The Category Dichotomy for Ideals

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Abstract

We prove that there is an ideal on ω that is not Katětov below nwd and does not have restrictions above $\mathcal{ED}$. We also prove that in the Laver model every tall P-ideal is Katětov-Blass above $\mathcal{ED}_{\mathsf{fin}}$ and that it is consistent that every Q^+ ideal is meager.

1 Introduction

The theory of ideals on countable sets is now a fundamental part of set theory and has countless applications in infinite combinatorics. For this reason, it is essential to have tools to classify and order the ideals. One such tool is the *Katětov order* (which was introduced in [38] by Katětov to study convergence in topological spaces) which has proven to be highly fruitful. When restricted to maximal ideals, it coincides with the more well-known *Rudin-Keisler order*, which has been greatly studied in the literature (see [23]). The Katětov order becomes particularly interesting when restricted to ideals with a certain degree of definability, such as Borel or analytic. As it is often the case, when restricting to definable objects, we find a richer structure that would not be expected to be present in the general case. One such result is the *Category Dichotomy*[1] of the fourth author, which is the following[2]:

Theorem 1 (H. [32]) ***Category Dichotomy for Borel Ideals***
Let $\mathcal{I}$ be a Borel ideal on ω. One of the following holds:

1. $\mathcal{I} \leq_K \mathsf{nwd}$.

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[1]We point out that the two alternatives of the Category Dichotomy are not mutually exclusive.
[2]The undefined notions will be reviewed in the next section.

2. *There is $X \in \mathcal{I}^+$ such that $\mathcal{ED} \leq_K \mathcal{I} \restriction X$.*

We can now ask whether it is possible to extend this dichotomy to a larger class of ideals. The first question is whether the dichotomy holds for all of them. In other words:

Problem 2 *Is every ideal on ω either Katětov below nwd or has a restriction above $\mathcal{ED}$?*

One would expect the answer to be negative, and it is easy to produce such examples assuming the *Continuum Hypothesis* (CH) However, finding an answer within ZFC is much harder. The main result of this article is to provide such an example. Upon examining the proof of the Theorem 1, it is evident that the argument can be extended to all ideals if we assume certain determinacy for transfinite games ($\mathsf{AD}(\mathbb{R})$ is more than enough). Furthermore, the third author and Jareb Navarro recently proved that the Category Dichotomy holds for all ideals in the Solovay model. In this way, the *Axiom of Choice (AC)* must play a role if we are to construct a counterexample to the dichotomy. While looking for the example, we explored the following classes of ideals:

1. Maximal ideals.
2. Ideals generated by MAD families.
3. P-ideals.
4. Ideals induced by independent families.
5. Ideals of nowhere dense sets of countable topological spaces.

We will prove that the desired example can be found in the last class; there is a countable topological space X such that the ideal of nowhere dense sets of X does not satisfy the Category Dichotomy. Such space already appears in the literature, it is the space constructed by the first author in [17], which he used to answer a question of Juhász.

Given Γ a class of ideals on ω, we will say that the *Category Dichotomy holds for* Γ if every ideal on Γ is either below nwd or has a restriction above $\mathcal{ED}$. The following table summarizes our results regarding the classes of ideals mentioned before:

Class of ideals	**Category Dichotomy**
Borel	True
Maximal ideals	Independent
Ideals generated by MAD families	Consistently false, unknown if consistently true
P-ideals	Independent
Non-meager ideals	Independent
Nowhere dense ideals induced by independent families	Consistently false, unknown if consistently true
Ideals of nowhere dense sets of countable topological spaces	False

Table 1. The Category Dichotomy for some classes of ideals

The paper is organized in the following way: first we review all the definitions and results regarding the Katětov order and definable ideals that will be used in the paper. In Section 4 we introduce the *uniformity number for a class of ideals*. Concretely, if Γ is a class of ideals on ω, by $\mathsf{non}(\Gamma)$ we denote the least cofinality of an ideal that is not in Γ. We review what is known regarding this invariants and make some small remarks. Sections 5 and 6 are respectively devoted to maximal ideals and ideals generated by MAD families. We prove that the Category Dichotomy for maximal ideals is equivalent to the non existence of Ramsey ultrafilters and the Category Dichotomy for ideals generated by MAD families is equivalent to the non existence of Cohen indestructible maximal almost disjoint families. Section 7 is dedicated to P-ideals. We prove that the Category Dichotomy for this class fails if either $\mathfrak{t} < \mathsf{cov}(\mathcal{M})$ or $\mathfrak{t} = \mathfrak{d}$. On the other hand, we show that it is true in the Laver model. In Section 8 we prove that it is consistent that every Q^+ ideal is meager, hence it is consistent that every non-meager ideal satisfies the Category Dichotomy. In Section 9 we look at ideals of the form $\mathsf{nwd}(X)$ where X is a countable topological space. The desired counterexample to the dichotomy can be found within this class. In Section 10 we look at the nowhere dense ideals of the spaces induce by independent families and prove that the Category Dichotomy may fail for this class.

2 Preliminaries and notation

For a set X, we denote by $\mathcal{P}(X)$ its power set. We say that $\mathcal{I} \subseteq \mathcal{P}(X)$ is an *ideal on* X if $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$, for every $A, B \subseteq X$, if $A \in \mathcal{I}$ and $B \subseteq A$ then

$B \in \mathcal{I}$ and if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$. On the other hand, $\mathcal{F} \subseteq \wp(X)$ is a *filter on* X if $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$, for every $A, B \subseteq X$, if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$ and if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$. An *ultrafilter* is a maximal filter that does not contain any finite set (for us, all ultrafilters are non principal). Given a family $\mathcal{B}$ of subsets of X, we define $\mathcal{B}^* = \{X \setminus B \mid B \in \mathcal{B}\}$. It is easy to see that if $\mathcal{F}$ is a filter then $\mathcal{F}^*$ is an ideal (called the *dual ideal of* $\mathcal{F}$) and if $\mathcal{I}$ an ideal then $\mathcal{I}^*$ is a filter (called the dual filter of $\mathcal{I}$). If $\mathcal{I}$ is an ideal on X, we let $\mathcal{I}^+ = \wp(X) \smallsetminus \mathcal{I}$ be the family of *$\mathcal{I}$-positive sets.* If $\mathcal{F}$ is a filter, we define $\mathcal{F}^+ = (\mathcal{F}^*)^+$; it is easy to see that $\mathcal{F}^+$ is the family of all sets that have non-empty (infinite) intersection with every element of $\mathcal{F}$. If $A \in \mathcal{I}^+$ then *the restriction of* $\mathcal{I}$ to A, defined as $\mathcal{I} \restriction A = \wp(A) \cap \mathcal{I}$, is an ideal on A.

The *cardinal invariants of the continuum* will play a fundamental role in this paper. We begin with a brief review of their definitions and key properties, focusing on those relevant to our discussion. For a more comprehensive treatment, we refer the reader to [4] and [3].

By $\mathfrak{c}$ we denote the size of the real numbers. Letting $f, g \in \omega^\omega$, define $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \omega$ and $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. A family $\mathcal{D} \subseteq \omega^\omega$ is a *dominating family* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$. The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family and the *dominating number* $\mathfrak{d}$ is the smallest size of a dominating family. It is easy to see that $\mathfrak{d}$ is also the least size of a dominating family with the more strict order $\leq$ (on the other hand, the least size of an unbounded family with $\leq$ is simply ω). The invariant $\mathfrak{d}$ can also be characterized using *interval partitions* (partitions of ω into intervals). Given P and R interval partitions, define $P \leq R$ if every interval of R contains at least one interval of P. It can be proved that the dominating number is equal to the least size of a dominating family of interval partitions (there is a similar characterization of $\mathfrak{b}$, but we do not mention it since it will not be used in the text).

Letting X be a topological space and $N \subseteq X$, we say N is *nowhere dense* if for every non empty open set $U \subseteq X$, we can find a non empty open set V such that $V \subseteq U$ and $V \cap N = \emptyset$. It is very easy to see that the closure of a nowhere dense set is also nowhere dense. A *meager set* is a countable union of nowhere dense sets. By $\mathsf{cov}(\mathcal{M})$ we denote the smallest size of a family of meager sets that covers ω^ω.

For any two sets A and B, we say $A \subseteq^* B$ (A is an almost subset of B) if $A \setminus B$ is finite. For $\mathcal{H} \subseteq [\omega]^\omega$ and $A, B \subseteq \omega$, we say that A *is a pseudointersection of* $\mathcal{H}$ if it is almost contained in every element of $\mathcal{H}$. On the other hand, B *is a pseudounion of* $\mathcal{H}$ if it almost contains every element of $\mathcal{H}$. We say $\mathcal{T} = \{A_\alpha \mid \alpha < \kappa\}$ is a *tower* if it is $\subseteq^*$-decreasing and has no infinite pseudointersection. On the other hand, we call $\mathcal{S} = \{A_\alpha \mid \alpha < \kappa\}$ an *increasing tower* if

$\{\omega \setminus A_\alpha \mid \alpha < \kappa\}$ is a tower. The *tower number* $\mathfrak{t}$ is the least length of a tower. It is known that $\mathfrak{t} \leq \mathfrak{b}$ and $\mathfrak{d}$ is larger than both $\mathfrak{b}$ and $\mathsf{cov}(\mathcal{M})$. Evidently, $\mathfrak{c}$ is larger than all the other cardinal invariants.

We say that $T \subseteq \omega^{<\omega}$ is a *tree* if it is closed under taking initial segments. If $s \in T$ we define $\mathsf{suc}_T(s) = \{n \mid s^\frown n \in T\}$ (where $s^\frown n$ is the sequence that has s as an initial segment and n in the last entry). If $T \subseteq \omega^{<\omega}$ we say that $f \in \omega^\omega$ is a *branch of* T if $f \restriction n \in T$ for every $n \in \omega$. The set of all branches of T is denoted by $[T]$. For every $n \in \omega$ we define $T_n = \{s \in T \mid |s| = n\}$. If $s \in \omega^{<\omega}$ then the *cone of* s is defined as $\langle s \rangle = \{f \in \omega^\omega \mid s \subseteq f\}$. Letting $\mathcal{X} \subseteq [\omega]^\omega$, we say a tree $T \subseteq \omega^{<\omega}$ is a $\mathcal{X}$*-branching tree* if $\mathsf{suc}_T(s) \in \mathcal{X}$ for every $s \in T$.

3 Preliminaries on ideals on countable sets

In this section, we gather the definitions and results on ideals in countable sets and the Katětov order that will be needed for the paper. Readers interested in learning more about these topics are encouraged to consult [32], [33], [30], [26], [10], [2], [52], [55], [24], [1], [40], [44] or [19] among others.

We will be mainly interested in filters and ideals on countable sets. Topology turns out to be extremely useful when studying them. We endow $\mathcal{P}(\omega)$ with the natural topology that makes it homeomorphic to 2^ω. In this way, the topology of $\mathcal{P}(\omega)$ has as a subbase the sets of the form $\langle n \rangle_0 = \{A \subseteq \omega \mid n \notin A\}$ and $\langle n \rangle_1 = \{A \subseteq \omega \mid n \in A\}$, for $n \in \omega$. We view filters and ideals as subspaces of $\mathcal{P}(\omega)$. All notions of Borel, analytic or meager are referred to this topology. The following Borel ideals will play a key role on the paper. For the next definitions, given $n \in \omega$, denote the column $C_n = \{(n, m) \mid m \in \omega\}$ and $\mathcal{C} = \{C_n \mid n \in \omega\}$. Given $f \in \omega^\omega$, denote $D(f) = \{(n, m) \in \omega \times \omega \mid m \leq f(n)\}$.

1. The ideal fin is the ideal of finite subsets of ω.

2. The *eventually different* ideal $\mathcal{ED}$ is the ideal on ω^2 generated by $\mathcal{C}$ and the graphs of functions from ω to ω.

3. The ideal $\mathcal{ED}_{\mathsf{fin}}$ is the restriction of $\mathcal{ED}$ to $\triangle = \{(n, m) \mid m \leq n\}$.

4. The ideal $\mathsf{fin}\times\mathsf{fin}$ is the ideal on ω^2 generated by $\mathcal{C} \cup \{D(f) \mid f \in \omega^\omega\}$.

5. The *nowhere dense ideal,* nwd is the ideal of nowhere dense subsets of the rational numbers.

We now list some of the main notions concerning ideals on countable sets.

Definition 3 *Let $\mathcal{I}$ be an ideal on ω (or any countable set).*

1. $\mathcal{I}$ is tall *if for every $X \in [\omega]^\omega$ there is $Y \in \mathcal{I}$ such that $Y \cap X$ is infinite.*

2. $\mathcal{I}$ *is* ω-hitting *if for every* $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ *there is* $Y \in \mathcal{I}$ *such that* $Y \cap X_n$ *is infinite for every* $n \in \omega$.

3. $\mathcal{I}$ *is* tight *if for every* $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$, *there is* $Y \in \mathcal{I}$ *such that* $Y \cap X_n$ *is infinite for every* $n \in \omega$.

4. $\mathcal{I}$ *is a* P-ideal *if every countable subfamily of* $\mathcal{I}$ *has a pseudounion in* $\mathcal{I}$.

5. $\mathcal{I}$ *is a* P^+-ideal *if every* $\subseteq$*-decreasing family* $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$ *has a pseudointersection in* $\mathcal{I}^+$.

6. $\mathcal{I}$ *is a* P^--ideal *if for every* $X \in \mathcal{I}^+$, *any* $\{X_n \mid n \in \omega\} \subseteq (\mathcal{I} \restriction X)^*$ *has a pseudointersection in* $\mathcal{I}^+$.

7. $\mathcal{I}$ *is a* Q^+-ideal *if for every* $X \in \mathcal{I}^+$ *and every partition* $\mathcal{P} = \{P_n \mid n \in \omega\}$ *of* X *into finite sets, there is* $A \in \mathcal{I}^+ \cap \wp(X)$ *such that* $|A \cap P_n| \leq 1$ *for every* $n \in \omega$. *Such* A *is called a* partial selector of $\mathcal{P}$ *and a* selector *in case it intersects each* P_n.

8. $\mathcal{I}$ *is* selective *if it is both* P^+ *and* Q^+.

9. $\mathcal{I}$ *is* weakly selective *if for every* $X \in \mathcal{I}^+$ *and* $\mathcal{P}$ *a partition of* X, *either* $\mathcal{P} \cap \mathcal{I} \neq \emptyset$, *or* $\mathcal{P}$ *has a (partial) selector in* $\mathcal{I}^+$.

10. $\mathcal{I}$ *is* $+$-Ramsey *if every* $\mathcal{I}^+$*-branching tree* T *has a branch in* $\mathcal{I}^+$.

11. $\mathcal{I}$ *is* Cohen indestructible (*or* $\mathbb{C}$-indestructible) *if* $\mathcal{I}$ *remains tall after adding a Cohen real to the universe.*

The notions dualize to filters. For example, a *P-filter* is a filter whose dual ideal is a P-ideal. The same convention applies to all the other properties listed above. We will switch between filters and ideals as needed, depending on what is most convenient at the time. An ultrafilter is called a *Ramsey ultrafilter* if its dual is a selective ideal. The following is a list of simple observations regarding the aforementioned notions:

1. ω-hitting ideals are tall and tight.
2. A tall P-ideal is ω-hitting.
3. An ideal is weakly selective if and only if it is P^- and Q^+.
4. Either P or P^+ implies P^-.
5. A $+$-Ramsey ideal is weakly selective.
6. If $\mathcal{U}$ is an ultrafilter, then $\mathcal{U}^*$ is ω-hitting.
7. $\mathcal{I}$ is tight if and only if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^+$, there is $Y \in \mathcal{I}$ such that $Y \cap X_n \neq \emptyset$ for every $n \in \omega$ (in order to prove the non trivial implication, it is enough to realize we can add to our original family all finite modifications of each X_n).

We now look at cardinal invariants associated to ideals on countable sets.

Definition 4 *Let $\mathcal{I}$ be a tall ideal on ω. Define:*

1. $\mathsf{non}^*(\mathcal{I})$ *is the smallest size of a family $\mathcal{H} \subseteq [\omega]^\omega$ such that for every $A \in \mathcal{I}$, there is $H \in \mathcal{H}$ such that $A \cap H$ is finite.*

2. $\mathsf{cof}(\mathcal{I})$ *is the smallest size of a cofinal family in $(\mathcal{I}, \subseteq)$.*

The invariant $\mathsf{non}^*(\mathcal{I})$ was introduced by Brendle and Shelah in [9], but with a different name. The notation used here follows [27]. It is easy to see that$\mathsf{non}^*(\mathcal{I}) \leq \mathsf{cof}(\mathcal{I})$.

Definition 5 *Let X, Y be two sets, $\mathcal{I}$ an ideal on X, $\mathcal{J}$ an ideal on Y and $f : X \longrightarrow Y$.*

1. *f is a* Katětov function *from $\mathcal{I}$ to $\mathcal{J}$ if for every $A \subseteq Y$, the following holds:*

   $$\text{If } A \in \mathcal{J}, \text{ then } f^{-1}(A) \in \mathcal{I}.$$

2. *f is a* Katětov-Blass function *from $\mathcal{I}$ to $\mathcal{J}$ if it is a Katětov function and it is finite to one.*

3. *f is a* Rudin-Keiser function *from $\mathcal{I}$ to $\mathcal{J}$ if for every $A \subseteq Y$, the following holds:*

   $$A \in \mathcal{J} \text{ if and only if } f^{-1}(A) \in \mathcal{I}.$$

4. *f is a* Rudin-Blass function *from $\mathcal{I}$ to $\mathcal{J}$ if it is a Rudin-Keisler function and it is finite to one.*

5. *$\mathcal{J} \leq_{\mathsf{K}} \mathcal{I}$ if there is a Katětov function from $\mathcal{I}$ to $\mathcal{J}$. The orders $\leq_{\mathsf{KB}}$, $\leq_{\mathsf{RK}}$ and $\leq_{\mathsf{RB}}$ are defined analogously.*

6. *$\mathcal{I}$ and $\mathcal{J}$ are* Katětov equivalent *(denoted as $\mathcal{I} =_{\mathsf{K}} \mathcal{J}$) if $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathsf{K}} \mathcal{I}$.*

The following is a list of easy facts about the Katětov order.

Lemma 6 *Let $\mathcal{I}$, $\mathcal{J}$ be ideals on ω and $X \subseteq \omega$.*

1. *If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{J}$.*

2. *$\mathsf{fin} \leq_{\mathsf{KB}} \mathcal{I}$.*

3. *$\mathcal{I}$ is Katětov equivalent to fin if and only if $\mathcal{I}$ is not tall.*

4. *If $X \in \mathcal{I}^+$, then $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{I} \restriction X$.*

5. *If* $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{J}$, *then* $\mathsf{non}^*(\mathcal{I}) \leq \mathsf{non}^*(\mathcal{J})$.

The following equivalence of the Katětov order is often useful:

Lemma 7 *Let* X, Y *be two sets,* $\mathcal{I}$ *an ideal on* X, $\mathcal{J}$ *an ideal on* Y *and* $f : X \longrightarrow Y$. *The following are equivalent:*

1. f *is a Katětov function from* $\mathcal{I}$ *to* $\mathcal{J}$.
2. *For every* $A \subseteq X$, *if* $A \in \mathcal{I}^+$, *then* $f[A] \in \mathcal{J}^+$.

We have the following:

Lemma 8

1. $\mathcal{ED} \leq_{\mathsf{KB}} \mathsf{fin}\times\mathsf{fin}$ *and* $\mathcal{ED} \leq_{\mathsf{KB}} \mathcal{ED}_{\mathsf{fin}}$.
2. $\mathcal{ED}$ *and* nwd *are Katětov incomparable.*
3. $\mathsf{fin}\times\mathsf{fin}$, $\mathcal{ED}_{\mathsf{fin}}$ *and* nwd *are Katětov incomparable.*

There are simple combinatorial characterizations of the Katětov order when one of the ideals is one of those we defined earlier.

Katětov relation	**Equivalent to**	**Reference**
$\mathcal{I} \leq_{\mathsf{K}} \mathsf{fin}$	$\mathcal{I}$ is not tall	[33]
$\mathcal{I} \not\leq_{\mathsf{K}} \mathsf{nwd}$	$\mathcal{I}$ is $\mathbb{C}$ indestructible	[33], [43]
$\mathcal{ED} \not\leq_{\mathsf{K}} \mathcal{I}$	Every partition of ω into sets in $\mathcal{I}$ has a selector in $\mathcal{I}^+$	[30]
$\mathcal{ED} \not\leq_{\mathsf{K}} \mathcal{I} \restriction X$ for all $X \in \mathcal{I}^+$	$\mathcal{I}$ is weakly selective	[30]
$\mathcal{ED}_{\mathsf{fin}} \not\leq_{\mathsf{KB}} \mathcal{I}$	Every partition of ω in finite pieces has a selector in $\mathcal{I}^+$	[49]
$\mathcal{ED}_{\mathsf{fin}} \not\leq_{\mathsf{KB}} \mathcal{I} \restriction X$ for all $X \in \mathcal{I}^+$	$\mathcal{I}$ is Q^+	[49]
$\mathsf{fin}\times\mathsf{fin} \not\leq_{\mathsf{K}} \mathcal{I}$	Every countable subfamily of $\mathcal{I}^*$ has a pseudointersection in $\mathcal{I}^+$	[49]
$\mathsf{fin}\times\mathsf{fin} \not\leq_{\mathsf{K}} \mathcal{I} \restriction X$ for all $X \in \mathcal{I}^+$.	$\mathcal{I}$ is P^-	[49]

Table 2. Combinatorial properties and the Katětov order

The ideals $\mathcal{ED}$, $\mathsf{fin}\times\mathsf{fin}$ and $\mathcal{ED}_{\mathsf{fin}}$ are related in the following way:

Proposition 9 *Let $\mathcal{I}$ be an ideal on ω such that $\mathcal{ED} \leq_K \mathcal{I}$. Either $\mathsf{fin}\times\mathsf{fin} \leq_K \mathcal{I}$ or there is $X \in \mathcal{I}^+$ such that $\mathcal{ED}_{\mathsf{fin}} \leq_K \mathcal{I} \restriction X$.*

Proof. We use the equivalences from Table 2. Since $\mathcal{ED} \leq_{\mathsf{K}} \mathcal{I}$, there is a partition $\mathcal{P} = \{P_n \mid n \in \omega\} \subseteq \mathcal{I}$ such that every selector of it is in $\mathcal{I}$. We now wonder if the family $\{\omega \setminus P_n \mid n \in \omega\} \subseteq \mathcal{I}^*$ has a pseudointersection in $\mathcal{I}^+$. If not, we get that $\mathsf{fin}\times\mathsf{fin} \leq_{\mathsf{K}} \mathcal{I}$. In case there is $X \in \mathcal{I}^+$ a pseudointersection, we conclude that $\mathcal{ED}_{\mathsf{fin}} \leq_{\mathsf{K}} \mathcal{I} \restriction X$. ■

In this way, the Category Dichotomy is equivalent to a trichotomy: Every Borel ideal is either below nwd, or has a restriction above $\mathsf{fin}\times\mathsf{fin}$ or has a restriction above $\mathcal{ED}_{\mathsf{fin}}$. This is how the dichotomy was proved in [32].

We now look at some variants of $\mathcal{ED}_{\mathsf{fin}}$. We will say an interval partition $P = \{P_n \mid n \in \omega\}$ is *increasing* if $|P_n| < |P_{n+1}|$ for every $n \in \omega$.

Definition 10 *Let $P = \{P_n \mid n \in \omega\}$ be an increasing partition. Define the ideal $\mathcal{ED}_P$ as the ideal generated by all selectors of P.*

Note that $\mathcal{ED}_{\mathsf{fin}}$ is of this form. It is easy to see that all these ideals are Katětov-Blass equivalent; however even more is true, as we will now see.

Definition 11 *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on ω. We say that $\mathcal{I}$* and *$\mathcal{J}$ are* isomorphic *if there is a bijection $f \in \omega^\omega$ that is a Rudin-Keisler function from $(\omega, \mathcal{I})$ to $(\omega, \mathcal{J})$.*

The definition of isomorphism between ideals found in the book [18] is more general than the one presented here. However, for tall ideals, this notion is equivalent to the one from the book, which is the case of interest in this work.

Proposition 12 *Let P and R be two increasing interval partitions. The ideals $\mathcal{ED}_P$ and $\mathcal{ED}_R$ are isomorphic.*

Proof. Write $P = \{P_n \mid n \in \omega\}$ and $R = \{R_n \mid n \in \omega\}$. For $s \in [\omega]^{<\omega}$, denote $\mathsf{cov}_P(s)$ as the least number of intervals from P that cover s. The number $\mathsf{cov}_R(s)$ is defined in the same way. To prove the proposition, it is enough to find a function $g \in \omega^\omega$ with the following properties:

1. g is bijective.
2. For every $n \in \omega$, we have that $\mathsf{cov}_R(g[P_n]) \leq 2$.
3. For every $m \in \omega$, we have that $\mathsf{cov}_P\left(g^{-1}(R_m)\right) \leq 2$.

Indeed, g will be the desired isomorphism since the image of every selector of P will be covered by at most two selectors of R, and the other way around. We will find g by finite approximations. Define $\mathbb{P}$ as the set of all q such that for every $n \in \omega$, the following conditions hold:

1. g is a partial injective function from ω to ω with finite domain.
2. If $P_n \subseteq \mathsf{dom}(g)$, then $\mathsf{cov}_R(q[P_n]) \leq 2$.
3. If $R_n \subseteq \mathsf{im}(g)$, then $\mathsf{cov}_P\left(q^{-1}(R_n)\right) \leq 2$.
4. If $P_n \nsubseteq \mathsf{dom}(g)$, then $\mathsf{cov}_R(q[P_n]) \leq 1$.
5. If $R_n \nsubseteq \mathsf{im}(g)$, then $\mathsf{cov}_P\left(q^{-1}(R_n)\right) \leq 1$.

We order $\mathbb{P}$ by reverse inclusion. Letting $n \in \omega$, define $D_n = \{q \in \mathbb{P} \mid P_n \subseteq \mathsf{dom}(q)\}$ and $E_n = \{q \in \mathbb{P} \mid R_n \subseteq \mathsf{im}(q)\}$. We claim that both sets are dense in $\mathbb{P}$. We verify it for D_n. Pick $q \in \mathbb{P}$ such that P_n is not contained in the domain of q. Choose m large enough such that $|P_n| \leq |R_m|$ and $R_m \cap \mathsf{im}(q) = \emptyset$. In case $P_n \cap \mathsf{dom}(q) = \emptyset$, extend q such that it maps P_n into R_m. In the other case, extend it such that it sends $P_n \setminus \mathsf{dom}(q)$ into R_m.

Since we only have countably many dense sets, a trivial application of the Rasiowa-Sikorski Lemma (see [42] or [37]) yields the desired isomorphism. ■

Meager ideals have very strong combinatorial properties, as we will now review.

Definition 13 *Let $\mathcal{I}$ be an ideal on ω and $P = \{P_n \mid n \in \omega\}$ a partition of ω into finite intervals. We say that P is a* Talagrand partition for $\mathcal{I}$ *if for every $X \subseteq \omega$, if X contains infinitely many elements of P, then $X \in \mathcal{I}^+$.*

The following is a classical theorem in the theory of ideals. The reader may consult [3] for a proof:

Theorem 14 (Talagrand, Jalali-Naini) *Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

1. *$\mathcal{I}$ is meager.*
2. *$\mathcal{I}$ has the Baire property.*
3. *$\mathcal{I}$ has a Talagrand partition.*

We have the following:

Proposition 15 *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. If $\mathcal{I}$ is analytic and $\mathcal{J} \leq_K \mathcal{I}$, then $\mathcal{J}$ is meager.*

Proof. Let $g \in \omega^\omega$ be a Katětov function from $(\omega, \mathcal{I})$ to $(\omega, \mathcal{J})$. Define $\mathcal{L} = \{A \mid g^{-1}(A) \in \mathcal{I}\}$, it follows that $\mathcal{J} \subseteq \mathcal{L}$ and since $\mathcal{I}$ is analytic, it is easy to see that $\mathcal{L}$ is analytic as well. In this way, $\mathcal{L}$ has the Baire property (see [39]) and by Theorem 14, it is meager. Since $\mathcal{J} \subseteq \mathcal{L}$, we conclude that $\mathcal{J}$ is also meager. ■

It is worth pointing out that the Katětov predecesors of a meager ideal are not necessarily meager. In fact, it can be shown that the meager ideals are cofinal in the Katětov order, so the hypothesis that $\mathcal{I}$ is analytic is essential.

The cardinal invariant $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})$ will play an important role in this work. We will need the following theorem, which was obtained by Meza, Minami and the fourth author.

Theorem 16 (H., Meza, Minami [34])

1. *$\mathsf{cov}(\mathcal{M}) = \mathsf{min}\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})\}$.*
2. *Let κ be an infinite cardinal. The following are equivalent:*
   - (a) *$\kappa < \mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})$.*
   - (b) *For every increasing interval partition $P = \{P_n \mid n \in \omega\}$ and $\mathcal{B}$ a family of size κ consisting of partial infinite selectors of P, there is a selector of P that has infinite intersection with every element of $\mathcal{B}$.*

4 Uniformity number of a class of ideals

Letting Γ a class of ideals on countable sets, we denote by $\mathsf{non}(\Gamma)$ the least cofinality of an ideal that is not in Γ. Of course, this definition only makes sense if there exists an ideal that is not in Γ, which is always the case in the interesting settings. The study of this type of cardinal invariants is quite useful, as it allows us to deduce properties of an ideal "for free" (or more formally, by merely knowing its cofinality). Although the notation used here is likely new, these invariants have been studied for quite some time. Below, we summarize some already known results.

Cardinal Invariant	**Uniformity number of**	**Reference**
$\mathfrak{b}$	Meager ideals	[4]
$\mathfrak{d}$	P^+-ideals	[4]
$\mathsf{cov}(\mathcal{M})$	+-Ramsey ideals	[29]

Table 3. Some uniformity numbers of classes

We are interested in the uniformity numbers for the classes of P^-, Q^+ and weakly selective ideals, which we denote by $\mathsf{non}(\mathsf{P}^-)$, $\mathsf{non}(\mathsf{Q}^+)$ and $\mathsf{non}(\mathsf{WS})$ respectively. The case of for $\mathsf{non}(\mathsf{P}^-)$ is very easy:

Proposition 17 $\mathfrak{d} = \mathsf{non}(\mathsf{P}^-)$.

Proof. Since $\mathsf{fin}\times\mathsf{fin}$ is not P^- and it has cofinality $\mathfrak{d}$, we get that $\mathsf{non}(\mathsf{P}^-) \leq \mathfrak{d}$. On the other hand, as noted above, every ideal of cofinality less than $\mathfrak{d}$ is P^+, so we conclude that $\mathfrak{d} \leq \mathsf{non}(\mathsf{P}^-)$. ■

For an ideal $\mathcal{I}$, denote the classes of ideals $K(\mathcal{I}) = \{\mathcal{J} \mid \mathcal{I} \nleq_{\mathsf{K}} \mathcal{J}\}$ and $KB(\mathcal{I}) = \{\mathcal{J} \mid \mathcal{I} \nleq_{\mathsf{KB}} \mathcal{J}\}$. The following notion was studied by Brendle and Flašková [8] and by Hong and Zhang in [28].

Definition 18 *Let $\mathcal{I}$ be an ideal on ω. The* exterior cofinality *(also called* generic existence number*) of $\mathcal{I}$ is defined as* $\mathfrak{ge}(\mathcal{I}) = \mathsf{min}\{\mathsf{cof}(\mathcal{J}) \mid \mathcal{I} \subseteq \mathcal{J}\}$.

The generic existence number was introduced to investigate the *generic existence* of certain special classes of ultrafilters. Specifically, it refers to the property that any filter with cofinality less than $\mathfrak{c}$ can be extended to an ultrafilter within that class. For further details, we refer the interested reader to the previously mentioned papers.

Lemma 19 *Let $\mathcal{I}$ be an ideal on ω. The following cardinal invariants are equal:*

1. $\mathfrak{ge}(\mathcal{I})$.
2. $\mathsf{min}\{\mathsf{cof}(\mathcal{J}) \mid \mathcal{I} \leq_K \mathcal{J}\}$.
3. $\mathsf{min}\{\mathsf{cof}(\mathcal{J}) \mid \mathcal{I} \leq_{KB} \mathcal{J}\}$.
4. $\mathsf{non}(K(\mathcal{I}))$.
5. $\mathsf{non}(KB(\mathcal{I}))$.

Proof. The proof of the equality between 1,2 and 3 can be found implicitly in Observation 3.1 of [8]. Finally, $\mathsf{non}(K(\mathcal{I}))$ is obviously equivalent to the invariant in point 2 and $\mathsf{non}(KB(\mathcal{I}))$ is clearly equivalent to the one in point 3. ■

We conclude that $\mathsf{non}(\mathsf{Q}^+) = \mathfrak{ge}(\mathcal{ED}_{\mathsf{fin}})$ and $\mathsf{non}(\mathsf{WS}) = \mathfrak{ge}(\mathcal{ED})$ (here we are using the fact that $\mathsf{cof}(\mathcal{I} \restriction X) \leq \mathsf{cof}(\mathcal{I})$ for an ideal $\mathcal{I}$ and $X \in \mathcal{I}^+$). In [8] several generic existence numbers are computed. In particular, we can find the following:

Proposition 20 (Brendle, Flašková [8]) *The following holds:*

1. $\mathsf{non}(\mathsf{WS}) = \mathsf{cov}(\mathcal{M})$.

2. $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}}) \leq \mathsf{non}(\mathsf{Q}^+)$.

The following was conjecture in [8]:

Conjecture 21 (Brendle, Flašková) *The cardinals* $\mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})$ *and* $\mathsf{non}(\mathsf{Q}^+)$ *are equal.*

The conjecture still remains unsolved. These two cardinal invariants certain look very similar, for example, we have the following:

Corollary 22 $\mathsf{cov}(\mathcal{M}) = \mathsf{min}\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})\} = \mathsf{min}\{\mathfrak{d}, \mathsf{non}(\mathsf{Q}^+)\}$.

Proof. We already know $\mathsf{cov}(\mathcal{M}) = \mathsf{min}\{\mathfrak{d}, \mathsf{non}^*(\mathcal{ED}_{\mathsf{fin}})\}$ by Theorem 16. To see the other equality, just note that $\mathsf{non}(\mathsf{WS})$ is the minimum between $\mathsf{non}(\mathsf{Q}^+)$ and $\mathsf{non}(\mathsf{P}^-)$. ■

5 The Category Dichotomy for Maximal ideals

It is very easy to characterize when the Category Dichotomy holds for maximal ideals (duals of ultrafilters).

Proposition 23 *The following are equivalent:*

1. *The Category Dichotomy for the class of maximal ideals.*
2. *There are no Ramsey ultrafilters.*

Proof. An ultrafilter $\mathcal{U}$ is Ramsey if and only if $\mathcal{U}^*$ is not Katětov above $\mathcal{ED}$. In this way, in order to prove the proposition, it is enough to show that no maximal ideal can be Katětov below nwd. In fact, by Proposition 15, the dual of an ultrafilter is not Katětov below any analytic ideal (alternatively, we could use Lemma 7 below).

■

It is very easy to build Ramsey ultrafilters with CH and a classic theorem of Kunen is that it is consistent that they do not exist (see [41]). In this way, the Category Dichotomy for the class of duals of ultrafilters is independent. The reader may consult [57], [51], [50], [58], [6], [14] and [11] to learn more about models where certain ultrafilters do not exist.

6 The Category Dichotomy for MAD families

We say $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite. A MAD *family* is a maximal almost disjoint family. We denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. A MAD family is *Cohen indestructible* if it remains maximal after adding a Cohen real, which is equivalent that its ideal is not Katětov below nwd. We now prove the following:

Proposition 24 *The following are equivalent:*

1. *The Category Dichotomy for the class of ideals generated by* MAD *families.*
2. *There are no Cohen indestructible* MAD *families.*

Proof. It is enough to prove that if $\mathcal{A}$ is a MAD family, then no restriction of $\mathcal{I}(\mathcal{A})$ is Katětov above $\mathcal{ED}$. This is true because $\mathcal{I}(\mathcal{A})$ is selective see [48], [59] or the survey [31]. Even more is true, no restriction of $\mathcal{I}(\mathcal{A})$ can be Katětov above a tall analytic ideal. This is because of a theorem of Mathias that an ultrafilter is Ramsey if and only if it intersects every tall analytic ideal. ■

While it is consistent that Cohen indestructible MAD families exist, it is unknown if it is consistent that they do not (this is a famous problem of Steprāns). In this way, the Category Dichotomy for the MAD families is consistently false, but we do not know if it may consistently be true. The reader may consult [33], [43], [10], [35] and [15] to learn more about indestructibility of MAD families and ideals.

7 The Category Dichotomy for P-ideals

The following result is known, we prove it for the sake of completeness:

Lemma 25 *Let $\mathcal{I}$ be an ideal on ω.*

1. *If $\mathcal{I}$ is tight, then $\mathcal{I} \nleq_K$ nwd.*
2. *If $\mathcal{I}$ is a P-ideal, then $\mathsf{fin}\times\mathsf{fin} \nleq_K \mathcal{I}$.*

Proof. First assume $\mathcal{I}$ is tight and $f : \mathbb{Q} \longrightarrow \omega$. We will prove that f is not a Katětov from $(\mathbb{Q},\mathsf{nwd})$ to $(\omega, \mathcal{I})$. If there is an open non empty set of $\mathbb{Q}$ whose image is in $\mathcal{I}$, there is nothing to do (see Lemma 7), so assume this is not the case. Let $\{U_n \mid n \in \omega\}$ be a base of open sets of $\mathbb{Q}$. Since each $f[U_n] \in \mathcal{I}^+$, we can find $A \in \mathcal{I}$ having infinite intersection with all of them. In this way $f^{-1}(A)$ is dense, so it is not nowhere dense.

Now assume $\mathcal{I}$ is a P-ideal and $f : \omega \longrightarrow \omega^2$, We will prove f is not a Katětov function from $(\omega, \mathcal{I})$ to $(\omega^2, \mathsf{fin}\times\mathsf{fin})$. For $n \in \omega$, denote the column $C_n = \{n\} \times \omega$. If there is $n \in \omega$ such that $f^{-1}(C_n) \notin \mathcal{I}$, there is nothing to do. In the other case, since $\mathcal{I}$ is a P-ideal, we can find $A \in \mathcal{I}$ such that $f^{-1}(C_n) \subseteq^* A$ for every $n \in \omega$. In this way, the image of the complement of A is in $\mathsf{fin}\times\mathsf{fin}$, hence f is not Katětov. ■

In this way, the Category Dichotomy for the class of P-ideals is equivalent to the statement that every P-ideal has a restriction above $\mathcal{ED}_{\mathsf{fin}}$ (equivalently, they are not Q^+). Note that item 2 in the lemma above is false for ω-hitting ideals. For example, take a maximal ideal extending $\mathsf{fin}\times\mathsf{fin}$.

Definition 26 *Let $\mathcal{T}$ be an increasing tower. Define $\mathcal{I}(\mathcal{T})$ as the ideal generated by $\mathcal{T}$. For $\mathcal{T}$ a tower, denote $\mathcal{F}(\mathcal{T})$ the filter it generates.*

Since the length of a (increasing) tower can not have countable cofinality, it follows that its ideal (filter) is a P-ideal (P-filter). We can now prove the following:

Theorem 27 *Both $\mathfrak{t} < \mathsf{cov}(\mathcal{M})$ and $\mathfrak{t} = \mathfrak{d}$ imply that there is a P-ideal that is Q^+. In this way, the Category Dichotomy for the class of P-ideals fails.*

Proof. If $\mathfrak{t} < \mathsf{cov}(\mathcal{M})$, then by Theorem 20, the ideal generated by an increasing tower is weakly selective and we are done. Now we assume $\mathfrak{t} = \mathfrak{d}$. Let $\mathcal{P} = \{P_\alpha \mid \alpha < \mathfrak{d}\}$ be a dominating family of interval partitions. We recursively define $\mathcal{B} = \{B_\alpha \mid \alpha < \mathfrak{d}\}$ such that for every $\alpha < \mathfrak{d}$:

1. $\mathcal{B}$ is a tower.
2. B_α is a partial selector of P_α. Moreover, either B_α is contained in the union the even intervals or P_α or in the union of the odd intervals.

This is easy to do (using that $\mathfrak{t} = \mathfrak{d}$). Note that item 2 above implies that $\mathcal{B}$ is actually a tower. We claim that $\mathcal{F}(\mathcal{B})$ is as desired. First, we claim that for every interval partition R, there is $B \in \mathcal{F}(\mathcal{B})$ that is a partial selector of R. Since $\mathcal{P}$ is a dominating family, we can find $\alpha < \mathfrak{d}$ such that every interval of P_α contains one of R. This implies that every interval of R can intersect at most two intervals of P_α. Since B_α is a partial selector of P_α and it never intersects two consecutive elements of P_α, it follows that B_α is a partial selector of R. Finally, we argue that $\mathcal{F}(\mathcal{B})$ is a Q^+-filter. Let $X \in \mathcal{F}(\mathcal{B})^+$ and Q a partition of X into finite pieces. Define $R = Q \cup \{\{n\} \mid n \in \omega \setminus X\}$, which is a partition of ω into finite pieces. In this way, there is $B \in \mathcal{F}(\mathcal{B})$ a partial selector of R. It is clear that $B \cap X \in \mathcal{F}(\mathcal{B})^+$ and is a partial selector of Q. ■

So the Category Dichotomy of P-ideals is consistently false, surprisingly, it is also consistently true, in fact, we will now prove that it holds in the Laver model. We first review some notions regarding Laver forcing. Let $T \subseteq \omega^{<\omega}$ be a tree and $s \in T$. We say *s is the stem of T* (denoted as $s = \mathsf{st}(T)$) if every node of T is comparable with s and it is maximal with this property. A tree $p \subseteq \omega^{<\omega}$ is a *Laver tree* if it consists of increasing sequences, it has a stem and if $t \in T$ extends the stem of T, then $\mathsf{suc}_T(s)$ is infinite. The set of Laver trees is denoted by $\mathbb{L}$ and given $p, q \in \mathbb{L}$, denote $p \leq q$ if $p \subseteq q$. Moreover, define $p \leq_0 q$ if $p \leq q$ and $\mathsf{st}(p) = \mathsf{st}(q)$. If $p \in \mathbb{L}$ and $s \in p$, define $p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}$. It is clear that $p_s \in \mathbb{L}$, it extends p and in case $\mathsf{st}(p) \subseteq s$, we have that $\mathsf{st}(p_s) = s$.

The *Laver generic real* will be denoted by $l_{gen} \in \omega^\omega$ as is the only element that is a branch of every tree in the generic filter. The *generic Laver* *interval partition* is defined as $P_{gen} = \{P_{gen}(n) \mid n \in \omega\}$ where $P_{gen}(n) = [l_{gen}(n), l_{gen}(n+1))$. This is an interval partition of almost all ω (we reiterate that for us, Laver trees consist of increasing functions).

Theorem 28 (See [3]) (Pure decision property) *Let $p \in \mathbb{L}$, X a finite set and $\dot{a}$ a $\mathbb{L}$-name such that $p \Vdash$ "$\dot{a} \in X$". There is $q \leq_0 p$ and $b \in X$ such that $q \Vdash$ "$\dot{a} = b$".*

We will need the following lemma:

Lemma 29 *Let $p \in \mathbb{L}$ with $s = \mathsf{st}(p) \in \omega^n$ and $\dot{a}$ such that $p \Vdash$ "$\dot{a} \in \dot{P}_{gen}(n)$". There is $q \leq_0 p$ such that one of the following holds:*

1. *There is a function $g : \mathsf{suc}_q(s) \longrightarrow \omega$ injective such that $q_{s^\frown i} \Vdash$ "$\dot{a} = g(i)$" for every $i \in \mathsf{suc}_q(s)$.*
2. *There is a sequence $\langle h_i \mid i \in \mathsf{suc}_q(s)\rangle$ such that for every $i \in \mathsf{suc}_q(s)$, we have that $h_i : \mathsf{suc}_q(s) \longrightarrow \omega$ is injective and $q_{s^\frown i^\frown j} \Vdash$ "$\dot{a} = h_i(j)$" for every $j \in \mathsf{suc}_q(s^\frown i)$.*

Proof. Note that since $s \in \omega^n$, then p has not decided $l_{gen}(n)$ or $l_{gen}(n+1)$. Moreover, if $s^\frown i^\frown j \in p$, then $p_{s^\frown i^\frown j} \Vdash$"$\dot{P}_{gen}(n) = [i, j)$". Now, for every $t \in p$ with $|t| = n+2$, we can apply the pure decision property to p_t and $\dot{a}$. In this way, we find $r \leq_0 p$ such that for every $t \in r$ with $|t| = n+2$, there is a_t such that $r_t \Vdash$"$\dot{a} = a_t$". We can now find $\overline{r} \leq_0 r$ such that for every $i \in \mathsf{suc}_{\overline{r}}(s)$, one of the following holds:

1. $a_t = a_z$ for every $t, z \in \overline{r}$ such that $|t| = |z| = n+2$ and $t(n) = z(n) = i$.
2. $a_t \neq a_z$ for every $t, z \in \overline{r}$ distinct such that $|t| = |z| = n+2$ and $t(n) = z(n) = i$.

If there are infinitely many $i \in \mathsf{suc}_{\overline{r}}(s)$ for which condition 2 holds, we can easily find and extension of $\overline{r}$ satisfying item 2 of the lemma. Assume that condition 1 holds for almost all $i \in \mathsf{suc}_{\overline{r}}(s)$. By pruning $\overline{r}$, we can assume that in fact this holds for all. For every $i \in \mathsf{suc}_{\overline{r}}(s)$, let a_i be the common value obtained in condition 1 above. We claim that for every $k \in \omega$, there are only finitely many $i \in \mathsf{suc}_{\overline{r}}(s)$ such that $a_i = k$. This is because if $i > k$, then $p_{s^\frown i} \Vdash$"$k < i = l_{gen}(n) \leq \dot{a}$". We can now easily obtain an extension of $\overline{r}$ for which item 1 in the lemma holds. ■

With the lemma at our disposal, we now prove the following:

Proposition 30 *Let $\mathcal{I}$ be a tall P-ideal, $p \in \mathbb{L}$ and $\dot{X}$ be a $\mathbb{L}$-name for a selector of P_{gen}. There is $q \leq_0 p$ and $B \in \mathcal{I}$ such that $q \Vdash$ "$\dot{X} \subseteq B$".*

Proof. By applying the previous lemma infinitely many times, without lost of generality, we may assume that for every $t \in p$ below the stem with $|t| = n$, one of the following holds:

1. There is a function $g^t : \mathsf{suc}_p(t) \longrightarrow \omega$ injective such that $p_{t^\frown i} \Vdash$"$\dot{a} = g^t(i)$" for every $i \in \mathsf{suc}_p(t)$.

2. There is a sequence $\langle h_i^t \mid i \in \mathsf{suc}_p(t) \rangle$ such that for every $i \in \mathsf{suc}_p(t)$, we have that $h_i^t : \mathsf{suc}_p(t^\frown i) \longrightarrow \omega$ is injective and $p_{t^\frown i^\frown j} \Vdash$"$\dot{a} = h_i^t(j)$" for every $j \in \mathsf{suc}_p(t^\frown i)$.

We will say $t \in p$ (below the stem) is of *Type 1* if condition 1 above holds and is of *Type 2* if condition 2 above is the one that is true. Let $t \in p$ below the stem. If t is of Type 1, denote $W(t) = \{im(g^t)\}$ and if it is of Type 2, denote $W(t) = \{im(h_i^t) \mid i \in \mathsf{suc}_p(t)\}$. Since $\mathcal{I}$ is a tall P-ideal, we can find a single $B \in \mathcal{I}$ such that B has infinite intersection with every element of each $W(t)$ for all $t \in p$ below the stem. Recursively, we can now find $q \leq_0 p$ such that for every $t \in q$ below the stem, we have the following:

1. If t is of Type 1, then the image of $g^t \restriction \mathsf{suc}_q(t)$ is contained in B.

2. If t is of Type 2, then for every $i \in \mathsf{suc}_q(t)$, we have that the image of $h_i^t \restriction \mathsf{suc}_q(t^\frown i)$ is contained in B.

In case the stem of p is empty, we get that $q \Vdash$"$\dot{X} \subseteq B$" and we are done. If the stem of p has size $m+1$, we get that $q \Vdash$"$\dot{X} \setminus \dot{l}_{gen}(m) \subseteq B$", so $q \Vdash$"$\dot{X} \subseteq B \cup \dot{l}_{gen}(m)$" and we are done. ■

To handle the iteration, we need a slightly stronger result. The proof of the following proposition is essentially the same as that of the previous one, with only a slight increase in notational complexity.

Proposition 31 *Let $\mathcal{I}$ be a tall P-ideal, $p \in \mathbb{L}$ and $\dot{H}$ be a $\mathbb{L}$-name for a function such that $p \Vdash$"$\forall n \in \omega(\dot{H}(n) \in [P_{gen}(n)]^{\leq n+1})$". There is $q \leq_0 p$ and $B \in \mathcal{I}$ such that $q \Vdash$"$\bigcup\limits_{n \in \omega} \dot{H}(n) \subseteq B$".*

We now recall the following notion:

Definition 32 *Let $\mathbb{P}$ be a partial order. We say $\mathbb{P}$ has the* Laver property *if for every $p \in \mathbb{P}$ and $\dot{f}$ a $\mathbb{P}$-name for a bounded function, there are $q \leq p$ and $S : \omega \longrightarrow [\omega]^{<\omega}$ such that or every $n \in \omega$ we have that $|S(n)| \leq n+1$ and $q \Vdash$"$\dot{f}(n) \in S(n)$".*

It is well-known that Laver forcing, as well as its iterations, have the Laver property (see [3] or [57]). With Proposition 31, we conclude the following:

Proposition 33 *Let $\mathcal{I}$ be a tall* P*-ideal and* $\dot{\mathbb{P}}$ *be an* $\mathbb{L}$*-name for a forcing notion with the Laver property.* $\mathbb{L} * \dot{\mathbb{P}}$ *forces that every selector of* P_{gen} *is in* $\mathcal{I}$*.*

By the *Laver model,* we mean a model obtained by iterating Laver forcing with countable support ω_2 many times over a model of CH. We can now prove the following:

Theorem 34 *In the Laver model, every tall* P*-ideal is Katětov above* $\mathcal{ED}_{\mathsf{fin}}$*. Hence, the Category Dichotomy for tall* P*-ideals holds.*

Proof. Follows by Proposition 33 and the fact that for every tall P-ideal in the final extension, we can find an intermediate extension in which it is also a tall P-ideal. ■

We do not know if we can extend Theorem 34 for ω-hitting ideals. While Proposition 30 is also true for them, it is not clear if Proposition 31 holds as well. We ask the following:

Problem 35

1. *Is it true that in the Laver model, every* ω*-hitting ideal is Katětov above* $\mathcal{ED}_{\mathsf{fin}}$*?*
2. *Is it consistent that the Category Dichotomy holds for the* ω*-hitting ideals?*

8 Every non meager ideal may be above $\mathcal{ED}_{\mathsf{fin}}$

In this short section, we will prove that it is consistent that every non meager ideal is Katětov-Blass above $\mathcal{ED}_{\mathsf{fin}}$ (in particular, it is not Q^+). Note that by Proposition 15, we have the following:

Corollary 36 *The following are equivalent:*

1. *The Category Dichotomy for non-meager ideals.*
2. *Every non-meager ideal has a restriction which is Katětov above* $\mathcal{ED}$*.*

The dual of a Ramsey ultrafilter is a counterexample for the Category Dichotomy for non-meager ideals. Interestingly, the dichotomy for this class is consistent. We need to recall the following important principle:

Filter Dichotomy

Let $\mathcal{I}$ be an ideal on ω. Either $\mathcal{I}$ is meager of there is an ultrafilter $\mathcal{U}$ such that $\mathcal{U}^* \leq_{\mathsf{RB}} \mathcal{I}$.

In [5] Blass and Laflamme proved that the Filter Dichotomy holds in the Miller and Blass-Shelah models. We need also to mention that Laflamme Zhou showed in [47] that the Filter Dichotomy implies that there are no Q-points. The main result of this section easily follows:

Theorem 37 *The Filter Dichotomy implies that every non meager ideal is Katětov-Blass above $\mathcal{ED}_{\mathsf{fin}}$. In particular, it implies the Category Dichotomy for non-meager ideals.*

Proof. Let $\mathcal{I}$ be a non meager ideal. By the Filter Dichotomy, there is $\mathcal{U}$ an ultrafilter such that $\mathcal{U}^* \leq_{\mathsf{RB}} \mathcal{I}$. By the theorem of Laflamme and Zhou mentioned above, we know that $\mathcal{U}$ is not a Q-point. In this way, $\mathcal{ED}_{\mathsf{fin}} \leq_{\mathsf{KB}} \mathcal{U}^* \leq_{\mathsf{RB}} \mathcal{I}$, hence $\mathcal{ED}_{\mathsf{fin}} \leq_{\mathsf{KB}} \mathcal{I}$. ■

The previous result provides an enlightening observation: If we seek to find, within ZFC, an ideal that serves as a counterexample to the Category Dichotomy, such an ideal cannot be non-meager and cannot be analytic. In this sense, it cannot be too large (non-meager) nor too small (analytic).

9 The Category Dichotomy for nowhere dense ideals

Let (X, τ) be a topological space. By $\mathsf{nwd}(X, \tau)$ (or simply $\mathsf{nwd}(X)$ if the topology is clear from context) we denote the ideal of nowhere dense subsets of (X, τ). As expected, our main interest is where the space is countable. In [45], [46] and [53] the authors study the nowhere dense ideals for different topologies with a number theoretic flavor. We will explore properties of a topological space that will ensure its ideal of nowhere dense sets fails Category Dichotomy. For the convenience of the reader, we now review the main topological notions that will be needed in this section.

Definition 38 *Let (X, τ) be a topological space, $b \in X$ and $\mathcal{B}$ a family of non-empty open sets of X.*

1. *(X, τ) is* zero dimensional *if it has a base of clopen sets.*
2. *$\mathsf{Clop}(X)$ denotes the family of clopen subsets of X.*
3. *Let $A \subseteq X$ be countable. A* converges to *b (denoted by $A \longrightarrow b$) if every open subset of b almost contains A.*
4. *(X, τ) is* Fréchet *if for every $a \in X$ and $Y \subseteq X$ such that $a \in \overline{Y}$, there is $A \in [Y]^{\omega}$ that converges to a.*
5. *$\mathcal{B}$ is a* π-base *if every non-empty subset of X contains an element of $\mathcal{B}$.*
6. *The* π-weight *of X is the smallest size of a π-base of X.*

7. *$\mathcal{B}$ is a* local π-base at b *if every neighborhood of* b *contains an element of* $\mathcal{B}$.

8. *The* π-character of b *is the smallest size of a* local π-base at b.

9. X has uncountable π-character everywhere *if every point of* X *has uncountable* π*-character.*

Recall that the closure of a nowhere dense set is also nowhere dense. In this way, $\mathsf{nwd}(X)$ is an ideal generated by closed sets.

Proposition 39 *Let* X *be a countable, Fréchet space with no isolated points. The ideal* $\mathsf{nwd}(X)$ *is meager.*

Proof. We may assume that $X = \omega$. Since our space is Fréchet and has no isolated points, for every $n \in \omega$, we can find $A_n \in [\omega]^{\omega}$ that converges to n. Now, find an interval partition $P = \{P_n \mid n \in \omega\}$ such that $P_n \cap A_i \neq \emptyset$ for every $n \in \omega$ and $i \leq n$. It follows that every $Y \subseteq \omega$ that contains infinitely many intervals of P, is dense. In this way, P is a Talagrand partition of $\mathsf{nwd}(X)$, and by Theorem 14, we conclude that $\mathsf{nwd}(X)$ is meager. ■

By the last remark in the previous section, it is natural to seek a counterexample to the Category Dichotomy within the class of nowhere dense ideals for countable topologies.

Proposition 40 (Topological Disjoint Refinement Lemma)
Let X *be a zero dimensional space with uncountable* π*-character everywhere. For every collection* $\{U_n \mid n \in \omega\}$ *of non-empty open sets, there is a pairwise disjoint family of non-empty clopen sets* $\{C_n \mid n \in \omega\}$ *such that* $C_n \subseteq U_n$ *for every* $n \in \omega$.

Proof. Since X is zero dimensional, we may actually assume that each U_n is clopen. Pick any $a_0 \in U_0$. Since $\{U_n \mid n \in \omega\}$ is not a local π-base at a_0, we can find $V_0 \in \mathsf{Clop}(X)$ with $a_0 \in V_0$ such that V_0 does not contain any of the U_n. Define $C_0 = U_0 \cap V_0$. Denote $U_n^0 = U_n \setminus C_0$. Pick any $a_1 \in U_1^0$. Since $\left\{U_n^0 \mid n \in \omega\right\}$ is not a local π-base at a_1, we can find $V_1 \in \mathsf{Clop}(X)$ with $a_1 \in V_1$ such that V_1 does not contain any of the U_n^0. Define $C_1 = U_1^0 \cap V_1$. We continue in this way and find each C_n. ■

With this result, we can prove:

Proposition 41 *Let* X *be a zero dimensional space that has uncountable* π*-character everywhere. The ideal* $\mathsf{nwd}(X)$ *is tight. In particular,* $\mathsf{nwd}(X) \nleq_{\mathsf{K}} \mathsf{nwd}$.

Proof. Let $\{Y_n \mid n \in \omega\} \subseteq \mathsf{nwd}(X)^+$, we need to find a nowhere dense set that intersects each Y_n. Since each Y_n is not nowhere dense, we may find a non-empty open set U_n such that Y_n is dense in U_n. We now apply the Topological Disjoint Refinement Lemma and find a pairwise disjoint family $\{C_n \mid n \in \omega\} \subseteq \mathsf{Clop}(X)$ such that $\emptyset \neq C_n \subseteq U_n$ for every $n \in \omega$. Since Y_n is dense in U_n, we may choose $y_n \in Y_n \cap C_n$. Let $D = \{y_n \mid n \in \omega\}$ and note that $D \cap C_n = \{y_n\}$ for each $n \in \omega$. In this way, D is a discrete set (hence nowhere dense) and it obviously has non-empty intersection with each Y_n. ■

We now want to find condition that guarantees that $\mathsf{nwd}(X)$ is not Katětov above $\mathcal{ED}$.

Lemma 42 *Let X be a Fréchet space with no isolated points, $\{N_n \mid n \in \omega\}$ is a family of nowhere dense sets, $a \in X$ and $D \subseteq X$ a dense set. There is $S \in [D]^\omega$ such that:*

1. *S converges to a.*
2. *$|S \cap N_n| \leq 1$ for every $n \in \omega$.*

Proof. Recall that the closure of a nowhere dense set is again nowhere dense, so we may assume each N_n is closed. Denote $M_n = \bigcup\limits_{i \leq n} N_i \in \mathsf{nwd}(X)$ and $M_n^{\mathsf{c}} = X \setminus M_n$, which is an open dense set. X is Fréchet, a is not an isolated point and D is dense, so we can find $B = \{b_n \mid n \in \omega\} \in [D]^\omega$ that converges to a. We may assume $b_n \neq a$ for every $n \in \omega$. Now, since $D \cap M_n^{\mathsf{c}}$ is dense (for each $n \in \omega$), we may find a countable $C_n \subseteq D \cap M_n^{\mathsf{c}}$ converging to b_n. Define $C = \bigcup\limits_{n \in \omega} C_n$. It is clear that $a \in \overline{C}$. Once again since X is Fréchet, we can find $E \in [C]^\omega$ such that $E \longrightarrow a$.

Claim 43 *$E \cap N_n$ is finite for every $n \in \omega$.*

Since C_n converges to b_n and $b_n \neq a$, it follows that $E \cap C_n$ is finite. Since $E \cap N_n \subseteq \bigcup\limits_{i<n} E \cap C_i$, the claim follows.

We can now easily find $S \in [E]^\omega$ that intersects each N_n in at most one point. ■

We now prove the following:

Proposition 44 *Let X be a countable Fréchet space with no isolated points. The ideal $\mathsf{nwd}(X)$ is weakly selective (it has no restrictions Katětov above $\mathcal{ED}$).*

Proof. Let $Y \in \mathsf{nwd}(X)^+$ and $\{N_n \mid n \in \omega\}$ be a partition of Y in nowhere dense sets. Since Y is not nowhere dense, we can find a non-empty open set

$U \subseteq \overline{Y}$. Note that U is on its own a Fréchet space with no isolated points and each N_n is nowhere dense in U.

Take an enumeration $U = \{a_n \mid n \in \omega\}$. We now apply Lemma 42 infinitely many times and for each $n \in \omega$, we find $A_n \subseteq Y$ with the following properties:

1. A_n converges to a_n.
2. $|A_n \cap N_m| \leq 1$ for every $m \in \omega$.

We can now extract $B \in [Y]^\omega$ such that for every $n \in \omega$, we have the following:

1. $B \cap A_n$ is infinite.
2. $B \cap N_n$ has at most one point.

Finally, note that $U \subseteq \overline{B}$, so B is not nowhere dense. ■

By combining Proposition 41 and Proposition 44, we conclude the following:

Theorem 45 *If X is a topological space with the following properties:*

1. *X is countable.*
2. *X is zero dimensional.*
3. *X is Fréchet.*
4. *X has uncountable π-character everywhere.*

Then, $\mathsf{nwd}(X) \nleq_K \mathsf{nwd}$ and $\mathcal{ED} \nleq_K \mathsf{nwd}(X) \restriction Y$ for every $Y \in \mathsf{nwd}(X)^+$. In other words, $\mathsf{nwd}(X)$ does not satisfy the Category Dichotomy.

Our quest for finding a counterexample for the Category Dichotomy will be complete if we can find a space with the aforementioned properties. Fortunately, a space with these qualities has already appeared in the literature. In [17], the first author proved the following theorem:

Theorem 46 (D.) *There is a topological space $\mathbb{D} = (\omega^{<\omega}, \tau_{\mathbb{D}})$ that is Fréchet, zero dimensional and with π-weight at least $\mathfrak{b}$.*

Although it is not explicitly stated in [17] that $\mathbb{D}$ has uncountable π-character everywhere, the proof that its π-weight is at least $\mathfrak{b}$ applies at any point. We briefly highlight the relevance and context of Theorem 46. In [36], the fourth author and Ramos García resolved an old problem of Malykhin by constructing a model in which every countable Fréchet topological group is second countable. Juhász subsequently questioned the role of algebra in this theorem. Since, in the context of topological groups, second countability and having countable π-weight are equivalent properties, Juhász raised the question of whether there exists (in ZFC) a countable Fréchet space with uncountable π-weight. This problem remained unsolved until the first author proved Theorem 46.

Corollary 47 *The Category Dichotomy fails for* $\mathsf{nwd}(\mathbb{D})$.

10 The Category Dichotomy for nowhere dense ideals induced by independent families

Although we have already seen an example of an ideal that does not satisfy the Category Dichotomy, it is worthwhile to explore which classes of ideals do satisfy it. Inspired by the results from the previous section, we aim to gain a deeper understanding of when the ideal of nowhere dense sets of a countable space satisfies the dichotomy. In this section, we will examine the ideals of spaces generated by independent families. We begin by recalling the relevant concepts.

In this section, for a given $A \subseteq \omega$, denote $A^0 = A$ and $A^1 = \omega \setminus A$. For a set $\mathcal{B}$, define $\mathbb{C}_{\mathcal{B}}$ as the set of all functions whose domain is a finite set of $\mathcal{B}$ and its range is contained in $\{0,1\}$ (in other words, $\mathbb{C}_{\mathcal{B}}$ is the standard forcing notion for adding a Cohen subset of $\mathcal{B}$ with finite conditions).

Definition 48 *Let* $\mathcal{B} \subseteq [\omega]^{\omega}$ *infinite. We say that* $\mathcal{B}$ *is an* independent family *if for every* $p \in \mathbb{C}_{\mathcal{B}}$*, we have that* $\bigcap\limits_{A \in dom(p)} A^{p(A)}$ *is infinite. The set* $\bigcap\limits_{A \in dom(p)} A^{p(A)}$ *will be denoted by* B^p.

A classical theorem of Fichtenholz and Kantorovich states that there exists a perfect independent family (see [37]). In particular, for any infinite cardinality up to $\mathfrak{c}$, there exists an independent family of that size. In recent years, there has been significant research on independent families. Readers interested in learning more may refer to the papers [56], [7], [13], [25], [16], [20], [22] or [21] for further details. The thesis [54] is a very good introduction to independent families and related topics.

Definition 49 *Let* $\mathcal{B}$ *be an independent family.*

1. *We say that* $\mathcal{B}$ separates points *if for every distinct* $m, n \in \omega$*, there is* $B \in \mathcal{B}$ *such that* $m \in B$ *and* $n \notin B$.
2. *The* envelope *of* $\mathcal{B}$ *is defined as* $\mathsf{Env}(\mathcal{B}) = \{B^p \mid p \in \mathbb{C}_{\mathcal{B}}\}$.

Letting $\mathcal{B}$ be an independent family that separates points, we can use $\mathsf{Env}(\mathcal{B})$ to generated a topology. The space $X(\mathcal{B}) = (\omega, \tau_{\mathcal{B}})$ is the topological space that has $\mathsf{Env}(\mathcal{B})$ as a base. The following lemma is well-known, the reader may consult [54] for a proof.

Lemma 50 *Let $\mathcal{B}$ be an independent family that separates points.*

1. *$X(\mathcal{B})$ is a Hausdorff, countable, zero dimensional space with no isolated points.*
2. *The following are equivalent:*
   (a) *$\mathcal{B}$ is countable.*
   (b) *$X(\mathcal{B})$ is second countable.*
   (c) *$X(\mathcal{B})$ is homeomorphic to the rational numbers.*
3. *Let $A \subseteq \omega$ such that $A \notin \mathcal{B}$. The following are equivalent:*
   (a) *$\mathcal{B} \cup \{A\}$ is independent.*
   (b) *Both A and $\omega \setminus A$ are dense in $X(\mathcal{B})$.*

In case $\mathcal{B}$ is a maximal independent family, the space $X(\mathcal{B})$ does not have disjoint dense sets. Such spaces are called *irresolvable.* The reader may consult [12] to learn more about irresolvable spaces and their cardinal invariants. To avoid constant repetition, **from now on we assume all our independent families separates points.**

Lemma 51 *Let $\mathcal{B}$ be an uncountable independent family. $X(\mathcal{B})$ has uncountable π-character everywhere. In fact, the π-character of every point is $|\mathcal{B}|$.*

Proof. Let $\kappa = |\mathcal{B}|$. Since $\mathsf{Env}(\mathcal{B})$ has the same size as $\mathcal{B}$, it follows that the π-character of any point is at least κ. We now proceed by contradiction, assume there is $a \in \omega$ and $\mathcal{U}$ a local π-base of a with $\mu = |\mathcal{U}| < \kappa$. Note that we may assume that $\mathcal{U} \subseteq \mathsf{Env}(\mathcal{B})$, so there is $P \subseteq \mathbb{C}_{\mathcal{B}}$ of size μ such that $\mathcal{U} = \{B^p \mid p \in P\}$. Since $\mu < \kappa$, there is $A \in \mathcal{B}$ such that $A \notin \mathsf{dom}(p)$ for every $p \in P$. Find $i < 2$ such that $a \in A^i$. It follows that A^i is an open neighborhood of a, so there should be $p \in P$ such that $B^p \subseteq A^i$. However, this contradicts that $\mathcal{B}$ is an independent family. ■

With Proposition 41 we conclude the following:

Corollary 52 *If $\mathcal{B}$ is an uncountable independent family, then $\mathsf{nwd}(X(\mathcal{B}))$ is tight. In particular, it is not Katětov below nwd.*

We introduce the following notation:

Definition 53 *Let $\mathcal{P} \subseteq [\omega]^{\omega}$. Define $\mathsf{Hit}(\mathcal{P})$ as the set of all $X \subseteq \omega$ that have infinite intersection with every element of $\mathcal{P}$.*

Note that if $\mathcal{F}$ is a filter, then $\mathsf{Hit}(\mathcal{F})$ is just $\mathcal{F}^{+}$. We will recall the following result, which is Proposition 6.24 in [4].

Proposition 54 *Let $\mathcal{P} \subseteq [\omega]^{\omega}$ of size less than $\mathfrak{d}$ and $\mathcal{D} \subseteq \mathsf{Hit}(\mathcal{P})$ a countable and decreasing family. There is $A \in \mathsf{Hit}(\mathcal{P})$ that is a pseudointersection of $\mathcal{D}$.*

We have the following characterization of the dominating number:

Theorem 55 *The cardinal $\mathfrak{d}$ is the least size of an independent family $\mathcal{B}$ such that $\mathsf{nwd}(X(\mathcal{B}))$ is not P^{-}.*

Proof. Let $\mathcal{B} \subseteq [\omega]^{\omega}$ be an independent family of size less than $\mathfrak{d}$, we will prove that $\mathsf{nwd}(X(\mathcal{B}))$ is P^{-}. Let $Y \in \mathsf{nwd}(X(\mathcal{B}))^{+}$ and $\mathcal{D} \subseteq (\mathsf{nwd}(X(\mathcal{B})) \restriction Y)^{*}$ countable and decreasing. We need to find a pseudointersection in $\mathsf{nwd}(X(\mathcal{B}))^{+}$. Since Y is somewhere dense, we may find $p \in \mathbb{C}_{\mathcal{B}}$ such that Y is dense in B^{p}. Let $\mathcal{P} = \{B^{q} \cap Y \mid q \in \mathbb{C}_{\mathcal{B}} \wedge p \subseteq q\}$. It follows that $\mathcal{D} \in \mathsf{Hit}(\mathcal{P})$ and $\mathcal{P}$ has size less than $\mathfrak{d}$. We can then invoke Proposition 54 and obtain $A \in \mathcal{P}^{+}$ a pseudointersection of $\mathcal{D}$. It follows that A is dense in B^{p}, so $A \in \mathsf{nwd}(X(\mathcal{B}))^{+}$.

We now need to find an independent family of size $\mathfrak{d}$ whose nowhere dense ideal is Katětov above $\mathsf{fin}\times\mathsf{fin}$. For every $n \in \omega$, we denote the column $C_n = \{n\} \times \omega$. Fix $\mathcal{A} = \{A_{\alpha} \mid \alpha < \mathfrak{d}\}$ an independent family such that for every $n \in \omega$, there are infinitely many elements of $\mathcal{A}$ for which n belongs and $\mathcal{D} = \{f_{\alpha} \mid \alpha < \mathfrak{d}\} \subseteq \omega^{\omega}$ a $\leq$-dominating family (not only with $\leq^{*}$). We fuse $\mathcal{A}$ and $\mathcal{D}$ as follows: for any $\alpha < \mathfrak{d}$, define $B_{\alpha} = \{(n,m) \mid n \in A_{\alpha} \wedge f_{\alpha}(n) < m\}$. Let $\mathcal{B} = \{B_{\alpha} \mid \alpha < \mathfrak{d}\}$.

Claim 56 *$\mathcal{B}$ is an independent family.*

Let $F, G \subseteq \mathfrak{d}$ finite and disjoint. We need to prove that $Y = \bigcap_{\alpha \in F} B_{\alpha} \cap \bigcap_{\beta \in G} (\omega \times \omega \setminus B_{\beta})$ is infinite. Since $\mathcal{A}$ is independent, we can find $k \in \bigcap_{\alpha \in F} A_{\alpha} \cap \bigcap_{\beta \in G} (\omega \setminus A_{\beta})$. It follows that the column C_k is almost contained in Y. This finishes the claim.

We will prove that each column C_n is nowhere dense. Let $p \in \mathbb{C}_{\mathcal{B}}$ and we look at the open set B^{p}. Pick any $\alpha < \mathfrak{d}$ such that $B_{\alpha} \notin \mathsf{dom}(p)$ and $n \in B_{\alpha}$. Define $q = p \cup \{(\alpha, 0)\}$. It follows that $B^{q} \subseteq B^{p}$ and $B^{q} \cap C_n$ is finite, so we are done.

Let $\alpha < \mathfrak{d}$, we will now prove that $G = \{(n,m) \mid m \leq f_{\alpha}(n)\}$ is nowhere dense. Let $p \in \mathbb{C}_{\mathcal{B}}$ and we look at the open set B^{p}. Pick any $\beta < \mathfrak{d}$ such that

$B_\beta \notin \mathsf{dom}(p)$ and $f_\alpha < f_\beta$ for each $\alpha \in \mathsf{dom}(p)$ (this is possible since $\mathcal{D}$ is a dominating family). Define $q = p \cup \{(\beta, 0)\}$. It follows that $B^q \subseteq B^p$ and $B^q \cap G = \emptyset$, so we are done.

Finally, by the previous two remarks and the fact that $\mathcal{D}$ is a dominating family, we conclude that $\mathsf{fin} \times \mathsf{fin} \subseteq \mathsf{nwd}(X(B))$. We rejected to verify that $\mathcal{B}$ separates points. However, if this was not the case, we can perform finite changes to countably many elements of $\mathcal{B}$ and make it separate points. The arguments described above still work after this finite modifications. ■

On the other hand, we have the following:

Proposition 57 *Let $\mathcal{B}$ be an independent family. If $|\mathcal{B}| <$ non*$(\mathcal{ED}_{\mathsf{fin}})$, then nwd$(\mathcal{B})$ is Q^+.*

Proof. Let $\mathcal{B} \subseteq [\omega]^\omega$ be an independent family of size $\kappa <$ non*$(\mathcal{ED}_{\mathsf{fin}})$, $Y \in \mathsf{nwd}(X(\mathcal{B}))^+$ and $\mathcal{P} = \{P_n \mid n \in \omega\}$ an interval partition of Y. We need to find a positive partial selector. We may assume the intervals of $\mathcal{P}$ are increasing in size. Since Y is not nowhere dense, we may find $p \in \mathbb{C}_{\mathcal{B}}$ such that Y is dense in B^p. Let $\mathcal{U} = \{B^q \cap Y \mid q \in \mathbb{C}_{\mathcal{B}} \wedge p \subseteq q\}$. For every $U \in \mathcal{U}$, we find f_U a partial infinite function such that for every $n \in \mathsf{dom}(f_U)$, we have that $f_U \in P_n \cap U$. Since $|\mathcal{U}| <$ non*$(\mathcal{ED}_{\mathsf{fin}})$, we can find a function $h \in \omega^\omega$ such that $h(n) \in P_n$ for every $n \in \omega$ and has infinite intersection with each f_U for $U \in \mathcal{U}$. It follows that the image of h is dense in B^p and it is a selector, so we are done. ■

We do not know the following:

Problem 58 *Is there an independent family $\mathcal{B}$ of size non*$(\mathcal{ED}_{\mathsf{fin}})$ such that nwd$(\mathcal{B})$ is not Q^+?*

We can now prove:

Theorem 59 *If $\omega_1 <$ cov$(\mathcal{M})$, then there is $\mathcal{B}$ an independent family such that nwd$(X(\mathcal{B}))$ does not satisfy the Category Dichotomy.*

Proof. Let $\mathcal{B}$ be an independent family of size ω_1. Since it is uncountable, by Corollary 52, we know that $\mathsf{nwd}(X(\mathcal{B})) \not\leq_{\mathsf{K}} \mathsf{nwd}$. Finally, the other possibility of the dichotomy is impossible since $\mathsf{cov}(\mathcal{M}) = \mathsf{min}(\mathfrak{d},$non*$(\mathcal{ED}_{\mathsf{fin}}))$ and by Proposition 57 and Proposition 55. ■

The following problem remains open:

Problem 60 *Is it consistent that the Category Dichotomy holds for the nowhere dense ideals of topologies generated by independent families?*

Building on the results of this section and the previous one, it is natural to ask the following questions:

Problem 61 *Let $\mathcal{B}$ be an uncountable independent family. When is $X(\mathcal{B})$ Fréchet? Is there such a family in* ZFC*?*

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