# GRUFF ULTRAFILTERS AND THE IDEAL OF SCATTERED SETS

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Abstract. We study gruff ultrafilters on the rational numbers and give some constructions under CH with some additional properties. We also compare gruff ultrafilters and P-points. To do this, it will be necessary to study the ideal of scattered sets of $\mathbb{Q}$.

## 1. Introduction and Preliminaries

A nonprincipal ultrafilter $\mathcal{G}$ is *gruff* if it has a base of perfect sets of $\mathbb{Q}$, i.e. closed and crowded (without insolated points) sets. Gruff ultrafilters were introduced in [10] by Eric van Douwen when he carried out an investigation about certain points in the Čech–Stone compactification of $\mathbb{Q}$ with the property that they actually generate an ultrafilter on $\mathbb{Q}$. The existence of gruff ultrafilters follows from different combinatorial principles, as $cov(\mathscr{M}) = \mathfrak{c}$ (in [10]), $\mathfrak{b} = \mathfrak{c}$ (in [7]) and $\diamondsuit(\mathfrak{b})$ (more recently in [11]). The question of whether the existence of gruff ultrafilters can be proved in ZFC is still open.

We start by giving a brief overview of the notions used in this paper. Throughout this section, $X$ will denote a countable set (typically $\omega$ or the set of rational numbers $\mathbb{Q}$). An *ideal* on $X$ is a set $\mathcal{I} \subseteq \mathcal{P}(X)\backslash\{X\}$ closed under subsets and finite intersections. For $\mathcal{A} \subseteq \mathcal{P}(X)$ the *dual family* is the set $\mathcal{A}^* = \{X\backslash A : A \in \mathcal{A}\}$. If $\mathcal{I}$ is an ideal on $X$ and $A \subseteq X$, we say that $A$ is $\mathcal{I}$ *positive* if $A \notin \mathcal{I}$. The collection of all $\mathcal{I}$ positive sets is denoted by $\mathcal{I}^+$. If $A \in \mathcal{I}^+$, the *restriction* of $\mathcal{I}$ to $A$ (denoted by $\mathcal{I} \restriction A$) is the set $\{I \cap A : I \in \mathcal{I}\}$. The restriction of $\mathcal{I}$ to $A$ is also an ideal. An ideal $\mathcal{I}$ on $X$ is *tall* if for every $A \in [X]^\omega$ exists $I \in \mathcal{I}$ such that $|I \cap A| = \omega$.

A set $\mathcal{F} \subseteq \mathcal{P}(X)$ is a *filter* if $\mathcal{F}^*$ is an ideal. If $\mathcal{F}$ is $\subseteq$-maximal we say that $\mathcal{F}$ is an *ultrafilter*. We say that an ultrafilter $\mathcal{U}$ is a *P-point* if for every $\mathcal{B} \in [\mathcal{U}]^\omega$ there exists $U \in \mathcal{U}$ almost included in each element of $\mathcal{B}$. Finally, we say that $\mathcal{B} \subseteq \mathcal{F}$ is a *filter base* if for every $F \in \mathcal{F}$ exists $B \in \mathcal{B}$ such that $B \subseteq F$.

Some of the ideals mentioned in this paper are:

(1) $fin = [\omega]^{<\omega}$ the ideal of the finite sets of $\omega$.

(2) $Disc = \{I \subseteq \mathbb{Q} : I \text{ is discrete}\}$.

(3) *conv* the ideal on $\mathbb{Q} \cap [0,1]$ generated by sequences in $\mathbb{Q} \cap [0,1]$ convergent in $[0,1]$.

(4) $nwd = \{I \subseteq \mathbb{Q} : I \text{ is nowhere dense}\}$.

(5) $mz = \{I \subset \mathbb{Q}(2^\omega) : \mu(\overline{I}) = 0\}$ where $\mu$ is the Lebesgue measure and $\mathbb{Q}(2^\omega)$ is the set $\{x \in 2^\omega : \exists n \forall m > n(x(m) = 0)\}$.

(6) $Scat = \{I \subseteq \mathbb{Q} : I \text{ is scattered}\}$.

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(7) Let $\mathcal{I}$ be an ideal on $X$ and let $\mathcal{J}$ be an ideal on $Y$. Define $\mathcal{I} \times \mathcal{J}$ an ideal on $X \times Y$ as follows:

$$A \in \mathcal{I} \times \mathcal{J} \Longleftrightarrow \{i \in X : A(i) \in \mathcal{J}\} \in \mathcal{I}^*,$$

where $A(n) = \{m \in Y : (n, m) \in A\}$.

(8) Let $\mathcal{I}$ be an ideal on $X$ and $\mathcal{J}_i$ be ideals on $Y_i$ $(i \in X)$. Define $\lim_{i\to\mathcal{I}} \mathcal{J}_i$ an ideal on $\prod_{i\in X} Y_i$ as follows:

$$A \in \lim_{i\to\mathcal{I}} \mathcal{J}_i \Longleftrightarrow \{i \in X : A(i) \in \mathcal{J}_i\} \in \mathcal{I}^*.$$

(9) $K_\sigma$ the ideal on $\omega^\omega$ $\sigma$-generated by compact sets, i.e, $I \in K_\sigma$ if and only if there exists $\{K_n : n \in \omega\}$ a countable collection of compact sets of $\omega^\omega$ such that $I \subseteq \bigcup_{n\in\omega} K_n$.

An important tool for classifying relations among ideals and filters is the Katětov order introduced in [21].

**Definition 1.** *Let $\mathcal{I}$ be an ideal on a countable set $X$ and $\mathcal{J}$ be an ideal on a countable set $Y$. We say that $\mathcal{I}$ is Katětov below $\mathcal{J}$ (denoted by $\mathcal{I} \leq_K \mathcal{J}$) if there exists $f : Y \longrightarrow X$ such that $f^{-1}[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$. Such $f$ is called Katětov morphism. If $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$, then we say that $\mathcal{I}$ and $\mathcal{J}$ are Katětov equivalent (denoted by $\mathcal{I} \cong_K \mathcal{J}$)*

Some immediate properties of Katětov order are listed here. Let $\mathcal{I}, \mathcal{J}$ be ideals on countable sets.

(1) $\mathcal{I} \leq_K fin$ if and only if $\mathcal{I}$ is not tall.
(2) If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I} \leq_K \mathcal{J}$.
(3) If $A \in \mathcal{I}^+$, then $\mathcal{I} \leq_K \mathcal{I} \restriction A$.
(4) $\mathcal{I}, \mathcal{J} \leq_K \mathcal{I} \times \mathcal{J}$.

If $\mathcal{I}$ is an ideal such that $\mathcal{I} \restriction A \leq_K \mathcal{I}$ for every $A \in \mathcal{I}^+$, then we say that $\mathcal{I}$ is *Katětov uniform.* To learn more about Katětov order see [3], [15] and [14] .

Destructibility and indestructibility of ultrafilters and ideals will play an important role in this paper. Let $V$ be a model of set theory, $\mathcal{U} \in V$ be an ultrafilter on $X$, and $W$ be some extension of $V$. If $V \cap [X]^\omega = W \cap [X]^\omega$, then $\mathcal{U}$ is ultrafilter in $W$. On the other hand, if $W$ contains new subsets of $X$, then $\mathcal{U}$ is no longer closed with respect to supersets in $W$ and is not even a filter. Therefore we are rather interested in the filter generated by $\mathcal{U}$ in $W$. If $\mathcal{U}$ generates an ultrafilter in $W$, we say that $\mathcal{U}$ is *preserved* in the extension, otherwise we say that it is *destroyed.* A similar phenomenon happens for tall ideals. If $\mathcal{I} \in V$ is a tall ideal on $X$ and $V \cap [X]^\omega = W \cap [X]^\omega$, then $\mathcal{I}$ is a tall ideal in $W$ but, if $W$ contains new subsets of $X$ we can not guarantee that $\mathcal{I}$ generates a tall ideal in $W$. Thus, if $\mathcal{I}$ generates a tall ideal in $W$, we say that $\mathcal{I}$ is preserved in the extension, otherwise we say that it is destroyed.

Given a forcing $\mathbb{P}$, we say that an ultrafilter $\mathcal{U} \in V$ is $\mathbb{P}$-*indestructible* if $\mathcal{U}$ is preserved in every generic extension of $V$ via $\mathbb{P}$. Otherwise, if $\mathcal{U}$ is always destroyed, we say that $\mathbb{P}$ *destroys* $\mathcal{U}$. The destructibility and indestructibility of tall ideals by $\mathbb{P}$ are defined analogously. To more about destructibility and indestructibility of ultrafilters and ideals see [16], [5], [4], [8] and [6].

Recall that a family $\mathcal{R} \subseteq \mathcal{P}(X)$ is a *reaping family* if for every $A \in [X]^\omega$ exists $R \in \mathcal{R}$ such that either $R \subseteq A$ or $R \cap A = \emptyset$. In such case, we say that $R$ *decides*

$A$. Note that $\mathcal{U}$ is ultrafilter if and only if every base of $\mathcal{U}$ is a reaping family. So, an ultrafilter $\mathcal{U}$ is $\mathbb{P}$-indestructible if and only if it is a reaping family in every extension by $\mathbb{P}$.

For any set $X$, we identify the family of finite sequences of $X$ with $X^{<\omega}$. Consequently, for $s \in \omega^{<\omega}$ with $|s| = k+1$ we can write $s = \langle s(0), ..., s(k)\rangle$. The *concatenation* of $s = \langle s(0), ..., s(k)\rangle$ with $n$ is the set $s^\frown\langle n\rangle = \langle s(0), ..., s(k), n\rangle$. A set $T \subseteq X^{<\omega}$ is a *tree*, if it is closed under initial segments, i.e., $t \in T$ and $s \subseteq t$ implies $s \in T$. Given $T \subseteq X^{<\omega}$ a tree, the *stem* of $T$ is the $\subseteq$-minimal element of $T$ with at least two immediate successors. For every $t \in T$, let $next_T(t)$ be the set $\{x \in X : t^\frown\langle x\rangle \in T\}$. The *splitting nodes of* $T$ is the set $split(T) = \{t \in T : |next_T(t)| = \omega\}$. Finally, the tree $T_s$ is defined by $T_s = \{t \in T : t \subseteq \wedge s \subseteq t\}$.

We say that a tree $p \subseteq 2^{<\omega}$ is a *perfect tree* if for every $s \in p$ exists $t \supseteq s$ such that $t^\frown\langle 0\rangle$ and $t^\frown\langle 1\rangle$ are both in $p$. *Sacks forcing* (denoted by $\mathbb{S}$) is the set of perfect trees of $2^{<\omega}$ ordered by inclusion (see [26]). This forcing notion adds a generic real defined as $g = \bigcap\{[p] : p \in G\}$ where $G$ is the generic filter on $\mathbb{S}$.

We say that a tree $T \subseteq \omega^{<\omega}$ is a *Miller tree* if for every $s \in T$ there exists $t \supseteq s$ such that $t^\frown\langle n\rangle \in T$ for infinite many $n \in \omega$. *Miller's forcing* (denoted by $\mathbb{PT}$) also known as *rational perfect set forcing*, is the set of Miller trees ordered by inclusion (see [25] and [18]).

## 2. Some properties of the ideal Scat

Recall that a set $I \subseteq \mathbb{Q}$ is *scattered* if every subset of $I$ has an insolated point; equivalently $I$ is scattered if it does not contain a crowded set. The collection of all scattered sets forms a tall ideal on $\mathbb{Q}$ denoted by *Scat*. Note that if $X$ is homeomorphic to $\mathbb{Q}$, then $\mathcal{I} = \{I \subseteq X : I \text{ is scattered}\}$ is isomorphic to *Scat*. Thus, the ideal *Scat* does not depend on which representation of $\mathbb{Q}$ is taken.

The rationals were characterized by W. Sierpiński (see [24]) as the unique (up to homeomorphism) countable first countable regular space without isolated points. Thus, Sierpiński's theorem gives us a useful characterization of *Scat* positive sets; $C \in Scat^+$ if and only if it contains a homeomorphic copy of $\mathbb{Q}$. Thus, $Scat \restriction A \leq_K Scat$ for every $A \in Scat^+$; in other words, *Scat* is Katětov uniform.

It is easy to see that *Scat* is a tall ideal, and since any scattered set is nowhere dense, we have that $Scat \leq_K nwd$.

**Definition 2.** *Let $\mathcal{I}$ be a tall ideal on a countable set $X$. Define the following cardinals associated with $\mathcal{I}$:*

*(1)* $add^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(A \not\subseteq^* I)\}$.
*(2)* $cov^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall Y \in [X]^\omega)(\exists A \in \mathcal{A})(|A \cap Y| = \aleph_0)\}$.
*(3)* $cof^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq^* A\}$.
*(4)* $non^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq [X]^\omega \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(|A \cap I| < \aleph_0)\}$.

$$
\begin{tikzcd}
 & & cov^*(\mathcal{I}) \arrow[dl, dash] \arrow[dr, dash] & & \\
\aleph_0 \arrow[r, dash] & add^*(\mathcal{I}) \arrow[dr, dash] & & cof^*(\mathcal{I}) \arrow[r, dash] \arrow[dl, dash] & 2^{\aleph_0} \\
 & & non^*(\mathcal{I}) & &
\end{tikzcd}
$$

It is well known that $\mathcal{I} \leq_K \mathcal{J}$ implies $cov^*(\mathcal{J}) \leq cov^*(\mathcal{I})$. The inequalities holding among these cardinals are summarized in the above diagram.

A countable base of $\mathbb{Q}$ shows that $non^*(Scat) = \aleph_0$, and in consequence, we have that $add^*(Scat) = \aleph_0$. What about $cov^*(Scat)$ and $cof^*(Scat)$?

**Proposition 3.** *There exists $\mathcal{A} \subseteq Disc$ of size $\mathfrak{c}$ such that any scattered set contains at most $\omega$ many elements of $\mathcal{A}$.*

*Proof.* For every $q \in \mathbb{Q}$ and $n \in \omega$ define the clopen set

$$B_n(q) = (q - \sqrt{2}/2^n, q + \sqrt{2}/2^n).$$

We will recursively construct $F : 2^{<\omega} \to [\mathbb{Q}]^{<\omega}$ and $\{r_n : n \in \omega\} \in [\omega]^\omega$ such that the following holds:

(I) $F(s) \cap F(t) = \emptyset$ whenever $s \neq t$.

(II) $r_n < r_{n+1}$ for every $n \in \omega$.

(III) For every $n \in \omega$, the collection $\{B_{r_n}(q) : q \in F(s) \wedge s \in 2^{\leq n}\}$ is a family of pairwise disjoint clopen sets.

Start by doing $F(\emptyset) = \{0\}$ and $r_0 = 0$. Choose $r_1 > r_0$ and $q_0$ such that $B_{r_1}(q_0) \subseteq B_{r_0}(0) \backslash B_{r_1}(0)$ and define $F(\langle 0 \rangle) = \{q_0\}$. On the other hand, denote by $\vec{1}$ the constant sequence 1 and define $F(\vec{1} \upharpoonright n) = \{10^n\}$ for every $n > 0$.

Let $n > 0$ and suppose that we already defined $r_i$ for every $i \leq n$ and $F(s)$ for every $s \in 2^{\leq n}$ satisfying all the conditions. For every $s \in 2^{\leq n}$ define $E_s$ as the set $\{t \in 2^{n+1} : s^\frown\langle 0 \rangle \subseteq t\}$. Now choose $r_{n+1} > r_n$ such that for every $s \in 2^{\leq n}$ and every $q \in F(s)$ exists $G(s,q) = \{p_{q,s,t} : t \in E_s\} \subseteq \mathbb{Q}$ such that the following happens:

(a) $B_{r_{n+1}}(p_{q,s,t}) \cap B_{r_{n+1}}(p_{q,s,t'}) = \emptyset$ whenever $t \neq t'$.

(b) $B_{r_{n+1}}(p_{q,s,t}) \subseteq B_{r_n}(q) \backslash B_{r_{n+1}}(q)$ for every $t \in E_s$.

For every $t \in 2^{n+1}$ (except for the constant sequence 1 of length $n+1$) define

$$F(t) = \{p_{q,s,t} : s^\frown\langle 0 \rangle \subseteq t \wedge q \in F(s)\}.$$

Observe that if $s \in 2^{<n}$ and $t \in E_s$, then for every $q \in F(s)$ exists $p \in F(t)$ that is at most distance $\sqrt{2}/2^{r_n}$ from $q$. This finishes the construction of $F$ and $\{r_n : n \in \omega\}$.

For every $x \in 2^\omega$ define $D(x) = \bigcup\{F(x \upharpoonright n) : x(n) = 1\}$.

**Claim 4.** *Let $x$ be an element of $2^\omega$. Then $D(x)$ is discrete.*

*Proof.* Let $q$ be an element of $D(x)$, so exists $n \in \omega$ such that $x(n) = 1$ and $q \in F(x \upharpoonright n)$. By condition (III) we have that $B_{r_n}(q)$ is an open neighborhood that separates $q$ from any element of $F(x \upharpoonright m)$ for $m \leq n$. On the other hand, since $x(n) = 1$, then by construction $F(t) \cap B_{r_n}(q) = \emptyset$ whenever $x \upharpoonright n+1 \subseteq t$. Thus, $B_{r_n}(q) \cap D(x) = \{q\}$. □

**Claim 5.** *Let $x \in 2^\omega$ such that $|x^{-1}(1)| = \omega$ and let $n \in \omega$ such that $x(n) = 0$. Then $F(x \upharpoonright n) \subseteq \overline{D(x)}$.*

*Proof.* Let $q$ be an element of $F(x \upharpoonright n)$. We will show that $D(x)$ contains a convergent sequence to $q$. Given $k \in \omega$, choose $m$ larger than $n$ such that $x(m+2) = 1$ and $\sqrt{2}/2^{r_m} < 1/k$. Thus, by construction there exists $p \in F(x \upharpoonright m+2)$ such that $p \in B_{r_m}(q) \subseteq \{p \in \mathbb{Q} : |q - p| < 1/k\}$. □

Now suppose that $S$ is scattered and that $E = \{x \in 2^\omega : F(x) \subseteq S\}$ has size at least $\omega_1$.

**Claim 6.** *$E$ is closed in $2^\omega$.*

*Proof.* Suppose that $z \in 2^\omega \setminus E$. Thus, there exists $n \in \omega$ such that $z(n) = 1$ and $F(z \restriction n) \nsubseteq S$ but then $\{x \in 2^\omega : (z \restriction n+1) \subseteq x\}$ is an open neighborhood of $z$ that is disjoint to $E$ . □

Since $E$ is closed (in particular compact) and uncountable, then it contains a Cantor set $C$ (see [22, page 31]). Let $T = \{x \restriction n : x \in C \wedge n \in \omega\}$ and $A = split(T)$. Note that $T$ is a perfect subtree of $2^{<\omega}$. The following claim finishes the proof.

**Claim 7.** *$F(A) = \bigcup_{s \in A} F(s)$ is a crowded set included in $S$.*

*Proof.* It is clear that $F(A) \subseteq S$. Now suppose that $q \in F(s)$ for some $s \in A$. We will show that $q$ is not an insolated point of $F(A)$. Since $C$ is a Cantor set, we can recursively construct $x$ an element of $C$ such that $s^\frown\langle 0\rangle \subseteq x$ and $N = \{n \in \omega : x \restriction n \in C' \wedge x(n) = 1\}$ is infinite. By claim 5, $D(x)$ contains a convergent sequence to $q$. Finally, since $N$ is infinite, it follows that $D(x) \cap F(A)$ contains a convergent sequence to $q$. □

□

**Corollary 8.** $cof^*(Scat) = cof^*(Disc) = \mathfrak{c}$.

Recall that an infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint* family (AD) if every two distinct elements of $\mathcal{A}$ have finite intersection and it is maximal (MAD) if it is maximal with that property. For an AD family $\mathcal{A}$ denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$. If $\mathcal{A}$ is a MAD family, we say that a forcing notion $\mathbb{P}$ destroys $\mathcal{A}$ if $\mathcal{A}$ is not longer maximal after forcing with $\mathbb{P}$ (see [4], [5]). Note that destructibility of a MAD family $\mathcal{A}$ is equivalent to destructibility of the ideal $\mathcal{I}(\mathcal{A})$.

In [13] Hrušák stated the following:

**Proposition 9.** *Let $\mathcal{A}$ be a MAD family. Then $\mathcal{A}$ is Miller indestructible if and only if $\mathcal{I}(\mathcal{A}) \nleq_K Scat$.*

Unfortunately this proposition is not true; later we will see that under CH it is posible to give a counterexample. However, inspired by Hrušák's result we ask whether the destructibility by Miller forcing can be characterized in terms of the ideal *Scat*. To answer this question we will need the following definition due to Brendle [4].

**Definition 10** (Brendle)**.** *Given a $\sigma$-ideal $\mathcal{I}$ on $\omega^\omega$, its trace ideal $tr(\mathcal{I})$ is an ideal on $\omega^{<\omega}$ defined by $a \in tr(\mathcal{I})$ if and only if $\{r : \exists^\infty n \in \omega (r \restriction n) \in a\} \in \mathcal{I}$.*

Although the use we will give to trace ideals in this paper is minimum, we consider it important to mention that they play an important role in the study of the destructibility of ideals (see [16], [4]). It turns out that for many forcings, the class of ideals on $\omega$ (or any countable set) which are destroyed can be understood in terms of the Katětov order and trace ideals. In particular, we have the following important result of Hrušák and Zapletal [16].

**Proposition 11** (Hrušák, Zapletal)**.** *Let $\mathcal{I}$ be a $\sigma$ ideal on $\omega^\omega$ such that $\mathbb{P}_\mathcal{I}$ is proper and has the continuous reading of names. If $\mathcal{J}$ is an ideal on $\omega$, then the following are equivalent:*

*(1) There is a condition $B \in \mathbb{P}_{\mathcal{J}}$ such that $B$ forces that $\mathcal{J}$ is not tall.*
*(2) There is $a \in tr(\mathcal{I})^+$ such that $\mathcal{J} \leq_K tr(\mathcal{I}) \restriction a$.*

Given a $\sigma$-ideal $\mathcal{I}$ on a Polish space $X$, we denote by $\mathbb{P}_{\mathcal{I}}$ to the set $Borel(X)/\mathcal{I}$ partial ordered by inclusion. In the case of Miller forcing, $\mathbb{PT}$ turns out to be forcing equivalent to $\mathbb{P}_{K_\sigma}$. It is well known that Sacks and Miller forcing have the the continuous reading of names (see [16]).

Using the previous proposition together with the fact that $tr(K_\sigma)$ is Katětov uniform we have the following lemma.

**Lemma 12.** *Let $\mathcal{I}$ be an ideal on a countable set. Then $\mathcal{I} \leq_K tr(K_\sigma)$ if and only if $\mathbb{PT}$ destroys $\mathcal{I}$.*

Thus, destructibility of ideals by Miller forcing can be understanding in terms of the ideal $tr(K_\sigma)$. Now we want to compare $Scat$ with $tr(K_\sigma)$.

**Proposition 13.** *Disc is not destroyed by $\mathbb{PT}$. In other words, $Disc \nleq_K tr(K_\sigma)$.*

*Proof.* We identify $\mathbb{Q}$ with $\mathbb{Q}(2^\omega)$. Let $G \subseteq \mathbb{PT}$ be a generic filter and fix $X \in [\mathbb{Q}]^\omega \cap V[G]$, so there are $Y \in [X]^\omega$ and $r \in 2^\omega$ such that $Y$ is a convergent sequence to $r$ in $V[G]$.

Case $r \in V$: For every $q \in \mathbb{Q}$ let $n_q = \min\{k \in \omega : (q \restriction k)^\frown \vec{0} = q\}$. On the other hand, for every $s \in 2^{<\omega}$ let $[s] = \{q \in \mathbb{Q} : s \subseteq q\}$, i.e, $[s]$ is the cone of $s$ as subset of $\mathbb{Q}$. Now define $Z = \{n \in \omega : Y \cap [(r \restriction n)^\frown \langle 1 - r(n)\rangle] \neq \emptyset\}$. Observe that since $Y$ is a convergent sequence to $r$, we can suppose that $|Y_n| \leq 1$ for every $n \in \omega$ where

$$Y_n = Y \cap [r \restriction n)^\frown \langle 1 - r(n)\rangle].$$

Let $\{n_k : k \in \omega\}$ be an increasing enumeration of $Z$ and define $h : \omega \longrightarrow \omega$ by

$$h(k) = \max\{n_q : q \in \bigcup_{j \leq n_{k+1}} Y_j\} + 1.$$

Since $\mathbb{PT}$ adds no dominating reals, exists $f \in V$ strictly increasing such that $f \nleq^* h$. Now for every $n \in \omega$ define $A_n = \{q \in \mathbb{Q}(2^\omega) : n_q \leq f(n)\}$ and let

$$A = \bigcup_{n \in \omega} (A_n \cap [r \restriction n)^\frown \langle 1 - r(n)\rangle]).$$

Note that $A \in V$ since $r$ and $f$ are both in $V$. On the other hand, since each $A_n$ is finite, it follows that $A$ is discrete. Finally, since $f \nleq^* h$, then $|A \cap Y| = \omega$.

Case $r \notin V$: Let $p$ be a forcing condition and $\dot{r}$, $\dot{Y}$ be names for $r$ and $Y$ such that $p \Vdash$ "$\dot{Y}$ is a convergent sequence to $\dot{r}$". Using a fusion argument if it would be necessary, we can assume that exists $\{r_s : s \in split(p)\} \subseteq 2^\omega \cap V$ such that for every $s \in split(p)$ if $s_n$ is the $\subseteq$-minimal element of $split(p)$ such that $s^\frown\langle n\rangle \subseteq s_n$, then the following happens:

(1) For each $k \in \omega$ and for all but finitely many $n \in next_p(s)$,

$$p_{s_n} \Vdash \text{“} r_s \restriction k = \dot{r} \restriction k \text{”}.$$

(2) $r_{s_n} \neq r_s$ and $r_{s_n} \neq r_{s_m}$ for each $n, m \in next_p(s)$.
(3) $\langle r_{s_n} : n \in next_p(s)\rangle$ converges to $r_s$.
(4) $p_{s_n}$ knows $y_{s_n}$ an element of $\dot{Y}$ such that

$$\triangle(y_{s_n}, r_{s_n}) > \triangle(r_{s_n}, r_s),$$

where $\triangle(s,t) = \min\{n \in \omega : s(n) \neq t(n)\}$.

Let $q$ be an extension of $p$ such that for every $s \in split(q)$, every $n \in next_q(s)$ and every $m \in next_q(s_n)$ we have that

$$\triangle(r_{s_n}, r_{(s_n)_m}) > \triangle(r_{s_n}, y_{s_n}).$$

Observe that $q$ can be obtained by removing a finite number of $s_n$ to each $s \in split(p)$. On the other hand, by condition (4) we have that for every $s \in split(q)$ and every $n \in next_q(s)$, if $k = \triangle(y_{s_n}, r_{s_n}) + 1$, then

$$[y_{s_n} \restriction k] \cap \{y_s : s \in split(q)\} = \{y_{s_n}\}.$$

Thus, $A = \{y_s : s \in split(q)\}$ is discrete and $q \Vdash$ "$|A \cap \dot{Y}| = \omega$". $\square$

**Corollary 14.** *Scat* $\not\leq_K$ $tr(K_\sigma)$.

*Proof.* It follows directly from $Disc \leq_K Scat$ and proposition 13. $\square$

**Proposition 15.** $tr(K_\sigma) \lneq_K Scat$.

*Proof.* It is enough to prove that $tr(K_\sigma) \leq_K Scat$. Once again, we identify $\mathbb{Q}$ with $\mathbb{Q}(2^\omega)$. Define $g : \omega^{<\omega} \longrightarrow 2^{<\omega}$ by induction on the lenght of the sequence:

(a) $g(\emptyset) = \emptyset$.

(b) $g(s^\frown\langle n\rangle) = g(s)^\frown\langle 0, 0, ..0\rangle^\frown\langle 1\rangle$, where $\langle 0, 0, ..0\rangle$ is the string of $n$ zeros.

In other words, if $s = \langle a_0, ..., a_n\rangle$, then $g(s) = 0^{a_0}{}^\frown 1^\frown 0^{a_1}{}^\frown 1^\frown \cdots {}^\frown 0^{a_n}{}^\frown 1$ where $0^{a_j}$ is the string of $a_j$ zeros. Observe that $g$ has the following properties:

(1) $s \subseteq t$ if and only if $g(s) \subseteq g(t)$.

(2) $g$ is injective.

(3) $g$ is onto the sequences ending in 1.

Let $\overline{g} : \omega^{<\omega} \longrightarrow \mathbb{Q}$ defined by

$$\overline{g}(s) = g(s)^\frown\vec{0}$$

where $\vec{0}$ is the constant sequence 0. Note that it follows from (2) and (3) that $\overline{g}$ is bijective.

Let $h$ be the inverse function of $g$. The following claim finishes the proof.

**Claim 16.** *Let $X$ be a crowded subset of $\mathbb{Q}$. Then $h[X]$ contains the set of splitting nodes of a Miller tree. In particular, $h$ is a witness of $tr(K_\sigma) \leq_K Scat$.*

*Proof.* It is equivalent to prove that there exists a Miller tree $T$ such that

$$\overline{g}[split(T)] \subseteq X.$$

We will recursively construct sets $T_n \subseteq \omega^{<\omega}$ such that $\overline{g}[T_n] \subseteq X$ for every $n \in \omega$. Let $t \in \omega^{<\omega}$ such that $\overline{g}(t) \in X$ and let $T_0 = \{t\}$. Assume that $n \geq 0$ and $T_0, ..., T_n$ have been constructed. For every $s \in T_n$ choose $X(s) \subseteq X$ and $E(s) \subseteq \omega^{<\omega}$ satisfying the following:

(I) $X(s)$ is a converget sequence to $\overline{g}(s)$.

(II) $g(s) \subset x$ for every $x \in X(s)$.

(III) $\overline{g}[E(s)] = X(s)$.

(I) and (II) can be done because $X$ is crowded and $\overline{g}[T_n] \subseteq X$; (III) is because $\overline{g}$ is bijective.

Observe that from (1) it follows that $s \subseteq r$ for every $r \in E(s)$. Moreover, if $r, r' \in E(s)$ and $r \neq r'$, then $r \cap r' = s$. Now let $T_{n+1} = \bigcup_{s\in T_n} E(s)$. This finishes the construction of $\{T_n : n \in \omega\}$.

Finally, consider $T' = \bigcup_{n\in\omega} T_n$ and let $T$ be the downwards closure of $T'$. It is clear that $T$ is a Miller tree with $split(T) = T'$, and by construction $\overline{g}[split(T)] \subseteq X$. □

□

Recall that given a proper ideal $\mathcal{I}$ on $X$ containing all the singletons from $X$, the *covering number* of $\mathcal{I}$ denoted by $cov(\mathcal{I})$, is the smalest number of sets in $\mathcal{I}$ with union $X$. On the other hand, observe that since any compact set of $\omega^\omega$ is included in a set of the form $\prod_{n\in\omega}[0, f(n)]$ for some $f \in \omega^\omega$, it is easy to see that $cov(K_\sigma) = \mathfrak{d}$. Finally, it is shown in [16] that

$$cov(K_\sigma) \leq cov^*(tr(K_\sigma)) \leq \max\{cov(K_\sigma), \mathfrak{d}\}.$$

Thus, $cov^*(tr(K_\sigma)) = \mathfrak{d}$ and hence we have that $cov^*(Scat) \leq \mathfrak{d}$.

We finish this section by giving an example of a MAD family $\mathcal{A}$ that is Miller indestructible but $\mathcal{I}(\mathcal{A}) \leq_K Scat$. All the following definitions and results of the section are due to Brendle, Guzmán, Hrušák and Raghavan [17].

**Definition 17.** *An ideal $\mathcal{I}$ is called Shelah-Steprāns if for every $X \in (\mathcal{I}^{<\omega})^+$ there is $Y \in [X]^\omega$ such that $\bigcup Y \in \mathcal{I}$. We say that $\mathcal{I}$ is nowhere Shelah-Steprāns if no restriction of $\mathcal{I}$ is Shelah-Steprāns.*

The property of being Shelah-Steprāns is upward closed in the Katětov order.

**Lemma 18.** *Let $\mathcal{I}, \mathcal{J}$ be ideals on countable sets. If $\mathcal{I}$ is Shelah-Steprāns and $\mathcal{I} \leq_K \mathcal{J}$, then $\mathcal{J}$ is Shelah-Steprāns.*

It is easy to see that ideals as $nwd$ and $tr(K_\sigma)$ are Shelah-*Steprāns*. Moreover, since they are also Katětov uniform it follows that they are nowhere Shelah-*Steprāns*.

**Proposition 19.** *Let $\mathcal{I}, \mathcal{J}$ be ideals such that $\mathcal{I}$ is nowhere Shelah-Steprāns and $\mathcal{J} \not\leq_K \mathcal{I}$. Then CH implies that there is a MAD family $\mathcal{A} \subseteq \mathcal{J}$ such that $\mathcal{I}(\mathcal{A}) \not\leq_K \mathcal{I}$.*

Since $Scat \not\leq tr(K_\sigma)$ and $tr(K_\sigma)$ is nowhere Shelah-*Steprāns*, then by the proposition 19 there is a MAD family $\mathcal{A} \subseteq Scat$ such that $\mathcal{I}(\mathcal{A}) \not\leq_K tr(K_\sigma)$. Since $\mathcal{I}(\mathcal{A}) \not\leq_K tr(K_\sigma)$, then $\mathcal{I}(\mathcal{A})$ is Miller indestructible and since $\mathcal{A} \subseteq Scat$, then $\mathcal{I}(\mathcal{A}) \leq_K Scat$.

## 3. The ideal Scat'

As we mentioned in the introduction, given a forcing notion $\mathbb{P}$, an ultrafilter $\mathcal{U}$ is $\mathbb{P}$-indestructible if and only if it is a reaping family in each extention by $\mathbb{P}$. In the case of Sacks forcing, indestructibility of reaping families can be characterized in terms of colorings from $2^{<\omega}$ into two colors.

The following proposition is only one special consequence of the full Halpern-Läuchli theorem.

**Proposition 20.** *Given a coloring $c : 2^{<\omega} \longrightarrow 2$, there exists a perfect tree $p \in \mathbb{S}$ and an infinite subset $A$ of $\omega$ such that $c$ is constant on $p \restriction A$.*

Let us remark that the full theorem is much stronger than the proposition. The reader can consult [12], [27], [28] and [9] to learn more about the Halpern-Läuchli theorem.

**Definition 21** (Yiparaki)**.** *A family $\mathcal{R} \subseteq [\omega]^\omega$ is called a Halpern-Läuchli if for every $c : 2^{<\omega} \to 2$ there are $p \in \mathbb{S}$ and $A \in \mathcal{R}$ such that $c$ is constant on $p \restriction A$.*

By considering colorings $c$ which are constant on the levels of the tree $2^{<\omega}$ we can easily see that every Halpern-Läuchli family is a reaping family. In fact, this property characterizes reaping families which are indestructible by the Sacks forcing.

**Theorem 22.** *(see [8]) $\mathcal{R} \subseteq [\omega]^\omega$ is a Halpern-Läuchli family if and only if it is a reaping family in every extension by Sacks forcing.*

When dealing with an ideal $\mathcal{I}$, we will say that $\mathcal{I}$ is HL as a shortcut for the statement that $\mathcal{I}^+$ is a Halpern–Läuchli family.

Before continuing, it will be necessary to establish some notation for the rest of the paper. We will write $\mathbb{Q}$ instead of $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$. For every $n \in \omega$, we use $I_n$ to denote the clopen set $(n\sqrt{2}, (n+1)\sqrt{2})$ . Finally, let $Scat'$ be the ideal on $\mathbb{Q}$ generated by $Scat \cup \{(0, n) : n \in \omega\}$.

**Observation 23.** *$A \in Scat'$ if and only if $(\forall^\infty n \in \omega)(A \cap I_n \in Scat)$.*

**Observation 24.** *$X \in (Scat')^+$ if and only if $(\exists^\infty n \in \omega)(X \cap I_n \notin Scat)$.*

It follows directly from the definition and the previous observation that $Scat'$ is Katětov uniform.

**Lemma 25.** *$Scat'$ is isomorphic to $fin \times Scat$.*

*Proof.* For every $n \in \omega$ let $f_n : I_n \longrightarrow \mathbb{Q}$ be an homeomorphism. Now consider $f : \mathbb{Q} \longrightarrow \omega \times \mathbb{Q}$ defined by $f(q) = (n, f_n(q))$ if and only if $q \in I_n$. Clearly $f$ is an homeomorphism (we are considering $\omega \times \mathbb{Q}$ with the product topology) and also satisfies $f[I_n] = \{n\} \times \mathbb{Q}$ for every $n \in \omega$. It is easy to see that $f$ is an isomorphism between $Scat'$ and $fin \times Scat$. □

From now on, we will also use $Scat'$ to refer to $fin \times Scat$, so depending on the context $A \in Scat'$ will mean that $A \subseteq \mathbb{Q}$ or $A \subseteq \omega \times \mathbb{Q}$.

Let $\pi : \omega \times \mathbb{Q} \longrightarrow \omega$ be the projection on the first coordinate. Given $A \subseteq \omega \times \mathbb{Q}$ and $n \in \omega$ define

$$A(n) = \{q \in \mathbb{Q} : (n, q) \in A\}.$$

**Lemma 26.** *Let $\mathcal{I}, \mathcal{J} \subset \mathcal{P}(\omega)$ be ideals, $\mathcal{I} \leq_K \mathcal{J}$. If $\mathcal{J}$ is HL, then so is $\mathcal{I}$.*

*Proof.* Let $f : \omega \longrightarrow \omega$ be a witness of $\mathcal{I} \leq_K \mathcal{J}$ and $W \supseteq V$ be an extension such that $\mathcal{J}^+$ is a reaping family in $W$. Let $A \in \mathcal{P}(\omega) \cap W$ and choose $R \in \mathcal{J}^+$ such that $R$ decides $f^{-1}[A]$. Then $f[R] \in \mathcal{I}^+$ and $f[R]$ decides $A$. □

**Corollary 27.** *Scat is HL.*

*Proof.* It follows because nwd is HL (see [8]) and the previous lemma. □

**Lemma 28.** *Let $\mathcal{I}$ be an ideal Katětov uniform over a countable set $X$ and let $\dot{x}$ be a $\mathbb{S}$-name for a subset of $X$. If $\mathcal{I}$ is HL, then for every $n \in \omega$ and $\{p_i : i \leq n\} \subseteq \mathbb{S}$ with mutually incompatible stems there are $E \in \mathcal{I}^+$ and $\{q_i \leq p_i : i \leq n\}$ such that every $q_i$ forces $E$ decides $\dot{x}$.*

*Proof.* Since $\mathcal{I}$ is a HL ideal there exists a condition $q_0 \leq p_0$ and $F_0 \in \mathcal{I}^+$ such that $q_0 \Vdash$ "$F_0$ *decides* $\dot{x}$". On the other hand, $\mathcal{I} \restriction F_0$ is a HL ideal and hence there are $q_1 \leq p_1$ and $F_1 \in (\mathcal{I} \restriction F_0)^+$ such that $q_1 \Vdash$ "$F_1$ *decides* $\dot{x}$". Note that $F_1 \in \mathcal{I}^+$ and hence $\mathcal{I} \restriction F_1$ is a HL ideal. Continuing this way a finite number of steps we can get a finite sequence $F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n$ of $\mathcal{I}$ positive subsets such $q_i \Vdash$ "$F_i$ *decides* $\dot{x}$" for each $i \leq n$. Finally, observe that the previous shows that if $E = F_n$, then $E \in \mathcal{I}^+$ and $q_i \Vdash$ "$E$ *decides* $\dot{x}$" for each $i \leq n$. $\square$

**Proposition 29.** *Let $\mathcal{I}$ be a Katětov uniform ideal over a countable set $X$. If $\mathcal{I}$ is HL, then so is $fin \times \mathcal{I}$.*

*Proof.* We will show that $fin \times \mathcal{I}$ is a reaping family in every extension via $\mathbb{S}$. Let $\dot{x}$ be a name for a subset of $\omega \times X$.

Using lemma 28 and a fusion argument we can recursively construct a condition $q \in \mathbb{S}$ and $E(|t|) \in \mathcal{I}^+$ ($t \in 2^{<\omega}$) such that there is a tree isomorphism $\varphi : 2^{<\omega} \to q$ and for each $t \in 2^{<\omega}$ the condition $q_{\varphi(t)} \Vdash$ "$E(|t|)$ *decides* $\dot{x}(|t|)$". Define a coloring $c : 2^{<\omega} \to 2$ by $c(t) = 1$ if and only if $q_{\varphi(t)} \Vdash$ "$E(|t|) \subseteq \dot{x}(|t|)$". By proposition 20 there are $p \in \mathbb{S}$ and $A \in [\omega]^\omega \cap V$ such that $c$ has constant value $i$ on $p \restriction A$. Note that $\bigcup_{n\in A}\{n\} \times E(n) \in (fin \times \mathcal{I})^+$. Let $q'$ be the downwards closure of $\varphi(p)$. It is clear that $q' \in \mathbb{S}$. Now if $i = 1$, then $q' \Vdash$ "$\bigcup_{n\in A}\{n\} \times E(n) \subseteq \dot{x}$" and if $i = 0$, then $q' \Vdash$ "$\bigcup_{n\in A}\{n\} \times E(n) \cap \dot{x} = \emptyset$". $\square$

**Corollary 30.** *$Scat'$ is HL.*

**Proposition 31.** *Let $\{\mathcal{J}_n : n \in \omega\}$ be a countable collection of HL Katetov uniform ideals. Then $\lim_{n\to fin} \mathcal{J}_n$ is HL.*

*Proof.* The proof is just a modification of the proof of proposition 29.

Assume that each $\mathcal{J}_n$ is an ideal on $X_n$ and let $\dot{x}$ be a $\mathbb{S}$-name for a subset of $\prod_{n\in\omega} X_n$. Let $q \in \mathbb{S}$ and $E(|t|) \in (\mathcal{J}_{|t|})^+$ ($t \in 2^{<\omega}$) such that there is a tree isomorphism $\varphi : 2^{<\omega} \to q$ and for each $t \in 2^{<\omega}$ the condition $q_{\varphi(t)}$ forces "$E(|t|)$ *decides* $\dot{x}(|t|)$". Let $c : 2^{<\omega} \to 2$ defined by $c(t) = 1$ if and only if $q_{\varphi(t)} \Vdash$ "$E(|t|) \subseteq \dot{x}(|t|)$". Let $p \in \mathbb{S}$ and $A \in [\omega]^\omega \cap V$ such that $c$ has constant value $i$ on $p \restriction A$ and let $q'$ be the downwards closure of $\varphi(p)$. Thus, $E = \bigcup_{n\in A}\{n\} \times E(n) \in (\lim_{n\to fin} \mathcal{J}_n)^+$, $q' \in \mathbb{S}$ and if $i = 1$, then $q' \Vdash$ "$E \subseteq \dot{x}$" and if $i = 0$, then $q' \Vdash$ "$E \cap \dot{x} = \emptyset$". $\square$

**Definition 32.** *For every $\alpha < \omega_1$, define a countable set $X_\alpha$ and an ideal $fin^\alpha$ on $X_\alpha$ as follows:*

*(1) $X_0 = \{0\}$ and $fin^0 = \{\emptyset\}$.*
*(2) $X_{\alpha+1} = \omega \times X_\alpha$ and $fin^{\alpha+1} = fin \times fin^\alpha$.*
*(3) If $\alpha$ is a limit ordinal, define $X_\alpha = \prod_{\beta<\alpha} X_\beta$ and*

$$fin^\alpha = \lim_{\beta\to bnd(\alpha)} fin^\beta$$

*where $bnd(\alpha)$ is the ideal of the bounded sets of $\alpha$.*

The following is well-known result.

**Lemma 33.** *Let $\alpha < \omega_1$ be a limit ordinal and let $B \subseteq \alpha$ be a strictly increasing sequence with limit $\alpha$. If $\mathcal{I} = bnd(\alpha) \restriction B$, then $fin^\alpha$ is Katětov equivalent to $\lim_{\beta\to\mathcal{I}} fin^\beta$.*

**Proposition 34.** *$fin^\alpha$ is HL for every $\alpha < \omega_1$.*

*Proof.* By induction on $\alpha$. It is clear that $fin$ is HL. Now if $\alpha = \beta + 1$, then by proposition 29, $fin^\alpha$ is HL. Finally, suppose that $\alpha$ is a limit ordinal. Let $B \subseteq \alpha$ be a strictly increasing sequence with limit $\alpha$. Note that $\mathcal{I} = bnd(\alpha) \restriction B$ is isomorphic to $fin$ and hence, by proposition 31, we have that $\lim_{\beta \to \mathcal{I}} fin^\beta$ is HL. Finally, by lemma 33, we have that $fin^\alpha$ is HL. □

We finish the section mentioning that it is easy to define examples of ideals which are not HL. Given $c : 2^{<\omega} \longrightarrow 2$, for each $p \in \mathbb{S}$ define

$$H_c(p) = \{n \in \omega : c \textit{ is constant on } p \restriction \{n\}\}.$$

Let $\mathcal{I}_c$ be the (possibly improper) ideal generated by $\{H_c(p) : p \in \mathbb{S}\}$. It is easy to see that an ideal $\mathcal{I}$ is not HL if and only if there exists $c : 2^{<\omega} \longrightarrow 2$ such that $\mathcal{I}_c \subseteq \mathcal{I}$.

## 4. Gruff ultrafilters and P-points

In this section, we present some constructions of gruff ultrafilters under CH with some additional properties. We start by giving some auxiliary results. The first lemma is just a modification of a result of D. Fernández and M. Hrušák (see [11]).

**Lemma 35.** *Let $\langle X_n : n \in \omega \rangle$ be a sequence of $Scat'$ positive sets such that $X_m \backslash X_n \in Scat'$ whenever $n \leq m$. Then exists $Y \in (Scat')^+$ such that $Y \backslash X_n \in Scat'$ for every $n \in \omega$.*

*Proof.* We will construct an increasing sequence $\langle k_n : n \in \omega \rangle$ of natural numbers and non-scattered sets $B_n \subseteq I_{k_n}$ as follows: Choose $k_0 \in \omega$ such that $X_0 \cap I_{k_0} \notin Scat$ and let $B_0 = C_0 \cap I_{k_0}$. Suppose we have picked $k_0, ..., k_n$ and $B_0, ..., B_n$. Choose $k_{n+1} > k_n$ large enough such that $X_{n+1} \cap I_{k_{n+1}} \notin Scat$ and $(X_{n+1} \backslash X_j) \cap I_m \in Scat$ for each $j \leq n$ and each $m \geq k_{n+1}$. Define $B_{n+1} = C_{n+1} \cap I_{k_{n+1}}$.

Finally, let $Y = \bigcup_{n \in \omega} B_{k_n}$. It is clear that $Y \in (Scat')^+$. On the other hand, if $n \in \omega$, then $(Y \backslash X_n) \cap I_j \in Scat$ for every $j \geq k_n$. Thus, $Y \backslash X_n \in Scat'$. □

The following two lemmas were first proven by E. van Dowen in [10], we include here the proof of one of them for completeness. For a complete proof of both lemmas see [7].

**Lemma 36.** *If $X \subseteq \mathbb{Q}$ is crowded, then it contains a perfect set.*

As a concecuence of the previous lemma, we have that if $X \in (Scat')^+$, then it contains a perfect unbounded set. In particular, it follows from lemmas 35 and 36 that if $\mathcal{F}$ is a countable filter base included in $(Scat')^+$, then there exists a perfect unbounded set $P$ such that $P \backslash F \in Scat'$.

**Lemma 37.** *Let $\mathcal{F}$ be a filter base consisting of perfect unbounded sets which extends the filter of co-bounded clopen sets. For every $R \subseteq \mathbb{Q}$ and every $F \in \mathcal{F}$ define,*

$$K_R(F) = \bigcup \{L \subseteq F : L \textit{ is crowded and } L \subseteq \overline{L \cap R}\}.$$

*Let $A \subseteq \mathbb{Q}$. Then either for $R = A$ or $R = \mathbb{Q} \backslash A$ the collection*

$$\mathcal{F} \cup \{K_R(F) : F \in \mathcal{F}\}$$

*is a free filter base consisting of perfect unbounded sets.*

*Proof.* We will divide the proof into some easy claims.

**Claim 38.** *For every $F \in \mathcal{F}$ the set $K_R(F)$ is either empty or perfect.*

*Proof.* Assume $K_R(F)$ is non-empty. Then it is crowded, being a union of crowded sets. It also satisfies $K_R(F) \subseteq \overline{K_R(F) \cap R}$ and therefore

$$K_R(F) \in \{L \subseteq F : L \textit{ is crowded and } L \subseteq \overline{L \cap R}\}.$$

On the other hand, since $\overline{K_R(F)} \subseteq F$ (because $F$ is closed), then

$$\overline{K_R(F)} \subseteq \overline{K_R(F) \cap R} \subseteq \overline{\overline{K_R(F)} \cap R}.$$

Thus,

$$\overline{K_R(F)} \in \{L \subseteq F : L \textit{ is crowded and } L \subseteq \overline{L \cap R}\}$$

and hence $K_R(F) = \overline{K_R(F)}$. □

Note that since every $F \in \mathcal{F}$ is crowded, then either $F \cap A$ contains a crowded set or $F \cap (\mathbb{Q}\backslash A)$ contains a crowded set. Thus, for each $F \in \mathcal{F}$ there exists $R \in \{A, \mathbb{Q}\backslash A\}$ such that $K_R(F)$ is perfect.

**Claim 39.** *For every $F \in \mathcal{F}$ there is $R \in \{A, \mathbb{Q}\backslash A\}$ such that $K_R(F)$ is unbounded.*

*Proof.* First observe that $K_R(G) \subseteq K_R(H)$ whenever $G \subseteq H$.

Let $F \in \mathcal{F}$ and suppose that $K_R(F)$ is bounded for each $R \in \{A, \mathbb{Q}\backslash A\}$. Let $H \in \mathcal{F}$ such that $H \subseteq F\backslash(K_A(F) \cup K_{\mathbb{Q}\backslash A}(F))$. Thus, both $K_A(H)$ and $K_{\mathbb{Q}\backslash A}(H)$ are empty, which is impossible. □

**Claim 40.** *There exists $R \in \{A, \mathbb{Q}\backslash A\}$ such that $K_R(F)$ is unbounded for every $F \in \mathcal{F}$.*

*Proof.* Suppose that the claim is not true, then there are $F, H \in \mathcal{F}$ such that $K_A(F)$ and $K_{\mathbb{Q}\backslash A}(H)$ are both bounded. Let $G \in \mathcal{F}$ be such that $G \subseteq F \cap H$. Then both $K_A(G)$ and $K_{\mathbb{Q}\backslash A}(G)$ are bounded which contradicts the previous claim. □

Let $R \in \{A, \mathbb{Q}\backslash A\}$ such that $K_R(F)$ is unbounded for every $F \in \mathcal{F}$. To prove that $\mathcal{F} \cup \{K_R(F) : F \in \mathcal{F}\}$ is a filter base it suffies to prove that $\{K_R(F) : F \in \mathcal{F}\}$ is a filter base because $K_R(F) \subseteq F$ for each $F \in \mathcal{F}$. Let $B \in [\mathcal{F}]^{<\omega}$ and choose $F \in \mathcal{F}$ such that $F \subseteq \bigcap B$. Thus, $K_R(F) \subseteq \bigcap\{K_R(H) : H \in B\}$. □

As a consequence of the proof of the previous lemma we have the following corollary.

**Corollary 41.** *Let $\mathcal{F}$ be a filter base consisting of perfect unbounded sets which extends the filter of co-bounded clopen sets. Let $P$ be a perfect unbounded set such that $K_P(F)$ is unbounded for every $F \in \mathcal{F}$. Then $\mathcal{F} \cup \{K_P(F) : F \in \mathcal{F}\}$ is a free filter base consisting of perfect unbounded sets and $K_P(F) \subseteq P \cap F$ for every $F \in \mathcal{F}$.*

In [11] Fernández and Hrušák proved that $\diamondsuit(\mathfrak{d})$ implies that there exists a gruff ultrafilter $\omega_1$-generated. Since $\diamondsuit(\mathfrak{d})$ is true in Sacks model (see [19]), then it follows from their argument that, when iterating Sacks forcing, a gruff ultrafilter appears in an intermediate model. The following theorem shows that actually there exists a gruff ultrafilter in the ground model.

**Theorem 42** (CH)**.** *There exists a gruff ultrafilter that is Sacks indestructible.*

*Proof.* Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of $\mathcal{P}(\mathbb{Q})$ and $\{c_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all colorings of $2^{<\omega}$ into two colors. We will recursively construct sets $\mathcal{F}_\alpha$ satisfying the following conditions for every $\alpha < \mathfrak{c}$ :

(1) $\mathcal{F}_\alpha$ is a countable filter base consisting of perfect unbounded sets.

(2) $\mathcal{F}_\alpha \subseteq \mathcal{F}_{\alpha+1}$.
(3) There is $F \in \mathcal{F}_{\alpha+1}$ such that $F$ decides $A_\alpha$.
(4) There are $F \in \mathcal{F}_{\alpha+1}$ and $q \in \mathbb{S}$ such that $c_\alpha$ is constant on $q \restriction F$.

We start by definig $F_0 = \{[n, \infty) : n \in \omega\}$. Assume that $0 < \alpha < \mathfrak{c}$ and $\mathscr{F}_\beta$ has been constructed for every $\beta < \alpha$. If $\alpha$ is a limit ordinal let $\mathscr{F}_\alpha = \bigcup_{\beta<\alpha} \mathscr{F}_\beta$. It is clear that $\mathcal{F}_\alpha$ satisfies all the conditions. Now suppose that $\alpha = \beta + 1$. Since $\mathscr{F}_\beta$ is a countable filter base, then by lemma 35 there exists $X \in (Scat')^+$ such that $X \backslash F \in Scat'$ for every $F \in \mathscr{F}_\beta$. Let $R \in \{A_\alpha, \mathbb{Q} \backslash A_\alpha\}$ such that $X' = X \cap R \in (Scat')^+$. Now since $Scat' \restriction X'$ is HL there are $q \in \mathbb{S}$ and $X'' \in (Scat')^+$ such that $X'' \subseteq X'$ and $c_\alpha$ is constant on $q \restriction X''$. Let $P$ be a perfect unbounded set included in $X''$, so by corollary 41 we have that $\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \{K_P(F) : F \in \mathcal{F}_\beta\}$ is a filter base and it satisfies all the desired. This finishes the construction of the $\mathcal{F}_\alpha$'s.

Finally, let $\mathcal{G}$ be the filter generated by $\bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. It is clear that $\mathcal{G}$ is a gruff ultrafilter, and by condition (4), $\mathcal{G}$ is also a Halpern-Läuchli family. Thus, $\mathcal{G}$ is a gruff ultrafilter Sacks indestructible. □

Let $\mathcal{U}$ be an ultrafilter on $X$ and $\mathcal{V}$ be an ultrafilter on $Y$, define $\mathcal{U} \times \mathcal{V}$ as the set
$$\{A \subseteq X \times Y : \{x \in X : A(x) \in \mathcal{V}\} \in \mathcal{U}\}.$$
It is easy to see that $\mathcal{U} \times \mathcal{V}$ is an ultrafilter on $X \times Y$. On the other hand, it follows directly from the definition of $\mathcal{U} \times \mathcal{V}$ that $\pi[A] \in \mathcal{U}$ for each $A \in \mathcal{U} \times \mathcal{V}$; in particular, $\pi$ shows that $\mathcal{U}^* \leq_K (\mathcal{U} \times \mathcal{V})^*$. Now consider $\omega \times \mathbb{Q}$ equipped with product topology (which is homeomorphic to $\mathbb{Q}$) and suppose that $\mathcal{U}$ is an ultrafilter on $\omega$ and that $\mathcal{G}$ is a gruff ultrafilter. If $\mathcal{B}$ is a filter base of perfect sets for $\mathcal{G}$, then
$$\{F \in \mathcal{U} \times \mathcal{G} : \forall n \in \omega(F(n) = \emptyset \text{ or } F(n) \in \mathcal{B})\}$$
is a filter base of perfect sets for $\mathcal{U} \times \mathcal{G}$. Thus, if gruff ultrafilters exist, then for any ultrafilter $\mathcal{U}$ exists a gruff ultrafilter such that $\mathcal{U}^* \leq_K \mathcal{G}^*$.

**Proposition 43.** *Assume that gruff ultrafilters exist. Then there exists a gruff ultrafilter Sacks destructible. In particular $\mathfrak{d} = \mathfrak{c}$ and $\diamondsuit(\mathfrak{d})$ imply that there exists a gruff ultrafilter Sacks destructible.*

*Proof.* Let $\mathcal{I}$ be an ideal that is not HL, and let $\mathcal{U}$ be an ultrafilter that extends $\mathcal{I}^*$. Let $\mathcal{G}$ be a gruff ultrafilter such that $\mathcal{U}^* \leq_K \mathcal{G}^*$ and suppose that $\mathcal{G}$ is Sacks indestructible. Thus, $\mathcal{G}^*$ is HL and hence, $\mathcal{U}^*$ is HL. Finally, since $\mathcal{I} \subseteq \mathcal{U}^*$, then $\mathcal{I}$ is HL, which is a contradiction. □

The next definition was introduced by J. Baumgartner in [2].

**Definition 44.** *Let $\mathcal{I}$ an ideal on $X$ and $\mathcal{U}$ be an ultrafilter on $Y$. We say that $\mathcal{U}$ is a $\mathcal{I}$-ultrafilter if for every $f : Y \longrightarrow X$ there is $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$. Equivalently, $\mathcal{U}$ is a $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \nleq_K \mathcal{U}^*$.*

It follows directly from the definition that if $\mathcal{U}$ is a $\mathcal{I}$-ultrafilter and $\mathcal{I} \leq_K \mathcal{J}$, then $\mathcal{U}$ is a $\mathcal{J}$-ultrafilter.

**Corollary 45.** *Assume that gruff ultrafilter exist. Then for every ideal $\mathcal{I}$ there exists a gruff ultrafilter that is not $\mathcal{I}$-ultrafilter.*

Many standard combinatorial properties of ultrafilters can be characterized in this way by Borel ideals of a low complexity. For example, it is known that if $\mathcal{U}$ is an ultrafilter on $\omega$, then $\mathcal{U}$ is P-point if and only if $fin \times fin \nleq_K \mathcal{U}^*$ if and only if

$conv \not\leq_K \mathcal{U}^*$. Since $conv \subseteq nwd$, then $conv \leq_K nwd$ and therefore, any P-point is a nowhere dense ultrafilter.

It is shown in [20, Prop. 5.5.5] that gruff ultrafilters cannot be P-points. However, we can ask whether there are gruff ultrafilters that are nowhere dense ultrafilters.

**Lemma 46** (Baumgartner [2])**.** *Assume Martin's Axiom for $\sigma$-centered partial orderings. Let $\mathcal{F}$ be a family of Scat positive sets such that $\mathcal{F}$ is closed under finite intersections and $|\mathcal{F}| < \mathfrak{c}$. Then for every $f : \mathbb{Q} \longrightarrow 2^\omega$ there is $B \subseteq \mathbb{Q}$ such that $f(B) \in mz$ and $B \cap F \in Scat^+$ for every $F \in \mathcal{F}$.*

**Proposition 47** (CH)**.** *There exists a gruff ultrafilter that is $mz$-ultrafilter.*

*Proof.* The proof will be just a modification of the proof of theorem 42. Let $\{A_\alpha : \alpha < \mathfrak{c}\} = \mathcal{P}(\mathbb{Q})$ and $\{f_\alpha : \alpha < \mathfrak{c}\} = \mathbb{Q}(2^\omega)^{\mathbb{Q}}$. We will construct sets $\mathcal{F}_\alpha$ such that for each $\alpha < \mathfrak{c}$ we have the following:

(1) $\mathcal{F}_\alpha$ is a countable filter base consisting of perfect unbounded sets.
(2) $\mathcal{F}_\alpha \subseteq \mathcal{F}_{\alpha+1}$.
(3) There is $F \in \mathcal{F}_{\alpha+1}$ such that $F$ decides $A_\alpha$.
(4) There is $F \in \mathcal{F}_{\alpha+1}$ such that $f_\alpha(F) \in mz$.

If either $\alpha = 0$ or $\alpha$ is a limit ordinal, define $\mathcal{F}_\alpha$ as in the proof theorem 42. So, assume $\alpha = \beta + 1$. Let $P$ be a perfect unbounded set such that $P$ decides $A_\alpha$ and $P \backslash F \in Scat'$ for every $F \in \mathcal{F}_\beta$. Let $\overline{\mathcal{F}}$ the closure under finite intersections of $\mathcal{F}_\beta \cup \{P\}$. Since $\mathcal{F}_\beta$ is countable, then $\overline{\mathcal{F}}$ is also countable. Now use lemma 46 to get $B \subseteq \mathbb{Q}$ such that $f_\alpha(B) \in mz$ and $B \cap F \in Scat^+$ for every $F \in \overline{\mathcal{F}}$. Note that $B \cap P$ must be a $Scat'$ positive set, since $[n, \infty) \in \overline{\mathcal{F}}$ for every $n \in \omega$. Finally, let $P'$ be a perfect unbounded set included in $P \cap B$ and define $\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \{K_{P'}(F) : F \in \mathcal{F}_\beta\}$.

Let $\mathcal{G}$ be the filter generated by $\bigcup_{\alpha<\mathfrak{c}} \mathcal{F}_\alpha$. Then by construction $\mathcal{G}$ is a gruff ultrafilter $mz$-ultrafilter. □

**Corollary 48** (CH)**.** *There exists a gruff ultrafilter that is nwd-ultrafilter.*

*Proof.* Since $mz \subseteq nwd$, then $mz \leq_K nwd$. Thus, any $mz$-ultrafilter is a $nwd$-ultrafilter. □

The previous corollary tells us that under CH we can construct gruff ultrafilters that are "close" to P-points. Inspired on this an on the proof of theorem 42 we have the following definition.

**Definition 49.** *Let $\mathcal{G}$ be a gruff ultrafilter. We say that $\mathcal{G}$ is a $P_{Scat'}$-point if for every $\mathcal{F} \in [\mathcal{G}]^\omega$ exists $P \in \mathcal{G}$ such that $P \backslash F \in Scat'$ for every $F \in \mathcal{F}$.*

**Corollary 50.** *CH implies that $P_{Scat'}$-point exist.*

The following proposition gives us an interesting relation between P-points and $\mathrm{P}_{Scat'}$-points.

**Proposition 51.** *Let $\mathcal{G}$ be $P_{Scat'}$-point on $\omega \times \mathbb{Q}$. Then $\pi[\mathcal{G}] = \{\pi[A] : A \in \mathcal{G}\}$ is a P-point.*

*Proof.* Let $\{U_n : n \in \omega\}$ be a countable subset of $\pi[\mathcal{G}]$. For every $n \in \omega$ define $A_n = \pi^{-1}[U_n] = U_n \times \mathbb{Q}$ and note that each $A_n$ is an element of $\mathcal{G}$. Let $P$ be an element of $\mathcal{G}$ such that $P \backslash A_n \in Scat'$ for every $n \in \omega$. We can suppose that $P$ is perfect unbounded and therefore $P \cap (\{n\} \times \mathbb{Q}) \notin Scat$ for every $n \in \omega$. Now if

$\pi[P]\backslash U_n$ were infinite for some $U_n$, then $P\backslash\pi^{-1}[U_n] \notin Scat'$. Thus, $\pi[P]$ is almost included in each $U_n$. $\square$

Although in general P$_{Scat'}$-point is a stronger property than P-point, we can show that both properties turn out to be equivalent for gruff ultrafilters which are products.

**Proposition 52.** *Let $\mathcal{U}$ be an ultrafilter on $\omega$ and $\mathcal{G}$ be a gruff ultrafilter. Then $\mathcal{U}\times\mathcal{G}$ is $P_{Scat'}$-point if and only if $\mathcal{U}$ is P-point.*

*Proof.* The only thing left is to prove that if $\mathcal{U}$ is P-point, then $\mathcal{U}\times\mathcal{G}$ is $P_{Scat'}$-point. Let $\{A_n : n\in\omega\}$ be a countable subset of $\mathcal{U}\times\mathcal{G}$. Without loss of generality the collection of $A_n$'s is $\subseteq$-decreasing. For every $n\in\omega$ define $U_n = \{k\in\omega : A_n(k)\in\mathcal{G}\}$, it is clear that each $U_n$ belongs to $\mathcal{U}$. Let $U$ be an element of $\mathcal{U}$ almost included in each $U_n$ and let $\langle x_n : n\in\omega\rangle$ be a strictly increasing sequence of natural numbers such that $U\backslash x_n \subseteq U_n$. For every $n\in\omega$ define $B_n = \bigcup\{\{x\}\times A_n(x) : x\in [x_n, x_{n+1})\cap U\}$. Now consider $B=\bigcup_{n\in\omega} B_n$. Observe that $B\in\mathcal{U}\times\mathcal{G}$ because $\{k\in\omega : B(k)\in\mathcal{G}\} = U\backslash x_0\in\mathcal{U}$. Finally, $B\backslash A_n \subseteq \bigcup_{k<n} B_k \in Scat'$ for every $n\in\omega$. $\square$

We finish this section by proving that P-point and P$_{Scat'}$-point are not equivalent properties.

**Theorem 53** (CH)**.** *There exists $\mathcal{G}$ a gruff ultrafilter on $\omega\times\mathbb{Q}$ such that $\pi[\mathcal{G}]$ is P-point but $\mathcal{G}$ is not $P_{Scat'}$-point.*

*Proof.* Let $\{A_\alpha : \alpha<\mathfrak{c}\}$ be an enumeration of all subsets of $\mathbb{Q}$. For every $n\in\omega$ define

$$P_n = [n,\infty)\times\bigcup_{k\geq n} I_k$$

and let $\mathcal{F} = \{P_n : n\in\omega\}$.

We will recursively construct sets $\mathcal{F}_\alpha$ satisfying the following conditions for every $\alpha<\mathfrak{c}$:

(1) $\mathcal{F}\subseteq\mathcal{F}_\alpha$.

(2) $\mathcal{F}_\alpha$ is a countable filter base consisting of perfect unbounded sets.

(3) There exists $P\in\mathcal{F}_{\alpha+1}$ such that $P$ decides $A_\alpha$.

(4) There exists $U\in\pi[\mathcal{F}_{\alpha+1}]$ almost included in each element of $\pi[\mathcal{F}_\alpha]$.

(5) For every $P\in\mathcal{F}_\alpha$ there exists $n\in\omega$ such that for all but finitely many $m\in\pi[P]$ we have that $P(m)\cap(0,n\sqrt{2})\notin Scat$.

Start by doing $\mathcal{F}_0=\mathcal{F}$ and if $\alpha$ is a limit ordinal let $\mathcal{F}_\alpha = \bigcup_{\beta<\alpha}\mathcal{F}_\beta$. So, we can suppose that $\alpha=\beta+1$ and that $\mathscr{F}_\beta$ has already been constructed. Let $U$ be an infinite subset of $\omega$ almost included in each element of $\pi[\mathcal{F}_\beta]$ and define $\mathcal{F}'_\beta = \{P\cap U\times\mathbb{Q} : P\in\mathcal{F}_\beta\}$. Note that $\mathcal{F}'_\beta$ is a countable filter base consisting of perfect unbounded sets and satisfies condition (5).

For every $P\in\mathcal{F}'_\beta$, every $X\in[U]^\omega$ and every $R\in\{A_\alpha, A_\alpha^c\}$ define the following properties:

$$*(P,X,R) := \exists n\in\omega\exists Y\in[X]^\omega(\forall m\in Y)(P(m)\cap(0,n\sqrt{2})\cap R(m)\in Scat^+).$$

$$\triangle(P,X,R) := \exists n\in\omega(\forall^\infty m\in X)(P(m)\cap(0,n\sqrt{2})\cap R(m)\in Scat^+).$$

**Claim 54.** *For every $P \in \mathcal{F}'_\beta$ and every $X \in [U]^\omega$, exists $R \in \{A_\alpha, A_\alpha^c\}$ such that $*(P, X, R)$ holds.*

*Proof.* It is clear from condition (5). $\square$

**Claim 55.** *Let $P$, $X$ and $R$ be such that $*(P, X, R)$ is false. Then $*(F, Y, R^c)$ holds for every $F \in \mathcal{F}'_\beta$ and every $Y \in [X]^\omega$.*

*Proof.* Suppose that there are $F \in \mathcal{F}'_\beta$ and $Y \in [X]^\omega$ such that $*(F, Y, R^c)$ is false. Then $*(K, Y, R^c)$ and $*(K, Y, R)$ are both false for every $K \in \mathcal{F}'_\beta$ included in $P \cap F$, which contradicts the previous claim. $\square$

**Claim 56.** *There are $X \in [U]^\omega$ and $R \in \{A_\alpha, A_\alpha^c\}$ such that $\triangle(P, X, R)$ holds for every $P \in \mathcal{F}'_\beta$.*

*Proof.* By the previous claim we can assume that there are $R \in \{A_\alpha, A_\alpha^c\}$ and $Y \in [X]^\omega$ such that $*(P, Z, R)$ holds for every $P \in \mathcal{F}'_\beta$ and every $Z \in [Y]^\omega$.

Let $\{F_n : n \in \omega\}$ be an enumeration of $\mathcal{F}'_\beta$. Now recursively construct $\{Y_k : k \in \omega\}$ a collection of infinite subsets of $Y$ and $\{n_k : k \in \omega\}$ a subset of $\omega$ such that for every $k \in \omega$ we have the following:

(I) $Y_k \supseteq Y_{k+1}$.

(II) $(\forall m \in Y_k)(F_k(m) \cap (0, n_k\sqrt{2}) \cap R(m) \in Scat^+)$.

Let $X$ be an infinite subset of $Y$ almost included in each $Y_n$. Then by construction $\triangle(P, X, R)$ holds for every $P \in \mathcal{F}'_\beta$. $\square$

Let $X$, $R$ be as in the previous claim and enumerate $\mathcal{F}'_\beta$ as $\{F_n : n \in \omega\}$. Now consider $\{E_k : k \in \omega\}$ a sequence of perfect unbounded sets and $\{n_k : k \in \omega\}$ a sequence of natural numbers constructed as follows:

Let $n_0$ be a natural number and $Y_0$ be an infinite subset of $X$ such that:

(a) $X \setminus Y_0$ is finite.

(b)$(\forall m \in Y_0)(F_0(m) \cap (0, n_0\sqrt{2}) \cap R(m) \in Scat^+)$.

For every $m \in Y_0$ choose a perfect set $E_0(m) \subseteq F_0(m) \cap (0, n_0\sqrt{2}) \cap R(m)$ and define

$$E_0 = \bigcup_{m \in Y_0} \{m\} \times E_0(m).$$

Suppose we already defined $E_0, ..., E_k$ and $n_0, ..., n_k$. Now choose $n_{k+1} \in \omega$ and $Y_{k+1} \in [X]^\omega$ such that $X \setminus Y_{k+1}$ is finite and $P(m) \cap (0, n_{k+1}\sqrt{2}) \cap R(m) \in Scat^+$ for every $m \in Y_{k+1}$, where $P = (\bigcap_{i \leq k} F_i) \cap P_{n_k}$. Define $E_{k+1}$ in an analogous way to $E_0$. This finishes the construction of $\{E_k : k \in \omega\}$ and $\{n_k : k \in \omega\}$.

Note that $\{n_k : k \in \omega\}$ is strictly increasing and hence $\{E_k : k \in \omega\}$ is pairwise disjoint. So, $E = \bigcup_{n \in \omega} E_n$ is a perfect unbounded set included in $R$ and it also satisfies that for every $P \in \mathcal{F}_\beta$ exists $n \in \omega$ such that for all but finitely many $m \in \pi[E]$ we have that $P(m) \cap (0, n\sqrt{2}) \notin Scat$. Thus, it is easy to see that $\mathcal{F}_\alpha = \mathcal{F}_\beta \cup \{K_E(P) : P \in \mathcal{F}_\beta\}$ satisfies all the conditions (See collolary 41).

To finish the proof, let $\mathcal{G}$ be the filter generated by $\bigcup_{\alpha < \mathfrak{c}} \mathcal{F}_\alpha$. Clearly $\mathcal{G}$ is gruff and $\pi[\mathcal{G}]$ is a P-point. On the other hand, observe that by construction $\mathcal{F} \subseteq \mathcal{G}$ and there is not $P \in \mathcal{G}$ such that $P \setminus P_n \in Scat'$ for every $n \in \omega$. $\square$

**Corollary 57.** *CH implies that there are gruff ultrafilters on $\omega \times \mathbb{Q}$ that are not a product, i.e, they do not have the form $\mathcal{U} \times \mathcal{G}$ where $\mathcal{U}$ is an ultrafilter on $\omega$ and $\mathcal{G}$ is a gruff ultrafilter.*

## 5. The generic ultrafilter of $(fin \times Scat)^+$

This section is inspired by a result by A. Blass, N. Dobrinen and D. Raghavan about the generic ultrafilter adds by $(fin^2)^+$ in [1, theorem 36]. For the rest of the paper, we will use $\mathbb{P}$ to denote $(fin \times Scat)^+$ partially ordered by inclusion and $\mathbb{P}_{Scat'}$ to denote $\mathcal{P}(\omega \times \mathbb{Q})/(fin \times Scat)$ partially ordered by inclusion modulo the ideal $fin \times Scat$. Observe that $\mathbb{P}_{Scat'}$ is the separative quotient of $\mathbb{P}$ and therefore they are forcing equivalent; i.e, they produce the same extensions.

We call a forcing condition $X \in \mathbb{P}$ (or $X \in \mathbb{P}_{Scat'}$) *standard* if every nonempty section $X(n)$ is crowded, i.e, any nonempty section $X(n)$ is homeomorphic to $\mathbb{Q}$. It is easy to see that the collection of standard conditions is dense on $\mathbb{P}$.

Note that lemma 35 implies that $\mathbb{P}_{Scat'}$ is $\sigma$-closed and in particular, forcing with $\mathbb{P}$ adds no new reals. On the other hand, observe that the function $a : [\omega]^\omega \longrightarrow \mathbb{P}$ defined by $a[X] = \pi^{-1}[X]$ is a complete embedding of the forcing notion $([\omega]^\omega, \subseteq)$ into $\mathbb{P}$. In particular, if $\mathcal{G} \subseteq \mathbb{P}$ is generic, then $a^{-1}[\mathcal{G}] = \pi[\mathcal{G}]$ is $([\omega]^\omega, \subseteq)$-generic and therefore $\pi[\mathcal{G}]$ is selective.

**Lemma 58.** *If $\mathcal{G}$ is $\mathbb{P}$ generic, then $\mathcal{G}$ is gruff ultrafilter in $V[\mathcal{G}]$.*

*Proof.* Since $\mathbb{P}$ adds no reals, it is clear that $\mathcal{G}$ is an ultrafilter in $V[\mathcal{G}]$.

Let $A$ be a forcing condition and $\dot{X}$ a name for a subset of $\omega \times \mathbb{Q}$ such that $A \Vdash$ "$\dot{X} \in \dot{\mathcal{G}}$". Since $\mathbb{P}$ adds no reals, there are $B \leq A$ and $X \in \mathbb{P}$ such that $B \Vdash$ "$X = \dot{X}$". Note that $B \Vdash$ "$B \in \dot{\mathcal{G}}$". Now let $P$ be a perfect unbounded set included in $B \cap X$. Thus, $P \leq A$ and $P \Vdash$ "$P \subseteq \dot{X} \wedge P \in \dot{\mathcal{G}}$". This finishes the proof. □

**Lemma 59.** *Exists a sequence $\langle \mathcal{P}_n : n \in \omega \rangle$ of partions of $\mathbb{Q}$ such that every $\mathcal{P}_n$ consists in two dense subsets and for every $P \in \prod_{n\in\omega} \mathcal{P}_n$ exists a crowded set $C$ almost disjoint to each $P(n)$.*

*Proof.* For every $s \in 2^{<\omega}$ we will construct a dense subset $D_s$ of $\mathbb{Q}$ as follows: We start by doing $D_\emptyset = \mathbb{Q}$. If $s \in 2^{<\omega}$ and $D_s$ has already been defined, then divide $D_s$ into two dense subsets $D_{s^\frown 0}$ and $D_{s^\frown 1}$. Observe that:

(a) For every $n \in \omega$ we have that $\mathbb{Q} = \bigcup_{s\in 2^n} D_s$.
(b) If $s \subseteq t$, then $D_t \subseteq D_s$.
(c) If $s \not\subseteq t$ and $t \not\subseteq s$, then $D_s \cap D_t = \emptyset$.

For every $n \in \omega$ define $\mathcal{P}_n = \{P_n(0), P_n(1)\}$ where

$$P_n(0) = \bigcup\{D_s : s \in 2^{n+1} \wedge s(n) = 0\},$$

$$P_n(1) = \bigcup\{D_s : s \in 2^{n+1} \wedge s(n) = 1\}.$$

It is clear that every $\mathcal{P}_n$ is a partition of $\mathbb{Q}$ into two dense pieces.

Let $\langle P_n(i(n)) : n \in \omega \wedge i \in 2^\omega \rangle$ be and element to $\prod_{n\in\omega} \mathcal{P}_n$. Let $\langle X_n : n \in \omega \rangle$ be a decreasing sequence of dense subsets of $\mathbb{Q}$ getting as follows:

$$X_0 = D_{\langle 1-i(0)\rangle} = \mathbb{Q} \backslash D_{\langle i(0)\rangle} = \mathbb{Q} \backslash P_0(i(0)).$$

In general, if $n \geq 1$ then

$$X_n = X_{n-1} \backslash \bigcup\{D_s : s \in 2^{n+1} \wedge s(n) = i(n)\} = X_{n-1} \backslash P_n(i(n)).$$

Recursively construct a sequence of finite subsets of $\mathbb{Q}$ as follows: Choose $q \in X_0$ and define $F_0 = \{q\}$. If $n > 0$ and $F_n$ has already been defined, then for each $q \in F_n$

choose $p_q \in X_{n+1}$ such that $|q - p_q| < 1/2^{n+1}$. This can be done because $X_{n+1}$ is dense. Finally, if $C = \bigcup_{n\in\omega} F_n$, then $C$ is crowded and $C \cap P_n(i(n)) \subseteq F_n$ for every $n \in \omega$. $\square$

**Definition 60.** *Let $\mathcal{U}$ be an ultrafilter on a countable set $X$. We say that $\mathcal{U}$ is a weak P-point if for any countably many non-principal ultrafilters $\mathcal{V}_n \neq \mathscr{U}$, there exists $A \in \mathcal{U}$ such that $A \notin \mathcal{V}_n$ for every $n \in \omega$.*

It follows directly from the definition that any P-point is a weak P-point, however, it can be proved that there are weak P-points that are not P-points. Moreover, in contrast to P-points, weak P-points actually exists in ZFC. In [23], Kunnen proved that there are $2^{\mathfrak{c}}$ weak P-points that are not P-points.

As we mentioned before, gruff ultrafilters cannot be P-points, however, they can be weak P-points.

**Theorem 61.** *If $\mathcal{G}$ is $\mathbb{P}$ generic, then $\mathcal{G}$ is weak P-point in $V[\mathcal{G}]$.*

*Proof.* The proof is analogous to the proof of theorem 36 in [1] with obvious changes. Let $\{\mathcal{V}_n : n \in \omega\} \subseteq V[\mathcal{G}]$ be a countable collection of ultrafilters on $\omega \times \mathbb{Q}$. It will be useful to distinguish four types of ultrafilters $\mathcal{V} \neq \mathcal{G}$.

(0) $\pi[\mathcal{V}]$ is principal.
(1) $\pi[\mathcal{V}]$ is non-principal and distinct from $\mathcal{U} = \pi[\mathcal{G}]$.
(2) $\pi[\mathcal{V}] = \mathcal{U}$ y $\mathcal{V} \cap (fin \times Scat) \neq \emptyset$.
(3) $\pi[\mathcal{V}] = \mathcal{U}$ y $\mathcal{V} \cap (fin \times Scat) = \emptyset$.

We will show that if $i \in 4$ and all $\mathcal{V}_n$ belong to type $(i)$, then there exists $A \in \mathcal{U}$ such that $A \notin \mathcal{V}_n$ for every $n \in \omega$. It will suffice because in the general case we can divide $\{\mathcal{V}_n : n \in \omega\}$ in four sets, one for each type, find suitable sets $A$ for each of the four subsets and take the intersection of those $A$'s.

**Type (0):** Let $X$ be a forcing condition such that:
(a) $X$ is standard.
(b) $X \Vdash$"The $\dot{\mathcal{V}}_n$ are ultrafilters of type (0)".
(c) $X$ knows a function $f \in \omega^\omega \cap V$ such that $X \Vdash$"$\pi[\dot{\mathcal{V}}_n]$ is generated by $\{f(n)\}$" for every $n \in \omega$.
(d) $\pi[X] \subseteq ran(f)$.

Conditions $(a)$, $(b)$ and $(c)$ are easy because $D = \{X \in \mathbb{P} : X \text{ is standard}\}$ is dense and $\mathbb{P}$ adds no new reals. For $(d)$, if $Z = \pi[X]\backslash ran(f)$ is infinite then $Y = \pi^{-1}[Z] \cap X$ is an extension of $X$ and $Y \Vdash$ "$Y \in \dot{\mathcal{G}}\backslash\dot{\mathcal{V}}_n$" for each $n \in \omega$. So, $Z$ is finite and therefore $Y = X\backslash\pi^{-1}[Z]$ is a condition that satisfies $(d)$.

For the rest of the proof we will recursively construct a $\subseteq$-decreasing sequence of standard conditions $\langle X_n : n \in \omega\rangle$, an increasing sequence of natural numbers $\langle x_n : n \in \omega\rangle$ and a sequence of $Scat$ positive sets $\langle A_n : n \in \omega\rangle$ such that:
(I) If $i < j$, then $\{x_i\} \times A_i \subseteq X_j$.
(II) If $f(n) = x_i$, then $X_{i+1} \Vdash$ "$\{x_i\} \times (\mathbb{Q}\backslash A_i) \in \dot{\mathscr{V}}_n$".

Start by doing $X_0 = X$. Suppose that $k \in \omega$ and we already constructed $X_i$ for $i \leq k$ and also $x_j, A_j$ for $j < k$. We want to construct $X_{k+1}$, $x_k$ and $A_k$.

Choose $x_k$ any element of $\pi[X_k]$ larger than $x_{k-1}$ (to get $x_0$ simply choose any element of $\pi[X_0]$) and define $S = X_k(x_k)$. Observe that $S$ is homeomorphic to $\mathbb{Q}$

because $X_k$ is a standard condition. Let $\langle \mathcal{P}_n : n \in \omega \rangle$ be a sequence of partitions of $S$ as in the lemma 59. Now consider $N = \{n \in \omega : f(n) = x_k\}$. Since $X_k$ is an extension of $X$ then $X_k$ knows that for each $n \in N$ one of the following happens:

(1) $\{x_k\} \times \mathbb{Q} \backslash S \in \dot{\mathcal{V}}_n$.
(2) $\{x_k\} \times P_n(0) \in \dot{\mathcal{V}}_n$.
(3) $\{x_k\} \times P_n(1) \in \dot{\mathcal{V}}_n$.

Let $Y$ be an extension of $X_k$ that decides these options for each $n \in N$. Define $h \in 2^\omega$ as follows: If $n \in N$ and $Y \Vdash$ "$\{x_k\} \times P_n(i) \in \dot{\mathcal{V}}_n$", then $h(n) = i$. In another case define $h$ arbitrarily. Now consider $\langle P_n(h(n)) : n \in \omega \rangle$ and use lemma 59 to find $A_k$ a crowded subset of $S$ such that $A_k \cap P_n(h(n))$ is finite for each $n \in \omega$. Observe that by construction $Y \Vdash$ "$\{x_k\} \times (\mathbb{Q} \backslash A_k) \in \dot{\mathcal{V}}_n$" for each $n \in N$.

To finish the step $k$, obtain $X_{k+1}$ from $Y$ by removing all scattered sections and adjoining all the sets $\{x_j\} \times A_j$ for $j \leq k$. Observe that $X_{k+1}$ is a standard condition included in $X_k$.

This completes the construction of the sequences of $X_k$'s, $x_k$'s and $A_k$'s. Finally, define

$$Y = \bigcup_{k \in \omega} \{x_k\} \times A_k.$$

It is clear that $Y$ is a standard condition and $Y \subseteq X_k$ for each $k \in \omega$. We claim that $Y \Vdash$ "$(\omega \times \mathbb{Q}) \backslash Y \in \dot{\mathcal{V}}_n$" for every $n \in \omega$. Consider an arbitrary $n \in \omega$. If $f(n)$ is not of the form $x_k$ for some $k \in \omega$, we finished because $Y \cap (\{f(n)\} \times \mathbb{Q}) = \emptyset$ and $\{f(n)\} \times \mathbb{Q} \in \mathcal{V}_n$. Assume that $f(n) = x_k$. Then by construction, $X_{k+1} \Vdash$ "$\{x_k\} \times (\mathbb{Q} \backslash A_k) \in \dot{\mathcal{V}}_n$". Finally, since $Y$ is an extension of $X_{k+1}$ and $(\{x_k\} \times (\mathbb{Q} \backslash A_k)) \cap Y = \emptyset$, it follows that $Y \Vdash$ "$(\omega \times \mathbb{Q}) \backslash Y \in \dot{\mathcal{V}}_n$".

**Type (1):** Since $\mathcal{U}$ is selective in $V[\mathcal{G}]$, in particular it is a weak P-point. So there exists $A \in \mathcal{U}$ such that $A \notin \pi[\mathcal{V}_n]$ for every $n \in \omega$. Thus, $\pi^{-1}[A] \in \mathcal{G} \backslash \mathcal{V}_n$ for every $n \in \omega$.

**Type (2):** In $V[\mathcal{G}]$ choose $B_n$ such that $B_n \in \mathcal{V}_n \cap (fin \times Scat)$ for every $n \in \omega$. Fix $n \in \omega$ and $F_n$ a finite subset of $\omega$ such that $B_n(k) \in Scat$ whenever $k \notin F_n$. If $F = \bigcup_{k \in F_n} \{k\} \times B_n(k)$, then either $F \in \mathcal{V}_n$ or $B_n \backslash F \in \mathcal{V}_n$. If $F \in \mathcal{V}_n$, then $F_n \in \pi[\mathcal{V}_n]$ but this is impossible because $\pi[\mathcal{V}_n] = \mathcal{U}$ and $\mathcal{U}$ is non-principal. So, $B_n \backslash F \in \mathcal{V}_n$. Since $n$ was arbitrary, we can suppose that $B_n(k) \in Scat$ for every $n, k \in \omega$.

For every $k \in \omega$ define $A(k) = \bigcup_{n \leq k} B_n(k)$. It is clear that every $A(k)$ is scattered and therefore

$$A = \bigcup_{k \in \omega} \{n\} \times A(k) \in fin \times Scat.$$

Using a previous argument it is easy to see that $A \in \mathcal{V}_n$ for every $n \in \omega$. On the other hand, $A \notin \mathcal{U}$ since $\mathcal{U}$ is a subset of $(fin \times Scat)^+$. Thus, the complement of $A$ is in $\mathcal{U} \backslash \mathcal{V}_n$ for very $n \in \omega$.

**Type (3):** Let $X$ be a condition that forces that the $\dot{\mathcal{V}}_n$ are ultrafilters of type (3). Since $\mathbb{P}_{Scat'}$ is $\sigma$-closed we can extend $X$ to a condition $Y$ that forces, for each $n \in \omega$, a specific set $A_n$ in the ground model to be in $\dot{\mathcal{U}} \backslash \dot{\mathcal{V}}_n$. Observe that

$Y \setminus A_n \in fin \times Scat$ and therefore $Y \Vdash$ "$Y \setminus A_n \notin \dot{\mathcal{V}}_n$" for every $n \in \omega$. Thus, $Y \Vdash$ "$Y \in \dot{\mathcal{G}} \setminus \dot{\mathcal{V}}_n$" for every $n \in \omega$. □

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