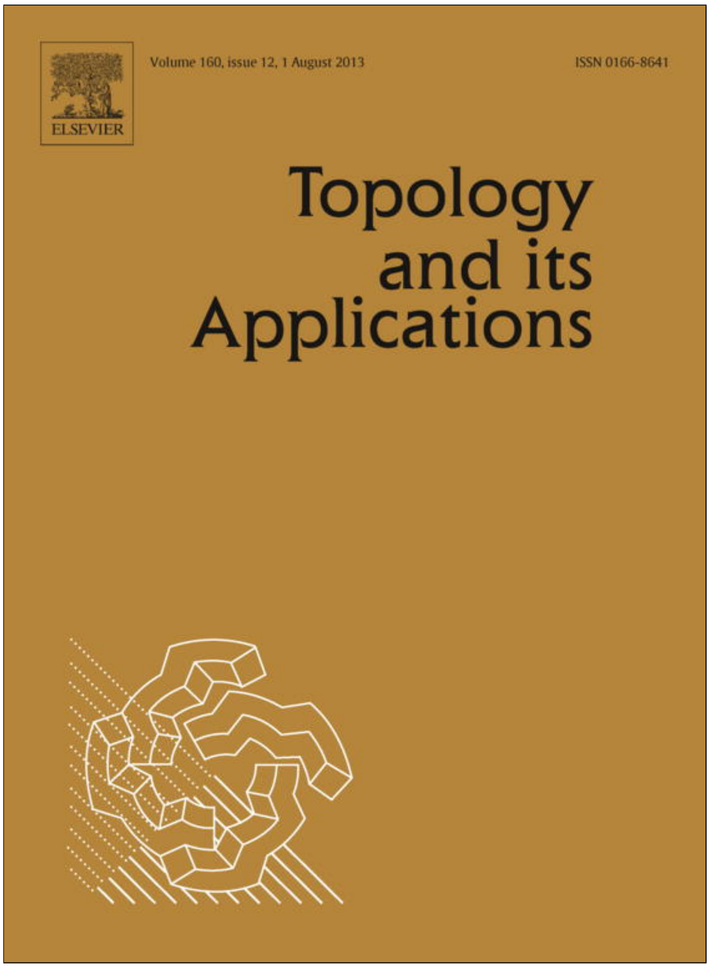
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n-Luzin gaps

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ABSTRACT

We introduce and study the notion of an n-Luzin gap, which is a natural generalization of a Luzin gap. We prove that under Martin's Axiom, every AD family $\mathcal{A}$ of size less than $\mathfrak{c}$ contains an n-Luzin gap or the corresponding Mrówka–Isbell space $\Psi(\mathcal{A})$ is normal.

0. Introduction

An infinite family $\mathcal{A} \subset \mathcal{P}(\omega)$ is *almost disjoint* (AD) if the intersection of any two distinct elements of $\mathcal{A}$ is finite. It is *maximal almost disjoint* (MAD) if it is not properly included in any larger AD family or, equivalently, if given an infinite $X \subseteq \omega$ there is an $A \in \mathcal{A}$ such that $|A \cap X| = \omega$. Given an almost disjoint family $\mathcal{A}$ and two subfamilies $\mathcal{B}, \mathcal{C}$ of $\mathcal{A}$ we say that a set $X \subseteq \omega$ *separates* $\mathcal{B}$ and $\mathcal{C}$ if $A \subseteq^* X$ for every $A \in \mathcal{B}$ and $A \cap X =^* \emptyset$ for every $A \in \mathcal{C}$.

One of the first constructions of almost disjoint families with special properties is the construction of Luzin [14] of an uncountable almost disjoint family $\mathcal{A}$ such that no two uncountable subfamilies of $\mathcal{A}$ can be separated. The ingenious property used in the proof deserves a name:

Definition 0.1. An almost disjoint family $\mathcal{A}$ is *Luzin* if it can be enumerated as $\{A_\alpha\colon \alpha < \omega_1\}$ so that $\forall \alpha < \omega_1\ \forall n \in \omega\ \{\beta < \alpha\colon A_\alpha \cap A_\beta \subseteq n\}$ is finite.

Abraham and Shelah [1] called (and so do we) an almost disjoint family $\mathcal{A}$ *inseparable* if no two uncountable subfamilies can be separated. It is easy to see that $\mathcal{A}$ is inseparable if and only if for every $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ the set $\bigcup \mathcal{B} \cap \bigcup \mathcal{C}$ is infinite. The point of Luzin's proof was that, Luzin families are inseparable. Abraham and Shelah proved that (1) assuming CH, there is an inseparable AD family which contains no Luzin subfamily, while (2) under $\mathsf{MA} + \neg\mathsf{CH}$ every inseparable AD family is a countable union of Luzin subfamilies.

Roitman and Soukup in [17] introduced the notion of an anti-Luzin family: An AD family $\mathcal{A}$ is an *anti-Luzin* family if for every $\mathcal{B} \in [\mathcal{A}]^{\aleph_1}$ there are $\mathcal{C}, \mathcal{D} \in [\mathcal{B}]^{\aleph_1}$ which can be separated (or equivalently, $\mathcal{A}$ does not contain uncountable inseparable families) and proved that assuming $\mathsf{MA} + \neg\mathsf{CH}$, every AD family is either anti-Luzin or contains an uncountable Luzin subfamily, and assuming $\upharpoonleft$,[3] there is an uncountable almost disjoint family which contains no uncountable anti-Luzin and no uncountable Luzin subfamilies.

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3 Recall that $\upharpoonleft$ is the following weakening of CH: There is a family $\mathcal{S} \subseteq [\omega_1]^\omega$ of size $\aleph_1$ such that every uncountable subset of ω_1 contains an element of $\mathcal{S}$.

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More recently, Dow [7] showed that PFA implies that every MAD family contains an uncountable Luzin subfamily. Dow and Shelah in [8] showed that Martin's Axiom does not suffice by showing that it is relatively consistent with $\mathsf{MA} + \neg\mathsf{CH}$ that there is a maximal almost disjoint family which is ω_1-*separated*, i.e. any disjoint pair of $\leqslant \omega_1$-sized subfamilies are separated.

To every almost disjoint family one can naturally associate the so-called *Mrówka–Isbell space*:

Definition 0.2. Given an AD family $\mathcal{A}$, define a space $\Psi(\mathcal{A})$ as follows: The underlying set is $\omega \cup \mathcal{A}$, all elements of ω are isolated and basic neighborhoods of $A \in \mathcal{A}$ are of the form $\{A\} \cup (A \setminus F)$ for some finite set F.

It follows immediately from the definition that $\Psi(\mathcal{A})$ is a separable, scattered, zero-dimensional, first countable, locally compact Moore space [16]. Normality of Ψ-spaces is characterized using separation as follows:

Proposition 0.3. *([20]) $\Psi(\mathcal{A})$ is normal if and only if $\mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ can be separated for every $\mathcal{B} \subseteq \mathcal{A}$.*

Slightly abusing notation we will call an AD family $\mathcal{A}$ *normal* if the space $\Psi(\mathcal{A})$ is normal. A natural choice would be to call $\mathcal{A}$ *completely separated*, but unfortunately a very similar term is already in use [19,9,5]. By the above proposition it follows that if $\Psi(\mathcal{A})$ is normal, then $2^{|\mathcal{A}|} = \mathfrak{c}$ so $\mathcal{A}$ must have size less than the continuum.

Luzin families are often referred to as Luzin gaps. However, that name has recently [21,10] been used to describe a weaker notion.

Definition 0.4. ([21]) A pair $\mathcal{A} = \{A_\alpha\colon \alpha < \omega_1\}$, $\mathcal{B} = \{B_\alpha\colon \alpha < \omega_1\}$ of subfamilies of $[\omega]^\omega$ is called a *Luzin gap* if there is an $m \in \omega$ such that

1. $A_\alpha \cap B_\alpha \subseteq m$ for all $\alpha < \omega_1$, and
2. $A_\alpha \cap B_\beta$ is finite yet $(A_\alpha \cap B_\beta) \cup (A_\beta \cap B_\alpha) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.

Every Luzin family $\mathcal{A}$ contains many Luzin gaps: given a pair $\{A_\alpha\colon \alpha < \omega_1\}$, $\{B_\alpha\colon \alpha < \omega_1\}$ of disjoint subfamilies of $\mathcal{A}$, there is an uncountable $X \subseteq \omega_1$ such that $\{A_\alpha\colon \alpha \in X\}$, $\{B_\alpha\colon \alpha \in X\}$ forms a Luzin gap. The basic property of a Luzin gap is that the two families $\mathcal{A}$ and $\mathcal{B}$ cannot be separated, and the property of being a Luzin gap is indestructible by forcing preserving ω_1 (see [21,10] or Section 1). Hence, the space $\Psi(\mathcal{A})$ cannot be normal (in any forcing extension preserving ω_1) for any AD family $\mathcal{A}$ containing a Luzin gap.

The following weakening of the notion of a Luzin gap is central for our considerations.

Definition 0.5. Let $n \in \omega$ and $\mathcal{B}_i = \{B^i_\alpha \mid \alpha \in \omega_1\}$ be disjoint subfamilies of an AD family $\mathcal{A}$ for $i < n$. We call $\langle \mathcal{B}_i \mid i < n\rangle$ an n-Luzin *gap* if there is $m \in \omega$ such that

1. $B^i_\alpha \cap B^j_\alpha \subseteq m$ for all $i \neq j$, $\alpha < \omega_1$ and
2. $\bigcup_{i \neq j}(B^i_\alpha \cap B^j_\beta) \nsubseteq m$ for all $\alpha \neq \beta < \omega_1$.

We say that $\mathcal{A}$ contains an n-Luzin gap if there is an n-Luzin gap $\langle \mathcal{B}_i \mid i < n\rangle$ where each $\mathcal{B}_i$ is a subfamily of $\mathcal{A}$. We will see that any family containing an n-Luzin gap is not normal, and our main theorem states that the converse is also true assuming Martin's Axiom:

Theorem 0.6. *Assume* MA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Assuming PFA the theorem can be strenghtened (see Theorem 3.8). We also show that the result does not follow from MA(σ-centered), as

Theorem 0.7. *It is consistent with* MA(σ*-centered*) *that there is an inseparable AD family of size ω_1 which does not contain n-Luzin gaps for any $n \in \omega$.*

The situation is reminiscent of ω_1-trees and Hausdorff gaps, an inseparable family that does not contain n-Luzin gaps for any $n \in \omega$ being the equivalent of a Suslin tree or a ccc destructible gap. A Suslin tree can be destroyed by two different means: (1) one can force with the tree an add an uncountable branch and (2) one can specialize the tree by a ccc forcing making it a union of countably many antichains. Similar situation occurs with ccc destructible Hausdorff gaps ([13] see [18]) a destructible Hausdorff gaps can be either (1) filled or (2) frozen, both by ccc forcing. Here, an inseparable family with no n-Luzin gaps can be either (1) forced normal or (2) frozen by forcing it to contain a Luzin gap, both by a ccc forcing.

An early (probably the first) example of a Ψ-space appears in [3]: A topology of the real line is refined by declaring all rational points isolated. To each irrational point a convergent sequence is chosen and the cofinite subsets of the given convergent sequence are declared basic open neighborhoods of the irrational number.

We call an almost disjoint family $\mathcal{A}$ $\mathbb{R}$-*embeddable* (see [11]) if there is an injection $e:\omega\to\mathbb{Q}$ such that for every $A\in\mathcal{A}$ there is an $r_A\in\mathbb{R}$ such that $e[A]$ converges to r_A and, moreover, $r_A\neq r_B$ whenever $A\neq B$. Evidently, this is equivalent that there is an injective and continuous $f:\Psi(\mathcal{A})\to\mathbb{R}$ such that $f(n)\in\mathbb{Q}$ for every $n\in\omega$. Using Tietze's theorem, it is easy to show that every normal family is $\mathbb{R}$-embeddable.

The notion of $\mathbb{R}$-embeddability together with a strengthening of the notion of an anti-Luzin family are some of the main tools used here.

Definition 0.8. An almost disjoint family $\mathcal{A}$ is *partially separated* if given a pair $\mathcal{B}=\{B_\alpha\colon\ \alpha<\omega_1\}$, $\mathcal{C}=\{C_\alpha\colon\ \alpha<\omega_1\}$ of disjoint subfamilies of $\mathcal{A}$ there is an uncountable $X\subseteq\omega_1$ such that the families $\{B_\alpha\colon\ \alpha\in X\}$, $\{C_\alpha\colon\ \alpha\in X\}$ are separated.

We call an AD family $\mathcal{A}$ *potentially* $\mathcal{P}$ (for a property $\mathcal{P}$) if there is a ccc forcing $\mathbb{P}$ such that $\Vdash_{\mathbb{P}}$ "$\mathcal{A}$ has $\mathcal{P}$". Similarly, we say that $\mathcal{A}$ is *indestructibly* $\mathcal{P}$, if $\mathcal{A}$ has property $\mathcal{P}$ in all ccc forcing extensions. We show that

Theorem 0.9. *The following are equivalent for an AD family $\mathcal{A}$:*

1. *$\mathcal{A}$ does not contain n-Luzin gaps for any $n\in\omega$,*
2. *$\mathcal{A}$ is potentially normal,*
3. *$\mathcal{A}$ is potentially $\mathbb{R}$-embeddable,*
4. *$\mathcal{A}$ is potentially partially separated.*

Dow and Shelah's [8] result mentioned above shows that it is consistent with MA that there is a MAD family which is potentially normal, while assuming PFA [7] all MAD families contain Luzin families, hence, also Luzin gaps. It is worth mentioning that Áviles and Todorčević also studied gaps of higher dimensions in [4], however their versions are strengthenings rather than weakenings of the classical notion.

1. Normality of AD families in ccc extensions

In the following, $\mathcal{A}$ will always be an AD family. Given $\mathcal{B},\mathcal{C}$ disjoint subsets of $\mathcal{A}$, we will define a forcing that adds a set separating $\mathcal{B}$ from $\mathcal{C}$. Let $\mathbb{S}_{\mathcal{B}\mathcal{C}}$ be the set of all $(s,\mathcal{F},\mathcal{G})$ such that,

1. $s\in{}^{<\omega}2$, $\mathcal{F}\in[\mathcal{B}]^{<\omega}$, $\mathcal{G}\in[\mathcal{C}]^{<\omega}$.
2. If $B\in\mathcal{F}$ and $C\in\mathcal{G}$ then $B\cap C\subseteq|s|$.

We say $(s,\mathcal{F},\mathcal{G})\leqslant(s',\mathcal{F}',\mathcal{G}')$ if and only if,

1. $s'\subseteq s$, $\mathcal{F}'\subseteq\mathcal{F}$, $\mathcal{G}'\subseteq\mathcal{G}$.
2. If $i\in\operatorname{dom}(s)\setminus\operatorname{dom}(s')$ then,
   a) If $i\in\bigcup\mathcal{F}'$ then $s(i)=1$.
   b) If $i\in\bigcup\mathcal{G}'$ then $s(i)=0$.

It is easy to prove that for all $n\in\omega$, $B\in\mathcal{B}$ and $C\in\mathcal{C}$ the following sets $\{(s,\mathcal{F},\mathcal{G})\mid |s|\geqslant n\}$, $\{(s,\mathcal{F},\mathcal{G})\mid B\in\mathcal{F}\}$ and $\{(s,\mathcal{F},\mathcal{G})\mid C\in\mathcal{G}\}$ are dense, so $\mathbb{S}_{\mathcal{B}\mathcal{C}}$ adds a set separating $\mathcal{B}$ from $\mathcal{C}$.

Lemma 1.1. *If $\mathcal{A}$ is partially separated, then $\mathbb{S}_{\mathcal{B}\mathcal{C}}$ is ccc.*

Proof. Let $\{p_\alpha\mid\alpha\in\omega_1\}$ be a set of conditions, and write $p_\alpha=(s_\alpha,\mathcal{F}_\alpha,\mathcal{G}_\alpha)$. Without loss of generality, we may assume that there are $n,m\in\omega$ such that $|\mathcal{F}_\alpha|=n$ and $|\mathcal{G}_\alpha|=m$ for every $\alpha\in\omega_1$. Let us enumerate $\mathcal{F}_\alpha=\{\mathcal{F}_\alpha(i)\mid i<n\}$ and $\mathcal{G}_\alpha=\{\mathcal{G}_\alpha(i)\mid i<m\}$.

Let $\mathcal{B}_0=\{\mathcal{F}_\alpha(0)\mid\alpha\in\omega_1\}$ and $\mathcal{C}_0=\{\mathcal{G}_\alpha(0)\mid\alpha\in\omega_1\}$. Since $\mathcal{A}$ is partially separated, there are $Z_0\in[\omega_1]^{\omega_1}$ and k_0 such that $\mathcal{F}_\alpha(0)\cap\mathcal{G}_\beta(0)\subseteq k_0$ for every $\alpha,\beta\in Z_0$. Now, let $\mathcal{B}_1=\{\mathcal{F}_\alpha(0)\mid\alpha\in Z_0\}$, $\mathcal{C}_1=\{\mathcal{G}_\alpha(1)\mid\alpha\in Z_0\}$ and find $Z_1\in[Z_0]^{\omega_1}$, $k_1\in\omega$ such that $\mathcal{F}_\alpha(0)\cap\mathcal{G}_\beta(1)\subseteq k_1$ for every $\alpha,\beta\in Z_1$. Repeating this process (mn times) we conclude there is $Z\in[\omega_1]^{\omega_1}$ and k such that $B_\alpha(i)\cap C_\beta(j)\subseteq k$ for every $\alpha,\beta\in Z$ and $i<n$, $j<m$.

For every $\alpha\in Z$, take s'_α such that $(s'_\alpha,\mathcal{F}_\alpha,\mathcal{G}_\alpha)\leqslant(s_\alpha,\mathcal{F}_\alpha,\mathcal{G}_\alpha)$ and $k<|s'_\alpha|$. Naturally, there are $s\in{}^{<\omega}2$ and $\alpha,\beta\in Z$ with the property that $s=s_\alpha=s_\beta$. We claim that $(s,\mathcal{F}_\alpha,\mathcal{G}_\alpha)$ and $(s,\mathcal{F}_\beta,\mathcal{G}_\beta)$ are compatible (and then, so are p_α and p_β). To prove this, we only need to realize that $(s,\mathcal{F}_\alpha\cup\mathcal{F}_\beta,\mathcal{G}_\alpha\cup\mathcal{G}_\beta)$ is a condition, but this is trivial since $k<|s'_\alpha|$. □

We will prove that $\mathbb{R}$-embedabbility implies partial separability next.

Proposition 1.2. *If $\mathcal{A}$ is $\mathbb{R}$-embeddable, then it is partially separated.*

Proof. Let $h : \Psi(\mathcal{A}) \to \mathbb{R}$ witness that $\mathcal{A}$ is $\mathbb{R}$-embeddable and take $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ disjoint subsets of $\mathcal{A}$. Fix D a countable base for $\mathbb{R}$ and for every $\alpha \in \omega_1$, find disjoint $U_\alpha, V_\alpha \in D$ such that $h(B_\alpha) \in U_\alpha$ and $h(C_\alpha) \in V_\alpha$. Choose also $m_\alpha \in \omega$ such that $h[B_\alpha \setminus m_\alpha] \subseteq U_\alpha$ and $h[C_\alpha \setminus m_\alpha] \subseteq V_\alpha$. Now, let $X \in [\omega_1]^{\omega_1}$ be such that there are $U, V \in D$ and m with the property that $U_\alpha = U$, $V_\alpha = V$ and $m_\alpha = m$ for all $\alpha \in X$. It is clear that if $\alpha, \beta \in X$ then $B_\alpha \cap C_\alpha \subseteq m$.

From the above we may conclude even more: note that being $\mathbb{R}$-embeddable is an indestructible property, so an $\mathbb{R}$-embeddable family is actually indestructibly partially separated.

Corollary 1.3. *The following are equivalent,*

1. *$\mathcal{A}$ is potentially $\mathbb{R}$-embeddable,*
2. *$\mathcal{A}$ is potentially indestructibly partially separated,*
3. *$\mathcal{A}$ is potentially normal.*

Proof. By the above comment it follows that 1 implies 2. Clearly 3 implies 1. In order to prove that 2 implies 3, let $\mathbb{P}$ be a ccc forcing such that $1_{\mathbb{P}}$ forces that an AD family $\mathcal{A}$ is indestructibly partially separated. Then, the forcing notions $\mathbb{S}_{\mathcal{BC}}$ will always be ccc (in any ccc extension) so we may iterate them and get a model where $\mathcal{A}$ is normal. □

As a consequence, assuming Martin's Axiom, small almost disjoint families which are potentially normal, are precisely those which are normal already.

Corollary 1.4. *Assume* MA. *Let $\mathcal{A}$ be an AD with $|\mathcal{A}| < \mathfrak{c}$, then $\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ is normal.*

Proof. Let $\mathcal{A}$ be potentially normal and of size less than $\mathfrak{c}$. We must prove that every $\mathcal{B}, \mathcal{C}$ disjoint subsets of $\mathcal{A}$ can be separated. Since we are assuming MA, it is enough to show that the forcing $\mathbb{S}_{\mathcal{BC}}$ is ccc (because we only need $|\mathcal{B}| + |\mathcal{C}| + \omega$ dense sets to do the job). Now, let $\mathbb{P}$ be a ccc forcing such that $\mathcal{A}$ is partially separated in $V[G]$ for every generic filter $G \subseteq \mathbb{P}$. Note that $\mathbb{S}_{\mathcal{BC}}$ is the same as $\mathbb{S}_{\mathcal{BC}}^{V[G]}$ and since $\mathcal{A}$ is partially separated, then it is ccc in $V[G]$. This implies that $\mathbb{S}_{\mathcal{BC}}$ is ccc in V (since any uncountable antichain in V would still be an uncountable antichain in $V[G]$). □

Assuming MA, we may get another equivalence of potential normality:

Corollary 1.5. *Assume* MA. *$\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ is indestructibly partially separated.*

Proof. Let $\mathcal{A}$ be potentially normal, let $\mathbb{P}$ be a ccc forcing and $G \subseteq \mathbb{P}$ a generic filter. We must prove that $\mathcal{A}$ is partially separated in $V[G]$. For this it is enough to see that every subfamily of $\mathcal{A}$ of size ω_1 is partially separated. To see this, in $V[G]$ choose $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$ and since $\mathbb{P}$ is ccc, then there is $\mathcal{A}'' \in V$ a subset of $\mathcal{A}$ of size ω_1 such that $\mathcal{A}' \subseteq \mathcal{A}''$. Since MA is true in V, $\mathcal{A}''$ is $\mathbb{R}$-embeddable, so it is partially separated in $V[G]$, hence so is $\mathcal{A}'$. □

The previous corollary cannot be proved in ZFC, as we will see in Section 3.

2. *n*-Luzin gaps

We start by proving some elementary facts about n-Luzin gaps.

Lemma 2.1. *If $\langle \mathcal{B}_i : i < n \rangle$ is an n-Luzin gap ($\mathcal{B}_i = \{B^i_\alpha \mid \alpha \in \omega_1\}$) then, for every $X \in [\omega_1]^{\omega_1}$ and every $k \in \omega$, there are $\alpha, \beta \in X$ such that*

$$\bigcup_{i \neq j} \left(B^i_\alpha \cap B^j_\beta\right) \not\subseteq k.$$

Proof. Let $m \in \omega$ testify that $\langle \mathcal{B}_i \mid i < n \rangle$ is n-Luzin. Without loss of generality $k > m$. First, find $Y \in [X]^{\omega_1}$ such that if $\alpha, \beta \in Y$ and $i < n$, then $B^i_\alpha \cap k = B^i_\beta \cap k$. Take $\alpha, \beta \in Y$ distinct. There are $i \neq j$ such that $B^i_\alpha \cap B^j_\beta \not\subseteq m$, but since $B^i_\alpha \cap k = B^i_\beta \cap k$ and $B^j_\beta \cap B^i_\beta \subseteq m \subseteq k$, B^i_α and B^j_β must intersect above k. □

With the aid of this lemma, we can prove the following:

Lemma 2.2. *If $\mathcal{A}$ is partially separated, then it does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. Let $\mathcal{A}$ be partially separated and take $\{B^n_\alpha\colon n \in \omega\}$ such that $B^i_\alpha \cap B^j_\alpha \subseteq m$ when $i \neq j$. Since $\mathcal{A}$ is partially separated. There are $X \in [\omega_1]^{\omega_1}$ and $k \in \omega$ such that $B^i_\alpha \cap B^j_\alpha \subseteq k$ for all $\alpha, \beta \in X$. Then, by the previous lemma, $\mathcal{A}$ cannot contain n-Luzin gaps. □

Since normal families are partially separated, we immediately conclude:

Corollary 2.3. *If $\mathcal{A}$ contains an n-Luzin gap, then it is not normal.*

Using this, we will be able to give a combinatorial reformulation of potential normality of AD families. First, we will introduce a forcing that makes $\mathcal{A}$ an $\mathbb{R}$-embeddable family. Instead of trying to embed $\Psi(\mathcal{A})$ into $\mathbb{R}$, we will try to embed it into the Cantor space ${}^{\omega}2$, identifying the rational numbers with the eventually 0 functions. It is easy to see that this suffices. Let $\mathcal{R}(\mathcal{A})$ be the set of all $(s, \mathcal{F})$ such that,

1. $s \in {}^{<\omega}\mathbb{Q}$ is injective and $\mathcal{F} \in [\mathcal{A}]^{<\omega}$.
2. If $A, B \in \mathcal{F}$ then $A \cap B \subseteq |s|$.

And $(s, \mathcal{F}) \leqslant (s', \mathcal{F}')$ if,

1. $s' \subseteq s$, $\mathcal{F}' \subseteq \mathcal{F}$.
2. If $i \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ and there is $A \in \mathcal{F}'$ such that $i \in A$ and $j = \max\{A \cap \mathrm{dom}(s')\}$ then $\Delta(s(i), s'(j)) \geqslant |s'|$ (where $\Delta(x, y)$ is the first n such that $x(n) \neq y(n)$).

Note that the A is unique since $(s', \mathcal{F}')$ is a condition.

Lemma 2.4. *If $\mathcal{R}(\mathcal{A})$ is ccc, then $\mathcal{A}$ is potentially $\mathbb{R}$-embeddable.*

Proof. Given $n \in \omega$, it is easy to prove that the set $D_n = \{(s, \mathcal{F}) \mid n < |s|\}$ is dense (this is due to the fact that if $A, B \in \mathcal{F}$ then $A \cap B \subseteq \mathcal{F}$, so we may extend the condition $(s, \mathcal{F})$ without changing $\mathcal{F}$). Also, if $A \in \mathcal{A}$ then the set $E_A = \{(s, \mathcal{F}) \mid A \in \mathcal{F}\}$ is dense. Given $(s, \mathcal{F})$ we first find $m \in \omega$ such that $X \cap Y \subseteq m$ for every $X \neq Y \in \mathcal{F} \cup \{A\}$ and then we extend $(s, \mathcal{F})$ to a condition $(s', \mathcal{F})$ such that $m < |s'|$. In this way, $(s', \mathcal{F} \cup \{A\})$ is below $(s, \mathcal{F})$.

Fix G a generic filter for $\mathcal{R}(\mathcal{A})$, we will prove that $\mathcal{A}$ is $\mathbb{R}$-embeddable in $V[G]$. Let $e = \bigcup_{(s,\mathcal{F}) \in G} s$ since the D_n are dense, then e is a function from ω to $\mathbb{R}$. We will show that if $A \in \mathcal{A}$ then $e[A]$ is a convergent sequence. For this, just note that if $A \in \mathcal{F}$ and $A \cap \mathrm{dom}(s) \neq \emptyset$ then $(s, \mathcal{F}) \Vdash$"if $x, y \in A \setminus \mathrm{dom}(s)$, then $\dot{e}(x) \restriction |s| = \dot{e}(y) \restriction |s|$".

Let us call $r_A \in {}^{\omega}2$ the limit of $e[A]$. It remains to be shown that $r_A \neq r_B$ whenever $A \neq B$. Let D_{AB} be the set of those $(s, \mathcal{F})$ that force r_A to be different from r_B. It is enough to show that this set is dense. Take a condition $(s, \mathcal{F})$, without loss of generality, we may assume that $A, B \in \mathcal{F}$ and $A \cap \mathrm{dom}(s)$, $B \cap \mathrm{dom}(s)$ are not empty. Now, it is easy to extend this condition in such a way that r_A and r_B belong to different clopen sets. □

We are finally ready to prove one of the main results of the paper.

Theorem 2.5. *$\mathcal{A}$ is potentially normal if and only if $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. If $\mathcal{A}$ contains an n-Luzin gap, then $\mathcal{A}$ still contains it in any forcing extension that preserves ω_1. Hence, we may conclude that $\mathcal{A}$ cannot be potentially normal. So, we only need to prove that if $\mathcal{A}$ does not contain n-Luzin gaps then it is potentially normal, or equivalently that it is potentially $\mathbb{R}$-embeddable. For this, we just need to see that $\mathcal{R}(\mathcal{A})$ is ccc.

Assume this is not the case, then there is a set $\{(s_\alpha, \mathcal{F}_\alpha) \mid \alpha \in \omega_1\}$ of pairwise incompatible conditions. We may assume that there is $s \in {}^{<\omega}\mathbb{R}$ such that $s_\alpha = s$ for all $\alpha \in \omega_1$ and $\{\mathcal{F}_\alpha \mid \alpha \in \omega_1\}$ forms a Δ-system with root R. Note that since $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are incompatible so are $(s, \mathcal{F}_\alpha \setminus R)$ and $(s, \mathcal{F}_\beta \setminus R)$. So, we may further assume that R is the empty set and all $\mathcal{F}_\alpha$ are of the same size, say n. We may also assume that if $i < n$ then $\mathcal{F}_\alpha(i) \cap m = \mathcal{F}_\beta(i) \cap m$ for all $\alpha, \beta \in \omega_1$.

Enumerate $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha(i) \mid i < n\}$ and let $\mathcal{B}_i = \{\mathcal{F}_\alpha(i) \mid \alpha \in \omega_1\}$. Note that, since each $(s, \mathcal{F}_\alpha)$ is a condition, then $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\alpha(j) \subseteq m$. Since $\mathcal{A}$ does not contain n-Luzin gaps, there are $\alpha \neq \beta$ such that if $i \neq j$ then $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\beta(j) \subseteq m$. We claim that $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are compatible, which will be a contradiction. Note that $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta)$ may fail to be a condition, since there could be $A, B \in \mathcal{F}_\alpha \cup \mathcal{F}_\beta$ such that $A \cap B \nsubseteq |s| = m$. However, in this case, A must be of the form $\mathcal{F}_\alpha(i)$ and B must be $\mathcal{F}_\beta(i)$ (because $(s, \mathcal{F}_\alpha)$ and $(s, \mathcal{F}_\beta)$ are conditions and $\mathcal{F}_\alpha(i) \cap \mathcal{F}_\beta(j) \subseteq m$ when $i \neq j$). However, since $\mathcal{F}_\alpha(i)$ and $\mathcal{F}_\beta(i)$ agree up to m, it is easy to extend $(s, \mathcal{F}_\alpha \cup \mathcal{F}_\beta)$ to a condition. □

Evidently, we may conclude,

Corollary 2.6. *If $\mathcal{A}$ is partially separated then it is potentially $\mathbb{R}$-embeddable.*

The reader might wonder why we are only considering ccc extensions instead of extensions preserving ω_1. It turns out both concepts are equivalent, as the next result show,

Corollary 2.7. *$\mathcal{A}$ is potentially normal if and only if there is a forcing notion $\mathbb{P}$ that does not collapse ω_1 and forces $\mathcal{A}$ to be normal.*

Proof. Note that if $\mathcal{A}$ contains an n-Luzin gap, then it still contains an n-Luzin gap in any forcing extension that preserves ω_1, so the result follows by the previous theorem. □

We may also prove the promised result,

Theorem 2.8. *Assume* MA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain n-Luzin gaps for any $n \in \omega$.*

Proof. The forward implication is clear, for the converse, just recall that under MA normality and potential normality are equivalent for families of size less than $\mathfrak{c}$. □

We will show that, under the Proper Forcing Axiom, we may "remove the n" from the previous result. Assume $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ are disjoint subfamilies of $\mathcal{A}$ and let $X = \{(B_\alpha, C_\alpha) \mid \alpha \in \omega_1\}$. For every $m \in \omega$ we define a coloring $c_m : [X]^2 \to 2$ by

$$c_m\big((B_\alpha, C_\alpha), (B_\beta, C_\beta)\big) = 1 \quad \text{iff} \quad (B_\alpha \cap C_\beta) \cup (B_\beta \cap C_\alpha) \subseteq m.$$

We may see X as a subset of the polish space ${}^\omega 2 \times {}^\omega 2$, so it carries a natural topology. In this way, note that $c^{-1}(\{0\}) \subseteq X^2$ is an open set. Let us recall the *Open Coloring Axiom* (see [22] and [15]),

OCA) If X is a separable metric space and $c : [X]^2 \to 2$ is such that $c^{-1}(\{0\})$ is open, then one of the following holds,
⋆) There is $M \in [X]^{\omega_1}$ that is monochromatic of color 0 (i.e. c restricted to $[M]^2$ is the constant 0).
⋆⋆) X may be cover by ω-monochromatic sets of color 1.

For us, it will be enough to observe that OCA implies that (given X is uncountable) there is always an uncountable monochromatic set in one of the colors.

The following result is a consequence of Theorem 13.5 of [22]. We present its short proof for the sake of completeness. Later (Theorem 3.8) we will see that it follows also from the P-ideal dichotomy.

Proposition 2.9. *If* OCA *is true, then every almost disjoint family is partially separated or contains a Luzin gap.*

Proof. Assume $\mathcal{A}$ is not partially separated, so there are disjoint subfamilies $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ of $\mathcal{A}$ such that for every $Y \in [\omega_1]^{\omega_1}$ and $n \in \omega$, there are $\alpha, \beta \in Y$ with the property that $B_\alpha \cap C_\beta \nsubseteq n$. We may assume there is $m \in \omega$ such that $B_\alpha \cap C_\alpha \subseteq m$ for all $\alpha \in \omega_1$.

Let X, and c_m be defined as above. The previous remark tells us that there are no uncountable 1-monochromatic sets, so OCA implies the existence of an uncountable 0-monochromatic set Y. Clearly $\{B_\alpha \mid \alpha \in Y\}$, $\{C_\alpha \mid \alpha \in Y\}$ is a Luzin gap. □

Corollary 2.10. *Assume* PFA. *Let $\mathcal{A}$ be an AD family. Then $\mathcal{A}$ is normal if and only if $|\mathcal{A}| < \mathfrak{c}$ and $\mathcal{A}$ does not contain Luzin gaps.*

We do not know whether a version of PFA (OCA) is necessary for the conclusion of the corollary,

Question 2.11. *Does the previous corollary hold assuming* MA*?*

It cannot be proved in ZFC, since it is consistent with MA(σ-centered) that there are 3-Luzin gaps that do not contain Luzin gaps. In order to prove this, first recall that a family $\mathcal{D} \subseteq \mathcal{P}(\omega)$ is *independent* if for any distinct $A_0, \ldots, A_n$, $B_0, \ldots, B_m \in \mathcal{D}$ the set $A_0 \cap \cdots \cap A_n \cap (\omega \setminus B_0) \cap \cdots \cap (\omega \setminus B_m)$ is infinite. We say that $\mathcal{D}$ *separates points* if for every distinct $n, m \in \omega$, there is $D \in \mathcal{D}$ such that $\{n, m\} \cap D$ has size 1.

Given $\mathcal{D}$ an independent family that separates points, we define the topological space $(\omega, \tau_{\mathcal{D}})$ which has $\mathcal{D} \cup \{\omega - D \mid D \in \mathcal{D}\}$ as a subbase.

Lemma 2.12. *(ω, τ_D) is homeomorphic to the rationals with the usual topology.*

Proof. This space is countable, first countable, zero-dimensional without isolated points, and this characterizes $\mathbb{Q}$ (this is an old result of Sierpiński, see [12]). □

To construct our 3-Luzin gap, we will first construct (in ZFC) a special type of a Luzin gap, which is interesting on its own,

Lemma 2.13. *There is a Luzin gap $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ such that $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable.*

Proof. Let $\mathcal{D} = \{D_n \mid n \in \omega\}$ and $\mathcal{E} = \{E_n \mid n \in \omega\}$ be disjoint families such that both separate points and $\mathcal{D} \cup \mathcal{E}$ is an independent family. As was remarked above, $(\omega, \tau_\mathcal{D})$ and $(\omega, \tau_\mathcal{E})$ are both homeomorphic to the rationals, and every open set of one topology is dense in the other. Identifying ω with $\mathbb{Q}$, we may view $\mathbb{R}$ as the metric completion of $(\omega, \tau_\mathcal{D})$ and $(\omega, \tau_\mathcal{E})$. Pick $\{r_\alpha \mid \alpha \in \omega_1\}$ a set of distinct irrationals, we will recursively build $\mathcal{B}$ and $\mathcal{C}$ such that,

1. In $(\omega, \tau_\mathcal{D})$, B_α is a convergent sequence to r_α and it is dense in $(\omega, \tau_\mathcal{E})$.
2. In $(\omega, \tau_\mathcal{E})$, C_α is a convergent sequence to r_α and it is dense in $(\omega, \tau_\mathcal{D})$.
3. $B_\alpha \cap C_\alpha = \emptyset$ while $B_\alpha \cap C_\beta$, $B_\beta \cap C_\alpha$ are non-empty finite sets for every $\beta < \alpha$.

It is clear that if the recursion could be carried out, we would have constructed the desired family. Assume B_ξ, C_ξ had been constructed for every $\xi < \alpha$, let's find B_α and C_α. Let $\{U_n \mid n \in \omega\}$ be a local base for r_α with $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$ in $(\omega, \tau_\mathcal{D})$ and $\{V_n \mid n \in \omega\}$ a base in $(\omega, \tau_\mathcal{E})$. Enumerate $\alpha = \{\xi_n \mid n \in \omega\}$ and we recursively build $B_\alpha = \{x_n \mid n \in \omega\} \cup \{y_n \mid n \in \omega\}$ such that:

1. $x_n, y_n \in U_n$,
2. $x_n \in V_n \setminus \bigcup_{m<n} C_{\xi_m}$,
3. $y_n \in C_{\xi_n} \setminus \bigcup_{m<n} C_{\xi_m}$.

It is easy to do that, since each U_n is dense in $(\omega, \tau_\mathcal{E})$ and all the V_n and C_{ξ_n} are dense in $(\omega, \tau_\mathcal{D})$. C_α is built in the same way, just making sure for it to be disjoint with B_α. □

Proposition 2.14. MA(σ*-centered*) *is consistent with the existence of a* 3*-Luzin gap without Luzin subgaps.*

Proof. We will see that there is such a family after adding a Cohen real. Let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be a Luzin gap with $B_a \cap C_\alpha = \emptyset$ such that both $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable. Assume D is a Cohen real. In $V[D]$, define $\mathcal{B}_1 = \{B_\alpha \cap D \mid \alpha \in \omega_1\}$, $\mathcal{B}_2 = \{B_\alpha \setminus D \mid \alpha \in \omega_1\}$, we will prove that $\langle \mathcal{B}_1, \mathcal{B}_2, \mathcal{C} \rangle$ is the family we are looking for. It is easy to see that it is indeed a 3-Luzin gap, so it remains to show that it has no Luzin gaps.

In $V[D]$, let $m \in \omega$ and $\mathcal{X} = \{X_\alpha \mid \alpha \in \omega_1\}$, $\mathcal{Y} = \{Y_\alpha \mid \alpha \in \omega_1\}$ be disjoint subfamilies of $\mathcal{A}$ such that $X_\alpha \cap Y_\alpha \subseteq m$. We may assume $\mathcal{X}$ is a subset of $\mathcal{B}_1, \mathcal{B}_2$ or $\mathcal{C}$ (similarly for $\mathcal{Y}$). However, since $\mathcal{B}$ and $\mathcal{C}$ are $\mathbb{R}$-embeddable and every member of $\mathcal{B}_1$ is disjoint from every member of $\mathcal{B}_2$, then we only need to consider the case where $\mathcal{X}$ is a subset of $\mathcal{B}_1$ or $\mathcal{B}_2$ and $\mathcal{Y}$ is a subset of $\mathcal{C}$. For concreteness, we will assume $\mathcal{X} \subseteq \mathcal{B}_1$, while the other case is similar. Find a function $h : \omega_1 \to \omega_1 \times \omega_1$ such that $X_\alpha = B_{h(\alpha)_0} \cap D$ and $Y_\alpha = C_{h(\alpha)_1}$. We know there is an uncountable $W \in V$ and $s \in \mathbb{C}$ such that s knows $h \restriction W$, we may assume $m < |s| = l$. Let $\alpha, \beta \in W$ distinct such that $B_{h(\alpha)_0} \cap l = B_{h(\beta)_0} \cap l$ and $C_{h(\alpha)_1} \cap l = C_{h(\beta)_1} \cap l$. Let $r > l$ such that $B_{h(\alpha)_0} \cap C_{h(\beta)_1}, B_{h(\beta)_0} \cap C_{h(\alpha)_1} \subseteq r$ and choose s' any extension of s such that $r < |s'|$ and if $x \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)$ then $s'(x) = 0$. In this way, s' forces $X_\alpha \cap Y_\beta$, $X_\beta \cap Y_\alpha \subseteq m$ so $(\mathcal{X}, \mathcal{Y})$ is not a Luzin gap.

To finish the proof, assume MA holds in V, then MA(σ-centered) is still true after adding a Cohen real (by a theorem of Roitman, see [6, Theorem 3.3.8]). □

3. Schizophrenic AD families

Recall that $\mathcal{A}$ is *inseparable* if for every $\mathcal{B}, \mathcal{C} \in [\mathcal{A}]^{\omega_1}$ the set $\bigcup \mathcal{B} \cap \bigcup \mathcal{C}$ is infinite or equivalently, for every $m \in \omega$ there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $B \cap C \nsubseteq m$. Clearly, every uncountable subfamily of an inseparable family is inseparable and $\mathcal{A}$ is inseparable if and only if all of its subfamilies of size ω_1 are inseparable.

Let us introduce a forcing aiming to add a Luzin family to a given AD family $\mathcal{A}$. Assume that $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ and for every $p \in [\omega_1]^{<\omega}$ let m_p be the smallest integer such that $A_\alpha \cap A_\beta \subseteq m_p$ for all $\alpha, \beta \in p$ distinct. We define the poset $\mathcal{SR}(\mathcal{A}) = [\omega_1]^{<\omega}$ (see [17]) and we say $p \leqslant q$ if and only if,

1. $q \subseteq p$,
2. If $\alpha \in p \setminus q$ and there is $\beta \in q$ with $\alpha < \beta$, then $A_\beta \cap A_\alpha \nsubseteq m_q$.

Lemma 3.1. *([17]) If $\mathcal{SR}(\mathcal{A})$ is ccc, then $\mathcal{A}$ potentially contains a Luzin family.*

Proof. For every $\alpha \in \omega_1$ define $D_\alpha = \{p \mid p \nsubseteq \alpha\}$. It is easy to see that this set is dense, since if $p \subseteq \alpha$ then $p \cup \{\alpha\} \leqslant p$. Let G be a generic filter and in $V[G]$ define $\mathfrak{B} = \{A_\alpha \mid \alpha \in \bigcup G\}$, then $\mathfrak{B}$ is uncountable (since the forcing is ccc) and it is easy to see that it is indeed a Luzin family. □

Using this result, we may obtain the following characterization due to Roitman and Soukup [17].

Proposition 3.2. *([17]) $\mathcal{A}$ is inseparable if and only if every uncountable subfamily of $\mathcal{A}$ potentially contains a Luzin family.*

Proof. First, assume every uncountable subfamily of $\mathcal{A}$ potentially contains a Luzin family. Let $\mathcal{B}, \mathcal{C}$ be uncountable subfamilies of $\mathcal{A}$ and define $\mathcal{A}' = \mathcal{B} \cup C$. We know there is $\mathbb{P}$ a ccc forcing such that $1_{\mathbb{P}}$ forces that $\mathcal{A}'$ contains a Luzin family. Aiming for a contradiction, assume there is $m \in \omega$ such that $B \cap C \subseteq m$ for every $B \in \mathcal{B}$ and $C \in \mathcal{C}$. Let $G \subseteq \mathbb{P}$ be a generic filter and in $V[G]$ find $\mathcal{D} = \{X_\alpha \mid \alpha \in \omega_1\} \subseteq \mathcal{A}'$ be a Luzin family. Clearly, there is $\alpha \in \omega_1$ such that $X_\alpha \in \mathcal{B}$ and $\{X_\xi \mid \xi < \alpha\} \cap \mathcal{C}$ is infinite, but then the set $\{\xi < \alpha \mid X_\alpha \cap X_\xi \subseteq m\}$ is infinite, which contradicts that $\mathcal{D}$ is a Luzin family.

For the other implication, it is enough to prove that if $\mathcal{A}$ is inseparable of size ω_1, then $\mathcal{SR}(\mathcal{A})$ is ccc. We will proceed by contradiction, suppose $\{p_\alpha \mid \alpha \in \omega_1\}$ is an antichain, we may assume it forms a Δ-system with root r, every $p_\alpha \setminus r$ has size n and there is $m \in \omega$ such that $m_{p_\alpha} = m$ for all $\alpha \in \omega_1$. Furthermore, thinning our family, we may assume that for all α, every member of r is below every member of $p_\alpha \setminus r$ and if $\alpha < \beta$, then every member of $p_\alpha \setminus r$ is below every member of $p_\beta \setminus r$. Write $p_\alpha \setminus r = \{p_\alpha(i) \mid i < n\}$ and we may suppose there is $k > m$ such that $p_\alpha(i) \cap (k \setminus m) \neq \emptyset$ for all $\alpha \in \omega_1$. Thinning our family again, we may assume $p_\alpha(i) \cap k = p_\beta(i) \cap k$ for all $\alpha, \beta \in \omega_1$.

We will now see that there are $X_0, Y_0 \in [\omega_1]^{\omega_1}$ such that if $\alpha \in X_0$ and $\beta \in Y_0$ then $p_\alpha(0) \cap p_\beta(1) \nsubseteq m$. Suppose this is false, then for every $x > m$, at least one of the following sets $B_x = \{\alpha \mid x \in p_\alpha(0)\}$, $C_x = \{\alpha \mid x \in p_\alpha(1)\}$ is countable (and they are disjoint, since x is bigger than m). Let $\mathcal{B}$ be the set of all the $p_\alpha(0)$ such that $\alpha \notin \bigcup_{|B_x| \leqslant \omega} B_x$ and $\mathcal{C}$ be the set of all $p_\alpha(1)$ such that $\alpha \notin \bigcup_{|C_x| \leqslant \omega} C_x$. In this way, $\mathcal{B}$ and $\mathcal{C}$ are two uncountable subfamilies of $\mathcal{A}$. However, if $B \in \mathcal{B}$ and $C \in \mathcal{C}$ then $B \cap C \subseteq m$, which contradicts that $\mathcal{A}$ was inseparable.

Repeating this process several times, we find there are $X, Y \in [\omega_1]^{\omega_1}$ such that if $\alpha \in X$ and $\beta \in Y$ then $p_\alpha(i) \cap p_\beta(j) \nsubseteq m$ when $i \neq j$. However, we already knew that $p_\alpha(i) \cap p_\beta(i) \nsubseteq m$, since $p_\alpha(i) \cap k = p_\beta(i) \cap k$ and $p_\alpha(i) \cap (k \setminus m) \neq \emptyset$. This implies that $p_\alpha \cup p_\beta$ is a common extension of p_α and p_β, which is a contradiction. □

Definition 3.3. We say that an AD family $\mathcal{A}$ is *schizophrenic* if it is inseparable and contains no n-Luzin gaps for any $n \in \omega$.

Recall that by Theorem 2.5 an AD family which contains no n-Luzin gaps for any $n \in \omega$ is potentially normal, hence there is a ccc forcing that makes it normal. On the other hand, by the previous result if $\mathcal{A}$ is inseparable there is another ccc forcing one that freezes it by adding a Luzin gap, so it become indestructibly not normal!

Corollary 3.4. *If $\mathcal{A}$ is schizophrenic, then $\mathcal{R}(\mathcal{A})$ and $\mathcal{SR}(\mathcal{A})$ are two ccc forcings such that $\mathcal{R}(\mathcal{A}) \times \mathcal{SR}(\mathcal{A})$ is not ccc.*

In particular, MA implies that there are no schizophrenic families (another way to prove this, is to remember that MA implies that potentially normal entails indestructibly partially separated, and partially separated families does not contain Luzin gaps).

We will now show that the existence of schizophrenic families is consistent with ZFC.

We will show that the existence of schizophrenic AD families is independent of the Martin Axiom for σ-centered partial orders and from CH (note that this is exactly the same situation with Suslin trees). We will use the Cohen forcing $\mathbb{C} = {}^{<\omega}2$.

Lemma 3.5. *If $\dot{A}$ is a $\mathbb{C}$ name for an uncountable subset of ordinals, then there is $s \in \mathbb{C}$ and $X \in V$ uncountable such that $s \Vdash$ "$X \subseteq \dot{A}$". In other words, any new uncountable set of ordinals contains an old uncountable set of ordinals.*

Proof. For every $s \in \mathbb{C}$, let $A_s = \{a \mid s \Vdash \text{“}a \in A\text{”}\}$. Clearly, if $G \subseteq \mathbb{C}$ is generic, then $A = \bigcup_{s \in G} A_s$ and since A is uncountable, then one of the A_s must be uncountable. □

Now we will prove,

Theorem 3.6. *The existence of a schizophrenic family is consistent with* MA*(σ-centered).*

Proof. Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ be an inseparable family (take a Luzin family, for example) and let $D \subseteq \omega$ be a Cohen real over V. In $V[D]$ define $\mathcal{A} \restriction \mathcal{D}$ to be the set of all $A_\alpha \cap D$ with $\alpha \in \omega_1$. We will show that this is a schizophrenic family (note first that $\mathcal{A} \restriction \mathcal{D}$ is still an almost disjoint family).

Let us see that it is inseparable. In $V[D]$, let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$, and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be uncountable subfamilies of $\mathcal{A}$. In this way, we may define $h : \omega_1 \to \omega_1 \times \omega_1$ in such a way that $B_\alpha = A_{h(\alpha)_0} \cap D$ and $C_\alpha = A_{h(\alpha)_1} \cap D$ where $h(\alpha) = (h(\alpha)_0, h(\alpha)_1)$. By the previous lemma, there is $s \in \mathbb{C}$ and $X \in [\omega_1]^{\omega_1}$ (in V) such that s knows $h \restriction X$. We will find an extension of s that forces what we need.

Fix $m \in \omega$, we need to show that there are $\alpha, \beta \in \omega_1$ such that $B_\alpha \cap C_\beta = (A_{h(\alpha)_0} \cap A_{h(\beta)_1}) \cap D$ is not contained in m. Let $\mathcal{B}' = \{A_{h(\alpha)_0} \mid \alpha \in X\}$ and $\mathcal{C}' = \{A_{h(\alpha)_1} \mid \alpha \in X\}$ since $\mathcal{A}$ is inseparable, there are $\alpha, \beta \in X$ and $k > m, |s|$ such that $k \in A_{h(\alpha)_0} \cap A_{h(\beta)_1}$. If s' is any extension of s such that $s'(k) = 1$, then $s' \Vdash$ "$k \in B_\alpha \cap C_\beta$" and we are done.

Now, we will prove that $\mathcal{A} \restriction \mathcal{D}$ contains no n-Luzin gaps for any $n \in \omega$. Let $n \in \omega$ and assume for every $i < n$ we have $\langle B^i_\alpha \mid \alpha \in \omega_1 \rangle$ subfamilies of $\mathcal{A} \restriction \mathcal{D}$ such that there is $m \in \omega$ with the property that $B^i_\alpha \cap B^j_\alpha \subseteq m$ whenever $i \neq j$. As before, define a function $h : \omega_1 \to \omega_1^n$ such that $B^i_\alpha = A_{h(\alpha)_i} \cap D$ (with the same notation as before). Find $s \in \mathbb{C}$ that forces all of this, and an uncountable $X \in V$ such that s knows $h \restriction X$. Let $l = |s|$ and we may assume $m < l$.

Find $\alpha, \beta \in X$ distinct such that $A_{h(\alpha)_i} \cap l = A_{h(\beta)_i} \cap l$ for all $i < n$. Note that if $i \neq j$ then $A_{h(\alpha)_i} \cap A_{h(\beta)_j} \cap l = A_{h(\alpha)_i} \cap A_{h(\alpha)_j} \cap l \subseteq m$. Let $r > l$ such that $A_{h(\alpha)_i} \cap A_{h(\beta)_j} \subseteq r$ when $i \neq j$. Choose s' any extension of s such that $r < |s'|$ and if $x \in \mathrm{dom}(s') \setminus \mathrm{dom}(s)$ then $s'(x) = 0$. In this way, s' forces $B^i_\alpha \cap B^j_\beta \subseteq m$ for all $i \neq j$, so it is not an n-Luzin gap.

Again, by Roitman's theorem [6, Theorem 3.3.8], if we start with a model of MA then MA(σ-centered) holds in the extension. □

Inspired by the construction of a Suslin tree under $\diamondsuit$, we will construct a schizophrenic family under this axiom. Given $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ we defined the poset $\mathcal{R}(\mathcal{A})$ which elements are pairs of the form $(s, \mathcal{F})$ with $\mathcal{F} \in [\mathcal{A}]^{<\omega}$. Evidently, we could instead define $\mathcal{R}(\mathcal{A})$ as pairs $(z, \mathcal{G})$ with $\mathcal{G} \in [\omega_1]^{<\omega}$. We use this reformulation in the next theorem.

Theorem 3.7. *$\diamondsuit$ implies the existence of a schizophrenic family.*

Proof. Using $\diamondsuit$ we will construct an inseparable $\mathcal{A}$ such that $\mathcal{R}(\mathcal{A})$ is ccc. Fix two sequences $\mathcal{D}_1 = \langle (X_\alpha, Y_\alpha) \mid \alpha \in \omega_1 \rangle$ and $\mathcal{D}_2 = \langle D_\alpha \mid \alpha \in \omega_1 \rangle$ such that X_α and Y_α are disjoint subsets of α and D_α is a collection of finite subsets of $\mathbb{Q}^{<\omega} \times [\alpha]^{<\omega}$. The idea is that $\mathcal{D}_1$ guesses pairs of disjoints subsets of ω_1 and $\mathcal{D}_2$ guesses subsets of $\mathbb{Q}^{<\omega} \times [\omega_1]^{<\omega}$. More precisely, if X, Y are disjoint subsets of ω_1 then there are stationary many α such that $(X \cap \alpha, Y \cap \alpha) = (X_\alpha, Y_\alpha)$ and if $\mathcal{B}$ is collection of finite subsets of $\mathbb{Q}^{<\omega} \times [\omega_1]^{<\omega}$ then there are stationary many α such that $\mathcal{B} \cap (\mathbb{Q}^{<\omega} \times [\alpha]^{<\omega}) = D_\alpha$. We will use $\mathcal{D}_1$ to get the inseparability and $\mathcal{D}_2$ to show that $\mathcal{R}(\mathcal{A})$ is ccc.

Recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha \in \omega_1\}$ such that (denoting $\mathcal{A}_{<\alpha} = \{A_\xi \mid \xi < \alpha\}$ and $\mathcal{A}_{\leqslant\alpha} = \{A_\xi \mid \xi \leqslant \alpha\}$) if $\beta < \alpha$,

1. If $\mathcal{D}_\beta$ is a maximal antichain in $\mathcal{R}(\mathcal{A}_{<\beta})$ then it is still a maximal antichain in $\mathcal{R}(\mathcal{A}_{\leqslant\alpha})$.
2. If X_β and Y_β are infinite then for any $n \in \omega$ there are $\xi \in X_\beta$ and $\eta \in Y_\beta$ such that $A_\alpha \cap A_\xi$ and $A_\alpha \cap A_\eta$ are not contained in n.

We will first see that if the above construction can be carried out, then $\mathcal{A}$ is a schizophrenic family. We will prove the inseparability first. Assume X, Y are disjoint uncountable subsets of ω_1 and $n \in \omega$, we will see there are $\xi \in X$ and $\eta \in Y$ such that $A_\xi \cap A_\eta \nsubseteq n$. For the assumption of $\mathcal{D}_1$, there is β such that $(X \cap \beta, Y \cap \beta) = (X_\beta, Y_\beta)$ and X_β, Y_β are infinite. Since X is uncountable, there is $\alpha \in X$ such that $\beta < \alpha$. Using 2, there is $\eta \in Y_\beta \subseteq Y$ such that $A_\alpha \cap A_\eta \nsubseteq n$.

To prove that $\mathcal{R}(\mathcal{A})$ is ccc, let $\mathcal{B} \subseteq \mathcal{R}(\mathcal{A})$ be a maximal antichain. The set of β such that $\mathcal{B} \cap (\mathbb{Q}^{<\omega} \times [\beta]^{<\omega})$ is a maximal antichain in $\mathcal{R}(\mathcal{A}_{<\beta})$ contains a closed and unbounded set. So, there is a β such that $\mathcal{B} \cap (\mathbb{Q}^{<\omega} \times [\beta]^{<\omega}) = D_\beta$ and D_β is a maximal antichain of $\mathcal{R}(\mathcal{A}_{<\beta})$. Then using 1, we conclude that D_β is a maximal antichain in $\mathcal{R}(\mathcal{A})$ so $\mathcal{B} = D_\beta$ hence $\mathcal{B}$ is countable.

It remains to be seen that the construction can be carried out. Assume we have constructed everything up to $\alpha < \omega_1$.

Let $L_1 = \{\beta_n \mid n \in \omega\}$ enumerate the set of all $\beta < \alpha$ such that X_β and Y_β are infinite, also define $L_2 = \{\gamma_n \mid n \in \omega\}$ as the set of all $\gamma < \alpha$ such that $\mathcal{D}_\gamma$ is an infinite maximal antichain in $\mathcal{R}(\mathcal{A}_{<\gamma})$ and let $\mathcal{R}(\mathcal{A}_{<\alpha}) = \{(s_n, \mathcal{F}_n) \mid n \in \omega\}$, it will also be convenient to list $\alpha = \{\delta_n \mid n \in \omega\}$. Furthermore, we may assume that for every $\beta \in L_1, \gamma \in L_2, (s, F) \in \mathcal{R}(\mathcal{A}_{<\alpha})$ and $\delta \in \alpha$ there is $n \in \omega$ such that $\beta_n = \beta, \gamma_n = \gamma, (s_n, F_n) = (s, F)$ and $\delta_n = \delta$. For any $n \in \omega$ we will recursively define $\mathcal{P}_n$, m_n and A^n_α such that:

1. $\mathcal{P}_n \in [\mathcal{A}_{<\alpha}]^{<\omega}$, $m_n \in \omega$ and $A^n_\alpha \subseteq m_n$.
2. If $k < n$ then $\mathcal{P}_k \subseteq \mathcal{P}_n$, $m_k < m_n$ and $A^k_\alpha \sqsubseteq A^n_\alpha$ (where $\sqsubseteq$ denotes end extension).
3. If $A_\xi \in \mathcal{P}_k$ and $k < n$ then $A_\xi \cap A^n_\alpha \subseteq A^k_\alpha$.
4. $A_{\delta_n} \in \mathcal{P}_n$.
5. For every $n \in \omega$ there are $\xi \in X_{\beta_n}$ and $\eta \in Y_{\beta_n}$ such that $A^n_\alpha \cap A_\xi$, $A^n_\alpha \cap A_\eta \nsubseteq n$.
6. For every $n \in \omega$ either there is $B \in \mathcal{F}_n$ such that $B \cap A^n_\alpha \nsubseteq |s_n|$ or there is $(z, \mathcal{G}) \in \mathcal{R}(\mathcal{A}_{<\gamma_n})$ and $r \in^{<\omega} \mathbb{Q}$ with the property that:
   a) $s_n, z \subseteq r$ and $|r| \leqslant m_n$.
   b) If $B \in \mathcal{F}_n$ and $C \in \mathcal{G}$ then $B \cap C \subseteq |r|$.
   c) If $\xi \in \mathcal{G}$ then $A_\xi \in \mathcal{P}_n$ and $\mathcal{F} \subseteq \mathcal{P}_n$.

The idea is that A^n_α are finite approximations of A_α so at the end we will define $A_\alpha = \bigcup A^n_\alpha$. $\mathcal{P}_n$ are the elements we "promise" not to intersect anymore. In point 6 we are making sure that either $(s_n, \mathcal{F}_n \cup \{\alpha\})$ is not a condition of $\mathcal{R}(\mathcal{A}_{\leqslant\alpha})$

or it is compatible with an element of $\mathcal{R}(\mathcal{A}_{<\gamma_n})$ since if $(z,\mathcal{G}), r$ satisfy the conditions from above then $(r, \mathcal{G}\cup\mathcal{F}\cup\{\alpha\})$ will be a common extension of $(z,\mathcal{G})$ and $(s_n, \mathcal{F}_n \cup \{\alpha\})$.

Assume we have defined everything up to n, let's define $\mathcal{P}_n$, A^n_α and m_n. Let $\mathcal{P}'_n = \bigcup_{i<n} \mathcal{P}_i \cup \{A_{\delta_n}\}$, $A^{n\prime}_\alpha = \bigcup_{i<n} A^i_\alpha$. First, we find $\xi \in X_{\beta_n}$ and $\eta \in Y_{\beta_n}$ such that $A_\xi, A_\eta \notin \mathcal{P}'_n$ (this may be done easily since X_{β_n}, Y_{β_n} are infinite and $\mathcal{P}'_n$ is finite). Define $A^{n\prime\prime}_\alpha$ an end extension of $A^{n\prime}_\alpha$ such that $A^{n\prime\prime}_\alpha \cap A_\xi$, $A^{n\prime\prime}_\alpha \cap A_\eta \nsubseteq n$ and with the property that $A^{n\prime\prime}_\alpha \cap A_\nu \subseteq A^{n\prime}_\alpha$ for all $A_\nu \in \mathcal{P}'_n$. We must now take care of point 6. Take $(s_n, \mathcal{F}_n)$ and if there is $B \in \mathcal{F}_n$ such that $B \cap A^{n\prime\prime}_\alpha \nsubseteq |s_n|$ we just let $A^n_\alpha = A^{n\prime\prime}_\alpha$, $\mathcal{P}_n = \mathcal{P}'_n \cup \mathcal{F}_n$ and $m_n > \max(A^n_\alpha)$ and we are done. Moreover, if there is $B \in \mathcal{F}_n$ such that $B \notin \mathcal{P}'_n$ we just take A^n_α and end extension of $A^{n\prime\prime}_\alpha$ such that $B \cap A^n_\alpha \nsubseteq |s_n|$ and we define $\mathcal{P}_n = \mathcal{P}'_n \cup \mathcal{F}_n$ and $m_n > \max(A^n_\alpha)$ and the requirements are fulfilled.

So assume $B \cap A^{n\prime\prime}_\alpha \subseteq |s_n|$ for all $B \in \mathcal{F}_n$ and $\mathcal{F}_n \subseteq \mathcal{P}_n$. To simplify the notation name $s = s_n$ and $\mathcal{F} = \mathcal{F}_n$. With out loosing generality, we may assume $A^{n\prime\prime}_\alpha \subseteq |s|$ (if not, just extend s). Naturally, there is $\gamma_n \leqslant \nu < \alpha$ such that $(s,\mathcal{F}) \in \mathcal{R}(\mathcal{A}_{\leqslant\nu})$ so by our recursion hypothesis, there is $(z,\mathcal{G}) \in \mathcal{R}(\mathcal{A}_{<\gamma_n})$ that is compatible with $(s,\mathcal{F})$. Let r be such that $(r, \mathcal{F}\cup\mathcal{G})$ is a common extension and define $\mathcal{P}_n = \mathcal{P}'_n \cup \mathcal{G}$, $m_n > \max\{A^n_\alpha, |r|\}$ and $A^n_\alpha = A^{n\prime\prime}_\alpha$ and we are finally done. □

In particular, there may be schizophrenic families in models of CH, however we will now show that the continuum hypothesis is not sufficient for the existence of a schizophrenic family. This will be done with the aid of the *P-ideal dichotomy* (see [2] and [15]). Recall than an ideal $\mathcal{I}$ is a P-ideal if for every $\{Y_n \mid \in \omega\} \subseteq \mathcal{I}$ there is a $Y \in \mathcal{I}$ that contains mod fin every Y_n.

PID) If $\mathcal{I} \subseteq [\omega_1]^{\leqslant\omega}$ is a P-ideal then one of the following holds,
⋆) There is $X \in [\omega_1]^{\omega_1}$ such that $[X]^\omega \subseteq \mathcal{I}$.
⋆⋆) There is a partition $\omega_1 = \bigcup S_n$ such that $[S_n]^\omega \cap \mathcal{I} = \emptyset$ for every $n \in \omega$.

PID is known to be consistent with CH (see [2, Section 3]). Given $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$ and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ we call $X \subseteq \omega_1$ a *partial separator* if there is $m \in \omega$ such that if $\alpha, \beta \in X$ then $B_\alpha \cap C_\beta \subseteq n$. Therefore, $\mathcal{A}$ is partially separated if any two disjoint uncountable subsets have an uncountable partial separator. Applying the same ideas as in [2, Theorem 2.2] we prove:

Theorem 3.8. *If* PID *holds and $\mathcal{A}$ is an AD of size ω_1, then either $\mathcal{A}$ is partially separated or it contains a Luzin gap.*

Proof. Let $\mathcal{B} = \{B_\alpha \mid \alpha \in \omega_1\}$ and $\mathcal{C} = \{C_\alpha \mid \alpha \in \omega_1\}$ be two disjoint subsets of $\mathcal{A}$. We will prove that either there is a Luzin gap contained in $\mathcal{B}, \mathcal{C}$ or they have an uncountable partial separator. Given $X \subseteq \omega_1$ and $n \in \omega$ define

$$X_n(\alpha) = \bigl\{\xi \in X \cap \alpha \bigm| (B_\xi \cap C_\alpha) \cup (B_\alpha \cap C_\xi) \subseteq n\bigr\}.$$

Now, let $\mathcal{I}$ be the set of all $X \in [\omega_1]^{\leqslant\omega}$ such that if $\alpha \leqslant \sup(X)$ and $n \in \omega$, then $X_n(\alpha)$ is finite. It is easy to see that $\mathcal{I}$ is an ideal and for the moment assume it is a P-ideal. Using PID either there is an uncountable $X \subseteq \omega_1$ such that $[X]^\omega \subseteq \mathcal{I}$ or there is an uncountable S such that $[S]^\omega \cap \mathcal{I} = \emptyset$. We will see that if the first option holds, then $\mathcal{B}, \mathcal{C}$ contains a Luzin gap and if the second then we get an uncountable partial separator. Without loss of generality, we may assume that there is $n \in \omega$ such that $B_\alpha \cap C_\alpha \subseteq n$ for all $\alpha \in \omega_1$.

Assume first that there is $X \in [\omega_1]^{\omega_1}$ such that $[X]^\omega \subseteq \mathcal{I}$ and define $h : X \to [X]^{<\omega}$ by $h(\alpha) = X_n(\alpha) \subseteq \alpha$. By a standard use of the pressing down lemma, there is $S \subseteq X$ stationary (stationary in X, not necessary in ω_1) such that h is constant on S. It is immediate that $\{B_\alpha \mid \alpha \in S\}, \{C_\alpha \mid \alpha \in S\}$ form a Luzin gap.

Now assume there is an uncountable S such that $[S]^\omega \cap \mathcal{I} = \emptyset$, we want to show that S contains an uncountable partial separator for $\mathcal{B}$ and $\mathcal{C}$. Assume this is not the case, let $M \subseteq S$ be a maximal partial separator, so it is countable. Pick $\gamma \in S$ such that $M \subseteq \gamma$. Since $\gamma \notin M$ then there is $\alpha_m \in M$ such that $(B_{\alpha_m} \cap C_\gamma) \cup (B_\gamma \cap C_{\alpha_m})$ is not contained in m. Let $X = \{\alpha_m \mid m \in \omega\} \cup \{\gamma\} \subseteq S$, since $X \notin \mathcal{I}$ then there is $m \in \omega$ such that $X_m(\gamma)$ is infinite, but this is clearly a contradiction. So we conclude there must be an uncountable partial separator.

To finish the proof, we only need to show that $\mathcal{I}$ is indeed a P-ideal. Let $Y^0 \subseteq Y^1 \subseteq Y^2 \subseteq \cdots \in \mathcal{I}$ and $\alpha = \sup(\bigcup_{n\in\omega} Y^n)$. Let $\alpha + 1 = \{\alpha_n \mid n \in \omega\}$ and define the set:

$$\begin{array}{ll} & Y^0 \setminus Y^0_0(\alpha_0) \\ \bigcup & \bigl(Y^1 \setminus Y^1_1(\alpha_0) \cup Y^1_1(\alpha_1)\bigr) \\ \bigcup & \bigl(Y^2 \setminus Y^2_2(\alpha_0) \cup Y^2_2(\alpha_1) \cup Y^2_2(\alpha_2)\bigr) \\ \vdots & \vdots \end{array}$$

It is easy to see that this set belongs to $\mathcal{I}$ and it is a pseudounion of the Y^n. □

So we may conclude the following,

Corollary 3.9. *There is a model of CH without schizophrenic families.*

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