FORCING PROPERTIES OF BOOLEAN ALGEBRAS OF TYPE $\mathcal{P}(\omega)/\mathcal{I}$

G. CAMPERO-ARENA, O. GUZMÁN, M. HRUŠÁK,
AND D. MEZA-ALCÁNTARA

Abstract. We study forcing properties of the Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$, where $\mathcal{I}$ is a Borel ideal on ω. We show (Theorem 2.9) that (under a large cardinal hypothesis) $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals if and only if it has a dense σ-closed subset. For analytic P-ideals $\mathcal{I}$ we show (Theorem 3.3) that either $\mathcal{P}(\omega)/\mathcal{I}$ is ω^ω-bounding or it is not proper. We also investigate the existence of completely separable $\mathcal{I}$-MAD families.

1. Introduction

In recent years, a large body of work has been done on the structure of definable (Borel, analytic, co-analytic, ...) ideals on a countable set and their corresponding quotients (Brendle-Mejia [9], Farah [16], Fremlin [22], Hrušák-Zapletal [30], Hrušák [25, 26], Louveau-Veličković [35], Solecki [45] [46], Solecki-Todorčević [47], He-Hrušák-Rojas-Solecki [23], Chodounský-Guzmán-Hrušák [12]).

We contribute to this line of research by studying the quotient Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$ for definable ideals $\mathcal{I}$ as forcing notions. We build on work done by Farah in [17, 18, 16]; by Just and Krawczyk in [31]; by Balcar, Hernández and Hrušák in [5]; by Hrušák and Zapletal in [30]; by Kurilić and Todorčević in [37], [38], [39], [40]; by Steprāns in [48].

First let us briefly consider quotients $\mathcal{P}(\omega)/\mathcal{I}$ without any definability restrictions. It was pointed out to us by Alan Dow, that every

Date: February 12, 2023.

First and fourth authors thank support from UNAM-PAPIIT IN119320 grant. Second author was supported by a PAPIIT grant IA102222. Third author was supported by a CONACYT grant A1-S-16164 and grant IN101323. Fourth author was supported by a UNAM-DGAPA-PASPA grant.

2020 *Mathematics Subject Classification*: 03G05, 03E40, 03E17, 03E15.

Key words and phrases: Boolean algebras, Borel ideals, Forcing, Quotiens over definable ideals.

forcing of size at most $\mathfrak{c}$ is forcing equivalent to a quotient $\mathcal{P}(\omega)/\mathcal{I}$ for some ideal $\mathcal{I}$ on ω.

Theorem 1.1 (Dow)**.** *For every partial order $\mathbb{P}$ of size at most $\mathfrak{c}$ there is an ideal $\mathcal{I}$ on ω such that the algebra $\mathcal{P}(\omega)/\mathcal{I}$ can be densely embedded into the completion $RO(\mathbb{P})$ of $\mathbb{P}$.*

We present its simple proof for the sake of completeness. First we recall some well-known facts. Recall that a family $\mathcal{J}$ of elements of a Boolean algebra $\mathbb{B}$ is *independent* if $\bigwedge_{a\in E} a \wedge \bigwedge_{a\in E'} -a \neq \mathbf{0}$ for any pair of disjoint finite subsets E, E' of $\mathcal{J}$. An old theorem of Fichtenholz and Kantorovich [21] states that *there is an independent family of size $\mathfrak{c}$ in $\mathcal{P}(\omega)/\mathsf{fin}$*. It is a well known fact that the subalgebra $\mathbb{C}$ of any Boolean algebra $\mathbb{B}$ generated by an independent family $\mathcal{J} \subseteq \mathbb{B}$ is *free*, that is *any function f from $\mathcal{J}$ to any Boolean algebra $\mathbb{A}$ has a (unique) extension to a homomorphism $F : \mathbb{C} \to \mathbb{A}$.*

Another known fact is Sikorski's extension theorem (see [44]): *given a subalgebra $\mathbb{C}$ of a Boolean algebra $\mathbb{B}$ and a complete Boolean algebra $\mathbb{A}$, any homomorphism $F : \mathbb{C} \to \mathbb{A}$ has an extension to a homomorphism $\bar{F} : \mathbb{B} \to \mathbb{A}$.*

Proof. Given a partial order $\mathbb{P}$ of size at most $\mathfrak{c}$, let $\mathcal{J}$ be an independent family of size $\mathfrak{c}$ in $\mathcal{P}(\omega)/\mathsf{fin}$ and let $f : \mathcal{J} \to \mathbb{P}$ be any surjection. Now, according to the observations made above there is a homomorphism $F : \mathcal{P}(\omega)/\mathsf{fin} \to RO(\mathbb{P})$ extending f. Let $\mathcal{I} = F^{-1}(\mathbf{0})$. Then $\mathbb{B} = rng(F)$ is a dense subalgebra of $RO(\mathbb{P})$ containing $\mathbb{P}$, and $\mathbb{B}$ is isomorphic to $\mathcal{P}(\omega)/\mathcal{I}$. □

In fact, this proof has the following immediate corollary.

Corollary 1.2. *For every complete Boolean algebra $\mathbb{B}$ of size at most $\mathfrak{c}$ there is an ideal $\mathcal{I}$ on ω such that the algebra $\mathcal{P}(\omega)/\mathcal{I}$ is isomorphic to $\mathbb{B}$.*

Proof. Apply the previous proof to $\mathbb{P} = \mathbb{B}$. □

Assuming the Continuum Hypothesis, Louveau [34] completely characterized which Boolean algebras are isomorphic to algebras of the type $\mathcal{P}(\omega)/\mathcal{I}$. Recall that a Boolean algebra $\mathbb{B}$ is *weakly σ-complete* if it contains no (ω, ω)-gaps, i.e if given two countable subsets A, B of $\mathbb{B}$ such that $a \wedge b = \mathbf{0}$ for every $a \in A$ and $b \in B$, there is a $c \in \mathbb{B}$ such that C *separates* A and B, that is, $a \leq c$ for all $a \in A$ and $c \wedge b = \mathbf{0}$ for all $b \in B$. It is easy to see that every $\mathcal{P}(\omega)/\mathcal{I}$ is weakly σ-distributive. On the other hand, Louveau also proved the following result.

Theorem 1.3 ([34] Assuming CH)**.** *For every weakly σ-complete Boolean algebra $\mathbb{B}$ of size at most $\mathfrak{c}$ there is an ideal $\mathcal{I}$ on ω such that the algebra $\mathcal{P}(\omega)/\mathcal{I}$ is isomorphic to $\mathbb{B}$.*

This theorem is neither true in ZFC [13] nor characterizes CH [15].

The situation is quite different if we restrict our attention to definable ideals and their quotients. The first notable difference is that no c.c.c. forcing can be represented as $\mathcal{P}(\omega)/\mathcal{I}$ for a definable ideal $\mathcal{I}$. We prove this and that no such quotient can be complete from the following theorem, which characterizes ideals with the Baire property.

Theorem 1.4 (Jalali-Naini–Talagrand, see [1])**.** *An ideal $\mathcal{I}$ satisfes the Baire Property if and only if there is a partition $\{I_k : k \in \omega\}$ of ω in finite pieces, such that for every infinite $A \subseteq \omega$, $\bigcup_{k\in A} I_k$ is $\mathcal{I}$-positive.*

Hence, by taking $\{A_\alpha : \alpha < \mathfrak{c}\}$ an almost disjoint family of subsets of ω, $\mathcal{A} = \{[\bigcup_{k\in A_\alpha} I_k] : \alpha < \mathfrak{c}\}$ is an antichain in $\mathcal{P}(\omega)/\mathcal{I}$ of size $\mathfrak{c}$. Moreover, by considering the family $F = \{\sup C : C \subseteq \mathcal{A}\}$ we get a subset of the completion of $\mathcal{P}(\omega)/\mathcal{I}$ of size $2^{\mathfrak{c}}$. Hence $\mathcal{P}(\omega)/\mathcal{I}$ cannot contain F since its size is $\mathfrak{c}$.

Farah asked in [16] if there are infinitely many analytic P-ideals (arbitrary analytic, definable ideals) whose quotients are provably in ZFC pairwise non-isomorphic. Oliver [42] proved that there are $\mathfrak{c}$-many pairwise non-isomorphic quotients on Borel ideals, however, his method does not seems to produce quotients which are distinct as forcing notions. On the other hand, Steprāns [48], and Hrušák and Zapletal [30] have shown that there are many distinct, and even forcing non-equivalent, definable quotients $\mathcal{P}(\omega)/\mathcal{I}$. However, most of these are co-analytic or more complex. This prompted the following question:

Question 1.5 ([25])**.** *Are there uncountably many forcing non-equivalent quotients $\mathcal{P}(\omega)/\mathcal{I}$ for Borel ideals $\mathcal{I}$?*

In fact, only a handful of quotients over Borel ideals have been studied as forcing notions:

- $\mathcal{P}(\omega)/\mathsf{fin}$ is the prototypical example, as seen in [4, 3].
- If $\mathcal{I}$ is F_σ, then $\mathcal{P}(\omega)/\mathcal{I}$ is σ-closed by a theorem of Just-Krawczyk [31]. In fact, under CH, $\mathcal{P}(\omega)/\mathcal{I}$ is isomorphic to $\mathcal{P}(\omega)/\mathsf{fin}$ for every F_σ ideal $\mathcal{I}$. On the other hand, this consistently fails, see [9].
- The forcing $\mathcal{P}(\omega\times\omega)/\mathsf{fin}\times\mathsf{fin}$ was considered in [49], [8], [29] and [24], and it was shown in [24], that even though it is also σ-closed, it is consistently not forcing equivalent with $\mathcal{P}(\omega)/\mathsf{fin}$.

- The quotient $\mathcal{P}(\mathbb{Q})/\mathsf{nwd}$ was considered in [5],[30], [19] and [10], and it is known to be forcing equivalent to $\mathbb{C} * \dot{\mathbb{P}}$, where $\mathbb{C}$ is the Cohen forcing and $\dot{\mathbb{P}}$ is a $\mathbb{C}$-name for a σ-closed forcing.
- Kurilić and Todorčević in [37] studied the quotient $\mathcal{P}(\mathbb{Q})/L$-$\mathsf{scatt}$, where L-scatt is the co-analytic ideal of scattered linearly ordered subsets of the rationals $\mathbb{Q}$, and showed that it is forcing equivalent to $\mathbb{S} * \dot{\mathbb{P}}$, where $\mathbb{S}$ is the Sacks forcing and $\dot{\mathbb{P}}$ is an $\mathbb{S}$-name for a σ-closed forcing.
- This ideal is not to be confused with T-scatt, the co-analytic ideal of topologically scattered subsets of the rationals, i.e. the subsets of $\mathbb{Q}$ all of whose subsets have an isolated point. The quotient $\mathcal{P}(\mathbb{Q})/T$-$\mathsf{scatt}$ is, in fact, forcing equivalent with $\mathbb{M}$, Miller's *rational perfect set forcing* [41]. By Lemma 2.4 in [11], there is a dense embedding from $\mathbb{M} = \{A \subseteq \mathbb{Q} : A \text{ is crowded and closed}\}$ in $\mathcal{P}(\mathbb{Q})/T$-$\mathsf{scatt}$.
- Three quotients over analytic P-ideals have been identified:
  a) [17] $\mathcal{P}(\omega)/\mathcal{Z} \simeq \mathcal{P}(\omega)/\mathsf{fin} * \mathbb{B}(\mathfrak{c})$, where $\mathcal{Z}$ is the ideal of asymptotic density zero subsets of ω,
  $$\mathcal{Z} = \{A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0\},$$
  and $\mathbb{B}(\kappa)$ denotes the measure algebra for adding κ-many random reals,
  b) [30] $\mathcal{P}(\omega)/tr(\mathcal{N}) \simeq \mathbb{B}(\omega) * \dot{\mathbb{P}}$ for some forcing $\mathbb{P}$ not adding reals, and
  c) [30] the non-proper $\mathcal{P}(\omega)/\mathcal{J}$, with $\mathcal{J}$ an analytic P-ideal, described in Example 3.12 of [30].
  We will say more about non-proper quotient forcings in Section 4.
- Finally, the Mathias-Prikry forcing for destroying an F_σ P-ideal gives rise to a Borel quotient (see [30]).

Here, we continue the study of forcings of the type $\mathcal{P}(\omega)/\mathcal{I}$. In particular, we provide answers to questions from [29] and [30].

2. ω-distributive quotients

In this section we answer a question of Hrušák and Verner from [29], by showing that every definable quotient not adding sequences of ordinals has a σ-closed dense subset. This puts another restriction on the class of forcings which can be represented as definable quotients as, for instance, Baumgartner's forcing shooting club through a stationary set [2] cannot be represented in this way.

We say that a partially ordered set $\mathbb{P}$ is *ω-distributive* if for every sequence $\{U_n : n < \omega\}$ of dense open sets in $\mathbb{P}$, $\bigcap_n U_n$ is dense. We say that $\mathbb{P}$ is *σ-closed* if for every decreasing sequence $p_n \in \mathbb{P}$, there exists $p \in \mathbb{P}$ such that $p \leq p_n$ for all n. We say that $\mathbb{P}$ is *separative* if for every $p, q \in \mathbb{P}$, such that $p \not\leq q$, there is $r \leq p$ incompatible with q.

The following lemmas hold for $\mathbb{P} = \mathcal{P}(\omega)/\mathcal{I}$, with $\mathcal{I}$ a Borel ideal.

Lemma 2.1. *Let $\mathbb{P}$ be a separative ordered set with size at most $\mathfrak{c}$. Then the following conditions are equivalent.*

(1) $\mathbb{P}$ is ω-distributive
(2) $\mathbb{P}$ does not add sequences of ordinals
(3) $\mathbb{P}$ does not add reals
(4) $\mathbb{P}$ does not add sequences of reals

Proof. To see that (1) implies (2), let τ be a $\mathbb{P}$-name for a sequence of ordinals, and for each n, let U_n be an open dense set of conditions deciding $\tau(n)$. By our assumption, $\bigcap_n U_n$ is dense, then for every condition p there are $q \leq p$, and a sequence of ordinals f in V such that $q \Vdash \tau = f$.

Clearly, (2) implies (3).

By a suitable (definable) bijection, sequences of reals are codified by reals, which proves that (3) implies (4).

We now prove that if $\mathbb{P}$ is not ω-distributive, then $\mathbb{P}$ adds a new real. Let $\{U_n : n < \omega\}$ be a sequence of dense open sets in $\mathbb{P}$, such that $\bigcap_n U_n$ is not dense. Take $r \in \mathbb{P}$ such that $\{s \in \mathbb{P} : s \leq r\} \cap \bigcap_n U_n = \emptyset$. For each n, take a maximal antichain A_n of conditions in U_n below r. Let G be a $\mathbb{P}$-generic filter with $r \in G$. In $V[G]$, define $F = \{(n, p) : p \in G \cap A_n\}$. Clearly $r \Vdash F \in (\mathbb{P})^\omega$. Note that F can be seen as a sequence of reals. Also, for all $f \in (\mathbb{P})^\omega \cap V$, $r \Vdash F \neq f$, because otherwise, if some $s \leq r$ forces that $F = f$, then for all n, by separativity, there is $p_n \in A_n$ (namely $p_n = f(n)$) such that $s \leq p_n$, and consequently $s \in U_n$ for all n, which is a contradiction. □

In order to investigate the relation between being ω-distributive and containing a dense σ-closed set, the next lemma shows one implication is easy.

Lemma 2.2. *If there exists a σ-closed dense set $D \subseteq \mathbb{P}$, then $\mathbb{P}$ is ω-distributive.*

Proof. It follows from a classical Baire-category argument. Let $\{U_n : n < \omega\}$ be a family of open dense subsets of $\mathbb{P}$, and $A \in \mathbb{P}$. Then, taking $A \geq A_0 \geq D_0 \geq A_1 \geq D_1 \ldots$, with $A_n \in U_n$ and $D_n \in D$, by σ-closedness, there is $C \in D$ such that $C \leq D_n$ for all n. Hence, $C \leq A$ and $C \in \bigcap U_n$. □

To obtain the implication in the other direction, we need to work a bit more.

Definition 2.3 ([3])**.** Let $\mathbb{P}$ be a partially ordered set. We say that $\mathbb{P}$ has the *Base Tree property* (in short BT-property) if there is a dense subset D of P such that

- D is atomless, i.e. for every $d \in D$ there are incompatible $d_1, d_2 \in D$ below d,
- D is σ-closed, and
- $|D| \leq \mathfrak{c}$.

The following lemma relaxes conditions for a partially ordered set to have the BT-property.

Lemma 2.4. *If $\mathbb{P}$ is an atomless partially ordered set such that*

(1) $\mathbb{P}$ contains a dense subset D with $|D| \leq \mathfrak{c}$, and

(2) $\mathbb{P}$ contains a dense σ-closed subset E

then $\mathbb{P}$ has the BT-property.

Proof. Let κ be a big enough cardinal, and M an elementary submodel of $H(\kappa)$ such that

(1) $\mathbb{P}, D, E$ are in M,

(2) $M^\omega \subseteq M$,

(3) $|M| = \mathfrak{c}$, and

(4) $D \subseteq M$.

Let us define $R = M \cap E$. We will prove that R is the dense set that we are looking for. By (3), $|R| \leq \mathfrak{c}$. For a given $p \in \mathbb{P}$, take $d \in D$ with $q \leq p$. By (4), $q \in M$, and since M is an elementary submodel of $H(\kappa)$, there is $r \in R$ such that $r \leq q$, proving that R is dense. Since E is σ-closed, by (2) R is σ-closed. Finally, R is atomless because $\mathbb{P}$ is. $\square$

Corollary 2.2 in [3] claims that a partially ordered set $\mathbb{P}$ has the BT-property if and only if its Boolean completion $RO(\mathbb{P})$ has it. We now change our focus to completions of quotients of type $\mathcal{P}(\omega)/\mathcal{I}$. It is well known that the ordered sets $\mathcal{P}(\omega)/\mathcal{I}$, $RO(\mathcal{P}(\omega)/\mathcal{I})$ (the Boolean completion of $\mathcal{P}(\omega)/\mathcal{I}$), and $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ ordered by the $\mathcal{I}$-almost inclusion defined as

$$A \subseteq_{\mathcal{I}} B \text{ if and only if } A \setminus B \in \mathcal{I},$$

are forcing-equivalent. As usual, $\subseteq^*$ denotes the relation $\subseteq_{\mathsf{fin}}$. We will show that under large cardinal assumptions, $\mathcal{P}(\omega)/\mathcal{I}$ has the BT-property, if $\mathcal{P}(\omega)/\mathcal{I}$ is ω-distributive. For this, we consider the *Banach-Mazur* game played on a partially ordered set $\mathcal{G}(\mathbb{P})$ (see [50]) defined as follows. At step n, players *Empty* and *Non-empty* choose (respectively) p_n and q_n in $\mathbb{P}$ such that $p_0 \geq q_0 \geq p_1 \geq q_1 \geq \dots$. *Non-empty* wins if there is $r \in \mathbb{P}$ such that $r \leq q_n$ for all n. Recall the following result from [50].

Theorem 2.5 (Veličković). *Let $\mathbb{B}$ be a complete Boolean algebra containing a dense subset D with $|D| \leq \mathfrak{c}$. If* Non-empty *has a winning strategy in $\mathcal{G}(\mathbb{B})$, then $\mathbb{B}$ has a dense σ-closed subset.*

Lemma 2.6. *If $\mathcal{P}(\omega)/\mathcal{I}$ is ω-distributive, then* Empty *does not have a winning strategy in $\mathcal{G}(RO(\mathcal{P}(\omega)/\mathcal{I}))$.*

Proof. Let us assume that *Empty* has a winning strategy σ contained in $(\mathbb{P}(\mathcal{P}(\omega)/\mathcal{I}))^{<\omega}$. We may assume that for all $t \in \sigma$ if $|t|$ is even, $t \frown p \in \sigma$, and $t \frown q \in \sigma$, then $p = q$; and if $|t|$ is odd, then $t \frown p \in \sigma$, for all $p \leq t(|t|-1)$. Let us denote $p_0 = \sigma(\emptyset)$. Note that for all n, the family

$$U_n = \{p \in \mathcal{P}(\omega)/\mathcal{I} : (\exists t \in \sigma \cap (\mathcal{P}(\omega)/\mathcal{I})^{2n+1})(t \frown p \in \sigma)\}$$

is open and dense below p_0 (or below a $p_0' \leq p_0$ in $\mathcal{P}(\omega)/\mathcal{I}$, if the reader prefers). If $p \in \bigcap_n U_n$, then we can identify a branch x in σ such that $p \leq x(x \restriction 2n+1)$, contradicting that σ is a winning strategy for *Empty*. Hence $\bigcap_n U_n = \emptyset$, and so, $\mathcal{P}(\omega)/\mathcal{I}$ is not ω-distributive. □

Note that we can obtain a winning strategy for some player in $\mathcal{G}(RO(\mathcal{P}(\omega)/\mathcal{I}))$ from a given winning strategy for the same player in $\mathcal{G}(\mathcal{I}^+)$, and viceversa. Moreover, $\mathcal{G}(\mathcal{I}^+)$ is equivalent to the game $\mathcal{C}(\mathcal{I}^+)$ defined as follows: At step n, players *Empty* and *Non-empty* choose (respectively) A_n and B_n in $\mathcal{I}^+$ such that $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \dots$. *Non-empty* wins if there is $C \in \mathcal{I}^+$ such that $C \subseteq_{\mathcal{I}} B_n$ for all n. We will use this game and the following theorem to prove that $\mathcal{G}(RO(\mathcal{P}(\omega)/\mathcal{I}))$ is determined when $\mathcal{I}$ is a Borel ideal.

Let W be a subset of U^ω. The *game in U with payoff set W*, denoted by $\mathcal{CS}(W)$, is defined as follows: Players I and II alternately choose elements $a_0, b_0, a_1, b_1, \dots$ of U, and Player I is declared the winner if and only if the sequence $(a_0, b_0, a_1, b_1, \dots)$ belongs to W. Our game $\mathcal{C}(\mathcal{I}^+)$ is an example of this type of game, and the following theorem establishes some conditions that render it determined.

Theorem 2.7 (Martin). *(LC) Let U be a set, $A \subseteq U^\omega$ be a Borel set, X a Polish space, $f : A \to X$ a continuous function, and $B \subseteq X$ a universally Baire set. Then the game whose payoff set is $f^{-1}(B)$ is determined.* □

As usual, LC denotes a large cardinal hypothesis. This and other details can be consulted in Zapletal's book [51]. In our case, the payoff set will be co-analytic, and hence, it is a universally Baire set.

Lemma 2.8 (LC). *If $\mathcal{I}$ is a Borel ideal, then $\mathcal{G}(RO(\mathcal{P}(\omega)/\mathcal{I}))$ is determined.*

Proof. We now verify that $\mathcal{C}(\mathcal{I}^+)$ satisfies the hypothesis of Martin's theorem. Consider $U = \mathcal{I}^+$ and define $A = \{C \in U^\omega : (\forall n) C_{n+1} \subseteq C_n\}$. Hence, A is a Borel set. Consider $X = (\mathcal{P}(\omega))^\omega$, which is a Polish space. The identity function from A to X is clearly continuous. Define $B = \{C \in X : (\exists D \in \mathcal{I}^+)(\forall n) D \subseteq_{\mathcal{I}} C(n)\}$. B is an analytic subset of X and $\mathcal{C}(\mathcal{I}^+)$ is the game whose payoff set is $f^{-1}(X \setminus B)$. Note that $X \setminus B$ is co-analytic, and so is universally Baire. By Martin's theorem, $\mathcal{C}(\mathcal{I}^+)$ is determined, and hence, $\mathcal{G}(RO(\mathcal{P}(\omega)/\mathcal{I}))$ is determined. □

Theorem 2.9 (LC). *If $\mathcal{I}$ is a Borel ideal and $\mathcal{P}(\omega)/\mathcal{I}$ is ω-distributive, then $\mathcal{P}(\omega)/\mathcal{I}$ contains a dense σ-closed set.*

Proof. By Lemma 2.8 and Lemma 2.6, *Non-empty* has a winning strategy for $\mathcal{G}(RO(\mathcal{P}(\omega)/\mathcal{I}))$. By Theorem 2.5, $RO(\mathcal{P}(\omega)/\mathcal{I})$ contains a dense σ-closed subset. Note that $RO(\mathcal{P}(\omega)/\mathcal{I})$ is atomless and contains a dense subset of cardinality $\mathfrak{c}$, namely $\mathcal{P}(\omega)/\mathcal{I}$. By Lemma 2.4 and Corollary 2.2 in [3], $\mathcal{P}(\omega)/\mathcal{I}$ has the BT-property, in particular, $\mathcal{P}(\omega)/\mathcal{I}$ contains a dense σ-closed set. □

The conditions listed for $\mathcal{I}$ in Theorem 2.1 are not equivalent to $\mathcal{I}$ being σ-closed. From a σ-closed $\mathcal{I}$ on ω, we can construct an ideal $\mathcal{I}^\omega$ such that $\mathcal{P}(\omega \times \omega)/\mathcal{I}^\omega$ has a σ-closed dense subset, but is not σ-closed. We define $\mathcal{I}^\omega$ as follows. For all $A \subseteq \omega \times \omega$,

$A \in \mathcal{I}^\omega$ if and only if for every n, $\{m : \{(n, m) \in A\}\}$ belongs to $\mathcal{I}$.

For every n, $X_n = [n, \infty) \times \omega$ is $\mathcal{I}^\omega$-positive, and if for every n, $X \setminus X_n \in \mathcal{I}$, then $X \in \mathcal{I}^\omega$, which proves that the decreasing sequence $\{X_n : n \in \omega\}$ does not have an $\mathcal{I}^\omega$-positive lower bound. On the other hand, the family $\mathcal{D} = \bigcup_n \{X \in (\mathcal{I}^\omega)^+ : X \subseteq \{n\} \times \omega\}$ is a dense σ-closed subset of $(\mathcal{I}^\omega)^+$.

3. Dichotomy for quotients over analytic P-ideals

We now deal with the important family of analytic P- ideals. Recall that $\mathcal{I}$ is a *P-ideal* if for every countable subset $\mathcal{C}$ of $\mathcal{I}$, there is $B \in \mathcal{I}$ such that $C \subseteq^* B$, for all $C \in \mathcal{C}$. Solecki [46] characterized the analytic P-ideals as the ideals of the form

$$\mathcal{I} = Exh(\varphi) = \{A \subseteq \omega : \lim_n \varphi(A \setminus n) = 0\},$$

where φ is a *lower semicontinuous submeasure* (lscsm), i.e. a function from $\mathcal{P}(\omega)$ to $[0,\infty)$ satisfying that for all $A, B \subseteq \omega$,

- $\varphi(\emptyset) = 0$,
- $\max\{\varphi(A), \varphi(B)\} \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$, and
- $\varphi(A) = \lim_n \varphi(A \cap n)$.

We say that a family $\mathcal{A} \subseteq \mathcal{I}^+$ is a *strong partition* when $A \cap B = \emptyset$ for all $A \neq B \in \mathcal{A}$, and $I \in \mathcal{I}$ if and only if $I \cap A \in \mathcal{I}$, for all $A \in \mathcal{A}$. Strong partitions represent countable maximal antichains in $\mathcal{P}(\omega)/\mathcal{I}$.

An example of a strong partition for $\mathcal{Z}$ is the following: For $n \in \omega$, define $A_n = \{k : k \equiv 2^n - 1 \pmod{2^{n+1}}\}$. Note that the density of A_n is 2^{-n}, and if A is a set such that for all n, $A \cap A_n \in \mathcal{Z}$, the density of A can be bounded by ε, for all $\varepsilon > 0$, as follows. Let n be such that $2^{-n+1} < \varepsilon$. Then, the density of $\bigcup_{k=n+1}^{\infty} A_k$ is 2^{-n}, and for some big enough N_k, we know that the density of $(A \cap A_k) \setminus N_k$ is less than 2^{-n-k}. Then, for N sufficiently large, the density of $(A \cap \bigcup_{k=1}^{n} A_k) \setminus N$ is less than 2^{-n}. Hence, the density of $A \setminus N$ is less than ε.

This example illustrates the general characterization of strong partitions on analytic P-ideals.

Theorem 3.1. *If $\mathcal{I} = Exh(\varphi)$ for some lscsm φ and $P = \{A_n : n \in \omega\}$ is a partition of ω in $\mathcal{I}$-positive pieces, then P is a strong partition if and only if for all $\varepsilon > 0$ there are N, M such that $\varphi(\bigcup_{n \geq N} A_n \setminus M) < \varepsilon$.*

Proof. Let us prove that if for some fixed ε, $\varphi(\bigcup_{n\geq N} A_n \setminus M) \geq \varepsilon$, for all N, M, then P is not a strong partition. We recursively define two increasing sequences n_k and m_k ($k \in \omega$) such that

(1) $m_0 = 0 = n_0$,
(2) $\varphi([m_k, m_{k+1}) \cap \bigcup_{j=n_k}^{\infty} A_j) > \frac{\varepsilon}{2}$, and
(3) $\varphi(\bigcup_{j=n_k}^{n_{k+1}-1} A_j \setminus m_k) > \frac{\varepsilon}{2}$.

For (2), suppose n_k and m_k are defined. Since $\varphi(\bigcup_{j=n_k}^{\infty} A_j \setminus m_k) \geq \varepsilon$, by the lower semicontinuity of φ, there is $m_{k+1} > m_k$ such that $\varphi(\bigcup_{j=n_k}^{\infty} A_j \cap [m_k, m_{k+1})) > \frac{\varepsilon}{2}$. For (3), let n_{k+1} be the maximal $j > n_k$ such that $A_{j-1} \cap [m_k, m_{k+1}) \neq \emptyset$. Now we define $X_k = \bigcup_{j=n_k}^{n_{k+1}-1} A_j \cap [m_k, m_{k+1})$, and $X = \bigcup_k X_k$. Clearly, for all k, $\varphi(X_k) \geq \frac{\varepsilon}{2}$, and so $\varphi(X \setminus M) > \frac{\varepsilon}{2}$, for all M, proving that $X \in \mathcal{I}^+$. However, for all $m \in \omega$, $X \cap A_m \subseteq [m_k, m_{k+1})$, for the unique k such that $m_k \leq m < m_{k+1}$.

The argument for the other implication is a replica of the argument given in the previous example. Let $X \subseteq \omega$ be such that $X \cap A_j \in \mathcal{I}$ for all j, and let $\varepsilon > 0$ be fixed. Take N, M such that $\varphi(\bigcup_{n\geq N} A_n \setminus M) < \frac{\varepsilon}{2}$. Let $K \geq M$ be such that $\varphi(\bigcup_{j=0}^{N-1} A_j \setminus K) < \frac{\varepsilon}{2}$. Then, $\varphi(X \setminus K) \leq \varphi(\bigcup_{j=0}^{N-1} A_j \setminus K) + \varphi(\bigcup_{n\geq N} A_n \setminus K) < \varepsilon$. Hence, $X \in \mathcal{I}$. $\square$

For a partial ordered set $\mathbb{P}$, $p \in \mathbb{P}$ and $A \subseteq \mathbb{P}$, we denote

$$p||A = \{q \in A : p||q\}.$$

where $p||q$ means that p and q are *compatible*, i. e. there is $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$.

Definition 3.2. We say that $\mathbb{P}$ is $(\omega, \cdot, \omega_1)$-*distributive* if for every countable family $\mathcal{A}$ of maximal antichains in $\mathbb{P}$, there is a dense subset B of $\mathbb{P}$ such that for all $p \in B$ and all $A \in \mathcal{A}$, the set $p||A$ is countable.

It is clear that every proper forcing is $(\omega, \cdot, \omega_1)$-distributive.

Theorem 3.3. *Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathcal{P}(\omega)/\mathcal{I}$ is ω^ω-bounding or it is not $(\omega, \cdot, \omega_1)$-distributive (and hence, it is not proper).*

Proof. Let φ be a lscsm such that $\mathcal{I} = Exh(\varphi)$, and let us suppose that $\mathcal{P}(\omega)/\mathcal{I}$ is $(\omega, \cdot, \omega_1)$-distributive. Let $\dot{f}$ be a name for a real number in ω^ω. For n, let $\mathcal{A}_n$ be a maximal antichain deciding $\dot{f}(n)$. Let us deal with $\mathcal{I}^+$ instead of $\mathcal{P}(\omega)/\mathcal{I}$, which is also $(\omega, \cdot, \omega_1)$-distributive. Taking $X \in \mathcal{I}^+$ as the B in the definition of $(\omega, \cdot, \omega_1)$-distributivity, we see that the restrictions of $\mathcal{A}_n$ in X are strong partitions of X. Let us denote by $||\varphi \restriction Z|| = \lim_{k\to\infty} \varphi(Z \setminus k)$. Let $r = ||\varphi \restriction X||$, and for each n, let us recursively define $\mathcal{B}_n \in [\mathcal{A}_n \restriction X]^{<\omega}$, $a_n \in [\omega]^{<\omega}$, and an auxiliary C_n, as follows:

(1) $\mathcal{B}_0 \in [\mathcal{A}_0 \restriction X]^{<\omega}$ such that for $C_0 = \bigcup \mathcal{B}_0$, $||\varphi \restriction C_0|| > \frac{r}{2}$,
(2) $a_0 \in [C_0]^{<\omega}$, with $\varphi(a_0) \geq \frac{r}{2}$,
(3) $\mathcal{B}_{n+1} \in [\mathcal{A}_{n+1} \restriction X]^{<\omega}$ such that $C_{n+1} = C_n \cap \bigcup \mathcal{B}_{n+1}$ satisfies $||\varphi \restriction C_{n+1}|| > \frac{r}{2}$ and $\bigcup_{k=0}^n a_k \subseteq C_{n+1}$, and finally
(4) $a_{n+1} \in [C_{n+1}]^{<\omega}$ with $\varphi(a_{n+1}) > \frac{r}{2}$, and $\min(a_{n+1}) > \max(a_n)$.

Let us assume that this construction is possible, and define $Y = \bigcup_k a_k$. By (4), $Y \in \mathcal{I}^+$. Note that $\mathcal{B}_n$ is a finite maximal antichain below Y of conditions deciding $\dot{f}(n)$, for all n. Hence, Y forces that $\dot{f}$ is dominated by a ground-model function.

We now verify our construction. Since $\mathcal{A}_0 \restriction X$ is a strong partition of X, by 3.1, for $\varepsilon = \frac{r}{4}$, there are a finite subset $\mathcal{B}_0$ of $\mathcal{A}_0$, and an $N \in \omega$ such that $\varphi(\bigcup\{A \in \mathcal{A}_0 : A \notin \mathcal{B}_0\} \cap X \setminus N) < \varepsilon$. Then, $||\varphi \restriction C_0|| > \frac{r}{2}$. By lower semicontinuity, C_0 contains some finite subset a_0 with $\varphi(a_0) > \frac{r}{2}$. An analogous argument shows that we can choose a finite subset $\mathcal{B}_{n+1}$ of $\mathcal{A}_{n+1} \restriction X$ such that $||\varphi \restriction C_{n+1}|| > \frac{r}{2}$. We may add finitely-many pieces from $\mathcal{A}_{n+1} \restriction X$, if necessary, in order to achieve $\bigcup_{k=1}^n a_k \subseteq C_{n+1}$. □

One would conjecture that if $\mathcal{P}(\omega)/\mathcal{I}$ is not proper, then it is equivalent to the collapse forcing $RO({}^{\omega}\mathfrak{c})$, at least in a positive restriction. For analytic P-ideals, we confirm this fact under CH.

Recall that an antichain $\mathcal{A}$ is a *refinement* of an antichain $\mathcal{B}$ if every $A \in \mathcal{A}$ is contained in some $B \in \mathcal{B}$.

Definition 3.4. We say that an antichain $\mathcal{A}$ of $\mathcal{P}(\omega)/\mathcal{I}$ has *true cardinality* $\mathfrak{c}$ if for every $\mathcal{I}$-positive set X, the family $\{[A] : A \cap X \in \mathcal{I}^+\}$ is countable or has cardinality $\mathfrak{c}$. An ideal $\mathcal{I}$ has the property $RP(\mathcal{I})$ if every maximal antichain $\mathcal{B}$ in $\mathcal{P}(\omega)/\mathcal{I}$ has a refinement $\mathcal{A}$ which is a maximal antichain and has true cardinality $\mathfrak{c}$.

Recall the following theorem that compares a partially ordered set with the collapse forcing $\mathfrak{c}^{<\omega}$, which we will use in the next proof. See Theorem 14.17 in [33].

Theorem 3.5 (McAloon)**.** *Let $\mathbb{P}$ be a partially ordered set with $|\mathbb{P}| \leq \mathfrak{c}$. If there is a countable family $\mathcal{A}$ of maximal antichains of $\mathbb{P}$ such that for every $p \in \mathbb{P}$ there is $A \in \mathcal{A}$ such that $|p||A| = \mathfrak{c}$, then $\mathbb{P}$ is forcing equivalent to $\mathfrak{c}^{<\omega}$.*

Proposition 3.6. *If $\mathcal{I}$ satisfies $RP(\mathcal{I})$, and $\mathcal{P}(\omega)/\mathcal{I}$ is not $(\omega, \cdot, \omega_1)$-distributive, then there is $X \in \mathcal{I}^+$ such that $\mathcal{P}(X)/\mathcal{I} \restriction X$ is forcing-equivalent to $\mathfrak{c}^{<\omega}$.*

Proof. Using the non $(\omega, \cdot, \omega_1)$-distributivity, we may choose a sequence $\{A_n : n < \omega\}$ of maximal antichains in $\mathcal{P}(\omega)/\mathcal{I}$, and an $\mathcal{I}$-positive set X such that for every $\mathcal{I}$-positive $Y \subseteq_{\mathcal{I}} X$, there is n such that $[Y]||A_n$ is uncountable. By $RP(\mathcal{I})$, for each n we can choose a maximal antichain B_n having true cardinality $\mathfrak{c}$ and refining A_n. Hence, for all $Z \in \mathcal{P}(X)/\mathcal{I} \restriction X$ there is n such that $Z||A_n$ is uncountable, and then $Z||B_n$ is uncountable. Since B_n has true cardinality $\mathfrak{c}$, we conclude that $|Z||A_n| = \mathfrak{c}$. By McAloon's Theorem 3.5, $\mathcal{P}(X)/\mathcal{I} \restriction X$ is forcing-equivalent to $\mathfrak{c}^{<\omega}$. □

Corollary 3.7 (CH)**.** *Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathcal{P}(\omega)/\mathcal{I}$ is ω^ω-bounding or there is $X \in \mathcal{I}^+$ such that $\mathcal{P}(X)/\mathcal{I} \restriction X$ is forcing-equivalent to the collapse forcing $\mathfrak{c}^{<\omega}$.*

Proof. If follows directly from Theorem 3.3 and Proposition 3.6, and the trivial fact that under CH, every $\mathcal{I}$-MAD family is of true cardinality $\mathfrak{c}$. □

4. Completely separable $\mathcal{I}$-MAD families

Clearly, now the main question is if $RP(\mathcal{I})$ is true for all analytic P-ideals $\mathcal{I}$, or even for all hereditarily meager. In this direction, we adapt the notion of a completely separable MAD family ([14], [6], [28]), and a construction of such families due to Shelah [43] (also see [27]). Proposition 3.6 motivates us to study $RP(\mathcal{I})$ and complete separablilty in the general context of Borel ideals. This is interesting in itself, beyond the context of forcing.

By $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{I} \cup \mathcal{A}$, where $\mathcal{I}$ is an ideal and $\mathcal{A}$ is a family of (typically $\mathcal{I}$-positive) sets. We say that a family $\mathcal{A}$ of $\mathcal{I}$ positive sets is *$\mathcal{I}$-almost disjoint* (in short, $\mathcal{I}$-AD) if $A \cap B \in \mathcal{I}$ for all $A \neq B \in \mathcal{A}$. Note that if $\mathcal{A}$ is $\mathcal{I}$-AD, then the family of equivalence classes of elements of $\mathcal{A}$ is an antichain in $\mathcal{P}(\omega)/\mathcal{I}$. We say that an $\mathcal{I}$-AD family $\mathcal{A}$ is an $\mathcal{I}$-MAD family if it is a maximal one. Let us recall that an *interval partition* of ω is a set $\{P_n : n < \omega\}$ of consecutive intervals of natural numbers.

Definition 4.1. Let $\mathcal{I}$ be an ideal. We say that an $\mathcal{I}$-MAD family is *completely separable* if for every $\mathcal{I}(\mathcal{A})$-positive set X, there is $A \in \mathcal{A}$ such that $A \subseteq X$.

Proposition 4.2. *Let $\mathcal{I}$ be a hereditarily meager ideal and $\mathcal{A}$ an $\mathcal{I}$-AD family. Then $\mathcal{I}(\mathcal{A})$ is hereditarily meager.*

Proof. We first prove the case when $\mathcal{A}$ is an $\mathcal{I}$-MAD family. Let X be an $\mathcal{I}(\mathcal{A})$-positive set. We will show that $\mathcal{I}(\mathcal{A}) \restriction X$ satisfies Talagrand's characterization of meager ideals. Since X is $\mathcal{I}(\mathcal{A})$-positive, there is a countable subfamily $\{A_n : n < \omega\}$ of $\mathcal{A}$ such that $X \cap A_n \in \mathcal{I}^+$, for all n. By Theorem 1.4, there are interval partitions $P^n = \{P^n_m : m < \omega\}$ of A_n such that for every $R \subseteq X \cap A_n$, if $P^n_m \subseteq R$ for infinitely many $m \in \omega$, then R is $\mathcal{I} \restriction (X \cap A_n)$-positive. Here, we are considering that the interval partition P^n of $X \cap A_n$, has the form $P^n_m = X \cap A_n \cap [k^n_m, k^n_{m+1})$ for some increasing sequence $k^n_m \in \omega$ $(m < \omega)$. Recursively, we define k_n for $n < \omega$ as follows: Let $k_0 = 0$, and let $k_{n+1} > k_n$ be big enough so that for each $j \leq n$, there is m_j such that $P^j_{m_j} \subseteq A_j \cap [k_n, k_{n+1})$. Then, $P = \{X \cap [k_n, k_{n+1}) : n < \omega\}$ is an interval partition of X, such that if $R \subseteq X$ contains infinitely many pieces of P, then for every n, R contains infinitely many pieces of P^n, showing that $R \cap A_n$ is $\mathcal{I} \restriction (X \cap A_n)$-positive, for every n. Hence, such R is $\mathcal{I}(\mathcal{A})$-positive.

For the general case, we can extend $\mathcal{A} \restriction X$ to an $\mathcal{I} \restriction X$-MAD family $\mathcal{A}'$ on X, and we may conclude by noting that $\mathcal{I}(\mathcal{A}) \restriction X \subseteq \mathcal{I} \restriction X(\mathcal{A}')$, and $\mathcal{I} \restriction X(\mathcal{A}')$ is meager. □

Definition 4.3. Let $\mathcal{S}$ be a set of infinite subsets of ω. We say that $\mathcal{S}$ is a *block-splitting family* if for every interval partition $\{P_n : n < \omega\}$, there exists $S \in \mathcal{S}$ such that the sets $\{n : P_n \subseteq S\}$ and $\{n : P_n \cap S = \emptyset\}$ are infinite.

We say that an ideal $\mathcal{I}$ is a P^+-ideal if for every *subseteq**-decreasing sequence $\{X_n : n < \omega\}$ of $\mathcal{I}$-positive sets, there is an $\mathcal{I}$-positive X such that $X \subseteq^* X_n$ for all n.

Lemma 4.4. *Let $\mathcal{I}$ be a hereditarily meager ideal, $\mathcal{S}$ a block-splitting family, $\mathcal{A}$ an $\mathcal{I}$-AD family and X an $\mathcal{I}(\mathcal{A})$-positive set. There exists $S \in \mathcal{S}$ such that $X \cap S$ and $X \setminus S$ are $\mathcal{I}(\mathcal{A})$-positive.*

Proof. Let us note that, if there is an $\mathcal{I}$-positive set $Y \subseteq X$ such that $\mathcal{A} \restriction Y$ is not an $\mathcal{I} \restriction Y$-MAD family, then there is a $W \subseteq Y$ such that $\mathcal{I}(\mathcal{A}) \restriction W = \mathcal{I} \restriction W$, and hence $\mathcal{I}(\mathcal{A}) \restriction W$ is a meager ideal. On the other hand, by 4.2, if $\mathcal{A} \restriction X$ is an $\mathcal{I}$-MAD family, then $\mathcal{I}(\mathcal{A}) \restriction X$ is a meager ideal. Hence, without loss of generality, we will assume that $\mathcal{I}(\mathcal{A}) \restriction X$ is a meager ideal. By 1.4, there is an interval partition $\{P_n : n < \omega\}$ of X such that for every $W \subseteq X$, if $P_n \subseteq W$ for infinitely many $n \in \omega$, then W is $\mathcal{I}(\mathcal{A}) \restriction X$-positive. Since $\mathcal{S}$ is a block-splitting family, there is $S \in \mathcal{S}$ such that the sets $\{n : P_n \subseteq S\}$ and $\{n : P_n \cap S = \emptyset\}$ are infinite, which proves that S and $X \setminus S$ are $\mathcal{I}(\mathcal{A}) \restriction X$-positive sets. □

Lemma 4.5. *Let $\mathcal{I}$ be a hereditarily meager P^+-ideal, and $\mathcal{A}$ an $\mathcal{I}$-AD family. Then $\mathcal{I}(\mathcal{A})$ is a P^+-ideal.*

Proof. Let $\{X_n : n < \omega\}$ be a decreasing sequence of $\mathcal{I}(\mathcal{A})$-positive sets. If there is an $\mathcal{I}$-positive pseudointersection B of $\{X_n : n < \omega\}$ such that $B \cap A \in \mathcal{I}$ for all $A \in \mathcal{A}$, then B is an $\mathcal{I}(\mathcal{A})$-positive pseudointersection of $\{X_n : n < \omega\}$.

Let us assume that for every $\mathcal{I}$-positive pseudointersection B of $\{X_n : n < \omega\}$, there is $A \in \mathcal{A}$ such that $B \cap A \in \mathcal{I}^+$. We will choose sequences of sets A_n, B_n and C_n as follows. B_0 is an $\mathcal{I}$-positive pseudointersection of $\{X_n : n \in \omega\}$, $A_n \in \mathcal{A}$ is such that $A_n \neq A_k$ for all $k < n$ and $C_n := A_n \cap B_n$ is $\mathcal{I}$-positive, and B_{n+1} an $\mathcal{I}$-positive pseudointersection of $\{X_k \setminus \bigcup_{j \leq n} A_j : k > n\}$ contained in X_n. This construction works since $X_n \setminus \bigcup_{j<n} A_j$ is $\mathcal{I}$-positive, and so, for each n, there is $A_n \in \mathcal{A}$ such that $A_n \neq A_k$ for all $k < n$, and $A_n \cap X_n \in \mathcal{I}^+$. Hence, $B = \bigcup B_n$ is an $\mathcal{I}(\mathcal{A})$-positive pseudointersection of $\{X_n : n < \omega\}$. □

Lemma 4.6. *Let $\mathcal{I}$ be a hereditarily meager ideal. If $\mathcal{A}$ is a completely separable $\mathcal{I}$-MAD family, then for every $\mathcal{I}(\mathcal{A})$-positive set X, the set $\{A \in \mathcal{A} : A \subseteq X\}$ has cardinality $\mathfrak{c}$.*

Proof. Since $\mathcal{I}(\mathcal{A})$ is hereditarily meager, there is an interval partition $\{P_n : n < \omega\}$ of X such that for every $x \in [\omega]^\omega$, $\bigcup_{n \in x} P_n$ is an $\mathcal{I}(\mathcal{A})$-positive set. Let t be a bijection from $2^{<\omega}$ onto ω, and define $X_y =$

$\bigcup_n P_{t(y\restriction n)}$, for all $y \in 2^\omega$. Then, the family $\{X_y : y \in 2^\omega\}$ is an AD-family of $\mathcal{I}(\mathcal{A})$-positive sets, each of them containing a set A_y from $\mathcal{A}$, by the complete separability of $\mathcal{A}$. It is clear that $A_y \neq A_w$, for all $y \neq w \in 2^\omega$. Hence, X contains at least (and also at most) $\mathfrak{c}$ sets from $\mathcal{A}$. □

Recall that the *block splitting number* $\mathfrak{bs}$ is the minimal size of a block spiltting family. By Kamburelis and Węglorz [32], it is known that $\mathfrak{bs} = \max\{\mathfrak{b}, \mathfrak{s}\}$.

Definition 4.7. Let $\mathcal{I}$ be an ideal. We denote the minimal size of an $\mathcal{I}$-MAD family by $\mathfrak{a}(\mathcal{I})$.

Proposition 4.8. *Let $\mathcal{I}$ be a hereditarily meager P^+-ideal. If $\mathfrak{bs} \leq \mathfrak{a}(\mathcal{I})$, then there is a completely separable $\mathcal{I}$-MAD family.*

Remark 4.9. *By a result of Farkas and Soukup* [20]*, if $\mathcal{I}$ is an analytic P-ideal, then $\mathfrak{b} \leq \mathfrak{a}(\mathcal{I})$. Hence, in this case, the hypothesis is reduced to $\mathfrak{s} \leq \mathfrak{a}(\mathcal{I})$, and it is fulfilled whenever $\mathfrak{s} \leq \mathfrak{b}$.*

Proof. Fix an enumerated block-splitting family $\mathcal{S} = \{S_\alpha : \alpha < \mathfrak{bs}\}$ of minimal size. For a given $\mathcal{I}$-AD family $\mathcal{A}$ and an $\mathcal{I}(\mathcal{A})$-positive set X, by 4.4, there is a minimal $\alpha < \mathfrak{bs}$ such that $X \cap S_\alpha$ and $X \setminus S_\alpha$ are $\mathcal{I}(\mathcal{A})$-positive. Hence, for such $\mathcal{A}$, X and α we can define a sequence $\tau_X^{\mathcal{A}}$ in 2^α such that $\tau_X^{\mathcal{A}}(\beta) = j$ if and only if $X \cap S_\beta^{1-j} \in \mathcal{I}(\mathcal{A})$. Note that if Y is an $\mathcal{I}(\mathcal{A})$-positive subset of X, then $\tau_Y^{\mathcal{A}}$ extends $\tau_X^{\mathcal{A}}$. Fix an enumeration $\{X_\alpha : \alpha < \mathfrak{c}\}$ of $[\omega]^\omega$. Recursively, we construct two sequences $\mathcal{A} = \{A_\alpha : \alpha \in \mathfrak{c}\} \subseteq [\omega]^\omega$ and $\{\sigma_\alpha : \alpha \in \mathfrak{c}\} \subseteq 2^{<\mathfrak{bs}}$ such that for all α,

1. $\mathcal{A}_\alpha = \{A_\beta : \beta < \alpha\}$ is an $\mathcal{I}$-AD family,
2. $\sigma_\alpha \nsubseteq \sigma_\beta$, for all $\beta < \alpha$,
3. if X_α is $\mathcal{I}(\mathcal{A}_\alpha)$-positive then $A_\alpha \subseteq X_\alpha$, and
4. $A_\xi \subseteq_{\mathcal{I}} S_\xi^{\sigma_\alpha(\xi)}$, for all $\xi < dom(\sigma_\alpha)$.

It is clear that if the construction works, then $\mathcal{A}$ is a completely separable $\mathcal{I}$-MAD family. Let us assume that $\mathcal{A}_\alpha$ and σ_β ($\beta < \alpha$) were already constructed, and also assume that X_α is $\mathcal{I}(\mathcal{A})$-positive (if not, take ω in its place). We recursively construct a family $\{X_s : s \in 2^{<\omega}\}$ of $\mathcal{I}(\mathcal{A}_\alpha)$-positive sets, a family $\{\eta_s : s \in 2^{<\omega}\}$ of sequences in $2^{<\mathfrak{bs}}$, with $dom(\eta_s) = \alpha_s$ satisfying

1. $X_\emptyset = X_\alpha$,
2. $\eta_s = \tau_{X_s}^{\mathcal{A}_\alpha}$, and
3. $X_{s\frown 0} = X_s \cap S_{\alpha_s}$ and $X_{s\frown 1} = X_s \setminus S_{\alpha_s}$.

Let us note that since $\alpha_s = dom\left(\tau^{\mathcal{A}_\alpha}_{X_s}\right)$, we have that $S_{\alpha_s} \cap X_s$ and $X_s \setminus S_{\alpha_s}$ are $\mathcal{I}(\mathcal{A}_\alpha)$-positive. Moreover, if $t \subseteq s$ then $\eta_t \subseteq \eta_s$ and $X_s \subseteq X_t$; and if s and t are incompatible then η_s and η_t are incompatible. For every $f \in 2^\omega$, let us define $\eta_f = \bigcup_n \eta_{f\restriction n}$. Since $\mathfrak{bs}$ has uncountable cofinality, η_f is in $2^{<\mathfrak{bs}}$, and moreover, if $f \neq g$, then η_f and η_g are incompatible. Since $\alpha < \mathfrak{c}$, there is $f \in 2^\omega$ such that there is no $\beta < \alpha$ such that $\eta_f \subseteq \sigma_\beta$. The sequence $\{X_{f\restriction n} : n < \omega\}$ is a decreasing sequence of $\mathcal{I}(\mathcal{A}_\alpha)$-positive sets. By 4.5, there is an $\mathcal{I}(\mathcal{A}_\alpha)$-positive set Y such that $Y \subseteq^* X_{f\restriction n}$ for all n. That is, for all n, $Y \setminus X_{f\restriction n} \in \mathsf{fin}$, consequently, for all $\xi \in dom(\eta_f)$, $Y \cap S_\xi^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A}_\alpha)$. This means that for all $\xi \in dom(\eta_f)$, there are a finite subset F_ξ of $\mathcal{A}_\alpha$ and $I_\xi \in \mathcal{I}$, such that $Y \cap S_\xi^{1-\eta_f(\xi)} \subseteq I_\xi \cup \bigcup F_\xi$. Let us define $\mathcal{D} = \{A_\beta : \sigma_\beta \subseteq \eta_f\} \cup \bigcup_{\xi \in dom(\eta_f)} F_\xi$. Note that $\mathcal{D}$ is a subset of $\mathcal{A}_\alpha$ with less than $\mathfrak{bs}$ elements, and since $\mathfrak{bs} \leq \mathfrak{a}(\mathcal{I}) \leq \mathfrak{a}(\mathcal{I} \restriction Y)$, $\mathcal{D}$ is not maximal. Let A_α be an infinite subset of Y, which is $\mathcal{I}$-AD with all sets in $\mathcal{D}$, and define $\sigma_\alpha = \eta_f$. It only remains to verify that A_α is $\mathcal{I}$-AD with A_β for all $\beta < \alpha$, but that is clearly the case when $A_\beta \in \mathcal{D}$. Suppose $A_\beta \notin \mathcal{D}$. In this case, $\sigma_\beta \not\subseteq \eta_f$. If $\xi = \Delta(\sigma_\alpha, \sigma_\beta)$, we have that $A_\alpha \subseteq_{\mathcal{I}} S_\xi^{\sigma_\alpha(\xi)}$ and $A_\beta \subseteq_{\mathcal{I}} S_\xi^{\sigma_\beta(\xi)}$. Since $\sigma_\alpha(\xi) = 1 - \sigma_\beta(\xi)$, A_α and A_β are $\mathcal{I}$-AD. □

We will now prove that a completely separable $\mathcal{I}$-MAD family exists if some cardinal characteristic condition plus a certain pcf/guessing principle are satisfied.

Lemma 4.10. *Let $\mathcal{I}$ be a hereditarily meager P^+-ideal. Let C be an infinite subset of ω and $\{C_n : n < \omega\}$ a partition of C in infinite pieces. There is a family $\mathcal{B}$ of $\mathfrak{b}$ infinite subsets of C such that if $\mathcal{A}$ is an $\mathcal{I}$-AD family and X is a subset of C for which there are a family $\{A_i : i < \omega\} \subseteq \mathcal{A}$ and a sequence $\{n_i : i < \omega\}$ such that*

(1) $X \cap A_i \in \mathcal{I}^+$,
(2) $A_i \subseteq C_{n_i}$, and
(3) $n_i \neq n_j$ if $i \neq j$,

then there is $B \in \mathcal{B}$ such that $B \cap X$ and $X \setminus B$ are $\mathcal{I}(\mathcal{A})$-positive.

Proof. Let $\mathcal{D} = \{f_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded family of increasing functions defined on C. Also, let $\mathcal{P} = \{P_\alpha : \alpha < \mathfrak{b}\}$ be an unbounded family of interval partitions of C, i.e. for every interval partition P of C, there is $\alpha < \mathfrak{b}$ such that for infinitely many I in P_α, there is J in P such that $J \subseteq I$. For every α and β in $\mathfrak{b}$, let $g_{\alpha\beta}$ be given by $g_{\alpha\beta}(j) = f_\alpha(k)$, for the maximal $k \geq 0$ such that $[j, k] \subseteq I$, for some $I \in P_\beta$. For each pair $\alpha, \beta \in \mathfrak{b}$, define $B_{\alpha\beta} = \{m \in C : \forall j(m \in C_j \rightarrow m \leq g_{\alpha\beta}(j))\}$. We now define the family $\mathcal{B} = \{B_{\alpha\beta} : \alpha, \beta \in \mathfrak{b}\}$. Let X, $\{A_i : i < \omega\}$ and $\{n_i : i < \omega\}$ be as in the hypothesis. We may assume that $X = \bigcup_i A_i$.

We first deal with the case in which $\mathcal{A} \restriction X$ is not an $\mathcal{I}$-MAD family. Let Y be an $\mathcal{I}$-positive set such that $Y \cap A \in \mathcal{I}$, for all $A \in \mathcal{A}$. Note that $\mathcal{I}(\mathcal{A}) \restriction Y = \mathcal{I} \restriction Y$. Since $\mathcal{I}$ is a P^+-ideal, we can find a subset D of Y such that $D \cap A_n$ is finite, for all n. Let $Q = \{I_n : n < \omega\}$ be an interval partition of D such that every set containing infinitely many pieces from Q is $\mathcal{I}$-positive. Let $R = \{J_n : n < \omega\}$ be an interval partition of ω such that for every n, there is $m(n)$ with $I_{m(n)} \subseteq \bigcup_{j \in J_n} C_j$. Let $\beta < \mathfrak{b}$ be such that $P_\beta = \{K_i : i < \omega\}$ is not dominated by R, i.e. the set $H = \{i : \exists n(i)(J_{n(i)} \subseteq K_i)\}$ is infinite. For every $i \in H$, define $h(i) = max(I_{m(n(i))})$, and let $\alpha < \mathfrak{b}$ be such that $f_\alpha \restriction H$ is not dominated by h, i.e. the set $K = \{i \in H : h(i) < f_\alpha(i)\}$ is infinite. Hence, for each $i \in K$, $I_{m(n(i))} \subseteq \bigcup_{j \in J_{n(i)}} \{r \in C_j : r \leq g_{\alpha\beta}(i)\}$, and then $B_{\alpha\beta} \cap D$ contains infinitely many intervals from Q. This proves that $B_{\alpha\beta} \cap D$ is a positive subset of X. On the other hand, $X \setminus B_{\alpha\beta}$ contains $X \setminus D$, which is an $\mathcal{I}(\mathcal{A})$-positive set.

Now we deal with the case in which $\mathcal{A} \restriction X$ is an $\mathcal{I}$-MAD family. By the maximality of $\mathcal{A} \restriction X$, we can find a sequence $\{A'_j : j < \omega\} \subseteq \mathcal{A}$ satisfying

(1) $A'_j \neq A_i$ for all i,
(2) $A'_j \neq A'_k$ if $j \neq k$, and
(3) $A'_j \cap X \in \mathcal{I}^+$.

Since $\mathcal{I}(\mathcal{A})$ is a P^+-ideal, for the sequence $X_n := \bigcup_{i \geq n} A_i$, there is an $\mathcal{I}(\mathcal{A})$-positive pseudointersection Y. Let us denote with D_n the set $A'_n \cap X$. Since $\mathcal{I}$ is a hereditarily meager ideal, for every n, there is an interval partition Q_n of $Y \cap D_n$, such that every set containing infinitely many pieces of Q_n is $\mathcal{I}$-positive. For all n, take an interval partition $\{R_n : n < \omega\}$ of Y in such a way that each interval J in R_n is large enough for $\bigcup_{j \in J} C_j$ to contain an interval I in Q_i, for all $i \leq n$. Let us fix enumerations for $Q_n = \{I(n, j) : j < \omega\}$ and $R_n = \{J(n, m) : m < \omega\}$, and a function $j(n, m, k)$, such that for all $n, m \in \omega$ and $k \leq n$, $I(k, j(n, m, k)) \subseteq \bigcup_{r \in J(n,m)} C_r$. Let $R = \{K_s : s < \omega\}$ be an interval partition in such a way that for every s and every $t \leq s$, there is $m(s, t) < \omega$ such that $J(t, m(s, t)) \subseteq K_s$. Let $\beta < \mathfrak{b}$ be such that $P_\beta = \{L_n : n < \omega\}$ is not dominated by R, i.e. the set $H = \{n \in \omega : \exists m(n)(K_{m(n)} \subseteq L_n)\}$ is infinite. For all $n \in H$, let $h(n)$ be the maximum of $\bigcup\{I(k, j(t, m(s(n), t), k) : t \leq s(n), k \leq m(s(n), t)\}$. Let $\alpha < \mathfrak{b}$ be such that $f_\alpha \restriction H \not\leq^* h$. Hence, the set

$$M = \bigcup\{I(k, j(t, m(s(n), t), k) : h(n) \leq f_\alpha(n), t \leq s(n), k \leq m(s(n), t)\}$$

is an $\mathcal{I}(\mathcal{A})$-positive set contained in $X \cap B_{\alpha\beta}$. Clearly, $X \setminus B_{\alpha\beta}$ is $\mathcal{I}(\mathcal{A})$-positive. □

By a simple modification of the proof above, we may conclude that under the lemma's hypothesis $RP(\mathcal{I})$ is satisfied.

The pcf/guessing principle mentioned before is defined as follows.

Definition 4.11. Let $\kappa \geq \mathfrak{b}$ be a cardinal number. By $P(\mathfrak{b}, \kappa)$ we denote the property that there is a family $\{U_\alpha : \omega \leq \alpha < \kappa\}$ such that

(1) $U_\alpha \subseteq \alpha$, and the order type of U_α is ω, for all $\alpha < \kappa$, and
(2) for every $X \subseteq \kappa$ with order type $\mathfrak{b}$, there is $\alpha < \sup X$ such that $|U_\alpha \cap X| = \omega$.

Shelah proved (in ZFC) that if $\mathfrak{b} \leq \kappa < \aleph_\omega$ then $P(\mathfrak{b}, \kappa)$ holds.

Theorem 4.12. *Let $\mathcal{I}$ be a hereditarily meager P^+ ideal. If*

(1) $\mathfrak{bs} \leq \mathfrak{a}(\mathcal{I})$, or
(2) $P(\mathfrak{b}, \mathfrak{s})$ and $\mathfrak{b} < \mathfrak{a}(\mathcal{I})$

then there is a completely separable $\mathcal{I}$-MAD family. In fact, $RP(\mathcal{I})$ holds.

Proof. Case 1 is a consequence of Proposition 4.8. For Case 2, let us additionally assume that Case 1 is not true. Hence, $\mathfrak{b} \leq \mathfrak{a}(\mathcal{I}) < \mathfrak{s}$, and so $\mathfrak{bs} = \mathfrak{s}$, i.e. there is a block-splitting family of size $\mathfrak{s}$. Let $\{U_\alpha(n) : n < \omega\}$ be an enumeration of U_α, and $\{P_\alpha : \alpha < \mathfrak{s}\}$ a partition of $\mathfrak{s}$ such that

- $|P_0| = \mathfrak{s}$ and $\omega \subseteq P_0$,
- for all $\alpha > 0$, $|P_\alpha| = \mathfrak{b}$ and $\alpha < \min(P_\alpha) < \sup(P_\alpha) \leq \alpha + \mathfrak{b}$.

Let $\{S_\alpha : \alpha \in P_0\}$ be a block-splitting family and $\{X_\alpha : \alpha < \mathfrak{c}\}$ an enumeration of $[\omega]^\omega$. Recursively, we construct three sequences $\{A_\alpha : \alpha \in \mathfrak{c}\}$, $\{\sigma_\alpha : \alpha \in \mathfrak{c}\}$, and $\{C_\alpha : \alpha \in \mathfrak{c}\}$, such that for all $\alpha < \mathfrak{c}$,

(i) $\mathcal{A}_\alpha = \{A_\xi : \xi < \alpha\}$ is an $\mathcal{I}$-AD family,
(ii) $\sigma_\alpha \in 2^{<\mathfrak{s}}$,
(iii) $C_\alpha : 2^{<\mathfrak{s}} \to \mathcal{P}(\omega)$,
(iv) $A_\alpha \subseteq C_\alpha(\sigma_\alpha \restriction \xi)^{\sigma_\alpha(\xi)}$, for all $\xi \in dom(\sigma_\alpha)$,
(v) $A_\alpha \subseteq X_\alpha$ if $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$, and
(vi) $\sigma_\alpha \nsubseteq \sigma_\beta$, for all $\beta < \alpha$.

First we define C_α, assuming that C_η, A_η and σ_η are defined for all $\eta < \alpha$. Let τ be in $2^{<\mathfrak{s}}$, let us say $\tau \in 2^\xi$. If $\xi \in P_0$, then define $C_\alpha(\tau) = S_\xi$. If $\xi \in P_\delta$ for some $\delta > 0$, recall that $\delta \leq \xi < \delta + \mathfrak{b}$. Let us focus on the sequence $\{\tau \restriction U_\delta(n) : n < \omega\}$. Let $R_{\alpha,\tau} = \{\gamma < \alpha : (\exists n < \omega)(\tau \restriction U_\delta(n) = \sigma_\gamma)\}$. Note that if $\gamma \in R_{\alpha,\tau}$, then there is a unique n

such that $\tau \restriction U_\delta(n) = \sigma_\gamma$. For all n, define $A_n^{\alpha,\tau} = A_\gamma$, if $\gamma \in R_{\alpha,\tau}$ and $\sigma_\gamma = \tau \restriction U_\delta(n-1)$, and define $A_n^{\alpha,\tau} = \emptyset$, if not. For each n, fix

$$B_n^{\alpha,\tau} = \bigcap_{i \leq n} (C_\alpha(\tau \restriction U_\delta(i)) \setminus A_n^{\alpha,\tau}) .$$

We will pick an enumerated family $\mathcal{D}_\alpha^\tau = \{D_\alpha^\tau(\nu) : \nu \in P_\delta\}$ in such a way that

- if $\xi > \min P_\delta$, then $\mathcal{D}_\alpha^\tau = \mathcal{D}_\alpha^{\tau \restriction \min P_\delta}$,
- if $R_{\alpha,\tau} = R_{\beta,\tau}$ for some $\beta < \alpha$, then $\mathcal{D}_\alpha^\tau = \mathcal{D}_\beta^\tau$, and
- in the remaining case, take $C_\alpha(\tau \restriction U_\delta(0))$ and $B_n^{\alpha,\tau} \setminus B_{n+1}^{\alpha,\tau}$ as the C and C_n (respectively) in the hypothesis of Lemma 4.10, and then pick the family $\mathcal{D}_\alpha^\tau$ as the family $\mathcal{B}$ given by this Lemma, and fix an enumeration for it, indexed by P_δ.

Now we define $C_\alpha(\tau) = D_\alpha^\tau(\xi)$.

We claim that for all $\beta < \alpha$ and $\eta \in dom(\sigma_\beta)$,

$$C_\beta(\sigma_\beta \restriction \eta) = C_\alpha(\sigma_\beta \restriction \eta).$$

Let us prove it by induction on η. By induction hypothesis, we have that $C_\alpha(\sigma_\beta \restriction U_\delta(i)) = C_\beta(\sigma_\beta \restriction U_\delta(i))$ for all i. On the other hand, note that clearly $R_{\beta,\sigma_\beta \restriction \eta} \subseteq R_{\alpha,\sigma_\beta \restriction \eta}$, but actually, the reverse inclusion is also true, because for every τ, if $\gamma \in R_{\alpha,\tau} \setminus R_{\beta,\tau}$ then $\gamma > dom(\sigma_\beta)$, and so $\tau \nsubseteq \sigma_\beta$. Hence $\mathcal{D}_\alpha^\tau = \mathcal{D}_\beta^\tau$ and the claim follows immediately from the definitions.

From the claim and an inductive argument based on condition (iv), we may deduce that

$$A_\beta \subseteq^* C_\alpha(\sigma_\beta \restriction \xi)^{\sigma_\beta(\xi)},$$

for all $\beta < \alpha$, and $\xi \in dom(\sigma_\beta)$.

Now we define σ_α. By recursion on ω, let T_n be the subset of $2^{<\mathfrak{s}}$ defined by $T_0 = \emptyset$ and $\tau \in T_{n+1}$ if and only if there is $s \in T_n$ such that

- $s = \tau \restriction |s|$,
- either $X \cap C_\alpha(\tau \restriction \xi)$ or $X \setminus C_\alpha(\tau \restriction \xi)$ belong to $\mathcal{I}(\mathcal{A}_\alpha)$, for all $|s| < \xi < |\tau|$, and
- $X \cap C_\alpha(\tau)$ and $X \setminus C_\alpha(\tau)$ are $\mathcal{I}(\mathcal{A}_\alpha)$-positive.

Since $T = \bigcup_n T_n$ has $\mathfrak{c}$ many branches, there is a branch B of T such that $\bigcup B \nsubseteq \sigma_\beta$ for all $\beta < \alpha$. Define $\sigma_\alpha = \bigcup B$. Let Y be an $\mathcal{I}(\mathcal{A}_\alpha)$-positive pseudointersection of $\{C_\alpha(\sigma_\alpha \cap T_n) : n < \omega\}$, i.e. $Y \in \mathcal{I}(\mathcal{A}_\alpha)^+$ and $Y \setminus C_\alpha(\sigma_\alpha \cap T_n) \in \mathcal{I}(\mathcal{A}_\alpha)$ for all n. Moreover, note that for all $\xi < dom(\sigma_\alpha)$, if $\sigma_\alpha \restriction \xi$ is not in T, then $Y \cap C_\alpha(\sigma_\alpha \restriction \xi)^{1-\sigma_\alpha(\xi)}$ is in $\mathcal{I}(\mathcal{A}_\alpha)$.

We claim that $\mathcal{A}_\alpha \restriction Y$ is not an $\mathcal{I}$-MAD family. To see it, let us consider the set

$$W = \{\xi < dom(\sigma_\alpha) : (\exists \beta < \alpha)(\xi = dom(\sigma_\beta) \vee \xi = dom(\sigma_\alpha \cap \sigma_\beta))\}.$$

We claim that W has less than $\mathfrak{b}$ many elements. Suppose not. Let W_0 be the set of the first $\mathfrak{b}$ elements of W. By $P(\mathfrak{b}, \mathfrak{s})$, there is $\delta < \sup W_0$ such that $U_\delta \cap W_0$ is infinite. Let ε be the minimum of P_δ. By its definition, the family $\mathcal{D}_\alpha^{\sigma_\alpha \restriction \varepsilon}$ splits Y, i.e. there is $\nu \in P_\delta$ such that $Y \cap D_\alpha^{\sigma_\alpha \restriction \varepsilon}(\nu)$ and $Y \setminus D_\alpha^{\sigma_\alpha \restriction \varepsilon}(\nu)$ are $\mathcal{I}(\mathcal{A}_\alpha)$-positive. Hence, $Y \cap C_\alpha(\sigma_\alpha \restriction \nu)$ and $Y \setminus C_\alpha(\sigma_\alpha \restriction \nu)$ are $\mathcal{I}(\mathcal{A}_\alpha)$-positive, in particular, $Y \cap C_\alpha(\sigma_\alpha \restriction \nu)^{1-\sigma_\alpha(\nu)}$ is $\mathcal{I}(\mathcal{A}_\alpha)$-positive, which is a contradiction, since $\varepsilon < \nu < dom(\sigma_\alpha)$.

For each $\xi \in W$, let $Z(\xi)$ be defined as follows:

- If there is $\beta < \alpha$ such that $\xi = dom(\sigma_\beta)$, then $Z(\xi) = \{A_\beta\}$.
- If not, then define $Z(\xi)$ as a finite subset of $\mathcal{A}_\alpha$ such that $Y \cap C_\alpha(\sigma_\alpha \restriction \xi)^{1-\sigma_\alpha(\xi)} \subseteq \bigcup Z(\xi)$. This finite set exists since $Y \cap C_\alpha(\sigma_\alpha \restriction \xi)^{1-\sigma_\alpha(\xi)}$ is in $\mathcal{I}(\mathcal{A}_\alpha)$.

Clearly, $\bigcup_{\xi \in W} Z(\xi)$ has less than $\mathfrak{b}$ many elements. We claim that for all $\beta < \alpha$, $Y \cap A_\beta$ is $\mathcal{I}$-almost contained in $Z(\xi)$ for some $\xi \in W$. This is clear when $\xi = dom(\sigma_\beta)$. In the other case, the claim follows from the fact that $A_\beta \subseteq C_\alpha(\sigma_\alpha \restriction \xi)^{1-\sigma_\alpha(\xi)}$. Since $\mathfrak{b} \leq \mathfrak{a}(\mathcal{I})$, $\mathcal{A}_\alpha \restriction Y$ cannot be an $\mathcal{I}$-MAD family. $\square$

5. Open questions

The existence of a completely separable MAD family is a famous problem of Erdös and Shelah [14]. We conjecture the same for quotients over analytic P-ideals. We list some interesting open questions.

Question 5.1. *Are there $\mathfrak{c}$-many non forcing equivalent quotients $\mathcal{P}(\omega)/\mathcal{I}$ with $\mathcal{I}$ Borel?*

Question 5.2. *Does $(\omega, \cdot, \omega_1)$-distributivity imply properness for $\mathcal{P}(\omega)/\mathcal{I}$ with $\mathcal{I}$ a Borel ideal?*

Question 5.3. *Is it true that if $\mathcal{I}$ is Borel and $\mathcal{P}(\omega)/\mathcal{I}$ is ω^ω-bounding, then one of the following conditions holds?*

(a) $\mathcal{P}(\omega)/\mathcal{I}$ does not add reals.
(b) There exists an $\mathcal{I}$-positive set X such that $\mathcal{I} \restriction X$ is a P-ideal.

Question 5.4. *Is it true (in ZFC) that if $\mathcal{I}$ is Borel and not proper, then there exists an $\mathcal{I}$-positive set X such that $\mathcal{P}(X)/(\mathcal{I} \restriction X)$ is forcing equivalent to $\mathfrak{c}^{<\omega}$?*

Question 5.5. *Let $\mathcal{I}$ be a Borel ideal. Does there exist (in ZFC) a completely separable $\mathcal{I}$-MAD family?*

Question 5.6. *Let $\mathcal{I}$ be a Borel ideal. Is RP($\mathcal{I}$) true (in ZFC)?*

Question 5.7. *Is it possible to avoid the large cardinals assumption in Theorem 2.9?*

References

[1] T. Bartoszynski, H. Judah. *Set Theory: On the Structure of the Real Line*, AK Peters / CRC Press. 1995.

[2] J.E. Baumgartner, L.A. Harrington, E.M. Kleinberg. *Adding a closed unbounded set*, The Journal of Symbolic Logic 41(2):481-482, 1976.

[3] Balcar, Bohuslav; Doucha, Michal; Hrušák, Michael *Base tree property*, Order 32 (2015), no. 1, 69-81.

[4] Balcar, Bohuslav; Pelant, Jan; Simon, Petr *The space of ultrafilters on $\mathbb{N}$ covered by nowhere dense sets*, Fund. Math. 110 (1980), no. 1, 11-24.

[5] B. Balcar, F. Hernández-Hernández, M. Hrušák, *Combinatorics of dense subsets of the rationals*, Fund. Math. 183, no. 1, 59-80, 2004.

[6] Balcar, Bohuslav; Simon, Petr *Disjoint refinement*. Handbook of Boolean algebras, Vol. 2, 333-388, North-Holland, Amsterdam, 1989.

[7] B. Balcar, P. Štepánek. *Teorie množin*. Academia, 2001.

[8] Blass, Andreas; Dobrinen, Natasha; Raghavan, Dilip, *The next best thing to a P-point*, J. Symb. Log. 80 (2015), no. 3, 866-900.

[9] J. Brendle, D.A. Mejia, *Rothberger gaps in fragmented ideals*, Fundamenta Mathematicae 227(1), 35-68, 2014.

[10] Brendle, Jörg; Montoya, Diana Carolina *A base-matrix lemma for sets of rationals modulo nowhere dense sets* Arch. Math. Logic 51 (2012), no. 3-4, 305-317.

[11] Cancino-Manríquez, Jonathan; Hrušák, Michael; Meza-Alcántara, David *Countable irresolvable spaces and cardinal invariants.* Topology Proc. 44 (2014), 189-196.

[12] Chodounský, David; Guzmán González, Osvaldo; Hrušák, Michael *Mathias-Prikry and Laver type forcing; summable ideals, coideals, and +-selective filters*, Arch. Math. Logic 55 (2016), no. 3-4, 493-504.

[13] van Douwen, Eric K.; van Mill, Jan *Subspaces of basically disconnected spaces or quotients of countably complete Boolean algebras*, Trans. Amer. Math. Soc. 259 (1980), no. 1, 121-127.

[14] Erdös, Paul; Shelah, Saharon *Separability properties of almost-disjoint families of sets*. Israel J. Math. 12 (1972), 207-214.

[15] Dow, A.; Frankiewicz, R.; Zbierski, P. *On closed subspaces of ω^**, Proc. Amer. Math. Soc. 119 (1993), no. 3, 993-997.

[16] I. Farah, *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers*, Mem. Amer. Math. Soc. 148, no. 702, 2000.

[17] I. Farah, *Analytic Hausdorff gaps* in DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 58, Amer. Math. Soc., 65-72, 2002.

[18] I. Farah, *Luzin gaps*, Trans. Amer. Math. Soc. 356(6), 2197-2239, 2004.

[19] Farah, Ilijas; Solecki, Slawomir *Two $F_{\sigma\delta}$ ideals*, Proc. Amer. Math. Soc. 131 (2003), no. 6, 1971-1975.
[20] B. Farkas, L. Soukup. More on cardinal invariants of analytic P-ideals. Comment. Math. Univ. Carolin. 50 (2009), no. 2, 281-295.
[21] G.M. Fichtenholz, L.V. Kantorovich. *Sur les opérations linéires dans l'espace des fonctions bornées*, Studia Math. 5:69-98, 1935.
[22] D.H. Fremlin, *The partially ordered sets of measure theory and Tukey's ordering*, Note di Matematica, 11 (1991), 177-214.
[23] He, Jialiang; Hrušák, Michael; Rojas-Rebolledo, Diego; Solecki, Slawomir *Tukey order among F_σ ideals*, J. Symb. Log. 86 (2021), no. 2, 855-870
[24] Hernández-Hernández, Fernando *Distributivity of quotients of countable products of Boolean algebras*, Rend. Istit. Mat. Univ. Trieste 41 (2009), 27-33 (2010)
[25] M. Hrušák, *Combinatorics of filters and ideals* in Set theory and its applications, Contemp. Math., 533, Amer. Math. Soc., 29-69, 2011.
[26] M. Hrušák, *Katětov order on Borel ideals*, Arch. Math. Logic 56, no. 7-8, 831-847, 2017.
[27] Hrušák, Michael *Almost disjoint families and topology.* Recent progress in general topology III, 601-638, Atlantis Press, Paris, 2014.
[28] Hrušák, Michael; Simon, Petr *Completely separable MAD families.* Open problems in Topology II, 179–184, Elsevier, 2007.
[29] Hrušák, Michael; Verner, Jonathan L. *Adding ultrafilters by definable quotients*, Rend. Circ. Mat. Palermo (2) 60 (2011), no. 3, 445-454.
[30] M. Hrušák, J. Zapletal, *Forcing with quotients*, Arch. Math. Logic 47, no. 7-8, 719-739, 2008.
[31] W. Just, A. Krawczyk, *On certain Boolean algebras $\mathcal{P}(\omega)/\mathcal{I}$* Trans. Amer. Math. Soc. 285, no. 1, 411-429, 1984.
[32] A. Kamburelis, B. Węglorz. *Splittings*, Arch. Math. Logic 35 (1996), no. 4, 263-277.
[33] Koppelberg, Sabine, *Handbook of Boolean algebras.* Vol. 1. Edited by J. Donald Monk and Robert Bonnet. North-Holland Publishing Co., Amsterdam, 1989.
[34] Louveau, Alain *Caractérisation des sous-espaces compacts de $\beta\mathbb{N}$*, Bull. Sci. Math. (2) 97 (1973), 259-263 (1974).
[35] A. Louveau, B. Veličković, *Analytic ideals and cofinal types*, Ann. Pure Appl. Logic 99, no. 1-3, 171-195, 1999.
[36] M. S. Kurilić, *Posets of isomorphic substructures of relational structures*, Zb. Rad. (Beogr.) 17(25) (2015), Selected topics in combinatorial analysis, 117-144.
[37] Kurilić, Miloš S.; Todorčević, Stevo *Posets of copies of countable non-scattered labeled linear orders*, Order 37 (2020), no. 1, 59-72.
[38] Kurilić, Miloš S.; Todorčević, Stevo *Stevo Copies of the random graph*, Adv. Math. 317 (2017), 526-552.
[39] Kurilić, Miloš S.; Todorčević, Stevo *The poset of all copies of the random graph has the 2-localization property*, Ann. Pure Appl. Logic 167 (2016), no. 8, 649-662.
[40] Kurilić, Miloš S.; Todorčević, Stevo *Forcing by non-scattered sets*, Ann. Pure Appl. Logic 163 (2012), no. 9, 1299-1308.

[41] Miller, Arnold W. *Rational perfect set forcing* Axiomatic set theory (Boulder, Colo., 1983), 143-159, Contemp. Math., 31, Amer. Math. Soc., Providence, RI, 1984.
[42] Oliver, M. R. *Contiunuum-many Boolean algebras of the form $\mathcal{P}(\omega)/\mathcal{I}$, $\mathcal{I}$* Borel. J. Symb. Log., 69 (2004) no. 3, 799–816.
[43] Shelah, Saharon *MAD saturated families and SANE player*. Canad. J. Math. 63 (2011), no. 6, 14161435.
[44] R. Sikorski. *Boolean Algebras*, Third Edition. Springer Verlag. 1969.
[45] S. Solecki. *Analytic ideals*, Bull. Symb.Log., (2) no. 3, (1996): 339–348.
[46] Solecki, Slawomir *Analytic ideals and their applications*, Ann. Pure Appl. Logic 99 (1999), no. 1-3, 51-72.
[47] Solecki, Slawomir; Todorčević, Stevo *Cofinal types of topological directed orders*, Ann. Inst. Fourier (Grenoble) 54 (2004), no. 6, 1877-1911 (2005).
[48] Steprāns, Juris *Many quotient algebras of the integers modulo co-analytic ideals*, Logic and its applications, 271-281, Contemp. Math., 380, Amer. Math. Soc., Providence, RI, 2005.
[49] Szymański, Andrzej; Zhou Hao Xua *The behaviour of ω^{2^*} under some consequences of Martin's axiom*. General topology and its relations to modern analysis and algebra, V (Prague, 1981), 577-584, Sigma Ser. Pure Math., 3, Heldermann, Berlin, 1983.
[50] B. Veličković. *Playful Boolean algebras*, Trans. Amer. Math. Soc., 296(1986):727-740.
[51] J. Zapletal. *Forcing idealized* Cambridge Tracts on Mathematics, 174, Cambridge University Press, 2008.

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México. Circuito Exterior S/N, Ciudad Universitaria, CDMX, 04510, México
E-mail address: gcampero@ciencias.unam.mx

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Morelia, México
E-mail address: oguzman@matmor.unam.mx

Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Morelia, México
E-mail address: michael@matmor.unam.mx

Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional Autónoma de México. Circuito Exterior S/N, Ciudad Universitaria, CDMX, 04510, México
E-mail address: dmeza@ciencias.unam.mx