On ℝ-embeddability of almost disjoint families and Akemann–Doner C*-algebras

by

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Abstract. An almost disjoint family $\mathcal{A}$ of subsets of $\mathbb{N}$ is said to be ℝ-embeddable if there is a function $f : \mathbb{N} \to \mathbb{R}$ such that the sets $f[A]$ are ranges of real sequences converging to distinct reals for distinct $A \in \mathcal{A}$. It is well known that almost disjoint families which have few separations, such as Luzin families, are not ℝ-embeddable. We study extraction principles related to ℝ-embeddability and separation properties of almost disjoint families of $\mathbb{N}$ as well as their limitations. An extraction principle whose consistency is our main result is:

• every almost disjoint family of size $\mathfrak{c}$ contains an ℝ-embeddable subfamily of size $\mathfrak{c}$.

It is true in the Sacks model. The Cohen model serves to show that the above principle does not follow from the fact that every almost disjoint family of size continuum has two separated subfamilies of size continuum. We also construct in ZFC an almost disjoint family where no two uncountable subfamilies can be separated but every countable subfamily can be separated from any disjoint subfamily.

Using a refinement of the ℝ-embeddability property called the controlled ℝ-embedding property we obtain the following results concerning Akemann–Doner C*-algebras which are induced by uncountable almost disjoint families:

• In ZFC there are Akemann–Doner C*-algebras of density $\mathfrak{c}$ with no commutative subalgebras of density $\mathfrak{c}$.
• It is independent from ZFC whether there is an Akemann–Doner algebra of density $\mathfrak{c}$ with no non-separable commutative subalgebra.

This completes an earlier result that there is in ZFC an Akemann–Doner algebra of density ω_1 with no non-separable commutative subalgebra.

1. Introduction. A family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is *almost disjoint* if any two distinct elements of $\mathcal{A}$ have finite intersection. The concept of almost disjointness was formally introduced by Sierpiński [33] and Tarski [35]

in 1928. However, already two years earlier Alexandroff and Urysohn [2] implicitly considered almost disjoint families defined as the ranges of sequences of rationals converging to distinct reals.

Hence, we say that an almost disjoint family $\mathcal{A}$ is $\mathbb{R}$-*embeddable* if there is a function (called an *embedding*) $f : \mathbb{N} \to \mathbb{R}$ such that the sets $f[A]$ for $A \in \mathcal{A}$ are the ranges of sequences converging to distinct reals (see e.g. [16, 13]). Two families $\mathcal{B}, \mathcal{C}$ of subsets of $\mathbb{N}$ are *separated* if there is $X \subseteq \mathbb{N}$ such that:

(1) If $B \in \mathcal{B}$ then $B \setminus X$ is finite.
(2) If $C \in \mathcal{C}$ then $C \cap X$ is finite.

Considering disjoint neighborhoods of two condensation points of the set consisting of the limits of converging sequences we see that uncountable $\mathbb{R}$-embeddable almost disjoint families contain many pairs of uncountable subfamilies which are separated. On the other hand, it is an old and beautiful result of Luzin [22] that there is an almost disjoint family $\mathcal{A}$ of size ω_1 such that no two uncountable subfamilies of $\mathcal{A}$ can be separated. We will call such uncountable families *inseparable*. To highlight the relationship between inseparable and $\mathbb{R}$-embeddable families, recall a dichotomy of [16] where it is shown that assuming the proper forcing axiom (PFA) every almost disjoint family of size ω_1 either is $\mathbb{R}$-embeddable or contains an inseparable subfamily, while Dow [8] showed that under the same assumption every maximal almost disjoint family (necessarily of size $\mathfrak{c}$) contains an inseparable subfamily.

An uncountable almost disjoint family $\mathcal{A}$ is called a *Q-family* ([1]) if for every $\mathcal{B} \subseteq \mathcal{A}$ the families $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ are separated (sometimes called a *separated* family). One of the earliest applications of Martin's axiom (MA) was to prove the consistency of the existence of Q-families, which is false under the continuum hypothesis CH by a counting argument (see [27] for a historical account). All Q-families are $\mathbb{R}$-embeddable and moreover they have a stronger uniformization type property: for every $\phi : \mathcal{A} \to \mathbb{R}$ there is $f : \mathbb{N} \to \mathbb{R}$ such that $f[A]$ is the range of a sequence converging to $\phi(A)$ (in other words $\lim_{n \in A}(f(n) - \phi(A)) = 0$) for each $A \in \mathcal{A}$ [13, Propositions 2.1, 2.3].

It is natural and useful (see e.g., [3, Theorem 2.39]) to consider versions of the above notions which are more cardinal specific. Let κ be a cardinal. Then:

• An almost disjoint family $\mathcal{A}$ has the κ-*controlled* $\mathbb{R}$-*embedding property* if for every $\phi : \mathcal{A} \to \mathbb{R}$ there is $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ and $f : \mathbb{N} \to \mathbb{R}$ such that $f[B]$ is the range of a sequence converging to $\phi(B)$ for every $B \in \mathcal{B}$.

([1]) Every Q-family, as a subset of $\mathcal{P}(\omega)$ with the Cantor set topology, is a *Q-set*, i.e. every subset is a relative G_δ [31].

- An almost disjoint family $\mathcal{A}$ of size κ is κ-*inseparable* if no two subfamilies of $\mathcal{A}$ of size κ can be separated.
- An almost disjoint family $\mathcal{A}$ is κ-*anti Lusin* if it has cardinality κ and for every subfamily $\mathcal{B} \subseteq A$ of cardinality κ there are two subfamilies $\mathcal{B}_0, \mathcal{B}_1 \subseteq \mathcal{B}$ of cardinality κ which can be separated [30].

This paper is a contribution to the study of extraction principles for almost disjoint families in the context of the above properties. Our main positive results concern the cardinality of the continuum $\mathfrak{c}$:

- It is consistent that every almost disjoint family of size $\mathfrak{c}$ contains an ℝ-embeddable subfamily of size $\mathfrak{c}$ (Theorem 31).
- It is consistent that every almost disjoint family of size $\mathfrak{c}$ has the ω_1-controlled ℝ-embedding property (Theorem 42).
- The above extraction principles are not consequences of every almost disjoint family of size $\mathfrak{c}$ containing a $\mathfrak{c}$-anti-Luzin subfamily (Theorems 14 and 17).

The first two extraction principles above are obtained in the iterated Sacks model. As a side product we also prove that in that model every partial function $f : X \to 2^{\mathbb{N}}$ for $X \subseteq 2^{\mathbb{N}}$ of cardinality $\mathfrak{c}$ is uniformly continuous on an uncountable $Y \subseteq X$ (Theorem 40). We do not know if the consistency of this property of functions can be deduced from known results as in [32] or [7] or from the fact that under PFA every function is monotone on an uncountable set (see [4]).

The third result above is obtained in the Cohen model from a result of Dow and Hart (Theorem 14) stating that in that model every almost disjoint family is $\mathfrak{c}$-anti-Luzin [9, Proposition 2.6] using Steprāns's characterization of $\mathcal{P}(\mathbb{N})/\text{Fin}$ in that model [34] and from the first of our negative results below:

- In the Cohen model there is an almost disjoint family of cardinality $\mathfrak{c}$ with no uncountable ℝ-embeddable subfamily (Theorem 17).
- In the Cohen model no uncountable almost disjoint family has the ω_1-controlled ℝ-embedding property (Theorem 18).

We should recall here that by a result of Avilés, Cabello Sánchez, Castillo, González and Moreno [3, Lemma 2.36] it is consistent (follows from MA) that $\mathfrak{c}$-inseparable families exist ($\mathfrak{c}$-inseparable families are called $\mathfrak{c}$-Luzin families in [9, 3]).

On the other hand, we also discover some ZFC limitations to other extraction principles:

- No almost disjoint family of size $\mathfrak{c}$ has the $\mathfrak{c}$-controlled ℝ-embedding property (Theorem 6).

- There is in ZF an inseparable family of cardinality ω_1 which has all possible separations (i.e., separating its countable parts from the rest of the family) (Corollary 11).

The second result is not only natural in the above context by showing that one cannot even consistently hope for extracting from every inseparable family an uncountable subfamily with even fewer separations (for example like Mrówka's family where one can only separate finite subfamilies from the rest of the family). It has also found a natural application in a construction of a thin-tall scattered operator algebra in [11]. Note that under the assumption of $\mathfrak{b} > \omega_1$ all inseparable families have the properties of our family from Corollary 11 (see [38, Theorem 3.3]).

Some of the above results concerning the $\mathbb{R}$-embeddability of almost disjoint families find immediate applications in the theory of C*-algebras. It was in the paper [1] of Akemann and Doner that certain C*-algebras were associated to an almost disjoint family $\mathcal{A}$ and a function $\phi : \mathcal{A} \to [0, 2\pi)$. We call these algebras Akemann–Doner algebras and denote them by $AD(\mathcal{A}, \phi)$. For the construction see Section 6 or [1, 6]. These algebras, initially for $\mathcal{A}$ and ϕ constructed only under CH in [1], were the first examples providing a negative answer to a question of Dixmier whether every non-separable C*-algebra must contain a non-separable commutative C*-subalgebra. Later Popa [29] found a different (and ZF) example, the reduced group C*-algebra of an uncountable free group. However, the latter C*-algebra is very complicated (e.g. it has no non-trivial idempotents [28] etc.) while Akemann–Doner algebras are approximately finite dimensional in the sense of [10], that is, there is a directed family of finite-dimensional C*-subalgebras whose union is dense in the entire C*-algebra. In [6] it was noted that employing an inseparable family $\mathcal{A}$ one can obtain in ZF a non-separable Akemann–Doner algebra with no non-separable commutative subalgebra. Such ZF examples must be obtained from almost disjoint families $\mathcal{A}$ of cardinality ω_1. This is because we have, for example, the above mentioned result of Dow and Hart that it is consistent that every almost disjoint family of cardinality $\mathfrak{c}$ is $\mathfrak{c}$-anti-Luzin. The cardinality of the almost disjoint family $\mathcal{A}$ is the density of the C*-algebra $AD(\mathcal{A}, \phi)$, that is, the minimal cardinality of a norm-dense set. Some natural questions arose, for example, whether one can have in ZF an Akemann–Doner algebra of density $\mathfrak{c}$ with no non-separable commutative subalgebra or whether it is consistent that every Akemann–Doner algebra of density $\mathfrak{c}$ has a commutative C*-subalgebra of density $\mathfrak{c}$. Here we answer these questions:

- In ZF there are Akemann–Doner C*-algebras of density $\mathfrak{c}$ with no commutative subalgebras of density $\mathfrak{c}$ (Theorem 45).

• It is independent from ZFC whether there is an Akemann–Doner algebra of density $\mathfrak{c}$ with no non-separable commutative subalgebra (Theorem 46 and the result of [1]).

In fact, we also prove in Theorems 47 and 48 that the existence of non-separable commutative C*-subalgebras in every Akemann–Doner algebra does not follow from the negation of CH.

The structure of the paper is as follows: in Section 2 we prove some preliminary ZFC results concerning ℝ-embeddability, Section 3 is devoted to the construction of an inseparable almost disjoint family where all countable parts can be separated from the remaining part of the family, Section 4 is devoted to the results mentioned above that hold in the Cohen model and Section 5 to the results that hold in the Sacks model. Finally, Section 6 concerns the consequences of the previous results for Akemann–Doner C*-algebras.

The set-theoretic terminology and notation are standard and can be found in [17]. The knowledge of C*-algebras required to follow Section 6 does not exceed a linear algebra course concerning 2×2 matrices. Any additional background can be found in [25].

All almost disjoint families are assumed to be infinite and consist of infinite sets. $A \subseteq^* B$ means that $B \setminus A$ is finite. We use ℕ, ℝ, ℚ for the non-negative integers, reals and rationals respectively. When we view elements of ℕ as von Neumann ordinals, i.e. subsets and/or elements of each other, then we use ω for ℕ. The cardinality of ℝ is denoted by $\mathfrak{c}$. If κ is a cardinal and X is a set, then $[X]^\kappa$ denotes the family of all subsets of X of cardinality κ. In particular $[A]^2$ is the set of all pairs $\{a, b\}$ of elements of A. Elements of A^n for $n \in \omega$ are n-tuples in A, i.e., $t = (t(0), t(2), \ldots, t(n-1))$. We consider $2^{<\omega} = \bigcup_{n \in \omega} 2^n$ with inclusion as a tree; we also consider its subtrees T and then $[T]$ denotes the set of all branches of T. The terminology concerning Cohen forcing ℂ and Sacks forcing 𝕊 is recalled at the beginning of Sections 4 and 5 respectively.

2. Preliminaries

2.1. ℝ-embeddability of almost disjoint families. Recall the definition of an ℝ-embeddable almost disjoint family from the introduction. A useful tool for describing properties of almost disjoint families are the associated Ψ-spaces [15]. The Ψ-space corresponding to an almost disjoint family $\mathcal{A} \subseteq \wp(\mathbb{N})$, whose points are identified with $\mathbb{N} \cup \mathcal{A}$, is denoted by $\Psi(\mathcal{A})$.

LEMMA 1. *Suppose that* $\mathcal{A}$ *is an almost disjoint family. There is a* 1-1 *correspondence between continuous functions* $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ *and functions* $f : \mathbb{N} \to \mathbb{R}$ *for which* $x_A = \lim_{n \in A} f(n)$ *exists for each* $A \in \mathcal{A}$. *It is given by* $f = \phi{\upharpoonright}\mathbb{N}$. *Then* $x_A = \phi(A)$ *for each* $A \in \mathcal{A}$.

LEMMA 2. *Let $\mathcal{A} \subseteq \wp(\mathbb{N})$ be an almost disjoint family. Consider $\mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega}$ with the topology where $\mathbb{N}^{<\omega}$ is discrete and the basic neighborhoods of $x \in \mathbb{N}^{\omega}$ are of the form*

$$\{y \in \mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega} \mid y(n) = x(n) \text{ for all } n \in F\}$$

for any finite $F \subseteq \omega$. The following conditions are equivalent (to the property of being $\mathbb{R}$-embeddable):

(1) *There is a continuous $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ such that $\phi {\restriction} \mathcal{A}$ is injective.*
(2) *There is a continuous $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ such that $\phi {\restriction} \mathcal{A}$ is injective and $\phi[\mathcal{A}]$ has dense complement in $\mathbb{R}$.*
(3) *There is a continuous $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ such that $\phi {\restriction} \mathcal{A}$ is injective and $\phi[\mathcal{A}] \subseteq \mathbb{R} \setminus \mathbb{Q}$.*
(4) *There is a continuous $\phi : \Psi(\mathcal{A}) \to \mathbb{R}$ such that ϕ is injective, $\phi[\mathcal{A}] \subseteq \mathbb{R} \setminus \mathbb{Q}$ and $\phi[\mathbb{N}] \subseteq \mathbb{Q}$.*
(5) *There is a continuous $\phi : \Psi(\mathcal{A}) \to \mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega}$ such that ϕ is injective, $\phi[\mathcal{A}] \subseteq \mathbb{N}^{\omega}$ and $\phi[\mathbb{N}] \subseteq \mathbb{N}^{<\omega}$.*
(6) *There is a continuous $\phi : \Psi(\mathcal{A}) \to 2^{\omega}$ such that $\phi {\restriction} \mathcal{A}$ is injective.*

Proof. (1)$\Rightarrow$(2) We may assume that $\mathcal{A}$ is infinite. Let $U \subseteq \mathbb{R}$ be the interior of $\phi[\mathcal{A}]$. If it is empty, we are done. Otherwise let $E = \{e_n \mid n \in \mathbb{N}\} \subseteq U$ be countable and dense in U. A continuous $\phi' : \Psi(\mathcal{A}) \to \mathbb{R} \setminus E$ such that $\phi'[\Psi(\mathcal{A})] \subseteq \phi[\Psi(\mathcal{A})]$ will satisfy (2). Let $\{x_k^n \mid n, k \in \mathbb{N}\} \subseteq \phi[\mathcal{A}]$ be distinct where $x_0^n = e_n$ for each $n \in \mathbb{N}$ and such that $|x_k^n - x_{k+1}^n| < 1/(n+k)$. We may choose such x_n^ks since e_ns are in the interior of $\phi[\mathcal{A}]$. Let $A_k^n \in \mathcal{A}$ be such that $\phi(A_k^n) = x_k^n$ for each $n, k \in \mathbb{N}$. Find finite $G_k^n \subseteq A_k^n$ such that the sets $A_k^n \setminus G_k^n$ are all pairwise disjoint and $|\phi(i) - x_k^n| < 1/(n+k)$ for all $i \in A_k^n \setminus G_k^n$ with $n, k \in \mathbb{N}$.

Modify ϕ to obtain ϕ' in the following way: Put $\phi' {\restriction} (A_k^n \setminus G_k^n)$ to be constantly x_{k+1}^n for each $n, k \in \mathbb{N}$ and $\phi'(A_k^n) = x_{k+1}^n$ for each $n, k \in \mathbb{N}$, and let ϕ' be equal to ϕ at the remaining points of $\Psi(\mathcal{A})$.

Injectivity of $\phi' {\restriction} \mathcal{A}$ and the inclusion $\phi'[\Psi(\mathcal{A})] \subseteq \phi[\Psi(\mathcal{A})] \setminus E$ are clear. So we are left with the continuity. ϕ' is clearly continuous at each A_k^n for $n, k \in \mathbb{N}$. Let $A \in \mathcal{A}$ be distinct from all A_k^n. Then each intersection $A \cap A_k^n$ is finite. As $|\phi'(i) - \phi(i)| < 2/(n+k)$ for $i \in A_k^n$ with any $n, k \in \mathbb{N}$, it follows that $\lim_{i \in A} |\phi'(i) - \phi(i)| = 0$, that is,

$$\lim_{i \in A} \phi'(i) = \lim_{i \in A} \phi(i) = \phi(A) = \phi'(A).$$

(2)$\Rightarrow$(3) Choose a dense countable $E \subseteq \mathbb{R} \setminus \phi[\mathcal{A}]$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a homeomorphism such that $\eta[E] = \mathbb{Q}$ and consider $\phi' = \eta \circ \phi$.

(3)$\Rightarrow$(4) Take ϕ satisfying (3) and modify it on $\mathbb{N}$ to obtain ϕ' such that $\phi'(n)$s are distinct rationals for all $n \in \mathbb{N}$ and $|\phi(n) - \phi'(n)| < 1/n$ for all $n \in \mathbb{N}$.

(4)⇔(5) First we construct a certain bijection $\rho : \mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega} \to \mathbb{R}$ such that $\rho[\mathbb{N}^{<\omega}] = \mathbb{Q}$ and $\rho[\mathbb{N}^{\omega}] = \mathbb{R} \setminus \mathbb{Q}$. Define a family $(I_s \mid s \in \mathbb{N}^{<\omega})$ of open intervals with rational end-points with the following properties:

- $I_{\emptyset} = \mathbb{R}$.
- $\bigcup\{\overline{I_{s\frown n}} \mid n \in \mathbb{N}\} = I_s$.
- Each end-point of an interval I_s is an endpoint of another interval $I_{s'}$ with $|s| = |s'|$.
- The diameter of I_s is smaller than $1/|s|$ for $s \neq \emptyset$.
- For every $s \in \mathbb{N}^{<\omega}$ we have $I_{s\frown n} \cap I_{s\frown n'} = \emptyset$ for distinct $n, n' \in \mathbb{N}$.
- Every rational is used as an end-point of two (and necessarily only two adjacent, by the previous properties) of the intervals I_s for $s \in \mathbb{N}^{<\omega}$; and 0 is an end-point of two of I_ss for some $|s| = 1$.

First define ρ on $\mathbb{N}^{<\omega}$ by defining $\rho(s)$ by induction on $|s|$. Let $\rho(\emptyset) = 0$. If $|s| = 1$, then $\rho(s)$ is the right end-point of I_s if I_s consists of positive reals, and $\rho(s)$ is the left end-point of I_s if I_s consists of negative reals. If $|s| > 1$, then $\rho(s)$ is the left end-point of I_s. For $x \in \mathbb{N}^{\omega}$ let $\rho(x)$ be the only point of $\bigcap_{n \in \omega} I_{x \restriction n}$.

Note that ρ is continuous and that $\rho^{-1}(x_n) \to \rho^{-1}(x)$ if x_n is a sequence of rationals converging to an irrational x. This proves (4)⇔(5).

(5)⇒(6) First note that there is an $\eta : \mathbb{N}^{<\omega} \cup \mathbb{N}^{\omega} \to \mathbb{N}^{\omega}$ which is continuous and the identity on $\mathbb{N}^{\omega}$: send $s \in \mathbb{N}^{<\omega}$ to $s\frown 0^{\omega}$. Now note that there is $\zeta : \mathbb{N}^{\omega} \to 2^{\omega}$ which is continuous and injective. Use the composition of these functions to deduce (6) from (5).

(6)⇒(1) is clear. ∎

REMARK 3. Using Lemma 1 we obtain versions of the conditions from Lemma 2 for functions from $\mathbb{N}$ into $\mathbb{R}$. In particular the definition of an $\mathbb{R}$-embeddable almost disjoint family from the introduction which is a version of item (1) of Lemma 2 is equivalent to some versions in the literature, e.g. in [13] which contains versions of item (4) of Lemma 2.

The following is a simple condition that allows us to get $\mathbb{R}$-embeddability.

LEMMA 4. *Let $T \subseteq 2^{<\omega}$ be a tree, $Z \subseteq [T]$ and $\mathcal{A} = \{A_r \mid r \in Z\}$ an almost disjoint family of subsets of $\mathbb{N}$. Suppose there is a family $\{B_s \mid s \in T\} \subseteq [\mathbb{N}]^{\omega}$ with the following properties:*

(1) $B_t = \bigcup\{B_{t\frown i} \mid t\frown i \in T, \ i \in \{0, 1\}\}$ *for all $t \in T$.*
(2) $B_s \cap B_t$ *is finite whenever $s, t \in T$ are incompatible.*
(3) $A_r \subseteq \bigcap_{n \in \omega} B_{r \restriction n}$ *for every $r \in Z$.*

Then $\mathcal{A}$ is $\mathbb{R}$-embeddable.

Proof. Define $\phi : \Psi(\mathcal{A}) \to 2^{\omega}$ by putting $\phi(A_r) = r$ for all $r \in Z$ and $\phi(n) = s\frown 0^{\omega}$ if $n \in B_s$, $|s| \geq n$ and s is the first sequence in the lexicographic

order which satisfies the previous requirements. If there is no such $s \in 2^{<\omega}$, then put $\phi(n) = 0^\omega$. Clearly $\phi{\upharpoonright}\mathcal{A}$ is injective, so to check (1) of Lemma 2 we are left with the continuity. Fix $r \in Z$. We shall show that ϕ is continuous at A_r.

By (3), if $k \in A_r$, then $k \in B_{r{\upharpoonright}n}$ for every $n \in \omega$. Fix $n \in \omega$. So if we take $k \in \omega$ belonging to the set

$$(*) \qquad\qquad A_r \setminus \bigcup \{B_t \mid |t| = n,\, t \neq r{\upharpoonright}n\},$$

then the condition "$k \in B_s$ and $|s| \geq n$" implies $r{\upharpoonright}n \subseteq s$ by (1). By (2) the set in $(*)$ almost covers A_r, and so for almost all elements of $k \in A_r$ we have $r{\upharpoonright}n \subseteq \phi(k)$. As $n \in \omega$ was arbitrary, it follows that $\lim_{k \in A_r} \phi(k) = r = \phi(A_r)$, as required for the continuity. ∎

REMARK 5. Recursively one can construct a family of convergent sequences $(q_n^\alpha)_{n \in \mathbb{N}}$ of rational numbers for $\alpha < \mathfrak{c}$ in such a way that no tree $T \subseteq 2^{<\omega}$ and no collection $\{B_t \mid t \in T\}$ satisfies the hypothesis of Lemma 4 for any family of $\wp(\mathbb{N})$ obtained through a bijection between $\mathbb{N}$ and $\mathbb{Q}$ from $\{\{q_n^\alpha\}_{n \in \mathbb{N}} \mid \alpha < \mathfrak{c}\}$. It follows that the condition in Lemma 4 is not equivalent to $\mathbb{R}$-embeddability. This way one can also conclude that there are $\mathbb{R}$-embeddable almost disjoint families of subsets of $\mathbb{N}$ which are not equivalent to a family of branches of $2^{<\omega}$.

2.2. κ-controlled $\mathbb{R}$-embedding property. Recall the definition of the κ-controlled $\mathbb{R}$-embedding property from the introduction.

THEOREM 6. *No almost disjoint family $\mathcal{A}$ of size $\mathfrak{c}$ has the $\mathfrak{c}$-controlled $\mathbb{R}$-embedding property.*

Proof. Let $\mathcal{A}$ be an almost disjoint family of size $\mathfrak{c}$ consisting of infinite sets. Let $(M_\alpha)_{\alpha < \mathfrak{c}}$ be a well-ordered, continuous, increasing chain of sets satisfying:

(1) $|M_\alpha| \leq \max(|\alpha|, \omega)$ for each $\alpha < \mathfrak{c}$.
(2) $\mathbb{R}^\mathbb{N}, \wp(\mathbb{N}) \subseteq \bigcup_{\alpha < \mathfrak{c}} M_\alpha$.
(3) If $A \in M_\alpha \cap \wp(\mathbb{N})$ and $f \in M_\alpha \cap \mathbb{R}^\mathbb{N}$ and $\lim_{n \in A} f(n)$ exists, then it belongs to $M_{\alpha+1}$.

It should be clear that one can construct such a sequence $(M_\alpha)_{\alpha < \mathfrak{c}}$. Define $\phi : \mathcal{A} \to [0,1]$ so that $\phi(A) \in \mathbb{R} \setminus M_{\alpha(A)+1}$ for $A \in \mathcal{A}$, where

$$\alpha(A) = \min\{\alpha < \mathfrak{c} \mid A \in M_\alpha\}.$$

This can be arranged by (2) and by (1). Now suppose $\mathcal{A}' \subseteq \mathcal{A}$ has cardinality $\mathfrak{c}$ and let $f : \mathbb{N} \to \mathbb{R}$. By (2) there is $\alpha_0 < \mathfrak{c}$ such that $f \in M_{\alpha_0}$. Take $A \in \mathcal{A}'$ such that $\alpha(A) \geq \alpha_0$, which exists by (1) as $\mathcal{A}'$ has cardinality $\mathfrak{c}$. Then $A \in M_{\alpha(A)} \cap \wp(\mathbb{N})$ and $f \in M_{\alpha(A)} \cap [0,1]^\mathbb{N}$, so by (3), if $\lim_{n \in A} f(n)$

exists, then it belongs to $M_{\alpha(A)+1}$. But $\phi(A) \notin M_{\alpha(A)+1}$ by the definition of ϕ, so $\lim_{n \in A} f(n) \neq \phi(A)$. ∎

However, it is quite possible to have almost disjoint families of cardinality κ with the κ-controlled embedding property:

PROPOSITION 7 ([13, cf. 2.3]). *Let κ be a cardinal. Assume* MA_κ*. Then every subfamily $\mathcal{A}$ of cardinality κ of the Cantor family $\mathcal{C} = \{A_x \mid x \in 2^\omega\}$ $\subseteq \wp(2^{<\omega})$, where $A_x = \{x{\restriction}n \mid n \in \omega\}$ for $x \in 2^\omega$, has the following strong version of the κ-controlled embedding property: For every function $\phi : \mathcal{A} \to [0,1]$ there is a function $f : 2^{<\omega} \to [0,1]$ such that for all $A \in \mathcal{A}$,*

$$\lim_{s \in A} f(s) = \phi(F).$$

Proof. It is well known that under the above hypothesis all subsets of 2^ω of cardinality κ are Q-sets and that it implies that all subfamilies of the Cantor family of cardinality κ can be separated from the rest of the family, i.e. they are Q-families in our terminology from the introduction. It follows that $\Psi(\mathcal{A})$ is a normal topological space. As the non-isolated points of $\Psi(\mathcal{A})$ correspond to $\mathcal{A}$ and form a discrete closed subset of $\Psi(\mathcal{A})$, any function ϕ on them is continuous and extends by the Tietze extension theorem to a continuous $\widetilde{\phi} : \Psi(\mathcal{A}) \to [0,1]$. So put $f = \widetilde{\phi}{\restriction}2^{<\omega}$ and use Lemma 1 identifying $2^{<\omega}$ and $\mathbb{N}$. ∎

3. A Luzin family with all possible separations in ZFC. The main striking property of the family constructed by Luzin [22] is that it is inseparable. On the other hand, there is also an almost disjoint family $\mathcal{A}$ of size $\aleph_1$ such that every countable $\mathcal{B} \subseteq \mathcal{A}$ can be separated from $\mathcal{A} \setminus \mathcal{B}$ implicitly constructed in [18] (see also [21, 26]). Here we construct an almost disjoint family which satisfies both properties simultaneously. As both of these properties are hereditary with respect to uncountable subfamilies, this shows certain limitations to any further extraction principles.

To construct the almost disjoint family with the aboved-mentioned properties we need colorings of pairs of countable ordinals with properties first obtained by Todorčević [36] (cf. [37]). In fact, the concrete construction we choose, due to Velleman [39], is based on a family of finite subsets of ω_1. It was Morgan [24] who connected these two ideas. For functions $c : [\omega_1]^2 \to \mathbb{N}$ we will abuse notation and denote $c(\{\alpha, \beta\})$ by $c(\alpha, \beta)$.

THEOREM 8. *There is a sequence $(g_\alpha \mid \alpha < \omega_1) \subseteq \{0,1,2\}^\mathbb{N}$ and a coloring $c : [\omega_1]^2 \to \mathbb{N}$ satisfying the following:*

(1) *For all $\beta < \alpha < \omega_1$ for all $k > c(\beta, \alpha)$ we have $\{g_\beta(k), g_\alpha(k)\} \neq \{1,2\}$.*
(2) *For all $\beta < \alpha < \omega_1$ we have $g_\beta(c(\beta, \alpha)) = 1$ and $g_\alpha(c(\beta, \alpha))) = 2$.*
(3) *For all $\gamma < \beta < \alpha < \omega_1$ if $c(\gamma, \beta) > c(\alpha, \beta)$, then $c(\gamma, \beta) = c(\gamma, \alpha)$.*

(4) *For all $\alpha < \omega_1$ and all $m \in \mathbb{N}$ the set $\{\beta < \alpha \mid c(\beta, \alpha) < m\}$ is finite.*
(5) *For all $\alpha < \omega_1$ the sets and $g_\alpha^{-1}[\{1\}]$ and $g_\alpha^{-1}[\{2\}]$ are infinite.*

Proof. We choose the approach from [19, Section 5]. Thus our $c : [\omega_1]^2 \to \mathbb{N}$ is m of [19, Definition 5.1], i.e., $c(\alpha, \beta)$ is the minimal rank of an element $X \in \mu$ such that $\alpha, \beta \in X$ where μ is an (ω, ω_1)-cardinal, in particular μ is a well founded collection of finite subsets of ω_1 such that for every $X \in \mu$ of non-zero rank, $X = X_1 * X_2$, which means in particular that $X = X_1 \cup X_2$ and $X_1 \cap X_2 < X_1 \setminus X_2 < X_2 \setminus X_1$, where for two sets A, B of ordinals $A < B$ means that $\alpha < \beta$ for any $\alpha \in A$ and $\beta \in B$. For more details on the structure of μ see [19, Definition 1.1].

The functions g_α for $\alpha < \omega_1$ are defined as follows: for $n = 0$ we put $g_\alpha(0) = 0$ for any $\alpha < \omega_1$ and for any $n \in \mathbb{N}$ we set

$$g_\alpha(n+1) = \begin{cases} 0 & \text{if } \exists X_1 * X_2 \in \mu \ \mathrm{rank}(X_1) = \mathrm{rank}(X_2) = n, \ \alpha \in X_1 \cap X_2, \\ 1 & \text{if } \exists X_1 * X_2 \in \mu \ \mathrm{rank}(X_1) = \mathrm{rank}(X_2) = n, \ \alpha \in X_1 \setminus X_2, \\ 2 & \text{if } \exists X_1 * X_2 \in \mu \ \mathrm{rank}(X_1) = \mathrm{rank}(X_2) = n, \ \alpha \in X_2 \setminus X_1. \end{cases}$$

Here $X_1 * X_2$ is as in [19, Definition 1.1(5)]. First let us argue that the g_αs are well defined. By [19, Definition 1.1(6, 7)] each $\alpha \in \omega_1$ is in an element of rank zero of the (ω, ω_1)-cardinal μ. Now by Velleman's Density Lemma [19, 2.3] it follows that α is in an element of rank n of μ for any $n \in \mathbb{N}$. By [19, Definition 1.1(5)] each element X of μ of rank greater than zero is of the form $X_1 * X_2$, which means in particular that $X = X_1 \cup X_2$ and $X_1 \cap X_2 < X_1 \setminus X_2 < X_2 \setminus X_1$. Now suppose that $\alpha \in X = X_1 * X_2$ and $\alpha \in Y = Y_1 * Y_2$ and the ranks of X_1, X_2, Y_1, Y_2 are elements of μ of fixed rank $n \in \mathbb{N}$. By [19, Definition 1.1(3)] there is an order preserving $f_{Y,X} : X \to Y$, which by [19, Definition 1.1(3, 5)] must satisfy $f[X_1] = Y_1$ and $f[X_2] = Y_2$ and moreover $f{\restriction}(X \cap (\alpha + 1))$ is the identity on $X \cap (\alpha + 1)$ be the coherence lemma 2.1 of [19], so $f_{Y,X}(\alpha) = \alpha$ and $f[X_1 \cap X_2] = Y_1 \cap Y_2$, $f[X_1 \setminus X_2] = Y_1 \setminus Y_2$ and $f[X_2 \setminus X_1] = Y_2 \setminus Y_1$ and so the value of $g_\alpha(n+1)$ does not depend on whether we applied the definition of $g_\alpha(n + 1)$ to $X_1 * X_2$ or $Y_1 * Y_2$, which completes the proof of the claim that the g_αs are well defined.

Now we will prove (1) and (2) for $\alpha < \beta < \omega_1$ such that $c(\alpha, \beta) > 0$. For (1) let $n + 1 = k > \mathrm{rank}(X)$ such that $\alpha, \beta \in X \in \mu$. Let Y (which exists by the above-mentioned Density Lemma) be such that $X \subseteq Y \in \mu$ and $\mathrm{rank}(Y) = k$. Then $Y = Y_1 * Y_2$. By [19, Definition 1.1(5)] we have $X \subseteq Y_1$ or $X \subseteq Y_2$, so $\{g_\beta(k), g_\alpha(k)\} \neq \{1, 2\}$. (2) follows from the definition of c, i.e., from the minimality of the rank of $X \ni \alpha, \beta$, which is of the form $X_1 \cup X_2$ with $X_1 \setminus X_2 < X_2 \setminus X_1$ by [19, Definition 1.1(5)] and by the hypothesis that $c(\alpha, \beta) > 0$. Property (3) is [19, Corollary 5.4(2)]. Property (4) is [19, Proposition 5.3(a)].

To obtain (5), recall [19, Theorem 4.5] that $(g_\alpha^{-1}[\{1\}], g_\alpha^{-1}[\{2\}])_{\alpha<\omega_1}$ is a Hausdorff gap, so the sets must be infinite from some point on, so it is enough to remove possibly countably many $\alpha < \omega_1$ and renumber the remaining ones.

So it remains to remove the hypothesis $c(\alpha, \beta) > 0$ from (1) and (2). Note that what we have proved so far is valid for α, β, γ from any subset of ω_1, in other words we can pass to an uncountable subset X of ω_1 and consider only g_αs for $\alpha \in X$ and then renumber X as ω_1 in an increasing manner. So we need to argue that there is an uncountable $X \subseteq \omega_1$ such that $c(\alpha, \beta) > 0$ for all $\alpha < \beta$ and $\alpha, \beta \in X$. To obtain X apply the Dushnik–Miller theorem [17, Theorem 9.7] to a coloring $d : [\omega_1]^2 \to \{0, 1\}$ given by $d(\alpha, \beta) = \min\{1, c(\alpha, \beta)\}$ knowing that all elements of rank zero must have fixed finite cardinality. ∎

THEOREM 9. *There are families $(X_\alpha, Y_\alpha, A_\alpha, B_\alpha \mid \alpha < \omega_1)$ of subsets of $\mathbb{N}$ such that:*

(1) $X_\alpha = A_\alpha \cup B_\alpha$ *is infinite, $A_\alpha \cap B_\alpha = \emptyset$ for all $\alpha < \omega_1$.*
(2) $X_\beta \cap X_\alpha =^* \emptyset$ *for all $\beta < \alpha < \omega_1$.*
(3) $Y_\beta \subseteq^* Y_\alpha$ *for all $\beta < \alpha < \omega_1$.*
(4) $X_\beta \subseteq^* Y_\alpha$ *for all $\beta < \alpha < \omega_1$.*
(5) $X_\alpha \cap Y_\alpha = \emptyset$ *for all $\alpha < \omega_1$.*
(6) *For every $\alpha < \omega_1$ and every $k \in \mathbb{N}$, for all but finitely many $\beta < \alpha$ there is $l > k$ such that $l \in A_\beta \cap B_\alpha$.*

Proof. Define all the sets as subsets of $[\{0, 1, 2\}^{<\omega}]^2$ instead of $\mathbb{N}$. For $\alpha < \omega_1$ put $X_\alpha = A_\alpha \cup B_\alpha$, where

$$A_\alpha = \big\{\{g_\alpha \restriction (n + 1), s\} \mid s \in \{0, 1, 2\}^{n+1}, g_\alpha(n) = 1, s(n) = 2\}, n \in \mathbb{N}\big\},$$

$$B_\alpha = \{\{g_\alpha \restriction (n + 1), s\} \mid s \in \{0, 1, 2\}^{n+1}, g_\alpha(n) = 2, s(n) = 1\}, n \in \mathbb{N}\}.$$

Then (1) is clear by Theorem 8(5).

If $\beta < \alpha < \omega_1$ and $\{r, s\} \in X_\alpha \cap X_\beta$ for $n \in \mathbb{N}$ such that $|r| = |s| = n + 1$ and $g_\alpha \restriction (n + 1) \neq g_\beta \restriction (n + 1)$, then $\{r, s\} = \{g_\alpha \restriction (n + 1), g_\beta \restriction (n + 1)\}$ and $\{r(n), s(n)\} = \{1, 2\}$, which means that $n \leq c(\alpha, \beta)$ by Theorem 8(1, 2). So we obtain (2).

Note that if $\beta < \alpha < \omega_1$, then $\{g_\alpha \restriction (c(\alpha, \beta) + 1), g_\beta \restriction (c(\alpha, \beta) + 1)\} \in A_\beta \cap B_\alpha$ by Theorem 8(2), so by Theorem 8(4) we obtain (6).

For $\alpha < \omega_1$ define

$$Y_\alpha = \bigcup_{\beta<\alpha} \Big(X_\beta \setminus \bigcup_{i \leq c(\beta,\alpha)} [\{0, 1, 2\}^{i+1}]^2\Big).$$

It follows that $X_\beta \subseteq^* Y_\alpha$ if $\beta < \alpha < \omega_1$, so we have (4). Also $Y_\alpha \cap X_\alpha = \emptyset$ because $X_\beta \cap X_\alpha \subseteq \bigcup_{i \leq c(\beta,\alpha)}[\{0, 1, 2\}^{i+1}]^2$ by Theorem 8(1, 2), so also (5) holds.

If $\gamma < \beta < \alpha$ we have $c(\gamma, \beta) = c(\gamma, \alpha)$ with the possible exception for $\gamma < \beta$ in the set $D(\beta, \alpha) = \{\delta < \beta \mid c(\delta, \beta) \leq c(\beta, \alpha)\}$ by Theorem 8(3). Moreover $D(\beta, \alpha)$ is finite Theorem 8(4). So almost all summands in the definition of Y_β appear literally in the definition of Y_α. The remaining summands of Y_β are $X_\gamma \setminus \bigcup_{i \leq c(\gamma, \beta)} [\{0, 1, 2\}^i]^2$ for $\gamma \in D(\beta, \alpha)$. Each of them is almost equal to a summand of Y_α of the form $X_\gamma \setminus \bigcup_{i \leq c(\gamma, \alpha)} [\{0, 1, 2\}^i]^2$ for $\gamma \in D(\beta, \alpha)$, which proves that $Y_\beta \subseteq^* Y_\alpha$, which is (3), completing the proof. ∎

An example of the use of the partition of X_α above into A_α and B_α is given in the following proposition which has found an application in [11].

PROPOSITION 10. *There are families $(X'_\alpha, Y'_\alpha : \alpha < \omega_1)$ of subsets of $\mathbb{N}$ and bijections $f_\alpha : \mathbb{N} \times \mathbb{N} \to X'_\alpha$ such that:*

(1) $X'_\beta \cap X'_\alpha =^* \emptyset$ *for all $\beta < \alpha < \omega_1$.*
(2) $Y'_\beta \subseteq^* Y'_\alpha$ *for all $\beta < \alpha < \omega_1$.*
(3) $X'_\beta \subseteq^* Y'_\alpha$ *for all $\beta < \alpha < \omega_1$.*
(4) $X'_\alpha \cap Y'_\alpha = \emptyset$ *for all $\alpha < \omega_1$.*
(5) *For every $\alpha < \omega_1$ and every $k \in \mathbb{N}$, for all but finitely many $\beta < \alpha$ there are $m_1 < \cdots < m_k$ and $n_1 < \cdots < n_k$ such that*

$$f_\alpha(i, n_j) = f_\beta(j, m_i) \quad \text{for all } 1 \leq i, j \leq k.$$

Proof. Consider a pairwise disjoint family $\{I_l \mid l \in \mathbb{N}\}$ of finite subsets $\mathbb{N}$ where $I_l = \{l_{i,j} \mid 1 \leq i, j \leq l\} \cup \{r_l\}$. Define $X'_\alpha = \bigcup \{I_l \mid l \in X_\alpha\}$ and $Y'_\alpha = \bigcup \{I_l \mid l \in Y_\alpha\}$ where X_α, Y_α satisfy Theorem 9. It is clear that (1)–(4) are satisfied. Put $X''_\alpha = \bigcup \{I_l \setminus \{r_l\} \mid l \in X_\alpha\}$. Now for $\alpha < \omega_1$ let A_α and B_α be as in Theorem 9 and define recursively in $l \in X_\alpha$ for elements of $I_l \setminus \{r_l\}$ an injection $h_\alpha : X''_\alpha \to \mathbb{N} \times \mathbb{N}$ in such a way that if $l \in A_\alpha$, then there are $m_1 < \cdots < m_l$ such that $h_\alpha(l_{i,j}) = (j, m_i)$ for all $1 \leq i, j \leq l$, and if $l \in B_\alpha$, then there are $n_1 < \cdots < n_l$ such that $h_\alpha(l_{i,j}) = (i, n_j)$ for all $1 \leq i, j \leq l$. Now use the elements $\{r_l \mid l \in X_\alpha\}$ to extend h_α to a bijection $h'_\alpha : X'_\alpha \to \mathbb{N} \times \mathbb{N}$ and define $f_\alpha = (h'_\alpha)^{-1}$. Note that Theorem 9(6) gives $l > k$ such that $l \in A_\beta \cap B_\alpha$, and so (5) follows. ∎

We may note several interesting properties of the almost disjoint family $(X_\alpha \mid \alpha < \omega_1)$ from Theorem 9.

COROLLARY 11. *There is an almost disjoint family $\mathcal{A}$ which is inseparable (Luzin), but for every countable $\mathcal{B} \subseteq \mathcal{A}$, the families $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated.*

Proof. As two parts of a countable almost disjoint family can be separated, it is enough to separate the initial fragment $\{X_\beta \mid \beta < \alpha\}$ from the remaining part $\{X_\beta \mid \beta \geq \alpha\}$. Our family from Theorem 9 of course has such a separation Y_α, so it is enough to show that it is inseparable. For this,

note that Theorem 9(5) implies that given $\alpha < \omega_1$ and $k \in \mathbb{N}$, for all but finitely many $\beta < \alpha$ we have $\max(X_\beta \cap X_\alpha) > k$. This condition implies the inseparability of the family in the standard way as in the case of the Luzin family (cf. [15]). ∎

COROLLARY 12. *There is a Luzin family $(X_\alpha \mid \alpha < \omega_1)$ such that whenever $\mathcal{X} \subseteq \omega_1$ is uncountable and councountable, there is a Hausdorff gap $(A_\alpha^{\mathcal{X}}, B_\alpha^{\mathcal{X}})_{\alpha < \omega_1}$ for which $((X_\alpha \mid \alpha \in \mathcal{X}), (X_\alpha \mid \alpha \in \omega_1 \setminus \mathcal{X}))$ is its almost disjoint refinement.*

Proof. Take the families $(X_\alpha \mid \alpha < \omega_1)$ and $(Y_\alpha \mid \alpha < \omega_1)$ from Theorem 9. Using the non-existence of countable gaps in $\wp(\mathbb{N})/\text{Fin}$, for each $\alpha < \omega_1$ we can recursively construct a separation C_α^X of $(X_\beta \mid \beta \in \mathcal{X} \cap \alpha)$ and $(X_\beta \mid \beta \in \alpha \setminus \mathcal{X})$, i.e., $C_\alpha^X \subseteq \mathbb{N}$ such that:

- $X_\beta \subseteq^* C_\alpha^X$ if $\beta \in \alpha \cap \mathcal{X}$.
- $X_\beta \cap C_\alpha^X =^* \emptyset$ if $\beta \in \alpha \setminus \mathcal{X}$.
- $C_\beta^X \cap Y_\beta \subseteq^* C_\alpha^X$ if $\beta < \alpha$.
- $(Y_\beta \setminus C_\beta^X) \cap C_\alpha^X =^* \emptyset$ if $\beta < \alpha$.

Putting $A_\alpha^X = C_\alpha^X \cap Y_\alpha$, $B_\alpha^X = Y_\alpha \setminus C_\alpha^X$ we obtain a Hausdorff gap. ∎

4. $\mathbb{R}$-embeddability in the Cohen model. Cohen forcing $\mathbb{C}$ consists of elements of $\mathbb{N}^{<\omega}$ and is ordered by reverse inclusion. By the *Cohen model* we mean the model obtained by adding ω_2-Cohen reals with finite supports to a model of the Generalized Continuum Hypothesis (GCH). Given $X \subseteq \omega_2$ we define $\mathbb{C}_X$ as the forcing adding Cohen reals (with finite supports) indexed by X. The following lemma is well known:

LEMMA 13 (Continuous reading of names for Cohen forcing). *If $\dot{A}$ is a $\mathbb{C}$-name for a subset of $\mathbb{N}$, then there is a pair $(\langle \mathcal{B}_n \rangle_{n \in \mathbb{N}}, F)$ such that:*

(1) *Each $\mathcal{B}_n \subseteq \mathbb{N}^{<\omega}$ is a maximal antichain.*
(2) *If $s \in \mathcal{B}_{n+1}$ then there is $t \in \mathcal{B}_n$ such that $t \subseteq s$.*
(3) *$F : \bigcup_{n \in \mathbb{N}} \mathcal{B}_n \to 2$.*
(4) *If $c \in \mathbb{N}^\omega$ is Cohen over V, then*

$$\dot{A}[c] = \{n \mid \exists m \, ((c{\restriction}m) \in \mathcal{B}_n \ \& \ F(c{\restriction}m) = 1)\}.$$

Here by $\dot{A}[c]$ we denote the evaluation of the name $\dot{A}$ using the generic real c. If the conditions (1)–(4) hold, we will say that $(\langle \mathcal{B}_n \rangle_{n \in \mathbb{N}}, F)$ *codes* $\dot{A}$.

As a warm-up we present a direct proof of a result of Dow and Hart from [9] which was obtained there using an ingenious axiomatization of $\wp(\mathbb{N})/\text{Fin}$ in the Cohen model.

THEOREM 14 ([9]). *In the Cohen model, every almost disjoint family of size ω_2 is ω_2-anti-Luzin.*

Proof. It is enough to show that in the Cohen model, every almost disjoint family of size ω_2 contains two subfamilies of size ω_2 that are separated. Let $\mathcal{A} = \{\dot{A}_\alpha \mid \alpha \in \omega_2\}$ be a $\mathbb{C}_{\omega_2}$-name for an almost disjoint family. Since $\mathbb{C}_{\omega_2}$ has the countable chain condition, for every $\alpha \in \omega_2$ we can find $S_\alpha \in [\omega_2]^\omega$ such that each $\dot{A}_\alpha$ is in fact a $\mathbb{C}_{S_\alpha}$-name. By CH and the Δ-system lemma, (see [20, Lemma III.6.15]) we can find $X \in [\omega_2]^{\omega_2}$ such that $\{S_\alpha \mid \alpha \in X\}$ forms a Δ-system with root $R \in [\omega_2]^\omega$.

We may further assume that the root R is the empty set (if not, we simply turn to the intermediate model obtained by forcing with $\mathbb{C}_R$). Since $\mathbb{C}_{S_\alpha}$ is a forcing notion equivalent to $\mathbb{C}$, we may assume that for each $\alpha \in X$, $\dot{A}_\alpha$ is a $\mathbb{C}_{\{\alpha\}}$-name. Since V is a model of CH, we can find $X_1 \in [X]^{\omega_2}$ and a pair $(\langle \mathcal{B}_n \rangle_{n \in \omega}, F)$ that codes $\dot{A}_\alpha$ for every α in X_1. In other words, if $\alpha \in X_1$, then $\dot{A}_\alpha$ is forced to be equal to

$$\{n \mid \exists m \, ((\dot{c}_\alpha{\restriction}m) \in \mathcal{B}_n \,\&\, F(\dot{c}_\alpha{\restriction}m) = 1)\}$$

(where $\dot{c}_\alpha$ is the name of the αth Cohen real). We claim that there are $s, t \in \mathbb{N}^{<\omega}$ such that:

(1) s and t are incomparable nodes of the same length.
(2) There are no m, s', t' with the following properties:

    (a) $m > |s|, |t|$.
    (b) $s', t' \in \mathcal{B}_m$.
    (c) $s \subseteq s', t \subseteq t'$.
    (d) $F(s') = F(t') = 1$.

If there were no such s and t, by a simple genericity argument we would obtain two elements of $\mathcal{A}$ with infinite intersection, which is not possible (in fact, every pair of incomparable nodes can be extended to a pair of nodes satisfying these properties). In $V[G]$, define

$$\mathcal{C}_0 = \{\dot{A}_\alpha[c_\alpha] \mid \alpha \in X_1 \wedge s \subseteq c_\alpha\},$$
$$\mathcal{C}_1 = \{\dot{A}_\alpha[c_\alpha] \mid \alpha \in X_1 \wedge t \subseteq c_\alpha\}.$$

It is easy to see that both families are of size ω_2 and are separated by $\bigcup\{A \setminus m \mid A \in \mathcal{C}_0\}$. ∎

A stronger statement "Every almost disjoint family of size continuum contains an $\mathbb{R}$-embeddable subfamily of size continuum", is consistent but it is false in the Cohen model. We will prove the latter fact in the rest of this section and the former in the next section.

We denote by $\mathbb{T}$ the set of all finite trees $T \subseteq \mathbb{N}^{<\omega}$ such that all maximal nodes of T have the same height, denoted by $\mathrm{ht}(T)$. Given a tree $T \subseteq \mathbb{N}^{<\omega}$ we define $[T]^{2,=} = \{\{s, t\} \in [T]^2 \mid |s| = |t|\}$.

DEFINITION 15. Define $\mathbb{P}$ as the collection of all triples $p = (T_p, R_p, \phi_p)$ that satisfy the following properties:

(1) $T_p \in \mathbb{T}$.
(2) $R_p \subseteq [T_p]^{2,=}$.
(3) If $\{s, t\} \in R_p$ and $\{s', t'\} \in [T_p]^{2,=}$ with $s \subseteq s'$ and $t \subseteq t'$ then $\{s', t'\} \in R_p$.
(4) $\phi_p : T_p \to 2$.
(5) There is no $\{s, t\} \in R_p$ such that $\phi_p(s) = \phi_p(t) = 1$.

Given $p, q \in \mathbb{P}$ we say $p \leq_{\mathbb{P}} q$ if $T_q \subseteq T_p$, $R_q = R_p \cap [T_q]^{2,=}$, $\phi_q \subseteq \phi_p$.

Since $\mathbb{P}$ is a countable partial order, it is a forcing notion equivalent to Cohen forcing. We define $\dot\phi_{\text{gen}}$ to be $\bigcup\{\phi_p \mid p \in \dot{\mathbb{G}}\}$ (where $\dot{\mathbb{G}}$ is the name for a generic filter of $\mathbb{P}$). It is easy to see that $\dot\phi_{\text{gen}}$ is forced to be a function from $\mathbb{N}^{<\omega}$ to 2.

DEFINITION 16. We define $\mathbb{U}$ as the set of all sequences $(p, \langle s_\alpha \rangle_{\alpha \in F})$ with the following properties:

- $p \in \mathbb{P}$.
- $F \in [\omega_2]^{<\omega}$.
- $s_\alpha \in T_p$ for every $\alpha \in F$ (where $p = (T_p, R_p, \phi_p)$).

We define $(p, \langle s_\alpha \rangle_{\alpha \in F}) \leq (q, \langle t_\alpha \rangle_{\alpha \in G})$ if

- $p \leq_{\mathbb{P}} q$.
- $G \subseteq F$.
- $t_\alpha \subseteq s_\alpha$ for every $\alpha \in G$.

It is easy to see that $\mathbb{U}$ is forcing equivalent to $\mathbb{C}_{\omega_2}$. Moreover, $\mathbb{U}$ is forcing equivalent to first forcing with $\mathbb{P}$ and then adding ω_2-Cohen reals. Given $\alpha < \omega_2$ we define $\dot{A}_\alpha$ to be the set $\{n \mid \dot\phi_{\text{gen}}(\dot{c}_\alpha \restriction n) = 1\}$ (where $\dot{c}_\alpha$ is the name for the αth Cohen real). It is easy to see that $\dot{\mathcal{A}} = \{\dot{A}_\alpha \mid \alpha < \omega_2\}$ is forced to be an almost disjoint family of size ω_2.

THEOREM 17. *In the Cohen model, there is an almost disjoint family of size ω_2 that does not contain uncountable $\mathbb{R}$-embeddable subfamilies.*

Proof. Since $\mathbb{U}$ is forcing equivalent to $\mathbb{C}_{\omega_2}$, we can think of the Cohen model as the model obtained after forcing with $\mathbb{U}$ over a model of the Continuum Hypothesis. Let $\mathcal{A}$ be the almost disjoint family that was defined above. To reach a contradiction, assume that there is $\dot{\mathcal{B}} = \{\dot{A}_{\dot\alpha_\xi} \mid \xi \in \omega_1\}$ and $\dot{f}$ such that $\dot{f}$ is forced to be an embedding of $\Psi(\dot{\mathcal{B}})$ into 2^ω as in Lemma 2(5). For every $\xi \in \omega_1$, we may find $r_\xi = (p_\xi, \langle s_\eta^{r_\xi} \rangle_{\eta \in F_\xi}) \in \mathbb{U}$ and β_ξ with the following properties:

- $r_\xi \Vdash \dot\alpha_\xi = \beta_\xi$.
- $\beta_\xi \in F_\xi$.
- $s_{\beta_\xi}^{r_\xi} \frown 0, s_{\beta_\xi}^{r_\xi} \frown 1 \in T_{p_\xi}$ (where $p_\xi = (T_{p_\xi}, R_{p_\xi}, \phi_{p_\xi})$).

By the Δ-system lemma (see [20, Lemma III.2.6]) we may find $p \in \mathbb{P}$, $R \in [\omega_2]^{<\omega}$, $W \in [\omega_2]^{\omega_1}$ and $s \in \mathbb{N}^{<\omega}$ with the following properties:

- $p_\xi = p$ for every $\xi \in W$.
- $\{F_\xi \mid \xi \in W\}$ forms a Δ-system with root R.
- $s_\eta^{r_\xi} = s_\eta^{r_{\xi'}}$ for every $\xi, \xi' \in W$ and $\eta \in R$.
- $s = s_{\beta_\xi}^{r_\xi}$ for every $\xi \in W$.

It is easy to see that $\{r_\xi \mid \xi \in W\} \subseteq \mathbb{U}$ is a centered ([2]) family. Let $\{H_\alpha \mid \alpha \in \omega_1\} \subseteq [W]^2$ be a pairwise disjoint family. For every $\alpha \in \omega_1$ we find $r'_\alpha = (p'_\alpha, \langle u_\eta^{r'_\alpha}\rangle_{\eta \in F'_\alpha}) \in \mathbb{U}$, t_α and z_α with the following properties:

- $r'_\alpha \leq r_{\xi_1}, r_{\xi_2}$ where $H_\alpha = \{\xi_1, \xi_2\}$ and $\xi_1 < \xi_2$.
- $s^\frown 0 \subseteq u_{\beta_{\xi_1}}^{r'_\alpha}$.
- $s^\frown 1 \subseteq u_{\beta_{\xi_2}}^{r'_\alpha}$.
- $t_\alpha, z_\alpha \in \mathbb{N}^{<\omega}$ are incompatible.
- $r'_\alpha \Vdash t_\alpha \subseteq \dot{f}(\dot{A}_{\beta_{\xi_1}}) \wedge z_\alpha \subseteq \dot{f}(\dot{A}_{\beta_{\xi_2}})$.

The last condition can be obtained since $\dot{f}$ is forced to be injective when restricted to $\dot{\mathcal{B}}$ as in Lemma 2(6). Once again, we can find $W_0 \in [\omega_2]^{\omega_1}$, $p' \in \mathbb{P}$, $s_0, s_1 \in \mathbb{N}^{<\omega}$, $R'' \in [\omega_2]^{<\omega}$, t, z such that for every $\alpha \in W_0$ the following hold:

- $p'_\alpha = p'$.
- $t_\alpha = t$ and $z_\alpha = z$.
- $\{F'_\alpha \mid \alpha \in W_0\}$ forms a Δ-system with root R'.
- $u_\eta^{r'_\alpha} = u_\eta^{r'_\delta}$ for every $\alpha, \delta \in W_0$ and $\eta \in R'$.
- $s_0 = u_{\beta_{\xi_1}}^{r'_\alpha}$ and $s_1 = u_{\beta_{\xi_2}}^{r'_\alpha}$ for every $\alpha \in W_0$ where $H_\alpha = \{\xi_1, \xi_2\}$.

Once again, the set $\{r'_\alpha \mid \alpha \in W_0\} \subseteq \mathbb{U}$ is centered. Let $\mathcal{M}$ be a countable elementary submodel of some $\mathsf{H}(\kappa)$ (where κ is a sufficiently large cardinal) containing all objects that have been defined so far. Let $\gamma \in \mathcal{M} \cap W_0$ and $\delta \in W \setminus \mathcal{M}$. Find $m \in \mathbb{N}$ such that $s^\frown m \notin T_{p'}$, and let $\widehat{s}$ be a sequence extending $s^\frown m$ such that $|\widehat{s}| = |s_0| = |s_1|$. Then find $\widehat{r} = (\widehat{p}, \langle y_\eta^{\widehat{r}}\rangle_{\eta \in F})$ with the following properties:

- $\widehat{r} \leq r'_\gamma, r_\delta$.
- $T_{p'} \cup \{\widehat{s}\} \subseteq T_{\widehat{p}}$ (where $\widehat{p} = (T_{\widehat{p}}, R_{\widehat{p}}, F_{\widehat{p}})$).
- $F = F'_\gamma \cup F_\delta$.
- $y_{\beta_\delta}^{\widehat{r}} = \widehat{s}$.
- $\{s_0, \widehat{s}\}, \{s_1, \widehat{s}\} \notin R_{p'}$.

([2]) Recall that a family of infinite subsets of ω is *centered* if the intersection of any finitely many of its elements is infinite.

We claim that $\widehat{r}$ forces that $\dot{f}[\dot{A}_{\beta_\delta}]$ has infinitely many elements extending t and infinitely many elements extending z (recall that $\dot{f}[\dot{A}_{\beta_\delta}]$ is forced to be a collection of finite sequences); this will contradict the continuity of $\dot{f}$. Let $\widehat{r}_1 \leq \widehat{r}$ and $k \in \mathbb{N}$. It will be enough to prove that we can extend $\widehat{r}_1$ to a condition that forces that there is $l > k$ which is in $\dot{A}_{\beta_\delta}$ and its image under $\dot{f}$ is an extension of t whose height is greater than k (the case of z is similar). Let $\alpha \in \mathcal{M} \cap W_0$ be such that $\mathrm{supp}(\widehat{r}_1) \cap \mathcal{M}$ and $F'_\alpha \setminus R'$ are disjoint. Let $\widehat{r}_2$ be the greatest lower bound of $\widehat{r}_1 \cap \mathcal{M}$ and r'_α; note that $\widehat{r}_2 \in \mathcal{M}$. Let $e \in \omega$ be such that $s_0 ^\frown e$ has not been used and let v extending $s_0 ^\frown e$ be such that $|v| = |s_{\beta_\delta}^{\widehat{r}_1}|$ and $\widehat{r}_3$ such that $s_{\beta_{\xi_1}}^{\widehat{r}_3} = v$ (where $H_\alpha = \{\xi_1, \xi_2\}$) and $\{v, s_{\beta_\delta}^{\widehat{r}_1}\} \notin R_{\widehat{r}_3}$. Since $\widehat{r}_3, \dot{f} \in \mathcal{M}$ we can find $\widehat{r}_4 \in \mathcal{M}$ such that $\widehat{r}_4 \leq \widehat{r}_3$ and $l > k$ such that $\widehat{r}_4 \Vdash l \in \dot{A}_{\beta_{\xi_1}} \wedge t \subseteq \dot{f}(l)$. Since the support of $\widehat{r}_4$ is contained in $\mathcal{M}$, it is compatible with $\widehat{r}_1$. Since $\{v, s_{\beta_\delta}^{\widehat{r}_1}\} \notin R_{\widehat{r}_3}$, we can find a common extension that forces that l is in $\dot{A}_{\beta_\delta}$. ∎

The above family clearly does not have the ω_1-controlled ℝ-embedding property, but a much stronger fact concerning the ω_1-controlled ℝ-embedding property can be proved in the Cohen model.

THEOREM 18. *In the Cohen model, no uncountable almost disjoint family $\mathcal{A}$ has the ω_1-controlled ℝ-embedding property.*

Proof. Let $\{c_\alpha \mid \alpha < \omega_2\}$ be the sequence of Cohen reals generating the Cohen model. Let $\mathcal{A}$ be an uncountable almost disjoint family. For every $A \in \mathcal{A}$ there is a countable $X_A \subseteq \omega_2$ such that $A \in V[\{c_\alpha \mid \alpha \in X_A\}]$. Define $\phi : \mathcal{A} \to 2^\omega$ by $\phi(A) = c_{\alpha_A}$ where $\alpha_A \notin X_A$ and all α_A's are distinct.

Suppose that $f : \mathbb{N} \to 2^\omega$. There is a countable $Y \subseteq \omega_2$ such that $f \in V[\{c_\alpha \mid \alpha \in Y\}]$. As $\mathcal{A}$ is uncountable, there is $A \in \mathcal{A}$ such that $\alpha_A \notin Y$, so $\alpha_A \notin X_A \cup Y$. Hence $\lim_{n \in A} f(n) \neq c_{\alpha_A} = \phi(A)$, proving that $\mathcal{A}$ does not have the ω_1-controlled property. ∎

REMARK 19. The above proof remains valid for any finite support product of no less than 2^ω c.c.c. forcings in place of the Cohen forcing.

5. ℝ-embeddability in the Sacks model. By the *Sacks model* we mean the model obtained by adding ω_2-Sacks reals iteratively with countable support to a model of the GCH. Recall that a tree $p \subseteq 2^{<\omega}$ is a *Sacks tree* if every node of p can be extended to a splitting node. We denote by $\mathbb{S}$ the collection of all Sacks trees, ordered by inclusion. Given $\alpha \leq \omega_2$ we denote by $\mathbb{S}_\alpha$ the countable support iteration of $\mathbb{S}$ of length α. We will now prove that in the Sacks model, every almost disjoint family of size continuum contains an ℝ-embeddable family of the same size. We will need to recall some important notions and results on Sacks forcing. For more on this forcing notion the reader may consult [5], [14] and [23].

DEFINITION 20. Let $\alpha \leq \omega_2$ and $n, m \in \mathbb{N}$.

(1) Given $p, q \in \mathbb{S}$ we say that $(p, m) \leq (q, n)$ if the following hold:

    (a) $p \leq q$.
    (b) $n \leq m$.
    (c) $q \cap 2^n = p \cap 2^n$.
    (d) If $n < m$ then for every $s \in q_n$ there are distinct $t_0, t_1 \in p_m$ such that $s \subseteq t_0, t_1$.

(2) Given $p, q \in \mathbb{S}_\alpha$ and $F \in [\alpha]^{<\omega}$ we say that $(p, m) \leq_F (q, n)$ if the following hold:

    (a) $p \leq q$.
    (b) $n \leq m$.
    (c) if $\beta \in F$ then $p{\restriction}\beta \Vdash (p(\beta), m) \leq (q(\beta), n)$.

We will often use the following result:

LEMMA 21 (Fusion lemma [5]). *Let* $\alpha \leq \omega_2$ *and* $\{(p_i, F_i, n_i) \mid i \in \mathbb{N}\}$ *be a family such that for every* $i \in \mathbb{N}$ *the following hold:*

(1) $p_i \in \mathbb{S}_\alpha$.
(2) $F_i \in [\alpha]^{<\omega}$.
(3) $F_i \subseteq F_{i+1}$.
(4) $n_i < n_{i+1}$.
(5) $(p_{i+1}, n_{i+1}) \leq_{F_i} (p_i, n_i)$.
(6) $\bigcup_{j \in \mathbb{N}} F_j = \bigcup_{j \in \mathbb{N}} \operatorname{supp}(p_j)$.

Define p *such that* $\operatorname{supp}(p) = \bigcup_{j \in \mathbb{N}} \operatorname{supp}(p_j)$ *and if* $\beta \in \operatorname{supp}(p)$ *then* $p(\beta)$ *is an* $\mathbb{S}_\beta$*-name for the intersection of* $\{p_i(\beta) \mid \beta \in \operatorname{supp}(p_i)\}$. *Then* $p \in \mathbb{S}_\alpha$ *and* $p \leq p_i$ *for every* $i \in \mathbb{N}$.

If $p \in \mathbb{S}$ and $s \in 2^{<\omega}$ we define $p_s = \{t \in p \mid t \subseteq s \lor s \subseteq t\}$. Note that p_s is a Sacks tree if and only if $s \in p$.

DEFINITION 22. Let $p \in \mathbb{S}_\alpha$, $F \in [\operatorname{supp}(p)]^{<\omega}$ and $\sigma : F \to 2^n$. We define p_σ as follows:

(1) $\operatorname{supp}(p_\sigma) = \operatorname{supp}(p)$.
(2) For $\beta < \alpha$ the following hold:

    (a) $p_\sigma(\beta) = p(\beta)$ if $\beta \notin F$.
    (b) $p_\sigma(\beta) = p(\beta)_{\sigma(\beta)}$ if $\beta \in F$.

Similar to the previous situation, p_σ is not necessarily a condition of $\mathbb{S}_\alpha$. We will say that $\sigma : F \to 2^n$ is *consistent with* p if $p_\sigma \in \mathbb{S}_\alpha$. A condition p is (F, n)-*determined* if for every $\sigma : F \to 2^n$, either σ is consistent with p or there is $\beta \in F$ such that $\sigma{\restriction}(F \cap \beta)$ is consistent with p and $(p{\restriction}\beta)_{\sigma{\restriction}(F\cap\beta)} \Vdash \sigma(\beta) \notin p(\beta)$.

We say that $p \in \mathbb{S}_\alpha$ is *continuous* if for every $F \in [\mathrm{supp}(p)]^{<\omega}$ and for every $n \in \mathbb{N}$ there are G and m such that

- $G \in [\mathrm{supp}(p)]^{<\omega}$.
- $F \subseteq G$.
- $n < m$.
- p is (G, m)-determined.

We will need the following lemmas:

LEMMA 23 ([5]). *Let $p \in \mathbb{S}_\alpha$, $n \in \mathbb{N}$ and $F \in [\mathrm{supp}(p)]^{<\omega}$. There is $(q, m) \leq_F (p, n)$ such that q is (F, n)-determined.*

LEMMA 24 ([14]). *For every $p \in \mathbb{S}_\alpha$ there is a continuous $q \leq p$ such that q is continuous.*

Let p be a continuous condition. We say that $\{(F_i, n_i, \Sigma_i) \mid i \in \omega\}$ is a *representation of p* if the following hold:

- $F_i \in [\mathrm{supp}(p)]^{<\omega}$, $n_i \in \omega$.
- $F_i \subseteq F_{i+1}$ and $n_i < n_{i+1}$.
- $\mathrm{supp}(p) = \bigcup_{i \in \mathbb{N}} F_i$.
- p is (F_i, n_i)-determined for every $i \in \omega$.
- Σ_i is the set of all $\sigma : F_i \to 2^{n_i}$ such that σ is consistent with p.

Note that if $\{(F_i, n_i, \Sigma_i) \mid i \in \omega\}$ is a representation of p and $f : \omega \to \omega$ is an increasing function, then $\{(F_{f(i)}, n_{f(i)}, \Sigma_{f(i)}) \mid i \in \mathbb{N}\}$ is also a representation of p. It is also easy to see that if p is continuous with representation $\{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and $\sigma \in \Sigma_i$, then p_σ is also a continuous condition. Given a continuous condition $p \in \mathbb{S}_\alpha$ and a representation $R = \{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ of p, we define $[p]_R$ as the set of all $\langle y_\beta \rangle_{\beta \in \mathrm{supp}(p)} \in (2^\omega)^{\mathrm{supp}(p)}$ such that for every $i \in \omega$ the function $\sigma : F_i \to 2^{n_i}$ given by $\sigma(\beta) = y_\beta \upharpoonright n_i$ belongs to Σ_i.

LEMMA 25. *Let $p \in \mathbb{S}_\alpha$ be continuous. If $R = \{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and $R' = \{(F'_i, n'_i, \Sigma'_i) \mid i \in \mathbb{N}\}$ are two representations of p, then $[p]_R = [p]_{R'}$.*

Proof. To reach a contradiction, assume there is $\overline{y} = \langle y_\beta \rangle_{\beta < a} \in [p]_R \setminus [p]_{R'}$. Since $\overline{y} \notin [p]_{R'}$ there must be $i \in \omega$ such that the function $\sigma : F'_i \to 2^{n'_i}$ given by $\sigma(\beta) = y_\beta \upharpoonright m_i$ is not in Σ'_i, i.e. σ is not consistent with p. Since p is (F'_i, n'_i)-determined, there is $\beta \in F'_i$ such that $\sigma \upharpoonright (F'_i \cap \beta)$ is consistent with p but $p_{\sigma \upharpoonright (F'_i \cap \beta)} \Vdash \sigma(\beta) \notin p(\beta)$. Let $j \in \omega$ be such that $F'_i \subseteq F_j$ and $n'_i < n_j$. Since $\overline{y} \in [p]_R$ we know that the function $\tau : F_j \to 2^{n_j}$ given by $\tau(\xi) = y_\xi \upharpoonright n_j$ is consistent with p. It is clear that $p_{\tau \upharpoonright (F_j \cap \beta)} \leq p_{\sigma \upharpoonright (F'_i \cap \beta)}$ and $\sigma(\beta) \subseteq \tau(\beta)$ so $p_{\tau \upharpoonright (F_j \cap \beta)}$ forces that $\tau(\beta)$ is not in $p(\beta)$, which contradicts the fact that τ is consistent with p. ∎

In light of the previous result, we will omit the subscript and only write $[p]$ to refer to $[p]_R$ where R is any representation of p. It is easy to see that if $p \in \mathbb{S}_\alpha$ is a continuous condition then $[p]$ is a compact set and $p \Vdash \overline{s}_{\text{gen}} \restriction \text{supp}(p) \in [p]$ (where $\overline{s}_{\text{gen}}$ is the sequence of generic reals). Let $S \in [\omega_2]^\omega$ and $\sigma : F \to 2^{<\omega}$ where $F \in [S]^{<\omega}$. We define $\langle\sigma\rangle_S$ as the set $\{\langle y_\beta\rangle_{\beta\in S} \in (2^\omega)^S \mid \forall\beta \in F \ (\sigma(\beta) \subseteq y_\beta)\}$. Note that the family

$$\{\langle\sigma\rangle_S : F \in [S]^{<\omega} \text{ and } \sigma : F \to 2^{<\omega}\}$$

forms a basis for the topology of $(2^\omega)^S$.

The following result is well known:

LEMMA 26 (Continuous reading of names for Sacks forcing). *Let $\alpha < \omega_2$, $p \in \mathbb{S}_\alpha$ and $\dot{x}$ be an $\mathbb{S}_\alpha$-name such that $p \Vdash \dot{x} \in [\omega]^\omega$. There is a continuous condition $q \le p$ and a continuous function $F : [q] \to [\mathbb{N}]^\omega$ such that $q \Vdash F(\dot{s}_{\text{gen}} \restriction \text{supp}(q)) = \dot{x}$ (where $\dot{s}_{\text{gen}}$ is the name for the generic real).*

We will need the following notion:

DEFINITION 27. Let $\mathcal{C}, \mathcal{D}$ be two subfamilies of $\wp(\mathbb{N})$. We say that the pair $(\mathcal{C}, \mathcal{D})$ *is decisive* if one of the following two conditions holds: either

(1) $c \cap d$ is infinite for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$, or
(2) $c \cap d$ is finite for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

Note that if the second alternative holds and $\mathcal{C}$ and $\mathcal{D}$ are both compact, then there is an m such that $c \cap d \subseteq m$ for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

LEMMA 28. *Let p, q be two continuous conditions in $\mathbb{S}_\alpha$ such that $\text{supp}(p) = \text{supp}(q)$ and $F : (2^\omega)^{\text{supp}(p)} \to [\mathbb{N}]^\omega$ a continuous function. There are $p', q' \in \mathbb{S}_\alpha$ such that the following hold:*

(1) $p' \le p$ *and* $q' \le q$.
(2) $\text{supp}(p) = \text{supp}(q) = \text{supp}(p') = \text{supp}(q')$.
(3) *The pair $(F[[p']], F[[q']])$ is decisive.*

Proof. We consider several cases. The first case is that there are $p' \le p$ and $q' \le q$ with $\text{supp}(p) = \text{supp}(q) = \text{supp}(p') = \text{supp}(q')$ and $m \in \mathbb{N}$ such that $F(\overline{y}) \cap F(\overline{z}) \subseteq m$ for every $\overline{y} \in [p']$ and $\overline{z} \in [q']$.

In this case it is clear that the pair $(F[[p']], F[[q']])$ is decisive. The second case is that for every $p' \le p$, $q' \le q$ with $\text{supp}(p) = \text{supp}(q) = \text{supp}(p') = \text{supp}(q')$ and $m \in \mathbb{N}$ there are $\overline{y} \in [p']$, $\overline{z} \in [q']$ and $k > m$ such that $k \in F(\overline{y}) \cap F(\overline{z})$.

Let $\text{supp}(p) = \{\alpha_n \mid n \in \mathbb{N}\}$. We will now recursively build the two sequences $\{(p^n, m_n, F_n) \mid n \in \mathbb{N}\}$ and $\{(q^n, k_n, G_n) \mid n \in \mathbb{N}\}$ such that for every $n \in \mathbb{N}$ the following hold:

• $p^0 = p$ and $q^0 = q$, $F_0 = G_0 = \emptyset$.
• $F_n \in [\text{supp}(p)]^{<\omega}$, $F_n \subseteq F_{n+1}$ and $\alpha_n \in F_{n+1}$.

- $G_n \in [\mathrm{supp}(q)]^{<\omega}$, $G_n \subseteq G_{n+1}$ and $\alpha_n \in G_{n+1}$.
- $m_0 = k_0 = 0$.
- $m_n < m_{n+1}$ and $k_n < k_{n+1}$.
- p^n and q^n are continuous conditions.
- $\mathrm{supp}(p^n) = \mathrm{supp}(q^n) = \mathrm{supp}(p)$.
- $(p^{n+1}, m_{n+1}) \leq_{F_{n+1}} (p^n, m_n)$ and $(q^{n+1}, k_{n+1}) \leq_{G_{n+1}} (q^n, k_n)$.
- p^{n+1} is (F_{n+1}, m_{n+1})-determined and q^{n+1} is (G_{n+1}, k_{n+1})-determined.
- For every $\sigma : F_n \to 2^{m_n}$ and $\tau : G_n \to 2^{k_n}$ if σ is consistent with p^n and τ is consistent with q^n then there is $l > n$ such that $l \in F(\overline{y}) \cap F(\overline{z})$ for every $\overline{y} \in [p^n_\sigma]$ and $\overline{z} \in [q^n_\tau]$.

Assume we are at step $n + 1$. Since both p^n and q^n are continuous conditions, we can find $F_{n+1}, G_{n+1}, m_{n+1}$ and k_{n+1} with the following properties:

- $F_n \cup \{\alpha_n\} \subseteq F_{n+1}$ and $G_n \cup \{\alpha_n\} \subseteq G_{n+1}$.
- $m_n < m_{n+1}$, $k_n < k_{n+1}$.
- p^n is (F_{n+1}, m_{n+1})-determined and q^n is (G_{n+1}, k_{n+1})-determined.

Let $W = \{(\sigma_i, \tau_i)\}_{i < u}$ enumerate all pairs (σ, τ) with $\sigma : F_{n+1} \to 2^{m_{n+1}}$ and $\tau : G_{n+1} \to 2^{k_{n+1}}$. We recursively find a sequence $\{(p_1^i, q_1^i) \mid i < u + 1\}$ such that for every $i < u$ the following hold:

- $p^n = p_1^0$ and $q^n = q_1^0$.
- $(p_1^{i+1}, m_{n+1}) \leq_{F_{n+1}} (p_1^i, m_{n+1})$ and $(q_1^{i+1}, k_{n+1}) \leq_{G_{n+1}} (q_1^i, k_{n+1})$.
- p_1^i and q_1^i are continuous.
- $\mathrm{supp}(p_1^i) = \mathrm{supp}(q_1^i) = \mathrm{supp}(p)$.
- If σ_i is consistent with p_1^{i+1} and τ_i is consistent with q_1^{i+1} then there is $l > n$ such that $l \in F(\overline{y}) \cap F(\overline{z})$ for every $\overline{y} \in [(p_1^{i+1})_{\sigma_i}]$ and $\overline{z} \in [(q_1^{i+1})_{\tau_i}]$.

Assume we are at step i. In case either σ_i is not consistent with p_1^i or τ_i is not consistent with q_1^i we simply define $p_1^{i+1} = p_1^i$ and $q_1^{i+1} = q_1^i$. Assume σ_i is consistent with p_1^i and τ_i is consistent with q_1^i. By the hypothesis, there are $l > n$, $\overline{y} \in [(p_1^i)_{\sigma_i}]$ and $\overline{z} \in [(q_1^i)_{\tau_i}]$ such that $k \in F(\overline{y}) \cap F(\overline{z})$. Since F is a continuous function, we can find p_1^{i+1} and q_1^{i+1} with the following properties:

- σ_i is consistent with p_1^{i+1}.
- τ_i is consistent with q_1^{i+1}.
- For every $\overline{y_1} \in [(p_1^{i+1})_{\sigma_i}]$ and $\overline{z_1} \in [(q_1^{i+1})_{\tau_i}]$ we have $k \in F(\overline{y}) \cap F(\overline{z})$.
- p_1^{i+1} and q_1^{i+1} are continuous.
- $(p_1^{i+1}, m_{n+1}) \leq_{F_{n+1}} (p_1^i, m_{n+1})$ and $(q_1^{i+1}, k_{n+1}) \leq_{F_n} (q_1^i, k_{n+1})$.
- $\mathrm{supp}(p_1^i) = \mathrm{supp}(q_1^i) = supp(p)$.

We then define $p^{n+1} = p_1^{u+1}$ and $q^{n+1} = q_1^{u+1}$.

Let p' and q' be the respective fusion sequences. It is easy to see that $F[\overline{c}] \cap F[\overline{e}]$ is infinite for every $\overline{c} \in [p']$ and $\overline{e} \in [q']$. ∎

Note that if p is continuous and $\beta = \min(\text{supp}(p))$ then we may assume that $p(\beta)$ is a real Sacks tree (not only a name).

PROPOSITION 29. *Let $p \in \mathbb{S}_\alpha$ be a continuous condition, $F : [p] \to [\mathbb{N}]^\omega$ a continuous function and $\beta = \min \text{supp}(\alpha)$. Then there are $q \in \mathbb{S}_\alpha$ with representation $\{(F_i, m_i, \Sigma_i) \mid i \in \mathbb{N}\}$ such that the following hold:*

(1) $q \leq p$.
(2) $\text{supp}(q) = \text{supp}(p)$.
(3) $F_0 = \{\beta\}$.
(4) *For every $i \in \mathbb{N}$ and every $\sigma, \tau \in \Sigma_i$ such that $\sigma(\beta) \neq \tau(\beta)$, the pair $(F[[q_\sigma]], F[[q_\tau]])$ is decisive.*

Proof. Let $\text{supp}(p) = \{\alpha_n \mid n \in \mathbb{N}\}$ with $\alpha_0 = \beta$. We recursively build a sequence $\{(p^n, m_n, F_n) \mid n \in \mathbb{N}\}$ with the following properties:

- $p^0 = p$.
- $F_0 = \{\beta\}$ and $m_0 = 0$.
- Each p^n is continuous and $\text{supp}(p^n) = \text{supp}(p)$.
- $F_n \in [\text{supp}(p)]^{<\omega}$ and $\alpha_n \in F_n$.
- $(p^{n+1}, m_{n+1}) \leq_{F_n} (p^n, m_n)$.
- $m_n < m_{n+1}$.
- For every $\sigma, \tau : F_n \to 2^{m_n}$ such that $\sigma(\beta) \neq \tau(\beta)$ and both are consistent with p^n, the pair $(F[[p^n_\sigma]], F[[p^n_\tau]])$ is decisive.

Assume we are at step n. We first find F_{n+1} and $m_{n+1} > m_n$ such that $F_n \cup \{\alpha_n\} \subseteq F_{n+1}$ and p^n is (F_{n+1}, m_{n+1})-determined. Let W be the set of all pairs (σ, τ) such that $\sigma, \tau : F_{n+1} \to 2^{m_{n+1}}$, $\sigma(\beta) \neq \tau(\beta)$ and both are consistent with p^n. Enumerate $W = \{(\sigma_i, \tau_i) \mid i \leq l\}$. We recursively build $\{q_i \mid i \leq l\}$ with the following properties:

- Each q_i is (F_{n+1}, m_{n+1})-determined and continuous.
- $\text{supp}(q_i) = \text{supp}(p)$.
- $(q_0, m_n) \leq_{F_{n+1}} (p^n, m_{n+1})$.
- $(q_{i+1}, m_{n+1}) \leq_{F_n} (q_i, m_{n+1})$ for $i < l$.
- For each $i \leq l$, either
    (a) σ_i or τ_i is not consistent with q_i, or
    (b) the pair $(F[[(q_i)_{\sigma_i}]], F[[(q_i)_{\tau_i}]])$ is decisive.

Assume we are at step $i < l$. In case σ_{i+1} or τ_{i+1} is not consistent with q_i we simply define $q_{i+1} = q_i$. We now assume both σ_{i+1} and τ_{i+1} are consistent with q_i. By applying the previous lemma to $(q_i)_{\sigma_i}$ and $(q_i)_{\tau_i}$ we obtain continuous conditions r_0, r_1 with the following properties:

- $r_0 \leq (q_i)_{\sigma_i}$.
- $r_1 \leq (q_i)_{\tau_i}$.

- $\operatorname{supp}(r_0) = \operatorname{supp}(r_1) = \operatorname{supp}(p)$.
- the pair $(F[[r_0]], F[[r_1]])$ is decisive.

We now define r to be a Sacks tree with the following properties:

- $r_{\sigma_{i+1}(0)} = r_0(\beta)$.
- $r_{\tau_{i+1}(0)} = r_1(\beta)$.
- $r_s = q_i(\beta)_s$ for $s \in q_i(\beta)_{m_n}$ and $s \notin \{\sigma_{i+1}(\beta), \tau_{i+1}(\beta)\}$.

Let $\dot{u}$ be an $\mathbb{S}$-name with the following properties:

- $r_0{\restriction}(\beta + 1) \Vdash \dot{u} = \langle r_0(\xi) \rangle_{\xi > \beta}$.
- $r_1{\restriction}(\beta + 1) \Vdash \dot{u} = \langle r_1(\xi) \rangle_{\xi > \beta}$.
- $r' \Vdash$ "$\dot{u} = \langle q_i(\xi) \rangle_{\xi > \beta}$" for every $r' \le r{\restriction}(\beta + 1)$ that is incompatible with both $r_0{\restriction}(\beta + 1)$ and $r_1{\restriction}(\beta + 1)$.

Let $q_{i+1} = r^\frown \dot{u}$. It is easy to see that q_{i+1} has the desired properties. Finally, we define $p^{n+1} = q_l$. The fusion has the desired properties. ∎

Note that in the proof of the proposition, it was crucial that we are working with the iteration of Sacks forcing and not with the countable support product.

Let a be a countable subset of ω_2. We can define $\mathbb{S}_a$ as a countable support iteration of Sacks forcing with domain a. Clearly, $\mathbb{S}_a$ is isomorphic to $\mathbb{S}_\delta$ where δ is the order type of a. Note that if $p \in \mathbb{S}_{\omega_2}$ is a continuous condition, then it can be seen as a condition of $\mathbb{S}_{\operatorname{supp}(p)}$. With this remark, it is easy to prove the following:

PROPOSITION 30. *Let $p \in \mathbb{S}_\alpha$ be a continuous condition that has a representation $\{(F_i, n_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and $F : [p] \to [\mathbb{N}]^\omega$ a continuous function. Let α^* be the order type of $\operatorname{supp}(p)$ and $\pi : \operatorname{supp}(p) \to \alpha^*$ be the (unique) order isomorphism. There are $q \in \mathbb{S}_{\alpha^*}$ and a continuous function $H : [q] \to [\mathbb{N}]^\omega$ with the following properties:*

(1) $\operatorname{supp}(q) = \alpha^*$.
(2) *The set $\{(\pi[F_i], n_i, \pi\Sigma_i) \mid i \in \mathbb{N}\}$ is a representation of q (where $\pi\Sigma_i = \{\pi\sigma \mid \sigma \in \Sigma_i\}$).*
(3) *If $\overline{\pi} : (2^\omega)^{\operatorname{supp}(p)} \to (2^\omega)^{\alpha^*}$ denotes the natural homeomorphism induced by π, then $\overline{\pi}{\restriction}[p]$ is a homeomorphism and $F = H\overline{\pi}$.*

We will say that (p, F) and (q, H) are *isomorphic* if the above conditions hold.

THEOREM 31. *In the Sacks model, every almost disjoint family of size ω_2 contains an ℝ-embeddable subfamily of size ω_2.*

Proof. Let $\dot{\mathcal{A}} = \{\dot{A}_\alpha \mid \alpha \in \omega_2\}$ be an $\mathbb{S}_{\omega_2}$-name for an almost disjoint family. For every $\alpha < \omega_2$ we choose a pair (p_α, F_α) with the following properties:

(1) p_α is a continuous condition.
(2) $F_\alpha : [p_\alpha] \to [\mathbb{N}]^\omega$ is a continuous function.
(3) $p_\alpha \Vdash F_\alpha(\bar{r}_{\mathrm{gen}} \restriction \mathrm{supp}(p_\alpha)) = \dot{A}_\alpha$.

By the Δ-system lemma, we can assume that $\{\mathrm{supp}(p_\alpha) \mid \alpha \in \omega_2\}$ forms a Δ-system with root $R \in [\omega_2]^\omega$. Let $\delta \in \omega_2$ be such that $R \subseteq \delta$. By a pruning argument, we may assume that $R = \mathrm{supp}(p_\alpha) \cap \delta$ for every $\alpha < \omega_2$. Since $\mathbb{S}_{\omega_2}$ has the ω_2-chain condition, there is $p \in \mathbb{S}_{\omega_2}$ that forces the set $\{\alpha \mid p_\alpha \in \dot{G}\}$ to have size ω_2 (where $\dot{G}$ is the name of the generic filter). Note that we may assume that $p \in \mathbb{S}_\delta$ (by increasing δ if needed).

Let $G_0 \subseteq \mathbb{S}_\delta$ be a generic filter such that $p \in G_0$. We will now work in $V[G_0]$. Let $W = \{\alpha \mid (p_\alpha \restriction \delta) \in G_0\}$, which has size ω_2 by the nature of p. For every $\alpha \in W$, let p'_α be the $\mathbb{S}_\delta$-name such that $p_\alpha = (p_\alpha \restriction \delta)^\frown p'_\alpha$. Note that we may view each $p'_\alpha[G_0]$ as a condition of $\mathbb{S}_{\omega_2}$ where $\mathrm{supp}(p'_\alpha[G_0]) = \mathrm{supp}(p_\alpha) \setminus \delta$. Let $\bar{r} = \langle r_\beta \rangle_{\beta < \delta}$ be the generic sequence of reals added by G_0. We now define $H_\alpha : [p'_\alpha[G_0]] \to [\mathbb{N}]^\omega$ by $H_\alpha(\langle y_\beta \rangle) = F_\alpha((\bar{r} \restriction \mathrm{supp}(p_\alpha))^\frown \langle y_\beta \rangle)$, which is a continuous function. By a previous lemma, for each $\alpha \in W$ there is a continuous q_α and $\{(F_i^\alpha, m_i^\alpha, \Sigma_i^\alpha) \mid i \in \omega\}$ a representation of q_α with the following properties:

- $q_\alpha \le p'_\alpha[G_0]$.
- $\mathrm{supp}(q_\alpha) = \mathrm{supp}(p'_\alpha[G_0])$.
- $F_0^\alpha = \{\beta_\alpha\}$ where $\beta_\alpha = \min(\mathrm{supp}(p'_\alpha[G_0]))$.
- For every $i \in \omega$ and every $\sigma, \tau \in \Sigma_i^\alpha$ such that $\sigma(\beta_\alpha) \ne \tau(\beta)$, the pair $(H_\alpha[[(q_\alpha)_\sigma]], H_\alpha[[(q_\alpha)_\tau]])$ is decisive.

Let α^* be the order type of $\mathrm{supp}(q_\alpha)$. For each $\alpha \in W$ we find $q_\alpha^* \in \mathbb{S}_{\alpha^*}$ and $H_\alpha^* : [q_\alpha^*] \to [\mathbb{N}]^\omega$ such that (q_α, H_α) and (q_α^*, H_α^*) are isomorphic. We can then find γ, $q^* \in \mathbb{S}_\gamma$ with representation $\{(F_i, m_i, \Sigma_i) \mid i \in \mathbb{N}\}$ and a continuous function $H : [\gamma] \to [\mathbb{N}]^\omega$ such that the set $W' \subseteq W$ consisting of all α such that $\alpha^* = \gamma$, $q_\alpha^* = q^*$ and $H_\alpha^* = H$ has size ω_2.

We first note that for every $i \in \mathbb{N}$ and every $\sigma, \tau \in \Sigma_i$ such that $\sigma(0) \ne \tau(0)$, the pair $(H[[q_\sigma^*]], H[[q_\tau^*]])$ is decisive, and furthermore, $H(\bar{y}) \cap H(\bar{z})$ is finite for every $\bar{y} \in q_\sigma^*$ and $\bar{z} \in q_\tau^*$. Indeed, the pair is decisive since (q_α, H_α) and (q^*, H) are isomorphic; and the second part of the claim follows since any pair of conditions indexed by elements of W' have disjoint supports (and $\mathcal{A}$ is forced to be an almost disjoint family).

Given $s \in q^* \cap 2^{m_i}$ let $B_s = \bigcup\{\bigcup H[[q_\sigma]] \mid \sigma \in \Sigma_i \wedge \sigma(0) = s\}$. Note that if s and t are two different elements of $q^* \cap 2^{n_i}$ then B_s and B_t are almost disjoint. Let T be the downwards closure of the set $\{s \mid s \in q^* \cap 2^{m_i} \wedge i \in \mathbb{N}\}$. Note that if $\alpha \in W'$, then $q_\alpha \Vdash \dot{A}_\alpha \subseteq \bigcap_{s \subseteq \dot{r}_{\beta_\alpha}} B_s$ where $\dot{r}_{\beta_\alpha}$ denotes the name of the β_α-generic real. Applying Lemma 4 to T, we conclude that $\mathcal{A}$ contains an $\mathbb{R}$-embeddable subfamily of size ω_2. ∎

We do not know the answer to the following:

PROBLEM 32. *Let V be a model of the Continuum Hypothesis and let $\mathbb{S}^{\omega_2}$ be the countable support product of ω_2 copies of Sacks forcing. Does $\mathbb{S}^{\omega_2}$ force that "every almost disjoint family of size ω_2 contains an $\mathbb{R}$-embeddable subfamily of size ω_2"?*

The rest of this section is devoted to the study of the controlled version of $\mathbb{R}$-embeddability in the Sacks model. In Theorem 42 we obtain the maximal possible ω_1-controlled embedding property since no family of size $\mathfrak{c}$ can have the $\mathfrak{c}$-controlled $\mathbb{R}$-embedding property by Theorem 6. From now on, $\dot{s}_\alpha$ will mean the canonical $\mathbb{S}_{\alpha+1}$-name for the αth generic real, i.e., the αth Sacks real; and s_α denotes the value of $\dot{s}_\alpha$ in the generic extension.

DEFINITION 33. $e : 2^\omega \to 2^\omega$ is the function satisfying $e(x)(n) = x(2n)$ for every $n \in \omega$.

LEMMA 34. *Let $u \subseteq 2^{<\omega}$ be in $\mathbb{S}$ and $H : [u] \to 2^{\mathbb{N}}$ be a homeomorphism. Let $\alpha < \omega_2$. Whenever $p \in \mathbb{S}_{\omega_2}$ is such that $p{\restriction}\alpha \Vdash p(\alpha) = \check{u}$ and $p{\restriction}\alpha \Vdash \dot{x} \in 2^\omega$ for an $\mathbb{S}_\alpha$-name $\dot{x}$, there is an $\mathbb{S}_\alpha$-name $\dot{q}$ such that $(p{\restriction}\alpha)^\frown\dot{q} \in \mathbb{S}_{\alpha+1}$, $(p{\restriction}\alpha)^\frown\dot{q} \le p{\restriction}(\alpha+1)$ and*

$$(p{\restriction}\alpha)^\frown\dot{q} \Vdash \check{e} \circ \check{H}(\dot{s}_\alpha) = \dot{x}.$$

In particular $(p{\restriction}\alpha)^\frown\dot{q} \Vdash e(\dot{s}_\alpha) = \dot{x}$ if $p(\alpha) = 1_{\mathbb{S}}$.

Proof. Define $\dot{q}$ to be an $\mathbb{S}_\alpha$-name for the set

$$\{y{\restriction}n \mid y \in [u], \; \forall k \in \omega \; H(y)(2k) = x(k), \; n \in \omega\}.$$

This is an $\mathbb{S}_\alpha$-name for a perfect subtree of u and so $(p{\restriction}\alpha)^\frown\dot{q} \in \mathbb{S}_{\alpha+1}$ and $(p{\restriction}\alpha)^\frown\dot{q} \le p{\restriction}(\alpha+1)$. We also have $(p{\restriction}\alpha)^\frown\dot{q} \Vdash \dot{s}_\alpha \in \dot{q}$ and $e(H(z)) = x$ for every $z \in [q]$, so the lemma follows. ∎

LEMMA 35. *Let $\beta < \delta < \omega_2$ and suppose that $p \in \mathbb{S}_{\delta+1} \subseteq \mathbb{S}_{\omega_2}$ and $\dot{F}$ is an $\mathbb{S}_\delta$-name for a continuous function from 2^ω onto 2^ω such that $F^{-1}[\{x\}] \cap [p(\delta)]$ is perfect for every $x \in 2^\omega$ in any forcing extension. There is an $\mathbb{S}_\delta$-name $\dot{r}$ such that $p{\restriction}\delta^\frown\dot{r} \le p$ and*

$$p{\restriction}\delta^\frown\dot{r} \Vdash \dot{F}(\dot{s}_\delta) = \dot{s}_\beta.$$

Proof. Let $\dot{q}$ be an $\mathbb{S}_\delta$-name for the set

$$\bigcap_{u\in\mathbb{G}_\delta} F^{-1}[[u(\beta)]] \cap [p(\delta)] = F^{-1}\Big[\bigcap_{u\in\mathbb{G}_\delta} [u(\beta)]\Big] \cap [p(\delta)] = F^{-1}[\{s_\beta\}] \cap p(\delta).$$

It is a name for a perfect set, as pre-images of singletons under F are perfect in $p(\delta)$ in any forcing extension. Let $\dot{r}$ be a name such that $[r] = q$. So $p{\restriction}\delta^\frown\dot{r} \in \mathbb{S}_{\delta+1}$. Also $q \subseteq [p(\delta)]$, so $p{\restriction}\delta^\frown\dot{r} \le p$. If $z \in q$, then $F(z) = \dot{s}_\beta$. But $p{\restriction}\delta^\frown\dot{r} \Vdash \dot{s}_\delta \in [\dot{r}] = \dot{q}$, so the lemma follows. ∎

DEFINITION 36. $c_1 : \mathbb{S} \to 2^\omega$ is the following coding of perfect subtrees of $2^{<\omega}$ by the reals. Let $\tau : \mathbb{N} \to 2^{<\omega}$ be any fixed bijection. Then given $p \in \mathbb{S}$ we define $c_1(p)(n) = 1$ if and only if $\tau(n) \in p$. Moreover, c_2 will denote the decoding function, i.e., for $x \in 2^\omega$,

$$c_2(x) = \{\tau(n) \mid x(n) = 1, \, n \in \omega\}.$$

DEFINITION 37. Let $\{U_n \mid n \in \mathbb{N}\}$ be a fixed bijective enumeration of all clopen subsets of 2^ω. Suppose that $p \in \mathbb{S}$. Define $F_p : 2^\omega \to 2^\omega$ as follows: First by recursion define a strictly increasing sequence $(n_i)_{i \in \mathbb{N}}$ such that n_0 is minimal satisfying $U_{n_0} \cap [p] \neq \emptyset \neq [p] \setminus U_{n_0}$ and both U_{n_0} and $2^\omega \setminus U_{n_0}$ are intervals in the lexicographical order on 2^ω. Given $n_0, \ldots, n_k$ for $k \in \mathbb{N}$ let n_{k+1} be minimal such that $n_{k+1} > n_k$ and the following conditions hold for every $\sigma \in 2^{k+2}$:

(1) $\bigcap_{0 \le i \le k+1} U_{n_i}^{\sigma(i)}$ is an interval in the lexicographical order on 2^ω.
(2) $\bigcap_{0 \le i \le k+1} U_{n_i}^{\sigma(i)} \cap [p] \neq \emptyset$.
(3) $\mathrm{diam}(\bigcap_{0 \le i \le k+1} U_{n_i}^{\sigma(i)} \cap [p]) \le (2/3)^{k+1}$.

Here $U_n^1 = U_n$ and $U_n^0 = 2^\omega \setminus U_n$ for $n \in \mathbb{N}$. Finally, for $x \in 2^\omega$ and $i \in \omega$ we define

$$F_p(x)(i) = \chi_{U_{n_{2i}}}(x).$$

LEMMA 38. *Let $p \in \mathbb{S}$. Then $F_p^{-1}[\{x\}] \cap [p]$ is perfect for any $x \in 2^\omega$ in any forcing extension.*

Proof. The conditions (1)–(3) of Definition 37 guarantee the property in the statement of the lemma, and are preserved by any forcing. ∎

LEMMA 39. *The function $f : 2^\omega \times 2^\omega \to 2^\omega$ defined as*

$$f(x, y) = F_{c_2(x)}(y)$$

is continuous.

Proof. Let $\varepsilon > 0$. Let $\{U_n \mid n \in \mathbb{N}\}$ and $\{U_{n_i} \mid i \in \mathbb{N}\}$ for $p \in \mathbb{S}$ be as in Definition 37. Let $i_0 \in 2\mathbb{N}$ be such that $\sum_{i=i_0}^\infty 1/2^i < \varepsilon/2$. Given $p \in \mathbb{S}$ there is $m \in \omega$ such that if $p, p' \in \mathbb{S}$ are such that $c_1(p){\restriction}m = c_1(p'){\restriction}m$, then the constructions of $\{U_{n_i} \mid i < i_0\}$ for p and for p' agree. It follows that if $x_n \in 2^\omega$ is sufficiently close to $x \in 2^\omega$, then $|F_{c_2(x_n)}(z) - F_{c_2(x)}(z)| < \varepsilon/2$ for all $z \in 2^\omega$ (i.e., $F_{c_2(x_n)}$ converges uniformly to $F_{c_2(x)}$). So

$$|F_{c_2(x)}(y) - F_{c_2(x_n)}(y_n)| = |F_{c_2(x)}(y) - F_{c_2(x)}(y_n) + F_{c_2(x)}(y_n) - F_{c_2(x_n)}(y_n)|$$
$$\le |F_{c_2(x)}(y) - F_{c_2(x)}(y_n)| + |F_{c_2(x)}(y_n) - F_{c_2(x_n)}(y_n)| < \varepsilon$$

if $x, x_n y, y_n \in 2^\omega$, $n \in \mathbb{N}$ and $|y - y_n|$ and $|x - x_n|$ are sufficiently small by the continuity of $F_{c_2(x)}$ and the above-mentioned uniform convergence. ∎

THEOREM 40. *The following statement is true in the Sacks model: Suppose that $\{x_\xi \mid \xi < \omega_2\} \subseteq 2^\omega$ is a set of distinct reals and $\{y_\xi \mid \xi < \omega_2\} \subseteq 2^\omega$. Then there is a continuous $g : 2^\omega \to 2^\omega$ and $X \subseteq \omega_2$ of cardinality ω_1 such that $g(x_\xi) = y_\xi$ for all $\xi \in X$. In fact, there is a ground model continuous $\phi : 2^\omega \times 2^\omega \to 2^\omega$ such that $\phi(x_\xi, s_\delta) = y_\xi$ for all $\xi \in X$ and some $\delta < \omega_2$.*

Proof. As CH holds in all the intermediate models, we may assume (renumbering the pairs (x_θ, y_θ)) that there are conditions $p_\theta \in \mathbb{S}_{\omega_2}$ and $\mathbb{S}_{\omega_2}$-names $\dot{x}_\theta, \dot{y}_\theta$ for x_θ and y_θ respectively where $\theta < \omega_2$ is such that $p_\theta \Vdash \dot{x}_\theta \notin V^{\mathbb{S}_{\theta+1}}$. Using [12, Theorem 6], by passing to a stronger condition than p_θ we may assume that for each $\theta < \omega_2$ there is a continuous $h_\theta : 2^\omega \to 2^\omega$ such that $p_\theta \Vdash \check{h}_\theta(\dot{x}_\theta) = \dot{s}_\theta$, for all $\theta < \omega_2$ (as no ground model function h can satisfy $h(s_\theta) = x_\theta$ since $x_\theta \notin V^{\mathbb{S}_{\theta+1}}$). Using the CH in the ground model we can apply the stationary Δ-system lemma ([3]) for countable sets and obtain a stationary $A \subseteq \{\alpha \in \omega_2 : cf(\alpha) = \omega_1\}$ such that $\{\text{supp}(p_\theta) \mid \theta \in A\}$ forms a Δ-system with root $\Delta \subseteq \omega_2$ and all the conditions agree on Δ.

Next the pressing down lemma yields a stationary $A' \subseteq A$ such that there is $\alpha < \omega_2$ with $supp(p_\theta) \cap \theta \subseteq \alpha$ for all $\theta \in A'$. We will work for the rest of the proof in $V^{\mathbb{S}_\alpha}$, which will be treated as the ground model. By passing to a subset of A' of cardinality ω_2 we may assume that

$$(1) \qquad\qquad p_\theta \Vdash \check{h}(\dot{x}_\theta) = \dot{s}_\theta$$

for a fixed continuous $h : 2^\omega \to 2^\omega$ and all $\theta \in A'$ and $p_\theta(\theta)$ is a fixed perfect tree $u \subseteq 2^{<\omega}$ and the supports of p_θs for $\theta \in A'$ are pairwise disjoint and $\min(\text{supp}(p_\theta)) = \theta$ for all $\theta \in A'$. Also fix a homeomorphism $H : [u] \to 2^\omega$. Recursively construct a strictly increasing $(\theta_\xi \mid \xi < \omega_1) \subseteq A'$ such that

$$\text{supp}(p_{\theta_\xi})) \subseteq [\theta_\xi, \theta_{\xi+1})$$

and $\dot{y}_\xi$ is an $\mathbb{S}_{\theta_{\xi+1}}$-name for all $\xi < \omega_1$. Let $\delta < \omega_2$ be $\sup\{\theta_\xi \mid \xi < \omega_1\}$.

We will work with the iteration $\mathbb{S}_{\delta+1}$. In the model $V^{\mathbb{S}_{\delta+1}}$ the function g is defined by

$$g(x) = e \circ f(e \circ H(h(x)), s_\delta),$$

where f is as in Lemma 39. We will prove that there are uncountably many $\xi < \omega_1$ such that $g(x_{\theta_\xi}) = y_{\theta_\xi}$. By (1) it is enough to prove that given

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$p \in \mathbb{S}_{\delta+1}$ and $\eta < \omega_1$ there are $p' \leq p$ with $p' \in \mathbb{S}_{\delta+1}$ and $\eta < \xi < \omega_1$ such that

$$(2) \qquad p' \Vdash \dot{f}(e \circ H(\dot{s}_{\theta_\xi}), \dot{s}_\delta) = \dot{s}_{\theta_{\xi+1}}, \qquad e(\dot{s}_{\theta_{\xi+1}}) = \dot{y}_{\theta_\xi}.$$

Given $\eta < \omega_1$ let $\eta < \xi < \omega_1$ be such that the support of $p{\upharpoonright}\delta$ is included in θ_ξ, so we can assume that $p{\upharpoonright}\delta \in \mathbb{S}_{\theta_\xi}$ and so $p(\delta)$ is an $\mathbb{S}_{\theta_\xi}$-name. As $\mathrm{supp}(p_{\theta_\xi}) \subseteq [\theta_\xi, \theta_{\xi+1})$, the conditions p and p_{θ_ξ} are compatible. Let $p'' \in \mathbb{S}_{\delta+1}$ be obtained from p by replacing $1_\mathbb{S}$ by $p_{\theta_\xi}(\alpha)$ on any $\alpha \in [\theta_\xi, \theta_{\xi+1})$ so that $p'' \leq p, p_{\theta_\xi}$ and $p''(\theta_\xi) = u$. Now to obtain the desired $p' \leq p''$ satisfying (2) we will modify p'' on $\theta_\xi, \theta_{\xi+1}$ and δ using Lemmas 34 and 35.

By Lemma 34 there is an $\mathbb{S}_{\theta_\xi}$-name $\dot{q}$ such that $(p''{\upharpoonright}\theta_\xi)^\frown \dot{q} \in \mathbb{S}_{\theta_\xi+1}$, $(p''{\upharpoonright}\theta_\xi)^\frown \dot{q} \leq p''{\upharpoonright}(\theta_\xi + 1)$ and

$$(3) \qquad (p''{\upharpoonright}\theta_\xi)^\frown \dot{q} \Vdash \check{e} \circ \check{H}(\dot{s}_{\theta_\xi}) = \check{c}_1(p(\delta)).$$

Since $p''(\theta_{\xi+1}) = 1_\mathbb{S}$ and $\dot{y}_{\theta_\xi}$ is an $\mathbb{S}_{\theta_{\xi+1}}$-name by the last part of Lemma 34 there is an $\mathbb{S}_{\theta_{\xi+1}}$-name $\dot{o}$ such that $(p''{\upharpoonright}\theta_{\xi+1})^\frown \dot{o} \in \mathbb{S}_{\theta_{\xi+1}}$, $(p''{\upharpoonright}\theta_{\xi+1})^\frown \dot{o} \leq p''{\upharpoonright}(\theta_{\xi+1})$ and

$$(4) \qquad (p''{\upharpoonright}\theta_{\xi+1})^\frown \dot{o} \Vdash \check{e}(\dot{s}_{\theta_{\xi+1}}) = \dot{y}_{\theta_\xi}.$$

In $V^{\mathbb{S}_{\theta_\xi}}$ consider the continuous function $F_{p(\delta)}$ as in Definition 37. Apply Lemma 35 (its hypothesis is satisfied by Lemma 38) and find an $\mathbb{S}_\delta$-name $\dot{r}$ such that $(p''{\upharpoonright}\delta)^\frown \dot{r} \leq p''$ and

$$(5) \qquad (p''{\upharpoonright}\delta)^\frown \dot{r} \Vdash \dot{F}_{p(\delta)}(\dot{s}_\delta) = \dot{s}_{\theta_{\xi+1}}.$$

Define $p' \leq p$ in $\mathbb{S}_{\delta+1}$ by replacing in p''

- u by $\dot{q}$ on the θ_ξth coordinate,
- 1 by $\dot{o}$ on the $\theta_{\xi+1}$th coordinate,
- $p(\delta)$ by $\dot{r}$ on the δth coordinate.

It follows that $p' \in \mathbb{S}_\delta$, $p' \leq p'' \leq p$ and $p'{\upharpoonright}(\theta_\xi + 1) \leq (p''{\upharpoonright}\theta_\xi)^\frown \dot{q}$, $p'{\upharpoonright}(\theta_{\xi+1} + 1) \leq (p''{\upharpoonright}\theta_{\xi+1})^\frown \dot{o}$ and $p'{\upharpoonright}(\delta + 1) \leq (p''{\upharpoonright}\delta)^\frown \dot{r}$.

Note that (5) and (3) give

$$p' \Vdash \dot{F}_{c_2(\check{e}\circ\check{H}(s_{\theta_\xi}))}(\dot{s}_\delta) = \dot{F}_{p(\delta)}(\dot{s}_\delta) = \dot{s}_{\theta_{\xi+1}}.$$

which together with (4) gives the required (2). ∎

REMARK 41. It is proved in [12] that under the hypothesis of Proposition 40 there is a continuous $g : 2^\mathbb{N} \to 2^\mathbb{N}$ and either there is $X \subseteq \omega_2$ of cardinality ω_2 such that $g(x_\xi) = y_\xi$, or $g(y_\xi) = x_\xi$ for every $\xi \in X$. Note that if $x_\xi = s_\xi$ and $y_\xi = s_{\xi+1}$, where s_ξ denotes the ξth Sacks real for $\xi < \omega_2$, then there is no continuous $g : 2^\mathbb{N} \to 2^\mathbb{N}$ such that $g(x_\xi) = y_\xi$ for ω_2-many $\xi < \omega_2$. This follows from the fact that any continuous function is coded in some intermediate model.

THEOREM 42. *In the Sacks model every almost disjoint family of cardinality ω_2 has the ω_1-controlled embedding property.*

Proof. Work in the Sacks model. Let $\mathcal{A}$ be any almost disjoint family of cardinality $2^\omega = \omega_2$ and $\phi : \mathcal{A} \to 2^\omega$ any function. By Theorem 31 and Lemma 2 and Remark 3 there is a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ of cardinality ω_2 and a function $f : \mathcal{A}' \to 2^\omega$ such that the limits $x_A = \lim_{n \in A} f(n)$ exist for each $A \in \mathcal{A}'$ and are different for distinct $A \in \mathcal{A}'$. By Theorem 40 there is a subfamily $\mathcal{B} \subseteq \mathcal{A}'$ of cardinality ω_1 and a continuous $g : 2^\omega \to 2^\omega$ such that $g(x_A) = \phi(A)$ for all $A \in \mathcal{B}$. By the continuity of g we have $\phi(A) = g(x_A) = \lim_{n \in A} g(f(n))$ for all $A \in \mathcal{B}$. So $f' : \mathbb{N} \to 2^\omega$ given by $g \circ f$ witnesses the ω_1-controlled embedding property for $\mathcal{A}$ and ϕ. ∎

6. An application: Abelian subalgebras of Akemann–Doner C*-algebras. The application of our combinatorial results from the previous sections presented here is related to non-commutative C*-algebras defined by Akemann and Doner [1] with the help of almost disjoint families. Let us recall these constructions. In this section $\mathbb{C}$ will stand for the field of complex numbers. We consider the C*-algebra M_2 of all complex 2×2 matrices with the usual linear algebra operations and with the operator norm, that is,

$$\|M\| = \sup\{\|Mv\|_2 : \|v\|_2 = 1, \, v \in \mathbb{C}^2\},$$

where $\|(v_1, v_2)\|_2 = \sqrt{|v_1|^2 + |v_2|^2}$ for $(v_1, v_2) \in \mathbb{C}^2$. We denote by $\ell_\infty(M_2)$ the C*-algebra of all norm bounded sequences from M_2 with the supremum norm and the coordinatewise operations. Let $c_0(M_2)$ denote the C*-subalgebra of $\ell_\infty(M_2)$ consisting of sequences of matrices whose norms converge to zero.

For $\theta \in [0, 2\pi)$ define a 2×2 complex matrix of a rank one projection by

$$p_\theta = \begin{bmatrix} \sin^2 \theta & \sin \theta \cos \theta \\ \sin \theta \cos \theta & \cos^2 \theta \end{bmatrix}.$$

Given $A \subseteq \mathbb{N}$ and $\theta \in [0, 2\pi)$ define $P_{A,\theta} \in \ell_\infty(M_2)$ by

$$P_{A,\theta}(n) = \begin{cases} 0, & n \notin A, \\ p_\theta, & n \in A. \end{cases}$$

Given an almost disjoint family $\mathcal{A} \subseteq \wp(\mathbb{N})$ and a function $\phi : \mathcal{A} \to [0, 2\pi)$ the *Akemann–Donner algebra* $AD(\mathcal{A}, \phi)$ is the subalgebra of $\ell_\infty(M_2)$ generated by $c_0(M_2)$ and $\{P_{A, \phi(A)} \mid A \in \mathcal{A}\}$. As the distance between $P_{A,\theta}$ and $P_{A',\theta'}$ is at least 1 for any distinct infinite $A, A' \subseteq \mathbb{N}$ and any (possibly equal) $\theta, \theta' \in [0, 2\pi)$, such algebras are non-separable if $\mathcal{A}$ is uncountable. Clearly if $\mathcal{A}$ is uncountable and $\phi : \mathcal{A} \to [0, 2\pi)$ is constantly equal to θ, then $AD(\mathcal{A}, \phi)$ contains the non-separable commutative C*-algebra isomorphic to the algebrs $C_0(\Psi(\mathcal{A}))$ of all complex valued continuous functions on $\Psi(\mathcal{A})$

vanishing at infinity because $p_\theta^2 = p_\theta = p_\theta^*$ since it is a projection. However, as Akemann and Doner proved under CH, one can choose $\mathcal{A}$ so that for every injective $\phi : \mathcal{A} \to (0, \pi/6)$ the algebra $AD(\mathcal{A}, \phi)$ has no non-separable commutative subalgebra. In [6] the hypothesis of CH was removed by showing that a ZFC Luzin family $\mathcal{A}$ is sufficient for this result of Akemann and Doner. We have the following two lemmas, implicit in [1, 6]:

LEMMA 43. *Suppose that $\mathcal{A}$ is an almost disjoint family and $\phi : \mathcal{A} \to [0, 2\pi)$. If there is $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ and $f : \mathbb{N} \to [0, 2\pi)$ such that $\lim_{n \in B} f(n) = \phi(B)$ for every $B \in \mathcal{B}$, then $AD(\mathcal{A}, \phi)$ contains a commutative C^*-subalgebra of density κ.*

Proof. First define $P_f \in \ell_\infty(M_2)$ by $P_f(n) = p_{f(n)}$. For $B \in \mathcal{B}$ define $R_B \in \ell_\infty(M_2)$ by $R_B(n) = P_f \chi_B(n)$ for $n \in \mathbb{N}$, where χ_B is the characteristic function of B. The hypothesis about f implies that $R_B - P_{B, \phi(B)} \in c_0(M_2)$ for all $B \in \mathcal{B}$ and so R_B is in $AD(\mathcal{A}, \phi)$. The algebra generated by $\{R_B \mid B \in \mathcal{B}\}$ is commutative isomorphic to $C_0(\Psi(\mathcal{B}))$ and of density κ as required. $\blacksquare$

LEMMA 44. *Let $c \in \mathbb{R}$ be such that $\|p_0 - p_\theta\| < 1/4$ for $\theta \in [0, c]$. Suppose that $\mathcal{A}$ is an almost disjoint family and that $\phi : \mathcal{A} \to [0, c]$ is such that for no $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ is there an $f : \mathbb{N} \to [0, c]$ such that $\lim_{n \in B} f(n) = \phi(B)$ for every $B \in \mathcal{B}$. Then $AD(\mathcal{A}, \phi)$ does not contain any commutative C^*-subalgebra of density κ.*

Proof. This is a slight modification of an argument from [1] and modified in [6]. Let $\rho : \{p_\theta \mid \theta \in [0, 1/4]\} \to [0, 1/4]$ be defined by $\rho(p_\theta) = \theta$. Then ρ is a continuous map from a closed subset of the unit ball $\mathbb{B}_1$ of M_2 into $[0, 1/4]$. Use the Tietze extension theorem to find a continuous $\eta : \mathbb{B}_1 \to [0, 1/4]$ which extends ρ.

Suppose that $\mathcal{C}$ is a commutative subalgebra of $AD(\mathcal{A}, \phi)$ whose density is κ. As in [1] and [6], in a slightly different language, it follows from simultaneous diagonalization of commuting matrices that there are rank one projections $q(n) \in M_2$ such that $a(n)q(n) = q(n)a(n)$ for each $n \in \mathbb{N}$ and each $a \in \mathcal{C}$ and we may assume that $\|q(n) - p_0\|^2 \leq 1/2$ by [6, (2.1)]. It is easy to note that for each $a \in AD(\mathcal{A}, \phi)$ the limit $\lim_{n \in A} a(n)$ exists and is a multiple of $p_{\phi(A)}$. The density of $\mathcal{C}$ being κ means that there is $\mathcal{B} \subseteq \mathcal{A}$ of cardinality κ such that for each $B \in \mathcal{B}$ there is $a_B \in \mathcal{C}$ such that the limit $\lim_{n \in B} a_B(n)$ exists and is equal to $z_B p_{\phi(B)}$ for a non-zero complex number z_B. By the compactness of the unit ball in M_2, for each infinite $B' \subseteq B$ there is an infinite $B'' \subseteq B'$ such that $\lim_{n \in B''} q(n) = q'$ exists, and so it has to be a rank one projection which commutes with $\lim_{n \in A} a_B(n)$ which is $z_B p_{\phi(B)}$, so $p_{\phi(B)}$ and q' commute; but $\|q' - p_{\phi(B)}\| \leq 1/\sqrt{2} + 1/4 < 1$ and so $q' = p_{\phi(B)}$ (see e.g. [6, Lemma 3]). This means that actually $\lim_{n \in B} q(n)$ exists and is equal to $p_{\phi(B)}$ for each $B \in \mathcal{B}$. By the continuity of η we

have $\lim_{n \in B} \eta(q(n)) = \eta(p_{\phi(B)}) = \phi(B)$. Define $f : \mathbb{N} \to [0, 1/4]$ by $f(n) = \eta(q(n))$. So $\lim_{n \in B} f(n) = \phi(B)$ for every $B \in \mathcal{B}$, contradicting the hypothesis on $\mathcal{A}$. ∎

As corollaries we obtain:

THEOREM 45. *In* ZFC, *for every almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ there is $\phi : \mathcal{A} \to [0, 2\pi)$ such that the Akemann–Doner C^*-algebra $AD(\mathcal{A}, \phi)$ of density $\mathfrak{c}$ has no commutative subalgebras of density $\mathfrak{c}$.*

Proof. Fix an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$. By Theorem 6 there is $\phi : \mathcal{A} \to \mathbb{R}$ such that for no $\mathcal{B} \subseteq \mathcal{A}$ of cardinality $\mathfrak{c}$ there is $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n \in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. By applying a continuous injective mapping we may assume that $\mathbb{R}$ is replaced by $[0, c]$, where $c \in \mathbb{R}$ is as in Lemma 44. Now Lemma 44 implies that $AD(\mathcal{A}, \phi)$ has no commutative subalgebras of density $\mathfrak{c}$. ∎

THEOREM 46. *It is consistent that every Akemann–Doner algebra of density $\mathfrak{c}$ contains a nonseparable commutative subalgebra.*

Proof. We claim that the above statement holds in the Sacks model. By Theorem 42, given any almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ and a function $\phi : \mathcal{A} \to \mathbb{R}$ there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\lim_{n \in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. It follows from Lemma 43 that $AD(\mathcal{A}, \phi)$ contains a non-separable commutative subalgebra. ∎

THEOREM 47. *Let $c \in \mathbb{R}$ be such that $\|p_0 - p_\theta\| < 1/4$ for $\theta \in [0, c]$. It is consistent with the negation of* CH *that for every almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ there is $\phi : \mathcal{A} \to [0, c]$ such that the Akemann–Doner algebra $AD(\mathcal{A}, \phi)$ of density $\mathfrak{c}$ has no non-separable commutative subalgebra.*

Proof. Work in the Cohen model. Fix an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$. By Theorem 18 is there an $\phi : \mathcal{A} \to \mathbb{R}$ such that for no uncountable $\mathcal{B} \subseteq \mathcal{A}$ is there an $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n \in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. By applying a continuous mapping we may assume that $\mathbb{R}$ is replaced by $[0, c]$, where $c \in \mathbb{R}$ is as in Lemma 44. Now Lemma 44 implies that $AD(\mathcal{A}, \phi)$ has no commutative non-separable subalgebras. ∎

THEOREM 48. *Let $c \in \mathbb{R}$ be such that $\|p_0 - p_\theta\| < 1/4$ for $\theta \in [0, c]$. It is consistent with the negation of* CH *that there is an almost disjoint family $\mathcal{A}$ of cardinality $\mathfrak{c}$ such that for every $\phi : \mathcal{A} \to [0, c]$ the Akemann–Doner algebra $AD(\mathcal{A}, \phi)$ of density $\mathfrak{c}$ has no non-separable commutative subalgebra.*

Proof. Work in the Cohen model. Let $\mathcal{A}$ be an almost disjoint family of cardinality $\mathfrak{c}$ from Theorem 17. By Theorem 17 for no $\phi : \mathcal{A} \to \mathbb{R}$ is there an uncountable $\mathcal{B} \subseteq \mathcal{A}$ and $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n \in B} f(n) = \phi(B)$ for all $B \in \mathcal{B}$. Now Lemma 44 implies that $AD(\mathcal{A}, \phi)$ has no commutative non-separable subalgebras. ∎

These results complete an earlier result of [6] that there is in ZFC an almost disjoint family $\mathcal{A}$ (any inseparable family) such that for every $\phi : \mathcal{A} \to [0, c)$ the Akemann–Doner algebra of density ω_1 has no non-separable commutative subalgebra.

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