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IDEAL INDEPENDENT FAMILIES AND THE ULTRAFILTER NUMBER

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Abstract. We say that $\mathcal{I}$ is an ideal independent family if no element of $\mathcal{I}$ is a subset mod finite of a union of finitely many other elements of $\mathcal{I}$. We will show that the minimum size of a maximal ideal independent family is consistently bigger than both $\mathfrak{d}$ and $\mathfrak{u}$, this answers a question of Donald Monk.

§1. On a question of Monk. Let $\mathfrak{s}_{mm}$ be the minimal cardinality of a maximal ideal independent family i.e., an infinite family $\mathcal{B} \subseteq [\omega]^\omega$ such that no element of $\mathcal{B}$ is a subset mod finite of a union of finitely many other elements of $\mathcal{B}$ and $\mathcal{B}$ is maximal with respect to this property, i.e., for any $X \in [\omega]^\omega \setminus \mathcal{B}$, it cannot be added to $\mathcal{B}$ and remain ideal independent. This means that there is $F \in [\mathcal{B}]^{<\omega}$ such that either $X \subseteq^* \bigcup F$ or there is $B \in \mathcal{B} \setminus F$ with $B \subseteq^* X \cup \bigcup F$. We will compare $\mathfrak{s}_{mm}$ with the ultrafilter number and the dominating number, for the definition and basic properties of the usual cardinal invariants see Blass [4].

In May 2013 at a conference at the Ben-Gurion University of the Negev Donald Monk asked if $\mathfrak{s}_{mm}$ was equal to $\mathfrak{u}$ (this question was communicated to Arnold Miller by Juris Steprans). In this note, we will provide a negative answer to this question.

It is interesting to determine which ideals can be generated by ideal independent families. As far as we know, this topic has been mostly unexplored. The next two results tackle this problem. Although they will not be needed in this note, they are useful to get intuition on ideal independent families. Our hope is that this will motivate the study of which ideals can be generated by ideal independent families.

PROPOSITION 1. *If $\mathcal{I}$ is an ideal generated by a strictly $\subseteq^*$-ascending sequence $A_\alpha \subseteq \omega$ for $\alpha < \omega_1$, then $\mathcal{I}$ is not generated by an ideal independent family.*

PROOF. By hypothesis, $\mathcal{I} = \{B : \exists \alpha < \omega_1 \ B \subseteq^* A_\alpha\}$. Suppose $\mathcal{B} \subseteq \mathcal{I}$ generates $\mathcal{I}$, we will prove that $\mathcal{B}$ is not ideal independent. For each α choose $F_\alpha \subseteq \mathcal{B}$ finite such that $A_\alpha \subseteq^* \cup F_\alpha$. By a direct application of the Δ-system lemma, we can assume that $\{F_\alpha : \alpha \in \omega_1\}$ is a delta system. Now, for $\alpha \in \omega_1$ there is $\beta > \alpha$ such that $\bigcup F_\alpha \subseteq^* A_\beta$. But $A_\beta \subseteq^* \bigcup F_\beta$, so for any $B \in (F_\alpha \setminus F_\beta)$, $B \subseteq^* \cup F_\beta$ and this implies that $\mathcal{B}$ is redundant. $\square$

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Although many ideals can be generated by an ideal independent family, this is not the case for the prime ideals.

PROPOSITION 2. *A non-principal prime ideal $\mathcal{I}$ on ω cannot be generated by an ideal independent family.*

PROOF. Suppose $\mathcal{B}$ is an ideal independent family generating $\mathcal{I}$. Let $\{A_n : n < \omega\} \subseteq \mathcal{B}$ be distinct. By adding at most one thing to each A_n we may suppose $\cup_{n<\omega} A_n$ is ω. Let

$$B = \bigcup_n (A_{2n} \setminus \cup_{i<2n} A_i) \text{ and } C = \bigcup_n (A_{2n+1} \setminus \cup_{i<2n+1} A_i)$$

and note these are complementary sets. If $B \in \mathcal{I}$ then for some finite $F \subseteq \mathcal{B}$ we have $B \subseteq^* \cup F$. But this means $A_{2n} \subseteq^* \bigcup F \cup \bigcup_{i<2n} A_i$ for n large enough, which contradicts that $\mathcal{B}$ is ideal independent. We conclude that $A_{2n} \notin F$. The argument for $C \in \mathcal{I}$ is similar. $\square$

The next proposition answers Monk's question negatively. In the rational perfect set model $\mathfrak{d} = \omega_2$ and $\mathcal{U} = \omega_1$, see Miller [12] and Blass-Shelah [3].

THEOREM 3. $\max\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{s}_{mm}$.

PROOF. Given a maximal ideal independent family $\mathcal{I}$, it is easy to see that the following family of sets is a reaping family:

$$\{A \setminus \bigcup F : F \in [\mathcal{I}]^{<\omega} \wedge A \in \mathcal{I} \setminus F\}.$$

It remains to prove that $\mathfrak{d} \leq \mathfrak{s}_{mm}$. Assume otherwise that $\mathfrak{s}_{mm} < \mathfrak{d}$, and let $\mathcal{A}$ be a witness for this. Note that $\omega =^* \bigcup \mathcal{A}$, so we can assume that indeed the equality holds. Let $\{A_n : n \in \omega\} \subseteq \mathcal{A}$ be such that its union is ω, We may assume that $A_n \neq A_m$ whenever $n \neq m$. Define $C_0 = A_0$ and $C_{n+1} = A_{n+1} \setminus \bigcup_{i \leq n} A_i$. For each $F \in [\mathcal{A}]^{<\omega}$ and $B \in \mathcal{A} \setminus (F \cup \{A_i : i < \omega\})$, define a function as follows:

$$\varphi_{F,B}(n) = \min\{k \in \omega : (\exists j \geq n)(C_j \cap B \cap k \setminus \bigcup F \neq \emptyset)\}.$$

Since the family $\mathcal{A}$ is ideal independent, the functions $\varphi_{F,B}$ are always well defined. Let h_0 be an increasing function not dominated by

$$\{\varphi_{F,B} \ : \ F \in [\mathcal{A}]^{<\omega},\ B \in \mathcal{A} \setminus (F \cup \{A_i : i < \omega\})\}.$$

Define $D_n = C_n \setminus h_0(n)$. Now for each $F \in [\mathcal{A}]^{<\omega}$, whenever it is possible, define a function as follows:

$$\tilde{\varphi}_F(n) = \min\{k \in \omega : (\exists j \geq n)(D_j \cap k \setminus \bigcup F \neq \emptyset)\}.$$

This is defined for n, otherwise

$$\bigcup_{j \geq n} D_j = \bigcup_{j \geq n} (C_j \setminus h_0(j)) \subseteq \bigcup F.$$

But then for some $j \geq n$ such that $A_j \notin F$ we would have

$$A_j \subseteq^* \bigcup_{i<j} A_i \cup \bigcup F$$

which contradicts that the family is ideal independent.

Let $h_1 > h_0$ be an increasing function not dominated by any totally defined $\tilde{\varphi}_F$ for $F \in [\mathcal{A}]^{<\omega}$ and such that $C_n \cap [h_0(n), h_1(n))$ is nonempty for all n.

Let

$$Y = \bigcup_{n\in\omega} (C_n \cap [h_0(n), h_1(n))) = \bigcup_{n\in\omega} D_n \cap h_1(n).$$

Let's see that $\mathcal{A} \cup \{Y\}$ is an ideal independent family.

Claim 1. *For all $F \in [\mathcal{A}]^{<\omega}$, $Y \not\subseteq^* \bigcup F$.*

If the function $\tilde{\varphi}_F$ is not defined, then $Y \cap \bigcup F$ is finite. Otherwise, by the definition of the function $\tilde{\varphi}_F$, if $\tilde{\varphi}_F(n) \leq h_1(n)$, then for some $j \geq n$ we have $D_j \cap \tilde{\varphi}_F(n) \setminus \bigcup F \neq \emptyset$, which implies

$$\emptyset \neq D_j \cap h_1(n) \setminus \bigcup F \subseteq D_j \cap h_1(j) \setminus \bigcup F \subseteq Y.$$

Since this happens for infinitely j and the family $\{D_j : j \in \omega\}$ is disjoint, we are done.

Claim 2. *For any $F \in [\mathcal{A}]^{<\omega} \setminus \{\emptyset\}$ and $B \in \mathcal{A} \setminus F$, we have $B \not\subseteq^* Y \cup \bigcup F$.*

If $B = A_n$ for some n this is clear. Otherwise, by the definition of $\varphi_{F,B}$ and the choice of h_0, we have that if $\varphi_{F,B}(n) \leq h_0(n)$, then for some $j \geq n$,

$$\emptyset \neq C_j \cap B \cap \varphi_{F,B}(n) \setminus \bigcup F \subseteq C_j \cap B \cap h_0(j) \setminus \bigcup F.$$

If $m \in C_j \cap B \cap h_0(j) \setminus \bigcup F$, then $m \notin Y \cup \bigcup F$. Since this happens infinitely many times, we are done. $\square$

We will now show that $\mathfrak{s}_{mm}$ can be smaller than the continuum, in fact this holds in the side by side countable support Sacks model.

Theorem 4. *In the side by side countable support Sacks model there is a maximal ideal independent family of size ω_1. In this model the continuum can be made arbitrarily large but $\mathfrak{s}_{mm} = \mathcal{U} = \mathfrak{d} = \omega_1$.*

Proof. We are forcing with the countable support product of κ-many Sacks posets for any κ over a model of CH.

To get a maximal ideal independent family which remains maximal after forcing, It is enough work with the ω-product of Sacks forcing $\mathbb{P} = \mathbb{S}^\omega$. We will give a short explanation of why this is the case. Let κ be any cardinal number, $p \in \mathbb{S}^\kappa$ and τ an $\mathbb{S}^\kappa$-name such that $p \Vdash$ "$\tau \in [\omega]^\omega$." Let M be a countable elementary submodel of $H(\theta)$ (for a large enough θ) such that $\kappa, p, \tau \in M$. Since $\mathbb{S}^\kappa$ is proper, we may now find an M-genric condition $q \leq p$ such that the support of q is $M \cap \kappa$. Note that below q, it is the case that τ is an $\mathbb{S}^{M\cap\kappa}$-name and clearly $\mathbb{S}^{M\cap\kappa}$ is isomorphic to $\mathbb{S}^\omega$ since M is countable.

By Laver's combinatorial generalization of the Halpern–Lauchli Theorem [11], for any $\mathbb{P}$-name τ for a subset of ω and $p \in \mathbb{P}$ there is $q \leq p$ and $Z \in [\omega]^\omega$ such that either

$$q \Vdash \text{“}Z \subseteq \tau \text{ or } q \Vdash Z \cap \tau = \emptyset.\text{”}$$

As Laver points out this may be used to build a descending mod finite sequence $Z_\alpha \in [\omega]^\omega$ for $\alpha < \omega_1$ in the ground model with the property that they generate a Ramsey ultrafilter in the extension.

LEMMA 5. *Given* $(Y_n \in [\omega]^\omega : n < \omega)$ *pairwise disjoint in the ground model,* τ *a* $\mathbb{P}$*-name for a subset of* ω*, and* $p \in \mathbb{P}$*, there are* $W_n \in [Y_n]^\omega$ *and* $q \leq p$ *such that*

$$q \Vdash \text{“}\forall n \ (W_n \subseteq \tau \text{ or } W_n \cap \tau = \emptyset).\text{”} \qquad \square$$

PROOF OF LEMMA. Let $(f_n : \omega \to Y_n)_n$ be a sequence of bijections in the ground model and define $\tau_n = f_n^{-1}(\tau)$. Let G be generic with $p \in G$. We go to the generic extension $V[G]$. Since $\{Z_\alpha \mid \alpha \in \omega_1\}$ is a $\subseteq^*$-decreasing sequence that generates a (Ramsey) ultrafilter, we can find $\alpha < \omega_1$ such that:

$$\forall n \in \omega\big(\big(Z_\alpha \subseteq^* \tau_n^G\big) \vee \big(Z_\alpha \subseteq^* \omega \setminus \tau_n^G\big)\big).$$

(Note that we are using that ω_1 was not collapsed, which follows by the properness of $\mathbb{P}$). Define $U_n = f_n[Z_\alpha]$ (clearly U_n is a subset of Y_n). Note that either U_n is almost contained in τ^G or is almost disjoint from it. In this way, we can define a function $g : \omega \longrightarrow \omega$ such that for every $n \in \omega$, either $U_n \setminus g(n) \subseteq \tau^G \cap Y_n$ or $U_n \setminus g(n) \subseteq Y_n \setminus \tau^G$. Moreover, since $\mathbb{P}$ is ω^ω-bounding, we can find a ground model function h dominating h. Let $W_n = U \setminus f(n)$ (note that W_n is a ground model set). It follows that either $W_n \subseteq \tau^G \cap Y_n$ or $W_n \subseteq Y_n \setminus \tau^G$. Back to the ground model, we can find a condition $q \in \mathbb{P}$ extending p such that $q \Vdash$“ $\forall n((W_n \subseteq \tau \cap Y_n) \vee (W_n \subseteq Y_n \setminus \tau))$”. This finishes the proof of the lemma. $\square$

Now we continue with the proof of Proposition 4. Using the Continuum Hypothesis and the fact that $\mathbb{S}^\omega$ is proper, it is possible to construct a sequence $\{(p_\alpha, \tau_\alpha) \mid \alpha \in \omega_1\}$ where $p_\alpha \in \mathbb{S}^\omega$ and τ_α is an $\mathbb{S}^\omega$-name for a subset of ω such that if τ is an $\mathbb{S}^\omega$-name for a subset of ω and $p \in \mathbb{S}^\omega$, then there is $\alpha < \omega_1$ such that $p_\alpha \leq p$ and $p_\alpha \Vdash$ “$\tau_\alpha = \tau$.″ We may further assume that each (p_α, τ_α) appears uncountably many times. By recursion, construct an increasing family of countable ideal independent families $\mathcal{I}_\alpha$ for $\alpha \in [\omega, \omega_1)$. Start with a partition of ω, say $\mathcal{I}_\omega = \{A_n : n \in \omega\}$.

At stage $\alpha + 1$ proceed as follows: let $\{A_n : n \in \omega\}$ be a reenumeration of $\mathcal{I}_\alpha$, and then define $B_n = (A_n \setminus \cup_{i<n} A_i)$, construct $Y_n \in [B_n]^\omega$ such that Y_n are infinite pairwise disjoint, $B_n \setminus Y_n$ is infinite, and $Y_n \cap A_k$ is finite for $k \neq n$ (this is possible since $\mathcal{I}_\alpha$ is an ideal independent family).

By Lemma 5, there is $\{W_n \in [Y_n]^\omega : n < \omega\}$ and $q \leq p_\alpha$ such that

$$q \Vdash \text{“}\forall n \ (W_n \subseteq \tau_\alpha \text{ or } W_n \cap \tau_\alpha = \emptyset).\text{”}$$

Take $W = \cup_{n<\omega}(B_n \setminus W_n)$ and let $\mathcal{I}_{\alpha+1} = \mathcal{I}_\alpha \cup \{W\}$. At limit steps just take $\mathcal{I}_\alpha$ to be the union of all the previous constructed families. It is not hard to see that this family is indeed ideal independent. We claim that τ is forced by q to never be added to our ideal independent family. Let G be generic with $q \in G$.

If for some n, $W_n \subseteq \tau_\alpha^G$, then $W \cup \tau_\alpha^G$ covers B_n and hence τ^G, W, A_i for $i < n$ cover A_n.

If for all $n W_n \cap \tau_\alpha^G = \emptyset$, then $\tau^G \subseteq W$ since the B_n partition ω, and so the pair is redundant.

Hence $\mathcal{I} = \bigcup_{\alpha<\omega_1} \mathcal{I}_\alpha$ will be a maximal ideal independent family in the ground model which remains a maximal ideal independent family in the generic extension.

§2. Parametrized Diamonds. Recall the following definition by Vojtáš [15]

DEFINITION 6. We say $(A, B, \to)$ is an *invariant* if,

(1) $\to \subseteq A \times B$.
(2) For every $a \in A$ there is $b \in B$ such that $a \to b$.
(3) There is no $b \in B$ such that $a \to b$ for all $a \in A$.

We say that $D \subseteq B$ is dominating if for every $a \in A$ there is a $d \in D$ such that $a \to d$, so (2) means that B is dominating and (3) that no singleton is dominating. Given an invariant $(A, B, \to)$ we define it's *evaluation* by $\langle A, B, \to \rangle = min\{|D| : D \subseteq B \text{ and } D \text{ is dominating}\}$. An invariant $(A, B, \to)$ is called Borel if A, B and $\to$ are Borel subsets of a polish space. Most of the usual (but not all) invariants are actually Borel invariants. In [6] for any Borel invariant $(A, B, \to)$, a guessing principle $\diamondsuit(A, B, \to)$ is defined and it is proved that it implies $\langle A, B, \to \rangle \leq \omega_1$ and it holds in most of the natural models where this inequality holds. For our applications in this note, we need to work in a slightly more general framework than the one in [6]. The following is a particular case of the diamond principles introduced in [8] and [7]:

DEFINITION 7. We say an invariant $(A, B, \to)$ is an $L(\mathbb{R})$- *invariant* if A, B and $\to$ are subsets of Polish spaces and all three of them belong to $L(\mathbb{R})$.

Following [6] we define the following guessing principle for any $L(\mathbb{R})$-invariant $(A, B, \to)$.

DEFINITION 8. $\diamondsuit_{L(\mathbb{R})}(A, B, \to)$
For every $C : 2^{<\omega_1} \to A$ such that $C \restriction 2^\alpha \in L(\mathbb{R})$ for all $\alpha < \omega_1$ there is a $g : \omega_1 \to B$ such that for every $R \in 2^{\omega_1}$ the set $\{\alpha \mid C(R \restriction \alpha) \to g(\alpha)\}$ is stationary.

Exactly as in the Borel case, $\diamondsuit_{L(\mathbb{R})}(A, B, \to)$ implies $\langle A, B, \to \rangle \leq \omega_1$. Given two $L(\mathbb{R})$-invariants $\mathbb{A} = (A_-, A_+, _{\mathbb{A}}\to)$ and $\mathbb{B} = (B_-, B_+, _{\mathbb{B}}\to)$ we define the *sequential composition* $\mathbb{A};\mathbb{B} = (A_- \times Bor(B_-^{A_+}), A_+ \times B_+, \to)$ where $Bor(B_-^{A_+})$ denotes the set of codes of all Borel functions from A_+ to B_- and $(a_-, f) \to (a_+, b_+)$ if $a_{-\mathbb{A}} \to a_+$ and $f(a_+)_{\mathbb{B}} \to b_+$. It is easy to see that $\mathbb{A};\mathbb{B}$ is an $L(\mathbb{R})$-invariant and in [4] it is proved that $\langle \mathbb{A};\mathbb{B} \rangle = max\{\langle \mathbb{A} \rangle, \langle \mathbb{B} \rangle\}$.
As usual we will write $\mathfrak{d}$ instead of $(\omega^\omega, \omega^\omega, \leq^*)$ and $\mathfrak{r}_\sigma$ instead of the invariant $\big(([\omega]^\omega)^\omega, [\omega]^\omega, \text{is } \sigma\text{-reaped}\big)$ (we say that $\langle X_n \rangle_{n \in \omega} \in ([\omega]^\omega)^\omega$ is *σ-reaped by* $A \in [\omega]^\omega$ if for every $n \in \omega$, either $A \subseteq^* X_n$ or $A \subseteq^* \omega \setminus X_n$).

THEOREM 9. $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ *implies* $\mathfrak{s}_{mm} = \omega_1$.

PROOF. We need to define a function F into $[\omega]^\omega \times Bor\big((\omega^\omega)^{[\omega]^\omega}\big)$ such that for all $\alpha \in \omega_1$, $F \restriction 2^\alpha$ is in $L(\mathbb{R})$. For each $\alpha < \omega_1$, let $e_\alpha : \omega \to \alpha$ be an enumeration of α in $L(\mathbb{R})$. By a suitable coding, we can assume that the domain of F is the set

$$\bigcup_{\alpha \in \omega_1} [\omega]^\omega \times ([\omega]^\omega)^\alpha.$$

Given $(A, \vec{\mathcal{I}}) \in [\omega]^\omega \times ([\omega]^\omega)^\alpha$ proceed as follows. If $\mathcal{I} = \langle \mathcal{I}_{e_\alpha(n)} : n \in \omega \rangle$ is not an ideal independent family, define $F(A, \vec{\mathcal{I}}) = (\omega, e)$, where $e(X)$ for $X \in [\omega]^\omega$ is the enumeration of X. Otherwise, define $B_n^{\vec{\mathcal{I}}} = I_{e_\alpha(n)} \setminus \bigcup_{i<n} I_{e_\alpha(i)}$. For each n, let $Z_n^{\vec{\mathcal{I}}} \subseteq B_n^{\mathcal{I}}$ be an infinite subset such that for all $\beta \neq e_\alpha(n)$, $Z_n^{\mathcal{I}} \cap I_\beta$ is finite,[1] and let $\varphi_{\vec{\mathcal{I}},n}$ be a recursive enumeration of $Z_n^{\vec{\mathcal{I}}}$. Then define $A_n = \varphi_{\vec{\mathcal{I}},n}^{-1}[Z_n^{\vec{\mathcal{I}}} \cap A]$. Now define a function $f_{A,\vec{\mathcal{I}}} : [\omega]^\omega \to \omega^\omega$ as follows: if $X \in [\omega]^\omega$ reaps A_n for all n, then define

$$f_{A,\vec{\mathcal{I}}}(X)(n) = \min\{k \in \omega : X \setminus k \subseteq A_n \vee (X \setminus k) \cap A_n = \emptyset\}$$

Otherwise define $f_{A,\vec{\mathcal{I}}}(X)$ to be the identity function. Finally, the value of F in $(A,\vec{\mathcal{I}})$ is given by $F(A,\vec{\mathcal{I}}) = (\langle A_n : n \in \omega\rangle, f_{A,\vec{\mathcal{I}}})$. Let $g : \omega_1 \longrightarrow [\omega]^\omega \times \omega^\omega$ be a $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$-guessing sequence for F. We can assume that for all α the set A_α in $g(\alpha) = (A_\alpha, h_\alpha)$ is coinfinite. Recursively define an ideal independent family as follows:

(1) Start with a partition of ω into infinitely many infinite sets $\vec{\mathcal{I}}_\omega = \langle I_n : n \in \omega\rangle$.
(2) Suppose we have defined $\vec{\mathcal{I}}_\alpha = \langle I_\beta : \beta < \alpha\rangle$. Now define I_α as follows:

$$I_\alpha = \bigcup_{n\in\omega} B_n^{\vec{\mathcal{I}}_\alpha} \setminus \varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)].$$

Let $\vec{\mathcal{I}}_{\alpha+1}$ be the family $\langle I_\beta : \beta \leq \alpha\rangle$. Finally, let $\mathcal{I} = \langle I_\alpha : \alpha \in \omega_1\rangle$ be the family obtained by the above recursion. Let's see that $\mathcal{I}$ is a witness for $\mathfrak{s}_{mm}$.

CLAIM 1. *$\mathcal{I}$ is an ideal independent family. We proceed by induction of $\alpha \in \omega_1$. Clearly $\mathcal{I}_\omega$ is ideal independent. Assume $\vec{\mathcal{I}}_\alpha$ is an ideal independent family. Then $\vec{\mathcal{I}}_{\alpha+1}$ is ideal independent, let $H_1 = \{\mathcal{I}_\beta \mid \beta \in H\}$:*

(a) *$I_\alpha \not\subseteq^* \bigcup H_1$. Let $n \in \omega$ be such that H is contained in the set $\{e_\alpha(0), \dots, e_\alpha(n)\}$, so $\bigcup H_1 \subseteq \bigcup_{i\leq n} B_i^{\vec{\mathcal{I}}_\alpha}$. By the definition of I_α, $I_\alpha \setminus \bigcup_{i\leq n} B_i^{\vec{\mathcal{I}}_\alpha}$ is infinite.*
(b) *For all $\beta \in \alpha \setminus H$, $I_\beta \not\subseteq^* I_\alpha \cup \bigcup H_1$. Let n be such that $\beta = e_\alpha(n)$. By the choice of $Z_n^{\vec{\mathcal{I}}_\alpha}$, we have that for any $\gamma \in \alpha \setminus \{\beta\}$, $Z_n^{\vec{\mathcal{I}}_\alpha} \cap I_\gamma$ is finite, so in particular, $Z_n^{\vec{\mathcal{I}}_\alpha} \cap \bigcup H_1$ is finite. Also by the construction of I_α, $B_n^{\vec{\mathcal{I}}_\alpha} \cap I_\alpha \cap \varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)]$ is finite. This both facts together give $\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)] \setminus I_\alpha \cup \bigcup H_1$ is infinite. Since $\varphi_n^{\vec{\mathcal{I}}_\alpha}[A_\alpha \setminus h_\alpha(n)] \setminus I_\alpha \cup \bigcup H_1 \subseteq I_\beta \setminus I_\alpha \cup \bigcup H_1$, we are done.*

CLAIM 2. *$\mathcal{I}$ is maximal. Pick any $X \in [\omega]^\omega$. If g guesses $(X, \langle I_\alpha : \alpha \in \omega_1\rangle)$ in γ, then we have that $A_\gamma \sigma$-reaps $\langle X_n : n \in \omega\rangle$ and h_γ almost dominates the function $l = f_{X,\mathcal{I}_\gamma}(A_\gamma)$. There are two cases:*

(i) *There are infinitely many $n \in \omega$ such that $A_\gamma \subseteq^* X_n$. Pick n such that $l(n) \leq h_\gamma(n)$. Then $A_\gamma \setminus h_\gamma(n) \subseteq X_n$, so $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq X \cap B_n^{\vec{\mathcal{I}}_\gamma}$. Then by the definition of I_γ, $B_n^{\vec{\mathcal{I}}_\gamma} \subseteq I_\gamma \cup \varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq I_\gamma \cup X$, which implies $I_{e_\gamma(n)} \subseteq X \cup I_\gamma \cup \bigcup_{i<n} I_{e_\gamma(n)}$.*
(ii) *For almost all $n \in \omega A_\gamma \subseteq^* \omega \setminus X_n$. Then for almost all n, $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq Z_n^{\vec{\mathcal{I}}_\gamma} \setminus X$, so for almost all n, $X \cap Z_n^{\vec{\mathcal{I}}_\gamma} \subseteq I_\gamma$, and for finitely many n, $A_\gamma \subseteq^* X_n$, so $\varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq^* Z_n^{\vec{\mathcal{I}}} \cap X \subseteq B_n \cap X$, which implies $B_n \setminus X \subseteq^* B_n \setminus \varphi_n^{\vec{\mathcal{I}}_\gamma}[A_\gamma \setminus h_\gamma(n)] \subseteq I_\gamma$. Putting all this together we have that $X \subseteq^* I_\gamma \cup \bigcup_{i\leq k} B_i$, for some $k \in \omega$.* $\square$

[1] $Z_n^{\vec{\mathcal{I}}} \subseteq B_n^{\mathcal{I}}$ should be found in a recursive way and should depend only on $\vec{\mathcal{I}}$.

It is interesting that we are using $\mathfrak{r}_\sigma$ instead of $\mathfrak{r}$. We do not know if we can prove the result above using only $\mathfrak{r}$.

The following result was proved by Hiroaki Minami [13] for Borel invariants, however, the proof for $L(\mathbb{R})$-invariants is the same.

PROPOSITION 10. *Let* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_1 \rangle$ *a finite support iteration of ccc forcings and* $(A, B, \to)$ *be an* $L(\mathbb{R})$ *-invariant with the following property: For all* $\alpha < \omega_1$ *there is* $b \in B \cap V[G_{\alpha+1}]$ *such that* $a \to b$ *for all* $a \in A \cap V[G_\alpha]$. *Then* $\mathbb{P}_{\omega_1} \Vdash$ "$\diamondsuit_{L(\mathbb{R})}(A, B, \to)$."

With the previous proposition we can conclude the following:

COROLLARY 11. *There is a finite support iteration of ccc forcings of length* ω_1 *such that* $\mathbb{P}_{\omega_1} \Vdash$ "$\mathfrak{s}_{mm} = \omega_1$."

PROOF. Define $\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha$ for $\alpha < \omega_1$ as follows. Let $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{M}(\dot{\mathcal{U}}_\alpha) * \dot{\mathbb{H}}$" where $\dot{\mathcal{U}}_\alpha$ is the name of any ultrafilter, $\mathbb{M}(\dot{\mathcal{U}}_\alpha)$ is its Mathias forcing and $\mathbb{H}$ is Hechler forcing. It is well known that the Mathias forcing associated to an ultrafilter is σ-centered, and the same is true for the Hechler forcing, so we have an iteration of *ccc* forcings. Let us see that the second condition of the theorem holds for this iteration.

Pick any $(\langle X_n : n \in \omega \rangle, F) \in [\omega]^\omega \times Borel((\omega^\omega)^{[\omega]^\omega}) \cap V[G_\alpha]$, where G_α is the generic for $\mathbb{P}_\alpha$. We claim that $(\dot{x}_\alpha, \dot{f}_\alpha)$, the generic for $\dot{\mathbb{Q}}_\alpha$, bounds $(\langle X_n : n \in \omega \rangle, F)$. To see that $\dot{x}_\alpha \sigma$-reaps $\langle X_n : n \in \omega \rangle$, just note that $X_n \in \mathcal{U}_\alpha$ or $\omega \setminus X_n \in \mathcal{U}_\alpha$, and since the generic $\dot{x}_\alpha$ is almost contained in every element of $\mathcal{U}_\alpha$, it follows that $\dot{x}_\alpha \sigma$-reaps $\langle X_n : n \in \omega \rangle$. Now, $F(\dot{x}_\alpha)$ is a function in $V[G_\alpha][\dot{x}_\alpha]$, and the generic f_α is dominating over $V[G_\alpha][\dot{x}_\alpha]$, so in particular $F(\dot{x}_\alpha) \leq^* f_\alpha$, as required. Finally, by the previous proposition it follows that $\diamondsuit_{L(\mathbb{R})}(\mathfrak{r}_\sigma; \mathfrak{d})$ holds in $\mathbb{P}_{\omega_1}$ and then $\mathfrak{s}_{mm}$ is equal to ω_1 in the extension. $\square$

In [6] it is shown that for any Borel invariant $(A, B, \to)$, most countable support iterations of proper forcings that force $\langle A, B, \to \rangle \leq \omega_1$ will also force $\diamondsuit(A, B, \to)$. This is also true for $L(\mathbb{R})$-invariants with the same proof.

THEOREM 12. *Let* $\langle \mathbb{Q}_\alpha \mid \alpha \in \omega_2 \rangle$ *be a sequence of Borel proper partial orders where each* $\mathbb{Q}_\alpha$ *is forcing equivalent to* $\wp(2)^+ \times \mathbb{Q}_\alpha$ *and let* $\mathbb{P}_{\omega_2}$ *be the countable support iteration of this sequence. If* $(A, B, \to)$ *is an* $L(\mathbb{R})$*-invariant and* $\mathbb{P}_{\omega_2} \Vdash$ "$\langle A, B, \to \rangle \leq \omega_1$" *then* $\mathbb{P}_{\omega_2} \Vdash$ "$\diamondsuit_{L(\mathbb{R})}(A, B, \to)$."

We will need the following notion:

DEFINITION 13. By $\mathbb{P}_{fin}$ we denote the set of trees $p \subseteq \omega^{<\omega}$ with the following properties:

(1) p is a finitely branching tree.
(2) For every $n \in \omega$, there is $l \in \omega$ such that if $s \in T$ and $|s| \geq l$ then $|suc_T(s)| \geq n$

Given $p, q \in \mathbb{P}_{fin}$ define $p \leq q$ if $p \subseteq q$.

The forcing $\mathbb{P}_{fin}$ has been studied (in much more generality) by Laflamme (see [10]), Zapletal (see [16] section 4.4.3, where $\mathbb{P}_{fin}$ is a particular case of a fat tree forcing) and by Hrušák and Hernández (see [9]). Furthermore, the following is a particular case of Theorem 4.4.8 of [16]:

PROPOSITION 14 ([16]). $\mathbb{P}_{fin}$ *is proper,* ω^ω*-bounding and does not add splitting reals.*

Moreover, by theorem 3.4.1 of [16] it follows that $\mathbb{P}_{fin}$ preserves Ramsey ultrafilters. The following result is well known, we prove it for the sake of completeness:

LEMMA 15. *Forcing with $\mathbb{P}_{fin}$ makes $\omega^\omega \cap V$ a meager set.*

PROOF. Let $G \subseteq \mathbb{P}_{fin}$ be a generic filter. Let $r_{gen} = \bigcap G$, it is easy to see that $r_{gen} \in \omega^\omega$ (moreover, r_{gen} and G are interdefinable). For every $n \in \omega$, define $N_n = \{x \in \omega^\omega \mid \forall m > n(r_{gen}(m) \neq x(m))\}$ and let $M = \bigcup\limits_{n\in\omega} N_n$. It is easy to see that each N_n is a nowhere dense set, so M is a meager set. Furthermore, a simple genericity argument shows that $\omega^\omega \cap V \subseteq M$. $\square$

We can now prove the following:

THEOREM 16. *There is a model where $\mathfrak{s}_{mm} < non(\mathcal{M})$ (in particular, the inequality $\mathfrak{s}_{mm} < \mathfrak{i}$ is consistent).*

PROOF. We perform a countable support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \leq \omega_2 \rangle$ where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \mathbb{P}_{fin}$." We claim that this is the model we are looking for. Let $G \subseteq \mathbb{P}_{\omega_2}$ be a generic filter. Since $\mathbb{P}_{fin}$ is ω^ω-bounding, then $\mathfrak{d} = \omega_1$ holds in $V[G]$. By theorem of Zapletal, there is a Ramsey ultrafilter in $V[G]$ of character ω_1, so $\mathfrak{r}_\sigma = \mathfrak{u} = \omega_1$ holds as well. By the previous remarks, we conclude that $\diamondsuit_{\omega_1}(\mathfrak{r}_\sigma; \mathfrak{d})$ holds in $V[G]$, so $\mathfrak{s}_{mm}$ is equal to ω_1 in the extension. Finally, since $\mathbb{P}_{fin}$ makes the ground model meager, so $non(\mathcal{M}) = \omega_2$ holds in the extension (finally, recall that $non(\mathcal{M}) \leq cof(\mathcal{M}) \leq \mathfrak{i}$ by [1]). $\square$

§3. Final remarks. By the results in the previous sections, it might be conjectured that $\mathfrak{s}_{mm} = max\{\mathfrak{d}, \mathfrak{r}_\sigma\}$ (note this equality holds in all the Cohen, random, Hechler, Sacks, Laver, Mathias and Miller models), however, it can be proved that this is not the case: the inequality $\max\{\mathfrak{r}_\sigma, \mathfrak{d}\} < \mathfrak{s}_{mm}$ holds in Shelah's model from [2]. The proof that $\mathfrak{s}_{mm}$ is big in this model follows the same lines for proving that the almost disjoint number $\mathfrak{a}$, is the cardinality the continuum. Since giving this proof in detail is very technical, we just mention this result without proof.

There are still some interesting questions for which we don't know the answer:

QUESTION 17. *Is* $\mathfrak{u} \leq \mathfrak{s}_{mm}$?

QUESTION 18. *Is* $\mathfrak{s}_{mm} \leq \mathfrak{i}$?

We would like to remark that in [14] Shelah built a model of $\mathfrak{i} < \mathfrak{u}$ (see also [5]) so in that model one of the questions has a negative answer, but we do not know which one.

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