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Some Structural Aspects of the Katětov Order on Borel Ideals

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Abstract We prove that the Katětov order on Borel ideals (1) contains a copy of $\mathscr{P}(\omega)/\mathbf{Fin}$, consequently it has increasing and decreasing chains of lenght $\mathfrak{b}$; (2) the sequence $\mathbf{Fin}^{\alpha}$ ($\alpha < \omega_1$) is a strictly increasing chain; and (3) in the Cohen model, Katětov order does not contain any increasing nor decreasing chain of length $\mathfrak{c}$, answering a question of Hrušák (2011).

Keywords Borel ideals · Katětov order · Cohen model

1 Introduction

The *Katětov order* is defined on ideals on ω as follows: $\mathsf{I} \leq_K \mathsf{J}$ if there is a function $f \in \omega^\omega$ such that $f^{-1}(I) \in \mathsf{J}$ for all $I \in \mathsf{I}$. This order is a generalization of the better-known Rudin-Keisler order. It is a powerful tool for the study of some properties about ideals and filters, like Ramsey type properties, Fubini property, classes of ultrafilters, destructibility of ideals by forcing, among other (see [3, 7, 8]). Frequently, the combinatorial properties about ideals have definable critical ideals in the Katětov order. In this paper we study some structural aspects of the Katětov order restricted to the family of Borel ideals, as an

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order-type. Clearly, the Katětov order among the whole family of ideals is more complicated. Another fragment of it that has been studied is the family of ideals generated by maximal almost-disjoint families (see [6]). Two relevant structural properties of Katětov order on definable ideals, the named *category and measure dichotomies*, are proved by Hrušák in [5]. A more complete study about ideals on ω is available in [4].

The properties we are going to prove are described in the abstract and they correspond with the number of each section. The notation we use is standard, and mainly follows [10].

2 Summable Ideals in the Katětov Order

We now prove that the Katětov order on Borel ideals contains a copy of $\mathscr{P}(\omega)/\mathbf{Fin}$, ordered by $\subseteq^*$. More specifically, this copy is contained inside the family of summable ideals. This result is analogous to another obtained by Ilijas Farah (Theorem 1.12.1(c) in [2]) about the Rudin-Blass order. Recall that an ideal I is *summable* if there is a function f from ω to $[0, \infty)$ satisfying $\lim_{n\to\infty} f(n) = 0$, $\sum_{n\in\omega} f(n) = \infty$ and

$$\mathsf{I} = \mathsf{I}_f := \left\{ A \subseteq \omega : \sum_{n\in A} f(n) < \infty \right\}.$$

Clearly, by definition, summable ideals are tall F_σ P-ideals. Let us denote by Σ the family of summable ideals.

Theorem 1 *There is an order embedding φ from $\mathscr{P}(\omega)/\mathbf{Fin}$ into Σ.*

Proof Let us recursively construct two sequences of real numbers p_n and r_n ($n \in \omega$) as follows: $r_0 = 1$, $p_0 = 0$, $p_{n+1} \geq ((r_n + 1)p_n + 1)\, r_n^{-1}$ and $r_{n+1} = 2^{-n-1}(p_{n+1} - p_n)^{-1}$: By definning intervals $I_n = [p_n, p_{n+1} - 1]$ we have constructed a partition of ω in intervals satisfying $\min(I_{n+1}) = \max(I_n) + 1$ and

(1) $|I_n| r_n \geq |\bigcup_{j<n} I_j|$ and
(2) $|I_n| r_{n+1} \leq 2^{-n-1}$.

For each infinite subset A of ω, let us define a function $f_A : \omega \to (0, 1]$ such that for every $k < \omega$

$$f_A(k) = \begin{cases} r_n & \text{if } k \in I_n \text{ and } n \notin A \\ r_{n+1} & \text{if } k \in I_n \text{ and } n \in A \end{cases}$$

□

The Theorem follows immediately from claims below.

Claim For every infinite and coinfinite subset A of ω, I_{f_A} is a non-trivial tall ideal.

Proof (Proof of claim) Note that

$$\sum_{n<\omega} f_A(n) = \sum_{j<\omega} \sum_{n\in I_j} f_A(n) \geq \sum_{j\in\omega\setminus A} r_j |I_j| \geq \sum_{j\in\omega\setminus A} |\bigcup_{i<j} I_i| = \infty.$$

□

Claim If $A, B \in [\omega]^{\omega}$ and $A \subseteq^* B$ then $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$.

Proof (Proof of claim) Note that if $A \subseteq^* B$ then $f_B \leq^* f_A$ and then $\mathsf{I}_{f_A} \subseteq \mathsf{I}_{f_B}$, and so $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$. □

Claim If $A, B \in [\omega]^{\omega}$ and $|A \setminus B| = \aleph_0$ then $\mathsf{I}_{f_A} \not\leq_K \mathsf{I}_{f_B}$.

Proof (Proof of claim) Let φ be in ω^{ω} and let us prove that φ is not a witness for $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$. First note that for any $n < \omega$ there is $F_n \subseteq I_n$ such that $|F_n| \geq \frac{1}{2}|I_n|$ and, either $\varphi(x) < \min(I_n)$ for all $x \in F_n$ or $\varphi(x) \geq \min(I_n)$ for all $x \in F_n$. Then we have two cases: □

Case 1 *The family* $C = \{n \in A \setminus B : x \in F_n \rightarrow \varphi(x) < \min(I_n)\}$ *is infinite*. Note that by condition (1) and the pigeonhole principle, for any $n \in C$ there is $k_n \in \bigcup_{j<n} I_j$ such that $|\varphi^{-1}[\{k_n\}] \cap F_n| \geq \frac{1}{2r_n}$. Note that for any $n \in A \setminus B$, $\sum_{i \in \varphi^{-1}(k_n)} f_B(i) \geq r_n \cdot \frac{1}{2r_n} = \frac{1}{2}$. If $\{k_n : n \in C\}$ is finite, then it belongs I_{f_A}. In other case we can take an infinite $C' \subseteq C$ such that for every $j < \omega$, $|\{k_n : n \in C'\} \cap I_j| \leq 1$, Then, we have that $\bigcup_{n \in C'} \varphi^{-1}[\{k_n\}] \notin \mathsf{I}_{f_B}$ but $\{k_n : n \in C'\} \in \mathsf{I}_{f_A}$. Hence in this case, φ is not a witness for $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$.

Case 2 *The family* $D = \{n \in A \setminus B : x \in F_n \rightarrow \varphi(x) \geq \min(I_n)\}$ *is infinite*. Note that $Y = \bigcup_{n \in D} F_n$ is an I_{f_B} positive set and $J = \varphi'' Y \in \mathsf{I}_{f_A}$ since $\sum_{n \in J} f_A(n) \leq \sum_{y \in Y} f_A(\varphi(y)) = \sum_{n \in D} \sum_{y \in F_n} f_A(\varphi(y)) \leq \sum_{n \in D} r_{n+1}|F_n| \leq \sum_{n \in D} 2^{-n-1}$. Hence in case 2, φ is not a witness for $\mathsf{I}_{f_A} \leq_K \mathsf{I}_{f_B}$.

Corollary 1 Σ *ordered by the Katětov order contains increasing and decreasing chains of lenght* $\mathfrak{b}$*, and antichains of size* $\mathfrak{c}$*.*

3 The Ideals $\mathbf{Fin}^{\alpha}$ in the Katětov Order

We investigate an increasing chain in the Katětov order of lenght ω_1.

Definition 1 (**Katětov** [9], **also see** [1]) A countable set X_α and an ideal $\mathbf{Fin}^{\alpha}$ on X_α ($\alpha < \omega_1$) are defined by recursion as follows: $X_0 = \{0\}$, $\mathbf{Fin}^0 = \{\emptyset\}$, $X_{\alpha+1} = \omega \times X_\alpha$,

$$\mathbf{Fin}^{\alpha+1} = \{A \subseteq X_{\alpha+1} : (\exists m)(\forall n \geq m)\{r \in X_\alpha : (n, r) \in A\} \in \mathbf{Fin}^{\alpha}\},$$

and if α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha}\{\beta\} \times X_\beta$ and

$$\mathbf{Fin}^{\alpha} = \{A \subseteq X_\alpha : (\exists \beta < \alpha)(\forall \gamma \geq \beta)\{r \in X_\gamma : (\gamma, r) \in A\} \in \mathbf{Fin}^{\gamma}\}.$$

Proposition 1 *The sequence* $\{\mathbf{Fin}^{\alpha} : \alpha < \omega_1\}$ *is strictly* $\leq_K$*-increasing.*

Proof First note that the projection $\pi_{X_\alpha} : X_{\alpha+1} \rightarrow X_\alpha$ is a witness for $\mathbf{Fin}_{\alpha} \leq_K \mathbf{Fin}_{\alpha+1}$. We conclude that the sequence is $\leq_K$-increasing by showing that if α is limit and $\beta = \gamma + 1 < \alpha$ then $\mathbf{Fin}^{\beta} \leq_K \mathbf{Fin}^{\alpha}$. Let $\{\alpha_n : n < \omega\}$ be an enumeration of $\alpha \setminus \gamma$, and let φ_0 be a bijection from $\bigcup_{\delta<\gamma}\{\delta\} \times X_\delta$ onto $\{1\} \times X_1$, and for $0 < n < \omega$, let $\varphi_n : X_{\alpha_n} \rightarrow X_\gamma$ be a witness of $\mathbf{Fin}^{\gamma} \leq_K \mathbf{Fin}^{\alpha_n}$. Now we define the Katětov function desired by $\varphi(\delta, r) = (n, \varphi_n(r))$, if $\gamma \leq \delta = \alpha_n$, and $\varphi(\delta, r) = (1, \varphi_0(\delta, r))$ if $\delta < \gamma$. Let us prove that φ works.

Let A be in $\mathbf{Fin}^{\beta}$, and $k \in \omega$ such that $\{r \in X_{\gamma} : (m, r) \in A\} \in \mathbf{Fin}^{\gamma}$, for all $m \geq k$. Let ε be the maximum of the family $\{\alpha_1, \ldots, \alpha_k, \beta\}$. Then, for all ordinal $\varepsilon < \xi < \alpha$, $\{r \in X_{\xi} : (\xi, r) \in \varphi^{-1}(A)\} = \varphi_m^{-1}\{r \in X_{\gamma} : (m, r) \in A\}$, where $\xi = \alpha_m$. Since φ_m witnesses $\mathbf{Fin}^{\gamma} \leq_K \mathbf{Fin}^{\xi}$, we are done.

For the strictness, it will be sufficient to prove $\mathbf{Fin}^{\alpha+1} \not\leq_K \mathbf{Fin}^{\alpha}$ for all $\alpha < \omega_1$. Let us suppose not, and let α be the minimal with respect to the property $\mathbf{Fin}^{\alpha+1} \leq_K \mathbf{Fin}^{\alpha}$, and let $f : X_{\alpha} \to X_{\alpha+1}$ a witness for this. For simplicity, let us denote $\omega' = \omega$ if α is a succesor, and $\omega' = \alpha$ if not, and for $\beta \in \omega'$ let β' be equal to $\alpha - 1$ if α is a successor, and $\beta' = \beta$ if not. For $\beta < \omega'$ and $k < \omega$, let us define $Y_{(\beta,k)} = \{j \in X_{\beta'} : f(\beta, j) \in \{k\} \times X_{\alpha}\}$. □

Case 1 *The set* $B = \{\beta < \omega' : (\exists k < \omega) Y_{(\beta,k)} \in (\mathbf{Fin}^{\beta'})^+\}$ *is unbounded.* For some cofinal family C of B, we can find an increasing sequence $\langle k_{\beta} : \beta \in C\rangle$ of natural numbers, satisfying $Y_{(\beta,k_{\beta})} \in (\mathbf{Fin}^{\beta'})^+$ for all $\beta \in C$. In this case, for every $\beta \in C$, we can consider the function $f_{\beta} : Y_{(\beta,k_{\beta})} \to X_{\alpha}$ given by $f_{\beta}(r) = proj_{X_{\alpha}}(f(\beta, r))$. From $\mathbf{Fin}^{\beta'} \restriction Y_{(\beta,k_{\beta})} \geq_K \mathbf{Fin}^{\beta'}$ and the minimality of α, [1] f_{β} is not a Katětov function, and then, for all $\beta \in C$, we can find a $\mathbf{Fin}^{\beta'}$-positive set A_{β} such that $f_{\beta}(A_{\beta}) \in \mathbf{Fin}^{\alpha}$. Define $A = \bigcup_{\beta\in C}\{k_{\beta}\} \times f_{\beta}(A_{\beta})$. Clearly, $A \in \mathbf{Fin}^{\alpha+1}$ but $f^{-1}(A) \supseteq \bigcup_{\beta\in C}\{\beta\} \times A_{\beta} \in (\mathbf{Fin}^{\alpha})^+$. This is a contradiction.

Case 2 *B is bounded.* Let $\langle \beta_n : n < \omega\rangle$ a cofinal increasing sequence in α with $\beta_0 > \sup B$. For all $n < \omega$, the function $g_n : X_{\beta_n} \to X_{\alpha+1}$ given by $g_n(r) = f(\beta_n, r)$ is not a Katětov function, then there is a $\mathbf{Fin}^{\beta'_n}$-positive set A_n such that $g_n(A_n) \in \mathbf{Fin}^{\alpha+1}$. Define $C_n = g_n(A_n) \setminus ((n+1) \times X_{\alpha})$. Note that for all n, $g_n^{-1}(C_n) \in (\mathbf{Fin}^{\beta'_n})^+$, because $f^{-1}(\{k\} \times X_{\alpha}) \cap \{\beta_j\} \times X_{\beta'_j} \in \mathbf{Fin}^{\beta'_j}$ for all k and j. Hence, $C = \bigcup_n C_n \in \mathbf{Fin}^{\alpha+1}$ but $f^{-1}(C) \supseteq \bigcup_n \{\beta_n\} \times g_n^{-1}(C_n) \in (\mathbf{Fin}^{\alpha})^+$, a contradiction again.

4 Chains in Katětov Order on Borel Ideals and the Cohen Model

In [4], M. Hrušák asked if there are increasing or decreasing $\leq_K$-chains of Borel ideals with lenght $\mathfrak{c}$. This section is dedicated to prove that, consistently, this is not the case. Let $\mathbb{C}_{\omega_2}$ be the forcing for adding ω_2-many Cohen reals. We first prove some facts about families of $\aleph_2$-many $\mathbb{C}_{\omega_2}$-names. Let us recall that every automorphism φ of $\mathbb{C}_{\omega_2}$, induces an automorphism $\overline{\varphi}$ of $V^{\mathbb{C}_{\omega_2}}$ (the family of $\mathbb{C}_{\omega_2}$-names on V) recursively defined by $\overline{\varphi}(\dot{A}) = \{\langle \overline{\varphi}(\dot{a}), \varphi(p)\rangle : \langle \dot{a}, p\rangle \in \dot{A}\}$, satisfying that for any $\mathbb{C}_{\omega_2}$-generic filter G on V,

$$val_G(\dot{A}) = val_{\varphi(G)}(\overline{\varphi}(\dot{A})).$$

Note that $\overline{\varphi}^{-1} = \overline{\varphi^{-1}}$ for all automorphism φ of $\mathbb{C}_{\omega_2}$.

Lemma 1 *Let V a model of CH and $\{\dot{A}_{\alpha} : \alpha < \omega_2\}$ a family of $\mathbb{C}_{\omega_2}$-names for real numbers. Then, there exists an automorphism φ of $\mathbb{C}_{\omega_2}$and some $\alpha < \beta < \omega_1$ such that $\varphi^{-1} = \varphi$ and $\overline{\varphi}(\dot{A}_{\alpha}) = \dot{A}_{\beta}$.*

[1] It is an easy fact that for every ideal $\mathcal{I}$ and every $\mathcal{I}$-positive set X, the restriction $\mathcal{I} \restriction X = \{A \subseteq X : A \in \mathcal{I}\}$ is an ideal on X which is Katětov above $\mathcal{I}$.

Proof For every $\alpha < \omega_2$, let X_α be the support of $\dot{A}_\alpha$. By Fodor's lemma, there is a root $R \subseteq \omega_1$ and $Y \in [\omega_2]^{\omega_2}$ such that for any $\alpha < \beta \in Y$, $X_\alpha \cap X_\beta = R$. For each $\alpha \in Y$, let C_α be a sequence $\langle (D_n^\alpha, E_n^\alpha) : n < \omega \rangle$ satisfying:

1. $D_n^\alpha \cup E_n^\alpha$ is a maximal antichain in $\mathbb{C}_{X_\alpha}$, and
2. $p \in D_n^\alpha$ implies $\overline{p} \Vdash \dot{A}_\alpha(n) = 0$ and $p \in E_n^\alpha$ implies $\overline{p} \Vdash \dot{A}_\alpha(n) = 1$, where $\overline{p}$ is the obvious extension of p to $\mathbb{C}_{\omega_2}$.

Every X_α is a countable set and then, for every $\alpha, \beta \in Y$, $\mathbb{C}_{X_\alpha} \cong \mathbb{C}_\omega \cong \mathbb{C}_{X_\beta}$. Let us consider each C_α as a subset of $\mathbb{C}_\omega$. Since $V \models CH$ and there are $\mathfrak{c} = \omega_1$-many countable sequences of pairs of countable subsets of ω, there are some $\alpha < \beta \in Y$ such that $C_\alpha = C_\beta$, i.e. there is an isomorphism ψ from $\mathbb{C}_{X_\alpha}$ onto $\mathbb{C}_{X_\beta}$ such that for all $n < \omega$, $p \in D_n^\alpha$ iff $\psi(p) \in D_n^\beta$ and $p \in E_n^\alpha$ iff $\psi(p) \in E_n^\beta$. Let us define the requested automorphism by

$$\varphi(p)(\gamma) = \begin{cases} p(\gamma) & \text{if } \alpha \neq \gamma \neq \beta \\ \psi(p)(\beta) & \text{if } \gamma = \alpha \\ \psi^{-1}(p)(\alpha) & \text{if } \gamma = \beta. \end{cases}$$

It is clear by definition that $\varphi = \varphi^{-1}$ and $\overline{\varphi}(\dot{A}_\alpha) = \dot{A}_\beta$. □

Note that the automorphism φ also satisfies that if $\dot{f}$ is the name of a witness for $\dot{I} \leq_K \dot{\mathsf{J}}$, then $\overline{\varphi}(\dot{f})$ is a name for a witness for $\overline{\varphi}(\dot{\mathsf{I}}) \leq_K \overline{\varphi}(\dot{\mathsf{J}})$. By the classical procedure, we can consider Borel ideals as real numbers (the Borel codes in 2^ω) and the Katětov order becomes a preorder on real numbers, that satisfies the hypothesis of the next Theorem.

Theorem 2 *Let V be a model of CH, $\leq$ a preorder relation on 2^ω satisfying that for all $\dot{x}, \dot{y}$ $\mathbb{C}_{\omega_2}$-names for elements of 2^ω and all automorphism φ of $\mathbb{C}_{\omega_2}$, $V[G] \models \dot{x} \leq \dot{y}$ iff $V[\varphi(G)] \models \dot{x} \leq \dot{y}$, for all $\mathbb{C}_{\omega_2}$-generic filter G on V. Then, in $V[G]$, there are no increasing nor decreasing $\leq$-chains of lenght $\mathfrak{c}$.*

Proof Suppose that in $V[G]$ exists an increasing chain of lenght ω_2. Let $\{\dot{r}_\alpha : \alpha < \omega_2\}$ be a family of $\mathbb{C}_{\omega_2}$-names such that $\mathbb{C}_{\omega_2} \Vdash \dot{r}_\alpha < \dot{r}_\beta$ for $\alpha < \beta < \omega_2$. By Lemma 1 there are an automorphism φ of $\mathbb{C}_{\omega_2}$ and $\alpha < \beta < \omega_2$ such that $\overline{\varphi}(\dot{r}_\alpha) = \dot{r}_\beta$ and $\overline{\varphi}(\dot{r}_\beta) = \dot{r}_\alpha$. Then, for any $\mathbb{C}_{\omega_2}$-generic filter G on V, $val_G(\dot{r}_\alpha) < val_G(\dot{r}_\beta)$, obviously, the same holds for the $\mathbb{C}_{\omega_2}$-generic filter $\varphi(G)$, i.e. $val_{\varphi(G)}(\dot{r}_\alpha) < val_{\varphi(G)}(\dot{r}_\beta)$. However, since $\overline{\varphi}(\dot{r}_\alpha) = \dot{r}_\beta$, $val_G(\dot{r}_\alpha) = val_{\varphi(G)}(\overline{\varphi}(\dot{r}_\alpha)) = val_{\varphi(G)}(\dot{r}_\beta)$ and $val_G(\dot{r}_\beta) = val_{\varphi(G)}(\overline{\varphi}(\dot{r}_\beta)) = val_{\varphi(G)}(\dot{r}_\alpha)$, and hence,

$$val_{\varphi(G)}(\dot{r}_\beta) = val_G(\dot{r}_\alpha) < val_G(\dot{r}_\beta) = val_{\varphi(G)}(\dot{r}_\alpha),$$

which is a contradiction. Analogously the decreasing case can be proved. □

Corollary 2 *In the Cohen model, the Katětov order does not contain increasing nor decreasing chains of Borel ideals with lenght $\mathfrak{c}$.*

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