma$. Of course it follows from the above that $q$ forces that

$$
\begin{align*}
& {\left[\left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{a}_{\xi}\right\} \backslash \check{m}\right) \cup \check{F}\right] \cap \check{k}=\left[\left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{a}_{\eta}\right\} \backslash \check{m}\right) \cup \check{F}\right] \cap \check{k}}  \tag{1}\\
& {\left[\left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{b}_{\xi}\right\} \backslash \check{m}\right) \cup \check{F}\right] \cap \check{k}=\left[\left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{b}_{\eta}\right\} \backslash \check{m}\right) \cup \check{F}\right] \cap \check{k}} \tag{2}
\end{align*}
$$

for every distinct $\xi, \eta \in \Gamma$.
Now aim at finding $r \leq q$ and distinct $\xi, \eta \in \Gamma$ such that $r$ forces that $\dot{p}_{\xi}$ and $\dot{p}_{\eta}$ are compatible. Choose any distinct $\xi, \eta \in \Gamma$. Find $k^{\prime} \geq k$ such that

$$
\begin{aligned}
& \bigcup\left\{B_{\alpha}: \alpha \in a_{\xi}\right\} \cap \bigcup\left\{B_{\alpha}: \alpha \in b_{\eta}\right\} \backslash k^{\prime}=\emptyset \\
& \bigcup\left\{B_{\alpha}: \alpha \in a_{\eta}\right\} \cap \bigcup\left\{B_{\alpha}: \alpha \in b_{\xi}\right\} \backslash k^{\prime}=\emptyset
\end{aligned}
$$

The existence of such a $k^{\prime} \in \mathbb{N}$ follows from the fact that $\left(a_{\xi} \cup b_{\xi}\right) \cap\left(a_{\eta} \cup b_{\eta}\right)=\emptyset$ and $\mathcal{B}$ is an a.d.-family. Extend $q$ to $r \in\{0,1\}^{k^{\prime}}$ by all 0 s. This implies that $r$ forces that

$$
\begin{aligned}
& \left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{a}_{\xi}\right\} \cap \bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{b}_{\eta}\right\}\right) \backslash \check{k}=\emptyset \\
& \left(\bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{a} a_{\eta}\right\} \cap \bigcup\left\{\dot{A}_{\alpha}: \alpha \in \check{b}_{\xi}\right\}\right) \backslash \check{k}=\emptyset .
\end{aligned}
$$

and so it forces that $\dot{p}_{\xi}$ and $\dot{p}_{\eta}$ are compatible.
Now aim at finding $r \leq q$ and distinct $\xi, \eta \in \Gamma$ such that $r$ forces that $\dot{p}_{\xi}$ and $\dot{p}_{\eta}$ are not compatible. Pick $\xi \in \Gamma$ such that $\Gamma \cap \xi$ is infinite. Pick $\alpha^{\prime} \in a_{\xi}$. By the Luzin property of $\mathcal{B}$ there is $\eta \in \Gamma \cap \xi$ and $\beta^{\prime} \in b_{\eta}$ such that

$$
l=\max \left(B_{\alpha^{\prime}} \cap B_{\beta^{\prime}}\right)>\max (m, k) .
$$

Extend $q$ to $r \in\{0,1\}^{l+1}$ so that $r(l)=1$. This implies that $r$ forces that

$$
\check{l} \in\left(\bigcup\left\{B_{\alpha}: \alpha \in \check{a}_{\xi}\right\} \backslash \check{m}\right) \cap\left(\bigcup\left\{B_{\alpha}: \alpha \in \check{b}_{\eta}\right\} \backslash \check{m}\right)
$$

and so that it forces that $\dot{p}_{\xi}$ and $\dot{p}_{\eta}$ are not compatible.

## 4. A GRAPH INDUCED BY AN a.d.-FAMILY

It is clear that given an a.d.-family $\mathcal{A}$ the properties of $\mathbb{P}_{\mathcal{A}}$ considered in the previous section can be expressed in the language of the compatibility graph of $\mathbb{P}_{\mathcal{A}}$ We consider a graph $\mathcal{G}(\mathcal{A})$ (Definition (36) closely related to the compatibility graph of $\mathbb{P}_{\mathcal{A}}$ whose vertices are pairs of disjoint finite subsets of $\mathcal{A}$. We show that this graph can also express these properties of $\mathbb{P}_{\mathcal{A}}$ (Proposition 41) and so can be considered an alternative tool to characterize the phenomena in the induced Banach spaces considered in the following sections.
Definition 35. Let $A, B, C, D$ be subsets of $\mathbb{N}$, we defin

$$
(A, B) \bowtie(C, D)=((A \backslash B) \cap(D \backslash C)) \cup((B \backslash A) \cap(C \backslash D))
$$

Definition 36. Suppose that $\mathcal{A}$ is an a.d.-family. We define a graph $\mathcal{G}(\mathcal{A})$ whose vertices are elements of

$$
\mathcal{V}(\mathcal{A})=\left\{(a, b): a, b \in[\mathcal{A}]^{<\omega}: a \cap b=\emptyset\right\}
$$

and $\left((a, b),\left(a, b^{\prime}\right)\right)$ is an edge of $\mathcal{G}(\mathcal{A})$ if and only if

$$
(\bigcup a, \bigcup b) \bowtie\left(\bigcup a^{\prime}, \bigcup b^{\prime}\right) \neq \emptyset .
$$

Here the union of an empty family is understood to be empty.
In this section we show that the properties of the partial order $\mathbb{P}_{\mathcal{A}}$ considered in the previous section can be completely and naturally expressed in terms of cliques and independent sets of $\mathcal{G}(\mathcal{A})$. This provides another combinatorial manifestation of the properties of Banach spaces considered in the following sections.
Lemma 37. Let $A, B, C, D, F$ be subsets of $\mathbb{N}$ such that

- $A \cap B, C \cap D \subseteq F$,
- $A \cap F=C \cap F, B \cap F=D \cap F$.

Then
(1) $(A, B) \bowtie(C, D) \cap F=\emptyset$,
(2) $((A, B) \bowtie(C, D)) \backslash F=((A \cap D) \cup(B \cap C)) \backslash F$.

Proof. For (1) note that by the hypothesis $A \cap F=C \cap F, B \cap F=D \cap F$ we have $(A, B) \bowtie(C, D) \cap F=(A, B) \bowtie(A, B) \cap F$ and $(A, B) \bowtie(A, B)=\emptyset$.

For (2) by (1) we have $(A, B) \bowtie(C, D)=((A, B) \bowtie(C, D)) \cap(\mathbb{N} \backslash F)$. Now the hypothesis $A \cap B, C \cap D \subseteq F$ implies that $((A, B) \bowtie(C, D)) \cap(\mathbb{N} \backslash F)=$ $((A \cap D) \cup(B \cap C))) \cap(\mathbb{N} \backslash F)$.

Definition 38. Suppose that $\mathcal{A}$ is an a.d. family. We define

- $\triangleleft: \mathcal{V}(\mathcal{A}) \rightarrow \mathbb{P}_{\mathcal{A}}$.
- $\triangleright: \mathbb{P}_{A} \rightarrow \mathcal{V}(\mathcal{A})$
by

$$
\begin{gathered}
\triangleleft(a, b)=(\bigcup a \backslash \bigcup b, \bigcup b \backslash \bigcup a) \\
\triangleright(p)=\left(\left\{A \in \mathcal{A}: A \subseteq^{*} A_{p}\right\},\left\{B \in \mathcal{A}: B \subseteq^{*} B_{p}\right\}\right)
\end{gathered}
$$

[^1]Lemma 39. Suppose that $\mathcal{A}$ is an a.d. family. Then

- $\triangleright$ is surjective with countable fibers, $\triangleleft$ is injective,
- $\triangleright(\triangleleft(a, b))=(a, b)$ for any $(a, b) \in \mathcal{V}(\mathcal{A})$.

Proof. Countable fibers of $\triangleright$ is the consequence of the definition of $\subseteq^{*}$. The rest of the first part of the lemma follows from the second part which is clear from the definitions of $\triangleright$ and $\triangleleft$ because $A \subseteq^{*} \bigcup a \backslash \bigcup b$ as well as $B \subseteq^{*} \bigcup b \backslash \bigcup a$ if and only if $A \in a, B \in b$.

Lemma 40. Suppose that $\mathcal{A}$ is an a.d.-family. Then $\mathbb{P}_{\mathcal{A}}=\bigcup \mathbb{P}_{n}$, where $\triangleright \mid \mathbb{P}_{n}$ is injective and for every $n \in \mathbb{N}$ and $p, q \in \mathbb{P}_{n}$ we have that

$$
p \| q \text { if and only if }(\triangleright(p), \triangleright(q)) \in \mathcal{G}(\mathcal{A}) \text {. }
$$

Proof. Given $p \in \mathbb{P}_{\mathcal{A}}$ let $\triangleright(p)=\left(a_{p}, b_{p}\right)$ and let $k \in \mathbb{N}$ and $F, F^{\prime}, G, G^{\prime} \in[\mathbb{N}]^{<\omega}$ be such that

- $\left[\left(\bigcup a_{p}\right) \backslash k\right] \cap\left[\left(\bigcup b_{p}\right) \backslash k\right]=\emptyset$,
- $A_{p} \backslash k=\left(\bigcup a_{p}\right) \backslash k$,
- $B_{p} \backslash k=\left(\bigcup b_{p}\right) \backslash k$,
- $\left(\bigcup a_{p}\right) \cap k=F$,
- $\left(\bigcup b_{p}\right) \cap k=G$,
- $A_{p} \cap k=F^{\prime}$,
- $B_{p} \cap k=G^{\prime}$.

We claim that the injectivity and the equivalence from the lemma holds for $p, q$ for which such quintuples $\left(k, F, F^{\prime}, G, G^{\prime}\right)$ are the same. For such $p, q$ Lemma 37 applies to $\left(\bigcup a_{p}, \bigcup b_{p}\right)$ and $\left(\bigcup a_{q}, \bigcup b_{q}\right)$. So $\left(\left(a_{p}, b_{p}\right),\left(a_{q}, b_{q}\right)\right) \in \mathcal{G}(\mathcal{A})$ if and only if $\triangleleft\left(a_{p}, b_{p}\right) \bowtie \triangleleft\left(a_{q}, b_{q}\right)=\emptyset$ if and only if $\left(\left(\bigcup a_{p}\right) \backslash k,\left(\bigcup b_{p}\right) \backslash k\right) \bowtie\left(\left(\bigcup a_{q}\right) \backslash k,\left(\bigcup b_{q}\right) \backslash\right.$ $k)=\emptyset$. For such $p, q$ Lemma 37 applies to $\left(A_{p}, B_{p}\right)$ and $\left(A_{q}, B_{q}\right)$ as well. So $p \| q$ if and only if $\left(A_{p} \backslash k, B_{p} \backslash k\right) \bowtie\left(A_{q} \backslash k, B_{q} \backslash k\right)=\emptyset$. Since $A_{p} \backslash k=\left(\bigcup a_{p}\right) \backslash k$, $B_{p} \backslash k=\left(\bigcup b_{p}\right) \backslash k$ and $A_{q} \backslash k=\left(\bigcup a_{q}\right) \backslash k, B_{q} \backslash k=\left(\bigcup b_{q}\right) \backslash k$, we can conclude the equivalence of the lemma.

Also if $p, q$ have the same quintuples $\left(k, F, F^{\prime}, G, G^{\prime}\right)$ and $\triangleright(p)=\left(a_{p}, b_{p}\right)=$ $\left(a_{q}, b_{q}\right)=\triangleright(q)$, then $A_{p} \cap k=F^{\prime}=A_{q} \cap k, B_{p} \cap k=G^{\prime}=B_{q} \cap k$ and $A_{p} \backslash k=$ $\left(\bigcup a_{p}\right) \backslash k=\left(\bigcup a_{q}\right) \backslash k=A_{q} \backslash k$, and $B_{p} \backslash k=\left(\bigcup b_{p}\right) \backslash k=\left(\bigcup b_{q}\right) \backslash k=B_{q} \backslash k$, and so $p=q$ which allows us to conclude the injectivity.

Proposition 41. Suppose that $\mathcal{A}$ is an a.d.-family and $\kappa$ is a cardinal of uncountable cofinality.
(1) $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered if and only if $\mathcal{G}(\mathcal{A})$ is the union of countably many cliques.
(2) $\mathbb{P}_{\mathcal{A}}$ has precaliber $\kappa$ if and only if every subset of $\mathcal{G}(\mathcal{A})$ of cardinality $\kappa$ contains a clique of cardinality $\kappa$
(3) $\mathbb{P}_{\mathcal{A}}$ is c.c.c. if and only if $\mathcal{G}(\mathcal{A})$ does not contain an uncountable independent family.
(4) Every family in $\mathbb{P}_{\mathcal{A}}$ of essentially distinct elements is the union of countably many antichains if and only if every family $\mathcal{G} \subseteq \mathcal{G}(\mathcal{A})$ such that $a \neq a^{\prime}$ and $b \neq b^{\prime}$ for any distinct $(a, b),\left(a^{\prime}, b^{\prime}\right) \in \mathcal{G}$ is the union of countably many independent sets.

Proof. Note that two elements $p, q \in \mathbb{P}(\mathcal{A})$ are essentially distinct if and only if $a \neq a^{\prime}$ and $b \neq b^{\prime}$, where $p=(a, b)$ and $q=\left(a^{\prime}, b^{\prime}\right)$. The properties of being the union of countably many cliques, having a clique of cardinality $\kappa$ in any subgraph
of cardinality $\kappa$, not containing an uncountable independent set, being the union of countably many independent sets pass from a graph to any of its subgraphs and hold for a graph which is the union of countably many subgraphs each having the property. This and Lemma 40 imply the proposition.

## 5. Forcing $\mathbb{P}_{\mathcal{A}}$ and the geometry of the unit sphere of the Banach Space $\mathcal{X}_{\mathcal{A}}$

In this section given an a.d.-family $\mathcal{A}$ we characterize certain natural geometric properties of the unit sphere of the Banach space $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty}\right)$ using combinatorial (in fact, rather forcing theoretic) properties of the splitting partial order $\mathbb{P}_{\mathcal{A}}$ (Definition 5). These characterizations are Propositions 46, 47, 48 and refer to $\mathbb{P}_{\mathcal{A}}$ satisfying the c.c.c., having precaliber $\kappa$ for an uncountable cardinal $\kappa$ and being $\sigma$-centered. This leaves out the geometric properties corresponding to $\mathcal{A}$ being antiramsey or $L$ family (Definitions (7). These properties cannot be as global as the properties from Propositions 46, 47, 48 for elementary geometric reasons (spheres always contain of cardinality $\mathfrak{c}$ of small diameters). The results we obtain in this direction refer to any sequence which induces, in some sense, an uncountable subset of $\mathbb{P}_{\mathcal{A}}$ consisting of essentially distinct conditions (Propositions 53, 54). However, when we pass to an appropriate subspace of $\mathcal{X}_{\mathcal{A}}$ in the following section, we will be able to extract from these results a natural geometric condition.

Lemma 42. Let $\mathcal{A}$ be an a.d.-family. If $f \in \mathcal{X}_{\mathcal{A}}$, then for every $A \in \mathcal{A}$ there exists $\lim _{n \in A} f(n)$. For every $\varepsilon>0$ and every $f \in \mathcal{X}_{\mathcal{A}}$ the set $\left\{A:\left|\lim _{n \in A} f(n)\right|>\varepsilon\right\}$ is finite.
Proof. Given $A \in \mathcal{A}$ the subset of $\ell_{\infty}$ consisting of $f_{\mathrm{s}}$ such $\lim _{n \in A} f(n)$ exists is closed in the supremum norm and $\mathcal{X}_{\mathcal{A}}$ is included in it because each generator of $\mathcal{X}_{\mathcal{A}}$ is in this set.

If $\left\{A:\left|\lim _{n \in A} f(n)\right|>\varepsilon\right\}$ were infinite, $f$ could not be approximated in the supremum norm by linear combinations of the generators of $\mathcal{X}_{\mathcal{A}}$.

Definition 43. Suppose that $p=\left(A_{p}, B_{p}\right) \in \mathbb{P}_{\mathcal{A}}$, then $f_{p} \in \mathcal{X}_{\mathcal{A}}$ is defined as

$$
f_{p}(n)= \begin{cases}1 & \text { if } n \in A_{p} \\ -1 & \text { if } n \in B_{p} \\ 0 & n \in \mathbb{N} \backslash\left(A_{p} \cup B_{p}\right)\end{cases}
$$

Lemma 44. Suppose that $f \in \mathcal{X}_{\mathcal{A}}$ and $\varepsilon>0$. Then there is a condition $p(f, \varepsilon) \in \mathbb{P}_{\mathcal{A}}$ such that
(1) $A_{p(f, \varepsilon)} \supseteq\{n: f(n) \geq \varepsilon\}$
(2) $B_{p(f, \varepsilon)} \supseteq\{n: f(n) \leq-\varepsilon\}$

Proof. Find $A_{1}, \ldots A_{n} \in \mathcal{A}$ for some $n \in \mathbb{N}$ and finitely supported $f^{\prime} \in c_{0}$ such that $\|g-f\|_{\infty} \leq \varepsilon / 2$, where

$$
g=\sum_{1 \leq i \leq n} r_{i} 1_{A_{i}}+f^{\prime}
$$

for some $r_{i} \in \mathbb{R}$ for $1 \leq i \leq n$. As $g$ assumes finitely many values, $A_{p(f, \varepsilon)}=\{n$ : $g(n) \geq \varepsilon / 2\}$ and $B_{p(f, \varepsilon)}=\{n: g(n) \leq-\varepsilon / 2\}$ yield a condition $p(f, \varepsilon) \in \mathbb{P}_{\mathcal{A}}$. Let
us verify that it satisfies the lemma. If $f(n) \geq \varepsilon$ for $n \in \mathbb{N}$, then $g(n) \geq \varepsilon / 2$ and so $n \in A_{p(f, \varepsilon)}$. A similar argument works for $B_{p(f, \varepsilon)}$.

Lemma 45. Suppose that $\mathcal{A}$ is an a.d.-family and $p, q \in \mathbb{P}_{\mathcal{A}}$ and $f, g \in \mathcal{X}_{\mathcal{A}}$ and $0<\varepsilon \leq 1$. Then
(1) $p$ is compatible with $q$ if and only if $\left\|f_{p}-f_{q}\right\|_{\infty} \leq 1$.
(2) $\left\|f_{p}-f_{q}\right\|_{\infty}>1$ if and only if $\left\|f_{p}-f_{q}\right\|_{\infty}=2$ (if and only if $p$ is incompatible with $q$ ).
(3) If $p(f, \varepsilon)$ is compatible with $p(g, \varepsilon)$, then $\|f-g\|_{\infty} \leq 1+\varepsilon$

Proof. The first two clauses follow from the definitions of $\mathbb{P}_{\mathcal{A}}, f_{p}$ and $f_{q}$. For the third clause, suppose that $p(f, \varepsilon)$ is compatible with $p(g, \varepsilon)$. So $\{n: f(n) \geq \varepsilon\} \cap\{n$ : $g(n) \leq-\varepsilon\}=\emptyset$ and $\{n: g(n) \geq \varepsilon\} \cap\{n: f(n) \leq-\varepsilon\}=\emptyset$. It follows that for any $n \in \mathbb{N}$ we have either

- $\{f(n), g(n)\} \in(-\varepsilon, \varepsilon)^{2}$ or
- $\{f(n), g(n)\} \in[-1,-\varepsilon] \times[-1, \varepsilon)$ or
- $\{f(n), g(n)\} \in[\varepsilon, 1] \times(-\varepsilon, 1]$ or
- $\{f(n), g(n)\} \in[-1, \varepsilon) \times[-1,-\varepsilon]$ or
- $\{f(n), g(n)\} \in(-\varepsilon, 1] \times[\varepsilon, 1]$.

In all cases $|f(n)-g(n)| \leq 1+\varepsilon$ as required in (3).

The following is a version of Theorem 2.6 of [29]:
Proposition 46. Suppose that $\mathcal{A}$ is an a.d.-family and $\kappa$ is an uncountable cardinal. The following conditions are equivalent:
(1) $\mathbb{P}_{\mathcal{A}}$ admits an antichain of cardinality $\kappa$,
(2) The unit sphere of $\mathcal{X}_{\mathcal{A}}$ admits a 2-equilateral set of cardinality $\kappa$,
(3) The unit sphere of $\mathcal{X}_{\mathcal{A}}$ admits an r-equilateral set of cardinality $\kappa$ for some $r>1$,
(4) The unit sphere of $\mathcal{X}_{\mathcal{A}}$ admits an $(1+\varepsilon)$-separated set of cardinality $\kappa$ for some $\varepsilon>0$.

Proof. Suppose that (1) holds and $\left\{p_{\xi}: \xi<\kappa\right\}$ is an antichain in $\mathbb{P}_{\mathcal{A}}$. Then $\left\{f_{p_{\xi}}: \xi<\kappa\right\}$ is 2-equilateral subset of the sphere of $\mathcal{X}_{\mathcal{A}}$ by Lemma 45 The implications from (2) to (3) and from (3) to (4) are obvious. So assume (4), fix $\varepsilon>0$ and let $\left\{f_{\xi}: \xi<\kappa\right\}$ be a $(1+\varepsilon)$-separated set in in the unit sphere of $\mathcal{X}_{\mathcal{A}}$. Consider $p\left(f_{\xi}, \varepsilon / 2\right)$ from Lemma 44 for every $\xi<\kappa$. These conditions must form an antichain, because otherwise by Lemma 45 (3) we would have $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \leq 1+\varepsilon / 2$.

Proposition 47. Suppose that $\mathcal{A}$ is an a.d.-family and $\kappa$ is an uncountable cardinal. The following conditions are equivalent:
(1) $\mathbb{P}_{\mathcal{A}}$ has precaliber $\kappa$.
(2) For every $\left\{f_{\xi}: \xi<\kappa\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}}}$ and every $0<\varepsilon<1$ there is $\Gamma \subseteq \kappa$ of cardinality $\kappa$ such that the diameter of $\left\{f_{\xi}: \xi \in \Gamma\right\}$ is not bigger than $(1+\varepsilon)$.
(3) For every $\left\{f_{\xi}: \xi<\kappa\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}}}$ there is $0<\varepsilon<1$ and there is $\Gamma \subseteq \kappa$ of cardinality $\kappa$ such that the diameter of $\left\{f_{\xi}: \xi \in \Gamma\right\}$ is not bigger than $(1+\varepsilon)$.

Proof. Assume (1). Let $\left\{f_{\xi}: \xi<\kappa\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}}}$ and $0<\varepsilon<1$. By the precaliber $\kappa$ there is $\Gamma \subseteq \kappa$ such that $\left\{p\left(f_{\xi}, \varepsilon\right): \xi \in \Gamma\right\}$ is pairwise compatible. So by Lemma 44 (3) $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \leq 1+\varepsilon$ for all distinct $\xi, \eta \in \Gamma$. The implication from (2) to (3) is obvious.

Now assume (3). Let $\left\{p_{\xi}: \xi<\kappa\right\} \subseteq \mathbb{P}_{\mathcal{A}}$. By (3) there is $\Gamma \subseteq \kappa$ of cardinality $\kappa$ and $0<\varepsilon<1$ such that $\left\|f_{p_{\xi}}-f_{p_{\eta}}\right\|_{\infty} \leq 1+\varepsilon$ for any two distinct $\xi, \eta \in \Gamma$. It follows from Lemma 45(1), (2) that $\left\{p_{\xi}: \xi<\kappa\right\}$ are pairwise compatible.

Proposition 48. Suppose that $\mathcal{A}$ is an a.d.-family. The following conditions are equivalent:
(1) $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered,
(2) For every $0<\varepsilon<1$ the sphere of $\mathcal{X}_{\mathcal{A}}$ is the union of countably many sets $\mathcal{X}_{n}$ such that $\|x-y\|_{\infty} \leq 1+\varepsilon$ for all $x, y \in \mathcal{X}_{n}$ for all $n \in \mathbb{N}$.

Proof. Assume (1). Fix $0<\varepsilon<1$. Let $\mathbb{P}_{n}$ for $n \in \mathbb{N}$ be centered families such that $\bigcup_{n \in \mathbb{N}} \mathbb{P}_{n}=\mathbb{P}_{\mathcal{A}}$. Define

$$
\mathcal{X}_{n}=\left\{f \in S_{\mathcal{X}_{\mathcal{A}}}: p(f, \varepsilon) \in \mathbb{P}_{n}\right\}
$$

It follows that the sphere of $\mathcal{X}_{\mathcal{A}}$ is covered by the $\mathcal{X}_{n}$ 's. If $f, g \in \mathcal{X}_{n}$, then $p(f, \varepsilon), p(g, \varepsilon) \in \mathbb{P}_{n}$ and so $p(f, \varepsilon)$ and $p(g, \varepsilon)$ are compatible. So by Lemma 45 (3), we conclude that $\|f-g\|_{\infty} \leq 1+\varepsilon$ as required for (2).

Now assume (2). Define

$$
\mathbb{P}_{n}=\left\{p \in \mathbb{P}_{\mathcal{A}}: f_{p} \in \mathcal{X}_{n}\right\}
$$

Suppose that $p, q \in \mathbb{P}_{n}$. Since $\left\|f_{p}-f_{q}\right\|_{\infty} \leq 1+\varepsilon$, Lemma 45 (1) and (2) imply that $p$ is compatible with $q$ which completes the proof of the lemma.
Definition 49. Suppose that $f \in \mathcal{X}_{\mathcal{A}}$ and $0<\varepsilon<1 / 2$. Then we define $A_{p[f, \varepsilon]}=$ $A \backslash B$ and $B_{p[f, \varepsilon]}=B \backslash A$, where
(1) $A=\bigcup\left\{n \in \mathbb{N}: \exists C \in \mathcal{A} \lim _{k \in C} f(k) \geq 1-\varepsilon \& n \in C \& f(n)>1-2 \varepsilon\right\}$,
(2) $B=\bigcup\left\{n \in \mathbb{N}: \exists C \in \mathcal{A} \lim _{k \in C} f(k) \leq-1+\varepsilon \& n \in C \& f(n)<-1+2 \varepsilon\right\}$.

Lemma 50. Suppose $f \in \mathcal{X}_{\mathcal{A}}$ and $0<\varepsilon<1 / 2$. Then $p[f, \varepsilon]=\left(A_{p[f, \varepsilon]}, B_{p[f, \varepsilon]}\right) \in$ $\mathbb{P}_{\mathcal{A}}$.
Proof. Clearly the sets $A_{p[f, \varepsilon]}$ and $B_{p[f, \varepsilon]}$ are disjoint. So we need to prove that these sets are almost equal to the union of some finite subfamily of $\mathcal{A}$. Let us focus on $A_{p[f, \varepsilon]}$, the case of $B_{p[f, \varepsilon]}$ is analogous. It is enough to prove that the family of $C \in \mathcal{A}$ such that $\lim _{n \in C} f(n) \geq 1-\varepsilon$ is finite. This follows from Lemma 42,

Remark 51: Note that given $f \in \mathcal{X}_{\mathcal{A}}$ the pair $(\{n \in \mathbb{N}: f(n) \geq \varepsilon\},\{n \in \mathbb{N}$ : $f(n) \leq-\varepsilon\}$ or the pair $(\{n \in \mathbb{N}: f(n)>\varepsilon\},\{n \in \mathbb{N}: f(n)<-\varepsilon\}$ do not need to be the conditions of $\mathbb{P}_{\mathcal{A}}$. This is because $\varepsilon 1_{A}+\Sigma_{n \in \mathbb{N}}\left[(-1)^{n} / n\right] 1_{n}$ is in $\mathcal{X}_{\mathcal{A}}$.
Lemma 52. Suppose $f \in \mathcal{X}_{\mathcal{A}}$ and $0<\varepsilon<1 / 3$. Then $A_{p[f, \varepsilon]} \subseteq A_{p(f, \varepsilon)}$ and $B_{p[f, \varepsilon]} \subseteq B_{p(f, \varepsilon)}$.
Proposition 53. Let $0<\varepsilon<1 / 2$. Let $\mathcal{A}$ be an a.d.-family. The following conditions are equivalent:
(1) $\mathcal{A}$ is an antiramsey a.d.-family.
(2) Whenever $\left\{f_{\xi}: \xi<\omega_{1}\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}}}$ is such that $\left\{p\left[f_{\xi}, \varepsilon / 4\right]: \xi<\omega_{1}\right\}$ consists of essentially distinct conditions, then
(a) there are $\xi<\eta<\omega_{1}$ such that $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}>2-\varepsilon$,
(b) there are $\xi<\eta<\omega_{1}$ such that $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}<1+\varepsilon$.

Proof. First let us prove that (1) implies (2). Let $a_{\xi}, b_{\xi} \in[\mathcal{A}]<\omega, m_{\xi} \in \mathbb{N}, F_{\xi}, G_{\xi} \in$ $[\mathbb{N}]^{<\omega}$ be such that $A_{p\left(f_{\xi}, \varepsilon / 4\right)}=\left(\bigcup a_{\xi} \backslash m_{\xi}\right) \cup F_{\xi}$ and $B_{p\left(f_{\xi}, \varepsilon / 4\right)}=\left(\bigcup b_{\xi} \backslash m_{\xi}\right) \cup G_{\xi}$. By passing to an uncountable subset we may assume that $m_{\xi}=m, G_{\xi}=G$ and $F_{\xi}=F$ for some $m \in \mathbb{N}, F, G \in[\mathbb{N}]^{<\omega}$ and that $\left\{a_{\xi}: \xi<\omega_{1}\right\}$ forms a $\Delta$-system with root $\Delta_{1}$ and that $\left\{b_{\xi}: \xi<\omega_{1}\right\}$ forms a $\Delta$-system with root $\Delta_{2}$. Both of these $\Delta$-systems must be uncountable, because otherwise we would have $a_{\xi} \subseteq \Delta_{1}$ or $b_{\xi} \subseteq \Delta_{2}$ for uncountably many $\xi<\omega_{1}$ and this would contradict, by Lemma 52 the fact that all $p\left[f_{\xi}, \varepsilon / 4\right]$ are essentially distinct. It follows that we may assume that $\left\{p\left(f_{\xi}, \varepsilon / 4\right): \xi<\omega_{1}\right\}$ consists of essentially distinct conditions as well.

Let $\xi, \eta<\omega$ are such that $p\left[f_{\xi}, \varepsilon / 4\right]$ and $p\left[f_{\eta}, \varepsilon / 4\right]$ are incompatible. This means that there is $n \in A_{p\left[f_{\xi}, \varepsilon / 4\right]} \cap B_{p\left[f_{\eta}, \varepsilon / 4\right]}$ or there is $n \in A_{p\left[f_{\xi}, \varepsilon / 4\right]} \cap B_{p\left[f_{\eta}, \varepsilon / 4\right]}$. By Definition 49 in the first case we have $f_{\xi}(n)>1-2 \varepsilon / 4$ and $f_{\eta}(n)<-1+2 \varepsilon / 4$ and in the second case $f_{\xi}(n)>-1+2 \varepsilon / 4$ and $f_{\eta}(n)>1-2 \varepsilon / 4$. In any case $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}>2-\varepsilon$.

Let $\xi, \eta<\omega$ be such that $p\left(f_{\xi}, \varepsilon / 4\right)$ and $p\left(f_{\eta}, \varepsilon / 4\right)$ are compatible. Then $\| f_{\xi}-$ $f_{\eta} \|_{\infty} \leq 1+\varepsilon / 4$ by Lemma 45 (3).

Now let us prove that (2) implies (1). Consider a family $\left\{p_{\xi}: \xi<\omega_{1}\right\}$ od essentially distinct conditions of $\mathbb{P}_{\mathcal{A}}$. Let $a_{\xi}, b_{\xi} \in[\mathcal{A}]^{<\omega}, m_{\xi} \in \mathbb{N}, F_{\xi}, G_{\xi} \in[\mathbb{N}]^{<\omega}$ be such that $A_{p_{\xi}}=\left(\bigcup a_{\xi} \backslash m_{\xi}\right) \cup F_{\xi}$ and $B_{p_{\xi}}=\left(\bigcup b_{\xi} \backslash m_{\xi}\right) \cup G_{\xi}$. By passing to an uncountable subset we may assume that $m_{\xi}=m, G_{\xi}=G$ and $F_{\xi}=F$ for some $m \in \mathbb{N}, F, G \in[\mathbb{N}]^{<\omega}$. Let $p_{\xi}^{\prime}=\left(\bigcup a_{\xi} \backslash m, \bigcup b_{\xi} \backslash m\right)$. Note that $p_{\xi}^{\prime}$ and $p_{\eta}^{\prime}$ are compatible if and only if $p_{\xi}$ and $p_{\eta}$ are compatible and that $p_{\xi}$ 's are essentially distinct. Also note that $p\left[f_{p_{\xi}^{\prime}}, \varepsilon / 4\right]=p_{\xi}^{\prime}$ for every $\xi<\omega_{1}$. So (2) yields $\xi<\eta<\omega_{1}$ such that $\left\|f_{p_{\xi}}-f_{p_{\eta}}\right\|_{\infty}>2-\varepsilon$, which gives that $p_{\xi}$ and $p_{\eta}$ are incompatible by Lemma 45 (1), and also yields $\xi<\eta<\omega_{1}$ such that $\left\|f_{p_{\xi}}-f_{p_{\eta}}\right\|_{\infty}<1+\varepsilon$, which gives that $p_{\xi}$ and $p_{\eta}$ are compatible by Lemma 45 (2). This completes the proof of (1).

Proposition 54. Let $0<\varepsilon<1 / 2$. Let $\mathcal{A}$ be an a.d.-family. The following conditions are equivalent:
(1) $\mathcal{A}$ is an L-family.
(2) Whenever $\left\{f_{\xi}: \xi<\omega_{1}\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}}}$ is such that $\left\{p\left[f_{\xi}, \varepsilon / 4\right]: \xi<\omega_{1}\right\}$ consists of essentially distinct conditions. Then $\left\{f_{\xi}: \xi<\omega_{1}\right\}$ is a union of countably many $(2-\varepsilon)$-separated subfamilies.
Proof. First let us prove that (1) implies (2). For $n \in \mathbb{N}$ let $\Gamma_{n} \subseteq \omega_{1}$ be such that $\left\{p\left[f_{\xi}, \varepsilon / 4\right]: \xi \in \Gamma_{n}\right\}$ is pairwise incompatible for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} \Gamma_{n}=\omega_{1}$. Fixing $n \in \mathbb{N}$ and distinct $\xi, \eta \in \Gamma_{n}$ there is $k \in A_{p\left[f_{\xi}, \varepsilon / 4\right]} \cap B_{p\left[f_{\eta}, \varepsilon / 4\right]}$ or there is $k \in$ $A_{p\left[f_{\xi}, \varepsilon / 4\right]} \cap B_{p\left[f_{\eta}, \varepsilon / 4\right]}$. By Definition 49] in the first case we have $f_{\xi}(k)>1-2 \varepsilon / 4$ and $f_{\eta}(k)<-1+2 \varepsilon / 4$ and in the second case $f_{\xi}(k)>-1+2 \varepsilon / 4$ and $f_{\eta}(k)>1-2 \varepsilon / 4$. In any case $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}>2-\varepsilon$.

Now let us prove that (2) implies (1). Consider a family $\left\{p_{\xi}: \xi<\omega_{1}\right\}$ od essentially distinct conditions of $\mathbb{P}_{\mathcal{A}}$. Let $a_{\xi}, b_{\xi} \in[\mathcal{A}]^{<\omega}, m_{\xi} \in \mathbb{N}, F_{\xi}, G_{\xi} \in[\mathbb{N}]^{<\omega}$ be such that $A_{p_{\xi}}=\left(\bigcup a_{\xi} \backslash m_{\xi}\right) \cup F_{\xi}$ and $B_{p_{\xi}}=\left(\bigcup b_{\xi} \backslash m_{\xi}\right) \cup G_{\xi}$. By passing to a part of a countable decomposition of $\omega_{1}$ we may assume that $m_{\xi}=m, G_{\xi}=G$ and $F_{\xi}=F$ for some $m \in \mathbb{N}, F, G \in[\mathbb{N}]^{<\omega}$. Let $p_{\xi}^{\prime}=\left(\bigcup a_{\xi} \backslash m, \bigcup b_{\xi} \backslash m\right)$. Note that $p_{\xi}^{\prime}$ and $p_{\eta}^{\prime}$ are compatible if and only if $p_{\xi}$ and $p_{\eta}$ are compatible and that $p_{\xi}$ 's are
essentially distinct. Also note that $p\left[f_{p_{\xi}^{\prime}}, \varepsilon / 4\right]=p_{\xi}^{\prime}$ for every $\xi<\omega_{1}$. So (2) yields a decomposition $\omega_{1}=\bigcup_{n \in \mathbb{N}} \Gamma_{n}$ such that for each $n \in \mathbb{N}$ and distinct $\xi<\eta<\Gamma_{n}$ we have $\left\|f_{p_{\xi}}-f_{p_{\eta}}\right\|_{\infty}>2-\varepsilon$. This which gives that $p_{\xi}$ and $p_{\eta}$ are incompatible by Lemma 45 (1). This completes the proof of (1).

## 6. Subspaces of the Banach space $\mathcal{X}_{\mathcal{A}}$ for antiramsey and Luzin a.d.-FAMILIES

In this section we pass to certain subspaces $\mathcal{X}_{\mathcal{A}, \phi}$ of $\mathcal{X}_{\mathcal{A}}$ in the case of antiramsey and Luzin a.d.-families $\mathcal{A}$ and exploit Propositions 53 and 54 to obtain Banach spaces with interesting geometry of the unit sphere (Proposition 57 and 58). The point of passing to these subspace is to free onself from the hypothesis on essentially distinct elements of the sphere in Propositions 53 and 54 and replace it with a condition which can be expressed in the language of Banach spaces. Recall Definition 7 of an antiramsey a.d.-family.

Definition 55. Let $\kappa$ be an uncountable cardinal and $\mathcal{A}$ be an a.d.-family of cardinality at least $\kappa$. A function $\phi=\left(\phi_{1}, \phi_{-1}\right): \kappa \rightarrow \mathcal{A}$ is called a pairing of $\mathcal{A}$ if $\phi_{1}(\alpha) \neq \phi_{-1}(\alpha)$ for every $\alpha<\kappa$ and $\left\{\phi_{1}(\alpha), \phi_{-1}(\alpha)\right\} \cap\left\{\phi_{1}(\beta), \phi_{-1}(\beta)\right\}=\emptyset$ for any $\alpha<\beta<\kappa$.

Definition 56. Let $\kappa$ be an uncountable cardinal and $\mathcal{A}$ be an a.d.-family of cardinality $\kappa$ and $\phi=\left(\phi_{1}, \phi_{-1}\right): \kappa \rightarrow \mathcal{A}$ be a pairing of $\mathcal{A}$. We define the Banach space $\mathcal{X}_{\mathcal{A}, \phi}$ as the linear span of

$$
c_{0} \cup\left\{1_{\phi_{1}(\alpha)}-1_{\phi_{-1}(\alpha)}: \alpha<\kappa\right\}
$$

in $\ell_{\infty}$ with the supremum norm.
Proposition 57. Suppose that $\mathcal{A}$ is an antiramsey a.d.-family and suppose that $\phi: \kappa \rightarrow \mathcal{A}$ is a pairing of $\mathcal{A}$. Let $\varepsilon \leq 1 / 3$ and $\left\{f_{\xi}: \xi<\omega_{1}\right\}$ is a $(1-\varepsilon)$-separated subset of the sphere $S_{\mathcal{X}_{\mathcal{A}, \phi}}$. Then
(1) there are $\xi<\eta<\omega_{1}$ such that $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \leq 1+2 \varepsilon$ and
(2) there are $\xi<\eta<\omega_{1}$ such that $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \geq 2-8 \varepsilon$.

Proof. For every $\xi<\omega_{1}$ find $n_{\xi}, m_{\xi} \in \mathbb{N}$ and rationals $q_{1}^{\xi}, \ldots q_{n_{\xi}}^{\xi}$ satisfying $0<$ $\left|q_{i}^{\xi}\right| \leq 1$ for all $1 \leq i \leq n$ and increasing $\alpha_{1}^{\xi}<\cdots<\alpha_{n_{\xi}}^{\xi}<\omega_{1}$ and rationally valued $g_{\xi} \in c_{0}$ whose support is included in $m_{\xi}$ such that $\left\|f_{\xi}-h_{\xi}\right\|_{\infty}<\varepsilon$, and $\phi_{l}\left(\alpha_{\xi_{i}}\right) \cap \phi_{l^{\prime}}\left(\alpha_{\xi_{j}}\right) \subseteq m_{\xi}$ for all distinct pairs $\langle l, i\rangle$ and $\left\langle l^{\prime}, j\right\rangle$ for $1 \leq i, j \leq n$ and $l, l^{\prime} \in\{-1,1\}$, where

$$
h_{\xi}=g_{\xi}+\sum_{1 \leq i \leq n} q_{i}\left(1_{\phi_{1}\left(\alpha_{\xi_{i}}\right) \backslash m_{\xi}}-1_{\phi_{-1}\left(\alpha_{\xi_{i}}\right) \backslash m_{\xi}}\right) .
$$

By passing to an uncountable subset of $\kappa$ we may assume that there are $n, m \in \mathbb{N}$ and rationals $q_{1}, \ldots, q_{n}$ and finitely supported $g \in c_{0}$ such that $n_{\xi}=n, m_{\xi}=m$ and $g_{\xi}=g$ and $q_{i}=q_{i}^{\xi}$ for all $1 \leq i \leq n$.

Moreover, by the $\Delta$-system lemma, we may assume that there is $1 \leq k<n$ and $\left\{\alpha_{i}: i<k\right\} \subseteq \omega_{1}$ such that $\alpha_{i}=\alpha_{i}^{\xi}$ for all $i<k$ and $\left\{\alpha_{k}^{\xi}, \ldots, \alpha_{n}^{\xi}\right\} \cap\left\{\alpha_{k}^{\eta}, \ldots, \alpha_{n}^{\eta}\right\}=$ $\emptyset$ for every $\xi<\eta<\omega_{1}$. Indeed $k$ must be less than $n$ because otherwise $h_{\xi}=h_{\eta}$ and so $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}<2 \varepsilon$ which contradicts $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}>1-\varepsilon$ as $\varepsilon \leq 1 / 3$. Define

$$
A_{p_{\xi}}=\left\{n \in \mathbb{N}:\left(h_{\xi}-g\right)(n)>0\right\}, \quad B_{p_{\xi}}=\left\{n \in \mathbb{N}:\left(h_{\xi}-g\right)(n)<0\right\}
$$

It is easy to see that $p_{\xi}=\left(A_{p_{\xi}}, B_{p_{\xi}}\right) \in \mathbb{P}_{\mathcal{A}}$ because $h_{\xi}-g_{\xi}$ is constant equal to non-zero $\pm q_{i}$ on each $\phi_{l}\left(\alpha_{i}^{\xi}\right) \backslash m$ for each $l \in\{-1,1\}$ and these sets are pairwise disjoint and are almost equal to elements of the family $\mathcal{A}$.

Since $\mathbb{P}_{\mathcal{A}}$ is assumed to satisfy the c.c.c, there are $\xi<\eta<\omega_{1}$ such that $p_{\xi}$ and $p_{\eta}$ are compatible. This means that the supremum of the values of $h_{\xi}-h_{\eta}=$ $\left(h_{\xi}-g\right)-\left(h_{\eta}-g\right)$ is at $\operatorname{most~}_{\max }^{k \leq i \leq n} \boldsymbol{\|}\left\|q_{i}\right\|_{\infty} \leq 1$. In particular, $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \leq 1+2 \varepsilon$ as required in 1 ).

Also the triangle inequality and the hypothesis that $\left\|f_{\xi}-f_{\eta}\right\|_{\infty}>1-\varepsilon$ for any $\xi<\eta<\omega_{1}$ imply that $\left\|h_{\xi}-h_{\eta}\right\|_{\infty}>1-3 \varepsilon$, so $\max _{k \leq i \leq n}\left\|q_{i}\right\|_{\infty} \geq 1-3 \varepsilon$. Let $1 \leq i \leq n$ be such that
*)

$$
\left|q_{i}\right| \geq 1-3 \varepsilon
$$

Now for $\xi<\omega_{1}$ consider

$$
q_{\xi}=\left(\phi_{1}\left(\alpha_{i}^{\xi}\right) \backslash m_{\xi}, \phi_{-1}\left(\alpha_{i}^{\xi}\right) \backslash m_{\xi}\right)
$$

These are conditions of $\mathbb{P}_{\mathcal{A}}$ by the choice of $m_{\xi}$. By Definition of 55 the $q_{\xi}$ s are essentially different and so by the hypothesis on $\mathcal{A}$ there are $\xi<\eta<\omega_{1}$ such that $q_{\xi}$ and $q_{\eta}$ are incompatible. This means that there is $n \in n$ such that $\mid h_{\xi}-$ $\left.h_{\eta}\right)(n)\left|=\left|q_{i}-\left(-q_{i}\right)\right|=2\right| q_{i} \mid$ which gives that $\left\|h_{\xi}-h_{\eta}\right\|_{\infty} \geq 2-6 \varepsilon$ by $\left.{ }^{*}\right)$, and so $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \geq 2-8 \varepsilon$ as required in 2$)$.

Recall Definition 8 of an $L$-family.
Proposition 58. Suppose that $\mathcal{A}$ is an a.d.-family which is an L-family and $\kappa$ is an uncountable cardinal. Suppose that $\phi: \kappa \rightarrow \mathcal{A}$ is a pairing of $\mathcal{A}$. Let $\varepsilon \leq 1 / 3$ and $\left\{f_{\xi}: \xi<\omega_{1}\right\}$ be a subset of the sphere $S_{\mathcal{X}_{\mathcal{A}, \phi}}$ such that

$$
\left\|x-f_{\eta}\right\|_{\infty} \geq 1-\varepsilon \text { for any } x \in \operatorname{span}\left(c_{0} \cup\left\{f_{\xi}: \xi<\eta\right\}\right)
$$

for every $\eta<\omega_{1}$. Then $\left\{f_{\xi}: \xi<\omega_{1}\right\}$ is the union of countably many $(2-5 \varepsilon)$ separated sets.

Proof. For every $\xi<\omega_{1}$ find $n_{\xi}, m_{\xi} \in \mathbb{N}$ and rationals $q_{1}^{\xi}, \ldots q_{n \xi}^{\xi}$ satisfying $0<$ $\left|q_{i}^{\xi}\right| \leq 1$ for all $1 \leq i \leq n$ and increasing $\alpha_{1}^{\xi}<\cdots<\alpha_{n_{\xi}}^{\xi}<\kappa$ and rationally valued $g_{\xi} \in c_{0}$ whose support is included in $m_{\xi}$ such that $\left\|f_{\xi}-h_{\xi}\right\|_{\infty}<\varepsilon / 2$, and $\phi_{l}\left(\alpha_{\xi_{i}}\right) \cap \phi_{l^{\prime}}\left(\alpha_{\xi_{j}}\right) \subseteq m_{\xi}$ for all distinct pairs $\langle l, i\rangle$ and $\left\langle l^{\prime}, j\right\rangle$ for $1 \leq i, j \leq n$ and $l, l^{\prime} \in\{-1,1\}$, where

$$
h_{\xi}=g_{\xi}+\sum_{1 \leq i \leq n} q_{i}\left(1_{\phi_{1}\left(\alpha_{\xi_{i}}\right) \backslash m_{\xi}}-1_{\phi_{-1}\left(\alpha_{\xi_{i}}\right) \backslash m_{\xi}}\right) .
$$

For $k \in \mathbb{N}$ there are $\Gamma_{k} \subseteq \omega_{1}$ such that $\omega_{1}=\bigcup_{k \in \mathbb{N}} \Gamma_{k}$ and for each $k \in \mathbb{N}$ and each $\xi \in \Gamma_{k}$ there are $n, m \in \mathbb{N}$ and rationals $q_{1}, \ldots, q_{n}$ and finitely supported $g \in c_{0}$ such that $n_{\xi}=n, m_{\xi}=m$ and $g_{\xi}=g$ and $q_{i}=q_{i}^{\xi}$ for all $1 \leq i \leq n$. Moreover by refining further the partition into $\Gamma_{k}$ s and using the Hewitt-Marczewski-Pondiczery theorem we may assume that if $\xi, \eta \in \Gamma_{k}$ and $\alpha=\alpha_{i}^{\xi}=\alpha_{j}^{\eta}$, then $i=j$. It is enough to show for each $k \in \mathbb{N}$ that $\left\{f_{\xi}: \xi \in \Gamma_{k}\right\}$ is the union of countably many $(2-\varepsilon)$-separated sets. So fix $k \in \mathbb{N}$.

First we claim that for every distinct $\xi, \eta \in \Gamma_{k}$ there is $1 \leq i \leq n$ such that $\left|q_{i}\right| \geq 1-2 \varepsilon$ and $\alpha_{i}^{\xi} \neq \alpha_{i}^{\eta}$. Indeed, otherwise

$$
h_{\xi}-h_{\eta}=\sum_{i \in X} q_{i}\left(1_{\phi_{1}\left(\alpha_{\xi_{i}}\right) \backslash m}-1_{\phi_{-1}\left(\alpha_{\xi_{i}}\right) \backslash m}\right)-\sum_{i \in X} q_{i}\left(1_{\phi_{1}\left(\alpha_{\eta_{i}}\right) \backslash m}-1_{\phi_{-1}\left(\alpha_{\eta_{i}}\right) \backslash m}\right)
$$

where $X=\left\{1 \leq i \leq n: \alpha_{i}^{\xi} \neq \alpha_{i}^{\eta}\right\}$. Then $\left\{\phi_{j}\left(\alpha_{i}^{\xi}\right): i \in X\right\} \cap\left\{\phi_{j}\left(\alpha_{i}^{\eta}\right): i \in X\right\}=\emptyset$ for any $j \in\{1,-1\}$. Then the almost disjointness of $\mathcal{A}$ implies that there is finitely supported $h \in c_{0}$ such that $\left\|h_{\xi}-h_{\eta}-h\right\|_{\infty} \leq \max _{i \in X}\left\|q_{i}\right\|_{\infty}<1-2 \varepsilon$ which means that the distance from $\operatorname{span}\left(c_{0} \cup\left\{f_{\xi}\right\}\right)$ to $f_{\eta}$ is less than $1-\varepsilon$ contradicting the hypothesis of the proposition.

Given $\xi \in \Gamma_{k}$ consider $p_{\xi}=\left(A_{\xi}, B_{\xi}\right)$, where

$$
A_{\xi}=\bigcup\left\{\phi_{1}\left(\alpha_{i}^{\xi}\right) \backslash m:\left|q_{i}\right| \geq 1-2 \varepsilon\right\}, \quad B_{\xi}=\bigcup\left\{\phi_{-1}\left(\alpha_{i}^{\xi}\right) \backslash m:\left|q_{i}\right| \geq 1-2 \varepsilon\right\}
$$

By the above claim for every $\xi, \eta \in \Gamma_{k}$ the conditions $p_{\xi}, p_{\eta}$ are essentially distinct, so $\left\{p_{\xi}: \xi \in \Gamma_{k}\right\}$ is the union of countably many antichains in $\mathbb{P}_{\mathcal{A}}$ by the hypothesis of the proposition. However, if $p_{\xi}$ and $p_{\eta}$ are incompatible there is $n \in \mathbb{N}$ such that $n \in \phi_{1}\left(\alpha_{i}^{\xi}\right) \cap \phi_{-1}\left(\alpha_{i}^{\eta}\right) \backslash m$ or $n \in \phi_{1}\left(\alpha_{i}^{\eta}\right) \cap \phi_{-1}\left(\alpha_{i}^{\xi}\right) \backslash m$ and $\left|q_{i}\right| \geq 1-2 \varepsilon$. This means that $\left\|h_{\xi}-h_{\eta}\right\|_{\infty} \geq 2-4 \varepsilon$ and consequently $\left\|f_{\xi}-f_{\eta}\right\|_{\infty} \geq 2-5 \varepsilon$ as required in the proposition.

## 7. Renormings of Banach spaces $\mathcal{X}_{\mathcal{A}}$

Two norms on the same vector space are said to be equivalent if the identity considered as a function between the metric spaces induced by the norms is continuous in both directions. For Banach spaces $\left(\mathcal{X},\| \|_{1}\right)$ and $\left(\mathcal{X},\| \|_{2}\right)$ this reduces to the existence of constants $c, C>0$ such that $c\|x\|_{1} \leq\|x\|_{2} \leq C\|x\|_{1}$ for all $x \in \mathcal{X}$. Given a Banach space $\left(\mathcal{X},\| \|_{\mathcal{X}}\right)$ we consider its equivalent renorming $\left(\mathcal{X},\| \|_{T}\right)$, where $T: \mathcal{X} \rightarrow \mathcal{Y}$ is a linear bounded operator for some Banach space $\mathcal{Y}$ and

$$
\|x\|_{T}=\|x\|_{\mathcal{X}}+\|T(x)\|_{\mathcal{Y}}
$$

for every $x$ is $\mathcal{X}$. The equivalence of the norms $\left\|\|_{\mathcal{X}}\right.$ and $\| \|_{T}$ follows from the fact that $\|x\|_{\mathcal{X}} \leq\|x\|_{T} \leq(1+\|T\|)\|x\|_{\mathcal{X}}$ for every $x \in \mathcal{X}$. In particular we consider renormings of Banach spaces $\mathcal{X}_{\mathcal{A}}$ for an a.d.-family $\mathcal{A}$ with the norm of the form $\left\|\|_{T}\right.$, where $T$ is injective and has separable range. More concretely we consider the operator $T: \mathcal{X}_{\mathcal{A}} \rightarrow \ell_{2}$ given by

$$
T(f)=\left(\frac{f(n)}{\sqrt{2^{n+1}}}\right)_{n \in \mathbb{N}}
$$

for any $f \in \mathcal{X}_{\mathcal{A}}$. So the new equivalent norm $\left\|\|_{T}\right.$ denoted by $\| \|_{\infty, 2}$ on $\mathcal{X}_{\mathcal{A}}$ is

$$
\|f\|_{\infty, 2}=\|f\|_{\infty}+\sqrt{\sum_{n \in \mathbb{N}} \frac{f(n)^{2}}{2^{n+1}}} .
$$

For details see Section 2 of [24].
We characterize geometric properties of the unit spheres of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ by the combinatorial properties of the partial order $\mathbb{P}_{\mathcal{A}}$ (Propositions 63, 64, 65, 66). In Proposition 67we translate the dichotomy 31into the language of the Banach space $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$.

Recall from Section 2.1 that when we want to specify the norm || || in a given Banach spaces $\mathcal{X}$, its unit sphere is denoted by $S_{\mathcal{X},\| \|}$ First we need a few technical lemmas relating the distances in the spheres $S_{\mathcal{X},\| \|_{\mathcal{X}}}$ and $S_{\mathcal{X},\| \|_{T}}$ for a given norm $\|\| \mathcal{X}$ and the norm $\| \|_{T}$ described above and induced by a linear bounded operator $T: \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{X}$ is considered with the norm $\|\| \mathcal{X}$ and $\mathcal{Y}$ is a Banach space.

Lemma 59 (Lemma 9 [24). Let $(\mathcal{X},\| \|)$ be a Banach space. Suppose that $0<$ $a \leq\|x\| \leq\left\|x^{\prime}\right\| \leq b<1$ for some $a, b \in \mathbb{R}$ and $x, x^{\prime} \in \mathcal{X}$. Then

$$
\left\|x-x^{\prime}\right\| \leq b\left\|\frac{x}{\|x\|}-\frac{x^{\prime}}{\left\|x^{\prime}\right\|}\right\|+(b-a)
$$

Lemma 60. Suppose that $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $T: \mathcal{X} \rightarrow \mathcal{Y}$ is an injective operator. Then for $n \in \mathbb{N}$ there are subsets $\mathcal{X}_{n}$ of the unit sphere $S_{\mathcal{X},\| \|_{T}}$ and constants $0<c_{n}<1$ such that $S_{\mathcal{X},\| \|_{T}}=\bigcup_{n \in \mathbb{N}} \mathcal{X}_{n}$ and for any $x, x^{\prime} \in \mathcal{X}_{n}$ for any $n \in \mathbb{N}$

$$
\left\|x-x^{\prime}\right\|_{\mathcal{X}} \leq\left(1-c_{n}\right)\left\|\left(x /\|x\|_{\mathcal{X}}\right)-\left(y /\|y\|_{\mathcal{X}}\right)\right\|_{\mathcal{X}}+c_{n} / 4 .
$$

Proof. For $k \in \mathbb{N} \backslash\{0\}$ and $0 \leq i \leq 4 k-2$ let

$$
\mathcal{X}_{k, i}=\left\{x \in S_{\mathcal{X},\| \|_{T}}: \frac{i}{4 k} \leq\|x\|_{\mathcal{X}} \leq \frac{i+1}{4 k} \&\|T(x)\|_{\mathcal{Y}}>1 / k\right\}
$$

Since $T$ is injective $S_{\mathcal{X},\| \|_{T}}=\bigcup\left\{\mathcal{X}_{k, i}: k \in \mathbb{N} \backslash\{0\}, 0 \leq i \leq 4 k-2\right\}$. Applying Lemma 59 to $x, x^{\prime} \in \mathcal{X}_{k, i}$ we obtain

$$
\left\|x-x^{\prime}\right\|_{\mathcal{X}} \leq(1-1 / k)\left\|\left(x /\|x\|_{\mathcal{X}}\right)-\left(x^{\prime} /\left\|x^{\prime}\right\|_{\mathcal{X}}\right)\right\|_{\mathcal{X}}+1 / 4 k
$$

so by renumerating $\mathcal{X}_{k, i} \mathrm{~s}$ as $\mathcal{X}_{n} \mathrm{~s}$ we obtain the lemma.
Lemma 61. Suppose that $(\mathcal{X},\| \| \mathcal{X})$ and $(\mathcal{Y},\| \| \mathcal{Y})$ are Banach spaces and suppose that $T: \mathcal{X} \rightarrow \mathcal{Y}$ is a linear bounded operator and $x, x^{\prime} \in S_{\mathcal{X}}$ satisfy $\left\|x-x^{\prime}\right\|_{\mathcal{X}}=2-\varepsilon^{\prime}$ for some $\varepsilon^{\prime} \geq 0$ and $\|T(x)\|_{\mathcal{Y}},\left\|T\left(x^{\prime}\right)\right\|_{\mathcal{Y}} \leq 1-\delta$ for some $0<\delta<1$. Then

$$
\|x /\| x\left\|_{T}-x^{\prime} /\right\| x^{\prime}\left\|_{T}\right\|_{T} \geq 1+\varepsilon-\varepsilon^{\prime}
$$

where $\varepsilon=\frac{2}{2-\delta}-1$. In particular, if $\delta=2 / 3, \varepsilon^{\prime}=0$ and $\kappa$ is a cardinal, then the unit sphere of $\left(\mathcal{X},\| \|_{T}\right)$ admits a $\left(1+\frac{1}{2}\right)$-separated set of cardinality $\kappa$ if the unit sphere sphere of $(\mathcal{X},\| \|)$ admits a 2-separated set of cardinality $\kappa$ on which $T$ is bounded by $1 / 3$.
Proof. Consider $y=x /\|x\|_{T}$ and $y^{\prime}=x^{\prime} /\left\|x^{\prime}\right\|_{T}$. As

$$
\left\|y-y^{\prime}\right\|_{T}=\left\|y-y^{\prime}\right\|_{\mathcal{X}}+\left\|T(y)-T\left(y^{\prime}\right)\right\|_{\mathcal{Y}}
$$

it is enough to prove that $\left\|y-y^{\prime}\right\|_{\mathcal{X}} \geq 1+\varepsilon-\varepsilon^{\prime}$. We have

$$
\left\|y-y^{\prime}\right\|_{\mathcal{X}}=\left\|\frac{x}{\|x\|_{T}}-\frac{x^{\prime}}{\left\|x^{\prime}\right\|_{T}}\right\|_{\mathcal{X}} \geq\left\|x-x^{\prime}\right\|_{\mathcal{X}}-\left\|\frac{x}{\|x\|_{T}}-x\right\|_{\mathcal{X}}-\left\|\frac{x^{\prime}}{\left\|x^{\prime}\right\|_{T}}-x^{\prime}\right\|_{\mathcal{X}}
$$

Since we have $\|x\|_{\mathcal{X}}=\left\|x^{\prime}\right\|_{\mathcal{X}}=1$, we estimate

$$
\left\|\frac{x}{\|x\|_{T}}-x\right\|_{\mathcal{X}}=\left|\frac{1}{\|x\|_{\mathcal{X}}+\|T(x)\|_{\mathcal{Y}}}-1\right|\|x\|_{\mathcal{X}}=\left|\frac{1}{1+\|T(x)\|_{\mathcal{Y}}}-1\right|
$$

Since we have $\|T(x)\|_{\mathcal{Y}} \leq 1-\delta$, we obtain that

$$
\frac{1+\varepsilon}{2}=\frac{1}{2-\delta} \leq \frac{1}{1+\|T(x)\|_{\mathcal{Y}}} \leq 1
$$

and so

$$
\left\|\frac{x}{\|x\|_{T}}-x\right\|_{\mathcal{X}} \leq 1-\frac{1+\varepsilon}{2}=\frac{1-\varepsilon}{2}
$$

The same calculation works for $\left\|\frac{x^{\prime}}{\left\|x^{\prime}\right\|_{T}}-x^{\prime}\right\|_{\mathcal{X}}$, so using $\left\|x-x^{\prime}\right\|_{\mathcal{X}}=2-\varepsilon^{\prime}$ we conclude that

$$
\left\|y-y^{\prime}\right\|_{T} \geq\left\|y-y^{\prime}\right\|_{\mathcal{X}} \geq 2-\varepsilon^{\prime}-2\left(\frac{1-\varepsilon}{2}\right)=1+\varepsilon-\varepsilon^{\prime}
$$

as required.

If the range of $T: \mathcal{X} \rightarrow \mathcal{Y}$ is separable, then infinitary combinatorics can play a big role in the renorming $\left\|\|_{T}\right.$ of $\mathcal{X}$ due to the following trivial fact:

Lemma 62. Suppose that $\mathcal{X}, \mathcal{Y}$ are Banach spaces and $T: \mathcal{X} \rightarrow \mathcal{Y}$ has separable range. Then for every $\varepsilon>0$ there are subsets $\mathcal{Y}_{j}$ for $j \in \mathbb{N}$ of the unit sphere $S_{\mathcal{X},\| \|_{T}}$ such that for any $x, x^{\prime} \in \mathcal{Y}_{j}$ for any $j \in \mathbb{N}$

$$
\left\|T(x)-T\left(x^{\prime}\right)\right\|_{\mathcal{Y}} \leq \varepsilon
$$

Proof. For metric spaces the separability is equivalent to the Lindelöf property, so we can cover the range of $T$ by countably many balls of diameter $\varepsilon$. The $Y_{n}$ s are preimages of these balls under $T$ intersected with $S_{\mathcal{X},\| \|_{T}}$.

Proposition 63. Suppose that $\mathcal{A}$ is an a.d.-family such that $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered and $T: \mathcal{X}_{\mathcal{A}} \rightarrow \mathcal{Y}$ is a bounded linear and injective operator with separable range into a Banach space $\mathcal{Y}$. Then the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{T}\right)$ is the union of countably many open sets of diameters strictly less than 1.

Proof. For $m, k \in \mathbb{N}, k>0$ let $\mathcal{Z}_{m, k} \subseteq S_{\mathcal{X}_{\mathcal{A}},\| \|_{\infty}}$ be of $\left\|\|_{\infty}\right.$-diameter not bigger that $1+1 / k$ and such that $S_{\mathcal{X}_{\mathcal{A}},\| \|_{\infty}}=\bigcup_{m \in \mathbb{N}} \mathcal{Z}_{m, k}$ for every $k \in \mathbb{N} \backslash\{0\}$. The existence of these sets follows from Proposition 47. Let $\mathcal{X}_{n}$ and $c_{n}$ be as in Lemma 60 By Lemma 60 it is enough to partition each $\mathcal{X}_{n}$ into countably many sets of diameters strictly less than 1 . So fix $n \in \mathbb{N}$.

As $0<c_{n}<1$ we have $\left(4-3 c_{n}\right) /\left(4-4 c_{n}\right)>1$ and so there is $k \in \mathbb{N} \backslash\{0\}$ such that $1+1 / k$ is smaller than this value and so

$$
\left(1-c_{n}\right)(1+1 / k)<\frac{\left(1-c_{n}\right)\left(4-3 c_{n}\right)}{\left(4-4 c_{n}\right)}=1-\frac{3 c_{n}}{4} .
$$

If $x /\|x\|_{\infty}, x^{\prime} /\left\|x^{\prime}\right\|_{\infty} \in \mathcal{Z}_{m, k}$ and $x, x^{\prime} \in \mathcal{Y}_{j}$ for $j, m \in \mathbb{N}$, where $\mathcal{Y}_{j}$ is as in Lemma 62 for $\varepsilon=c_{n} / 4$, Lemma 60 implies that

$$
\left\|x-x^{\prime}\right\|_{T}=\left\|x-x^{\prime}\right\|_{\infty}+\left\|T(x)-T\left(x^{\prime}\right)\right\|_{\mathcal{Y}} \leq 1-\frac{3 c_{n}}{4}+c_{n} / 4+c_{n} / 4=1-c_{n} / 4
$$

As $\mathcal{X}_{n}=\bigcup_{j, m \in \mathbb{N}}\left(\left\{x \in S_{\mathcal{X}_{\mathcal{A}},\| \|_{T}}: x /\|x\| \in \mathcal{Z}_{m, k}\right\} \cap \mathcal{Y}_{j}\right)$, we have obtained the required partition of $\mathcal{X}_{n}$ which completes the proof of the proposition.

Proposition 64. Suppose that $\kappa$ is a cardinal of uncountable cofinality and $\mathcal{A}$ is an a.d.-family such that $\mathbb{P}_{\mathcal{A}}$ has precaliber $\kappa$ and $T: \mathcal{X}_{\mathcal{A}} \rightarrow \mathcal{Y}$ is a bounded linear and injective operator with separable range into a Banach space $\mathcal{Y}$. Then every set of unit vectors in $\left(\mathcal{X}_{\mathcal{A}},\| \|_{T}\right)$ of cardinality $\kappa$ contains a subset of cardinality $\kappa$ which has the diameter not bigger than $1-\varepsilon$ for some $\varepsilon>0$.

Proof. Let $\mathcal{U} \subseteq S_{\mathcal{X}_{\mathcal{A}},\| \|_{T}}$ be of cardinality $\kappa$. Let $\mathcal{X}_{n}$ and $c_{n}$ be as in Lemma 60 Since the $\mathcal{X}_{n}$ s cover $S_{\mathcal{X}_{\mathcal{A}},\| \|_{T}}$, by the uncountable cofinality of $\kappa$ there is $n \in \mathbb{N}$ such that $\mathcal{U} \cap \mathcal{X}_{n}$ has cardinality $\kappa$. Fix such an $n \in \mathbb{N}$ and let us prove that $\mathcal{U} \cap \mathcal{X}_{n}$ contains a subset of cardinality $\kappa$ which has the diameter not bigger than $1-\varepsilon$ for some $\varepsilon>0$ which is sufficient to prove the proposition.

As $0<c_{n}<1$ we have $\left(4-3 c_{n}\right) /\left(4-4 c_{n}\right)>1$ and so there is $k \in \mathbb{N} \backslash\{0\}$ such that $1+1 / k$ is smaller than this value. In particular we have

$$
\left(1-c_{n}\right)(1+1 / k)<\frac{\left(1-c_{n}\right)\left(4-3 c_{n}\right)}{\left(4-4 c_{n}\right)}=1-\frac{3 c_{n}}{4} .
$$

By passing to a subset of cardinality $\kappa$, by Proposition 47 we may assume that

$$
\left\{x /\|x\|_{\infty}: x \in \mathcal{U} \cap \mathcal{X}_{n}\right\}
$$

has $\left\|\|_{\infty}\right.$-diameter not bigger than $(1+1 / k)$, and so Lemma 60 implies that

$$
\left\|x-x^{\prime}\right\|_{\infty} \leq 1-\frac{3 c_{n}}{4}+c_{n} / 4=1-c_{n} / 2
$$

for any $x, x^{\prime} \in \mathcal{U} \cap \mathcal{X}_{n}$. Using Lemma 62 for $\varepsilon=c_{n} / 4$ and the uncountable cofinality of $\kappa$ we may find $\mathcal{V} \subseteq \mathcal{U} \cap \mathcal{X}_{n}$ of cardinality $\kappa$ such that $\left\|T(x)-T\left(x^{\prime}\right)\right\|_{\mathcal{Y}} \leq c_{n} / 4$ for every $x, x^{\prime} \in \mathcal{V}$. So for $x, x^{\prime} \in \mathcal{V}$ we have

$$
\left\|x-x^{\prime}\right\|_{T}=\left\|x-x^{\prime}\right\|_{\infty}+\left\|T(x)-T\left(x^{\prime}\right)\right\|_{\mathcal{Y}} \leq 1-\frac{c_{n}}{2}+c_{n} / 4=1-c_{n} / 4
$$

which completes the proof of the proposition.

Proposition 65. Suppose that $\mathcal{A}$ is an a.d.-family such that $\mathbb{P}_{\mathcal{A}}$ satisfies the c.c.c. and $T: \mathcal{X}_{\mathcal{A}} \rightarrow \mathcal{Y}$ is a bounded linear and injective operator with separable range into a Banach space $\mathcal{Y}$. Then the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{T}\right)$ does not admit an uncountable 1-separated set.

Proof. Let $\mathcal{U} \subseteq S_{\mathcal{X}_{\mathcal{A}},\| \|_{T}}$ be of cardinality $\omega_{1}$. We will find two elements of $\mathcal{U}$ which are $\left\|\|_{T}\right.$-distant by less than 1 . Let $\mathcal{X}_{n}$ and $c_{n}$ be as in Lemma 60, Since the $\mathcal{X}_{n} \mathrm{~s}$ cover $S_{\mathcal{X}_{\mathcal{A}},\| \|_{T}}$ there is $n \in \mathbb{N}$ such that $\mathcal{U} \cap \mathcal{X}_{n}$ is uncountable. Fix such an $n \in \mathbb{N}$ and let us prove that $\mathcal{U} \cap \mathcal{X}_{n}$ contains two elements $\left\|\|_{T}\right.$-distant by less than 1.

Using Lemma 62 for $\varepsilon=c_{n} / 4$ we may find an uncountable $\mathcal{V} \subseteq U \cap \mathcal{X}_{n}$ such that $\left\|T(x)-T\left(x^{\prime}\right)\right\| \mathcal{Y} \leq c_{n} / 4$ for every $x, x^{\prime} \in \mathcal{V}$. As $0<c_{n}<1$ we have $\left(4-3 c_{n}\right) /(4-$ $\left.4 c_{n}\right)>1$ and so there is $k \in \mathbb{N} \backslash\{0\}$ such that $1+1 / k$ is smaller than this value. In particular we have

$$
\left(1-c_{n}\right)(1+1 / k)<\frac{\left(1-c_{n}\right)\left(4-3 c_{n}\right)}{\left(4-4 c_{n}\right)}=1-\frac{3 c_{n}}{4}
$$

Lemma 60 implies that

$$
\left\|x-x^{\prime}\right\|_{\infty} \leq\left(1-c_{n}\right)\|x /\| x\left\|_{\infty}-x^{\prime} /\right\| x^{\prime}\left\|_{\infty}\right\|+c_{n} / 4
$$

for any $x, x^{\prime} \in \mathcal{V} \cap \mathcal{X}_{n}$. Now apply Proposition 47 for $\left\{x /\|x\|_{\infty}: x \in V \cap \mathcal{X}_{n}\right\}$ finding $x, x^{\prime} \in V \cap \mathcal{X}_{n}$ such that $\|x /\| x\left\|_{\infty}-x^{\prime} /\right\| x^{\prime}\left\|_{\infty}\right\|<1+1 / k$. So for these $x, x^{\prime} \in \mathcal{V}$ we have

$$
\left\|x-x^{\prime}\right\|_{T}=\left\|x-x^{\prime}\right\|_{\infty}+\left\|T(x)-T\left(x^{\prime}\right)\right\| \mathcal{Y} \leq 1-\frac{3 c_{n}}{4}+c_{n} / 4+c_{n} / 4=1-c_{n} / 4
$$

which completes the proof of the proposition.
Proposition 66. Suppose that $\mathcal{A}$ is an a.d.-family, Then the following implications hold.
(1) If $S_{\mathcal{X}_{\mathcal{A}}, \|} \|_{\infty, 2}$ is the union of countably many sets of diameters not bigger than 1 , then $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered.
(2) If every subset of $S_{\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}}$ of cardinality $\kappa$ of uncountable cofinality contains a further subset which has the diameter not bigger than 1 , then $\mathbb{P}_{\mathcal{A}}$ has precaliber $\kappa$.
(3) If $S_{\mathcal{X}_{\mathcal{A}}, \|} \|_{\infty, 2}$ does not admits an uncountable $(2-\varepsilon)$-separated set for some $\varepsilon>0$, then $\mathbb{P}_{A}$ satisfies the c.c.c.

Proof. In the parts (1), (2) and (3) it is enough to prove that there is $m \in \mathbb{N}$ such that

$$
\mathbb{Q}_{m}=\left\{p \in \mathbb{P}_{\mathcal{A}}:\{0, \ldots, m\} \cap\left(A_{p} \cup B_{p}\right)=\emptyset\right\}
$$

is $\sigma$-centered, has precaliber $\kappa$ and satisfies the c.c.c. respectively. This is because for a fixed $m \in \mathbb{N}$ the partial order $\mathbb{P}_{\mathcal{A}}$ consists of finitely many parts each isomorphic to $\mathbb{Q}_{m}$ which depend on the intersections of $A_{p}$ and $B_{p}$ with $\{0, \ldots, m\}$ for $p \in \mathbb{P}_{\mathcal{A}}$.

In the proofs of all parts above we will use the fact that $\|T(x)\| \leq 1 / m$ if $x \mid\{0, \ldots, m\}=0$, if $T$ is as in the definition of the norm $\left\|\|_{\infty, 2}\right.$ and $\| x \|_{\infty} \leq 1$. In particular, if $p, q \in \mathbb{Q}_{m}$, then Lemma 61 can be applied for $\delta=1-1 / m$ to the elements $f_{p}, f_{q}$ defined in Definition 43, i.e., by Lemma 45 (1), (2) we have

$$
\begin{equation*}
\left\|f_{p} /\right\| f_{p}\left\|_{\infty, 2}-f_{q} /\right\| f_{q}\left\|_{\infty, 2}\right\|_{\infty, 2} \geq \frac{2}{2-\delta}=\frac{2}{1+1 / m}=2-\frac{2}{m+1} \tag{*}
\end{equation*}
$$

if $p, q \in \mathbb{Q}_{m}$ are incompatible.
Also note that a finite set $\mathbb{P} \subseteq \mathbb{P}_{\mathcal{A}}$ is centered if and only if $\bigcup\left\{A_{p}: p \in \mathbb{P}\right\} \cap \bigcup\left\{B_{p}\right.$ : $p \in \mathbb{P}\}=\emptyset$ if and only if $\mathbb{P}$ is pairwise compatible.
(1) Let $\mathcal{X}_{n}$ 's form a cover of $S_{\mathcal{X}_{\mathcal{A}}, \|} \|_{\infty, 2}$ by sets of $\left\|\|_{\infty, 2}\right.$-diametres not bigger than 1 . For $n \in \mathbb{N}$ consider

$$
\mathbb{P}_{n}=\left\{p \in \mathbb{Q}_{3}: f_{p} /\left\|f_{p}\right\|_{\infty, 2} \in \mathcal{X}_{n}\right\}
$$

Then $\bigcup_{n \in \mathbb{N}} \mathbb{P}_{n}=\mathbb{Q}_{3}$ and by $\left(^{*}\right)$ each $\mathbb{P}_{n}$ is pairwise compatible and so it is centered.
(2) Given a set $\left\{p_{\xi}: \xi<\kappa\right\} \subseteq \mathbb{Q}_{3}$, consider $\left\{f_{p_{\xi}} /\left\|f_{p_{\xi}}\right\|_{\infty, 2}: \xi<\kappa\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}}, \|} \|_{\infty, 2}$. By the hypothesis there is $\Gamma \subseteq \kappa$ of cardinality $\kappa$ such that $\left\{f_{p_{\xi}} /\left\|f_{p_{\xi}}\right\|_{\infty, 2}: \xi \in \Gamma\right\}$ has the diameter not bigger than 1. As in the proof of (1) by $\left(^{*}\right)$ we obtain that $\left\{p_{\xi}: \xi \in \Gamma\right\}$ is pairwise compatible, and so as required.
(3) Fix $\varepsilon>0$. Let $m \in \mathbb{N}$ be such that $2 /(1+m)<\varepsilon$. Given a set $\left\{p_{\xi}: \xi<\right.$ $\left.\omega_{1}\right\} \subseteq \mathbb{Q}_{m}$, consider $\left\{f_{p_{\xi}} /\left\|f_{p_{\xi}}\right\|_{\infty, 2}: \xi<\kappa\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}}$. By the hypothesis there are $\xi<\eta<\omega_{1}$ such that $\left\|f_{p_{\xi}} /\right\| f_{p_{\xi}}\left\|_{\infty, 2}-f_{p_{\eta}} /\right\| f_{p_{\eta}}\left\|_{\infty, 2}\right\| \leq 2-\varepsilon<2-2 /(1+m)$. As in the proof of (1) by $\left(^{*}\right) p_{\xi}$ and $p_{\eta}$ are compatible, as required.

Proposition 67. Assume OCA. Whenever $\mathcal{A}$ is an a.d.-family, then either
(1) The unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ is the union of countably many sets of diameters less than 1 or
(2) for each $0<\varepsilon<1$ there is an uncountable $(1+\varepsilon)$-separated subset of the unit sphere of $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$.

Proof. As proved in Proposition [31, the Open Coloring Axiom implies that for every a.d.-family $\mathcal{A}$ the forcing $\mathbb{P}_{\mathcal{A}}$ is either $\sigma$-centered or fails to satisfy the c.c.c.

If $\mathbb{P}_{\mathcal{A}}$ admits an uncountable antichain, then $S_{\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}}$ admits a $(1+\varepsilon)$-separated set for every $0<\varepsilon<1$ by Proposition 66 (3).

If $\mathbb{P}_{\mathcal{A}}$ is $\sigma$-centered, then $S_{\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}}$ is the union of countably many sets of diameter less than 1 by Proposition 63,

The following proposition shows that the above dichotomy cannot be proved in ZFC alone.

Proposition 68. Let $\mathcal{A}$ be an antiramsey a.d.-family. Then for every injective bounded linear operator $T: \mathcal{X}_{\mathcal{A}} \rightarrow \mathcal{Y}$ with separable range and every Banach space $\mathcal{Y}$ and every $\varepsilon>0$ there is a set $\left\{x_{\xi}: \xi<\omega_{1}\right\} \subseteq S_{\mathcal{X}_{\mathcal{A}},\| \|_{T}}$ such that for every uncountable $\Gamma \subseteq \omega_{1}$

- there are $\xi, \eta \in \Gamma$ such that $0<\left\|x_{\xi}-x_{\eta}\right\|_{T}<1$ and
- there are $\xi, \eta \in \Gamma$ such that $\left\|x_{\xi}-x_{\eta}\right\|_{T} \geq 2-\varepsilon$ and

Proof. Let $\mathcal{A}=\left\{A_{\alpha}: \alpha<\omega_{1}\right\}$. Let $k \in \mathbb{N}$ be such that $2 /(k+1) \leq \varepsilon$.
By Lemma 62 there is an uncountable $\Theta \subseteq \omega_{1}$ such that $\left\|T\left(1_{A_{\alpha}}-1_{A_{\beta}}\right)\right\| \mathcal{Y} \leq 1 / k$ for every $\alpha, \beta \in \Theta$. Let $\left\{\alpha_{\xi}: \xi<\omega_{1}\right\}$ be the increasing enumeration of $\Theta$ and let $x_{\xi}^{\prime}=1_{A_{\alpha_{\xi}}}-1_{A_{\alpha_{\xi+1}}}$ and $x_{\xi}=x_{\xi}^{\prime} /\left\|x_{\xi}^{\prime}\right\|_{T}$. Let $\Gamma \subseteq \omega_{1}$ be uncountable.

Since $\mathbb{P}_{\mathcal{A}}$ satisfies the c.c.c. when $\mathcal{A}$ is antiramsey (Definition 7 Proposition 65] implies that there are $\xi, \eta \in \Gamma$ such that $0<\left\|x_{\xi}-x_{\eta}\right\|_{T}<1$.

Let $p_{\xi}=\left(A_{\alpha_{\xi}} \backslash A_{\alpha_{\xi+1}}, A_{\alpha_{\xi+1}} \backslash A_{\alpha_{\xi}}\right) \in \mathbb{P}_{\mathcal{A}}$. Note that $x_{\xi}^{\prime}=f_{p_{\xi}}$ (see Definition 431). Since $\mathcal{A}$ is antiramsey there are $\xi<\eta$ such that $p_{\xi}$ and $p_{\eta}$ are not compatible. By Lemma 45 (2) we obtain that $\left\|x_{\xi}^{\prime}-x_{\eta}^{\prime}\right\|_{\infty}=2$. Taking $\delta=1-1 / k$ in Lemma 61 (which is justified by the choice of $\Theta$ ) we obtain that

$$
\left\|x_{\xi}-x_{\eta}\right\|_{T} \geq \frac{2}{2-\delta}=\frac{2}{1+1 / k}=2-\frac{2}{k+1} \geq 2-\varepsilon
$$

as required.
8. Renorming of a subspace of $\mathcal{X}_{\mathcal{A}, \phi}$ for an antiramsey a.d.-family $\mathcal{A}$

In this section we consider a space of the form $\mathcal{X}_{\mathcal{A}, \phi}$ for an a.d.-family $\mathcal{A}$ and a pairing $\phi($ see Definitions 55 and 56) $)$ with the norm $\left\|\|_{\infty, 2}\right.$ defined at the beginning of the previous section.

Proposition 69. Suppose that $\kappa$ is an uncountable cardinal, $\mathcal{A}$ is an antiramsey a.d.-family of cardinality $\kappa$ and $\rho>0$. There is a pairing $\phi: \kappa \rightarrow \mathcal{A}$ and a subspace $\mathcal{Y}$ (both depending on $\rho$ ) of $\mathcal{X}_{\mathcal{A}, \phi}$ such that whenever $\left\{y_{\xi}: \xi<\omega_{1}\right\} \subseteq S_{\mathcal{Y},\| \|_{\infty, 2}}$ for each $\xi<\eta<\omega_{1}$ satisfies

$$
\left\|y_{\xi}-y_{\eta}\right\|_{\infty, 2} \geq 1-\varepsilon
$$

for some $\varepsilon \leq 1 / 9$, then
(1) there are $\xi<\eta$ such that $\left\|y_{\xi}-y_{\eta}\right\|_{\infty, 2}>2-24 \varepsilon-\rho$ and
(2) there are $\xi<\eta$ such that $\left\|y_{\xi}-y_{\eta}\right\|_{\infty, 2}<1$.

Proof. Let $T$ be as in the definition of $\left\|\|_{\infty, 2}\right.$. Let $m \in \mathbb{N}$ be such that $2 /(m+1)<\rho$. Then $T$ restricted to

$$
\mathcal{X}_{m}=\left\{x \in \mathcal{X}_{\mathcal{A}}: x \mid\{0, \ldots, m\}=0\right\}
$$

has norm not bigger than $1 / m$. Let $\phi: \kappa \rightarrow \mathcal{A}$ be a pairing such that $\phi_{1}(\alpha) \cap$ $\{0, \ldots, m\}=\phi_{-1}(\alpha) \cap\{0, \ldots, m\}$ for every $\alpha<\kappa$. Then the space $\mathcal{Y}$ generated by both

$$
\left\{x \in c_{0}: x \mid\{0, \ldots, m\}=0\right\} \text { and }\left\{1_{\phi_{1}(\alpha)}-1_{\phi_{-1}(\alpha)}: \alpha<\kappa\right\}
$$

is included in $\mathcal{X}_{m} \cap \mathcal{X}_{\mathcal{A}, \phi}$, so again $T$ restricted to it has norm not bigger than $1 / m$ and so Lemma 61 can be applied to $\mathcal{Y}$ and $\delta=1-1 / m$.

By passing to an uncountable set and using Lemma 62 we may assume that two conditions are valid, namely first that $\left\|T\left(y_{\xi}\right)-T\left(y_{\eta}\right)\right\|_{2} \leq \varepsilon$ and second that $\left|\left\|y_{\xi}\right\|_{\infty}-\left\|y_{\eta}\right\|_{\infty}\right| \leq \varepsilon$ for every $\xi, \eta \in \omega_{1}$. The first condition and the hypothesis on $\left\{y_{\xi}: \xi<\omega\right\}$ imply that $\left\|y_{\xi}-y_{\eta}\right\|_{\infty} \geq 1-2 \varepsilon$ for every $\xi<\eta<\omega_{1}$, as $\left\|y_{\xi}\right\|_{\infty}<\left\|y_{\xi}\right\|_{\infty, 2}=1$ and $\left\|y_{\eta}\right\|_{\infty}<\left\|y_{\eta}\right\|_{\infty, 2}=1$, by the injectivity of $T$. The second condition allows us to apply Lemma 59 for $b=\max \left(\left\|y_{\xi}\right\|_{\infty},\left\|y_{\eta}\right\|_{\infty}\right), a=$ $\max \left(\left\|y_{\xi}\right\|_{\infty},\left\|y_{\eta}\right\|_{\infty}\right)$ and $(b-a) \leq \varepsilon$ to conclude that

$$
b\left\|y_{\xi} /\right\| y_{\xi}\left\|_{\infty}-y_{\eta} /\right\| y_{\eta}\left\|_{\infty}\right\|_{\infty}+\varepsilon \geq\left\|y_{\xi}-y_{\eta}\right\|_{\infty} \geq 1-2 \varepsilon
$$

and so

$$
\left\|y_{\xi} /\right\| y_{\xi}\left\|_{\infty}-y_{\eta} /\right\| y_{\eta}\left\|_{\infty}\right\|_{\infty}>1-3 \varepsilon
$$

for any $\xi<\eta<\omega_{1}$. Moreover $3 \varepsilon \leq 1 / 3$, so by Proposition 57 there are $\xi<$ $\eta<\omega_{1}$ such that $\left\|y_{\xi} /\right\| y_{\xi}\left\|_{\infty}-y_{\eta} /\right\| y_{\eta}\left\|_{\infty}\right\|_{\infty}>2-24 \varepsilon$. Let $\varepsilon^{\prime} \geq 0$ be such that $\left\|y_{\xi} /\right\| y_{\xi}\left\|_{\infty}-y_{\eta} /\right\| y_{\eta}\left\|_{\infty}\right\|_{\infty}=2-\varepsilon^{\prime}$. Clearly $-\varepsilon^{\prime}>-24 \varepsilon$. Before applying 61 note that $\left(y_{\xi} /\left\|y_{\xi}\right\|_{\infty}\right) /\left\|y_{\xi} /\right\| y_{\xi}\left\|_{\infty}\right\|_{\infty, 2}=y_{\xi} /\left\|y_{\xi}\right\|_{\infty, 2}=y_{\xi}$ and similarly for $y_{\eta}$, So Lemma 61 yields

$$
\left\|y_{\xi}-y_{\eta}\right\|_{\infty, 2} \geq \frac{2}{2-\delta}-\varepsilon^{\prime}=\frac{2}{1+1 / m}-\varepsilon^{\prime}=2-\varepsilon^{\prime}-\frac{2}{m+1}>2-24 \varepsilon-\rho
$$

as required in the case of (1).
By the definition of an antiramsey a.d.-family, the partial order of $\mathbb{P}_{\mathcal{A}}$ satsifies the c.c.c. and so by Proposition 65 there are $\xi<\eta<\omega_{1}$ like in (2).

## 9. Questions

As the properties of the a.d.-families considered in this paper and the geometric phenomena on the induced Banach spaces are completely new, the results of this paper generate many questions which the authors have not answered. Below we mention several of such questions which seem most interesting and natural at this moment to the authors.

Question 70. Can we consistently improve the dichotomy 31 obtaining: for any a.d.-family $\mathcal{A}$ the partial order $\mathbb{P}_{\mathcal{A}}$ is either $\sigma$-centered or it contains an antichain of cardinality $|\mathcal{A}|$ ?

This would be equivalent to obtaining a dichotomy for Banach spaces $\left(\mathcal{X}_{\mathcal{A}},\| \|_{T}\right)$ : Banach spaces $\left(\mathcal{X}_{\mathcal{A}},\| \|_{T}\right)$ either contain a $(1+\varepsilon)$-separated set of cardinality equal to the density of the Banach space or the unit sphere of it is the union of countably many sets which have diameter strictly less than 1.

The defintion of a Luzin family (Definition 11) makes sense only for a.d.-families of cardinality $\omega_{1}$. However the strong consequences of being Luzin like being an $L$-family in principle could hold for a.d.-families of bigger cardinalities:

Question 71. Is it consistent that there are L-families of cardinalities bigger than $\omega_{1}$ ?

In fact one could introduce a notion symmetric to $\mathbb{P}_{\mathcal{A}}$ having precaliber $\omega_{1}$ :
Definition 72. An a.d.-family is called L-saturated if every uncountable subset of $\mathbb{P}_{\mathcal{A}}$ consisting of pairwise essentially distinct elements contains and uncountable antichain.

Question 73. Is it consistent that there are L-saturated a.d.-families of cardinalities bigger than $\omega_{1}$ ?

Note that in Questions 71 and 73 we can only hope for the consistency because it was proved in [9] that it is consistent that every a.d.-family of cardinality $\mathfrak{c}=\omega_{2}$ contains a subfamily of cardinality $\omega_{2}$ which is $\mathbb{R}$-embeddable which yields a big pairwise compatible set in $\mathbb{P}_{\mathcal{A}}$ by Proposition 27. We also do not know the answers to the following questions:

Question 74. Is it consistent that there is an a.d.-family such that $\mathbb{P}_{\mathcal{A}}$ has precaliber $\omega_{1}$ but $\mathbb{P}_{\mathcal{A}}$ is not $\sigma$-centered?

Question 75. Is it consistent that there is an a.d.-family of cardinality $\omega_{1}$ which is L-saturated but is not an L-family?

Question 76. Is there (consistently) an L-family which is not a Luzin family?
It is clear in the light of the results of the last three sections of this paper that any answer to the above questions would generate a result concerning Banach spaces.

The dichotomy 31 holds for Banach spaces of the form $\left(\mathcal{X}_{\mathcal{A}},\| \|_{\infty, 2}\right)$ under an additional axiom OCA. It is very natural to ask if this can be extended for bigger classes of Banach spaces, for example for all Banach spaces:

Question 77. Is it consistent that for every nonseparable Banach space the unit sphere either admits an uncountable (1+)-separated set or is the union of countably many sets of diameters strictly less than 1?

Question 78. Is there in ZFC a nonseparable Banach space $\mathcal{X}$ and $\delta>0$ such that for sufficiently small $\varepsilon>0$ every $(1-\varepsilon)$-separated set in $S_{\mathcal{X}}$ contains two elements distant by less than 1 and two elements distant by more than 1 ?

Question 79. Is there in ZFC a nonseparable Banach space $\mathcal{X}$ such that for sufficiently small $\varepsilon>0$ every $(1-\varepsilon)$-separated set in $S_{\mathcal{X}}$ contains two elements distant by less than $1+\delta$ and two elements distant by more than $1+2 \delta$ ?

Let us also repeat a relevant question from [13] (cf. Proposition 16)
Question 80. Is it consistent with $\mathrm{MA}+\neg \mathrm{CH}$ that there is an a.d.-family admitting a 3-Luzin gap but not admitting a 2-Luzin gap?

## References

1. C. Akemann and J. Doner, A nonseparable $C^{*}$-algebra with only separable abelian $C^{*}-$ subalgebras. Bull. London Math. Soc. 11 (1979), no. 3, 279-284.
2. T. Bice, P. Koszmider, A note on the Akemann-Doner and Farah-Wofsey constructions. Proc. Amer. Math. Soc. 145 (2017), no. 2, 681-687.
3. F. Cabello Sánchez, J. M. F. Castillo, W. Marciszewski, G. Plebanek, A. Salguero-Alarcón, Sailing over three problems of Koszmider. J. Funct. Anal. 279 (2020), no. 4, 108571.
4. M. Cúth, O. Kurka, B. Vejnar, Large separated sets of unit vectors in Banach spaces of continuous functions, Colloq. Math. 157 (2019), no. 2, 173-187.
5. R. Deville, G. Godefroy, V. Zizler, Smoothness and renormings in Banach spaces. Pitman Monographs and Surveys in Pure and Applied Mathematics, 64. Longman Scientific \& Technical, Harlow; copublished in the United States with John Wiley \& Sons, Inc., New York, 1993.
6. J. Elton and E. Odell, The unit ball of every infinite-dimensional normed linear space contains a $(1+\varepsilon)$-separated sequence, Colloq. Math. 44 (1981), 105-109.
7. R. Engelking, General topology. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.
8. M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, Banach space theory. The basis for linear and nonlinear analysis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.
9. O. Guzmán, Osvaldo; M. Hrušák, P. Koszmider, On $\mathbb{R}$-embeddability of almost disjoint families and Akemann-Doner $C^{*}$-algebras. Fund. Math. 254 (2021), no. 1, 15-47.
10. P. Hájek, T. Kania, T. Russo, Separated sets and Auerbach systems in Banach spaces. Trans. Amer. Math. Soc. 373 (2020), no. 10, 6961-6998.
11. F. Hernández-Hernández, M. Hrušák, $Q$-sets and normality of $\Psi$-spaces. Spring Topology and Dynamical Systems Conference. Topology Proc. 29 (2005), no. 1, 155-165.
12. F. Hernández-Hernández, M. Hrušák, M. Topology of Mrówka-Isbell spaces. Pseudocompact topological spaces, 253-289, Dev. Math., 55, Springer, Cham, 2018.
13. M. Hrušák, O. Guzmán, n-Luzin gaps. Topology Appl. 160 (2013), no. 12, 1364-1374.
14. M. Hrušák, Almost disjoint families and topology. Recent progress in general topology. III, 601-638, Atlantis Press, Paris, 2014.
15. T. Jech, Set theory. The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
16. W. Johnson, J. Lindenstrauss, Some remarks on weakly compactly generated Banach spaces. Israel J. Math. 17 (1974), 219-230.
17. W. Johnson, J. Lindenstrauss, Correction to: "Some remarks on weakly compactly generated Banach spaces" Israel J. Math. 32 (1979), no. 4, 382-383.
18. T. Kania, T. Kochanek, Uncountable sets of unit vectors that are separated by more than 1, Studia Math. 232 (2016), no. 1, 19-44.
19. P. Komjáth, V. Totik, Problems and theorems in classical set theory. Problem Books in Mathematics. Springer, New York, 2006.
20. P. Koszmider, Forcing minimal extensions of Boolean algebras. Trans. Amer. Math. Soc. 351 (1999), no. 8, 3073-3117.
21. P. Koszmider, Uncountable equilateral sets in Banach spaces of the form $C(K)$. Israel J. Math. 224 (2018), no. 1, 83-103.
22. P. Koszmider, N. J. Laustsen, A Banach space induced by an almost disjoint family, admitting only few operators and decompositions. Adv. Math. 381 (2021), 107613.
23. P. Koszmider, H. M. Wark, Large Banach spaces with no infinite equilateral sets. To appear in Bull. Lond. Math. Soc. arXiv: 2104.05335.
24. P. Koszmider, Banach spaces in which large subsets of spheres concentrate To appear in Journal of the Inst. of Math. Jussieu. arXiv: 2104.05335.
25. C. Kottman, Subsets of the unit ball that are separated by more than one. Studia Math. 53 (1975), no. 1, 15-27.
26. A. Kryczka, S. Prus, Separated sequences in nonreflexive Banach spaces. Proc. Amer. Math. Soc. 129 (2001), no. 1, 155-163.
27. N. N. Luzin. On subsets of the series of natural numbers. Izvestiya Akad. Nauk SSSR. Ser. Mat., 11, (1947), 403-410.
28. S. Mercourakis, G. Vassiliadis, Equilateral sets in infinite dimensional Banach spaces. Proc. Amer. Math. Soc. 142 (2014), no. 1, 205-212.
29. S. Mercourakis, G. Vassiliadis, Equilateral sets in Banach spaces of the form $C(K)$. Studia Math. 231 (2015), no. 3, 241-255.
30. F. Riesz, Über lineare Funktionalgleichungen, Acta Math. 41 (1916), 71-98.

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[^1]:    ${ }^{1}$ Note that $(A, B) \bowtie(C, D) \neq \emptyset$ is equivalent to $((A \backslash B) \cup(C \backslash D)) \cap((B \backslash A) \cup(D \backslash C)) \neq \emptyset$.

