Topology and its Applications 305 (2022) 107871

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

More on MAD families and P-points ☆

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A R T I C L E I N F O

Article history:
Received 29 July 2021
Received in revised form 6 October 2021
Accepted 6 October 2021
Available online 9 October 2021

Keywords:
MAD families
P-points
Franklin space

A B S T R A C T

We prove that, under the assumption of CH, for every MAD family $\mathcal{A}$ there is a P-point $\mathcal{U}$ such that the Franklin compact space associated to $\mathcal{A}$ is a Fréchet-Urysohn space ($\mathsf{FU}(\mathcal{U})$-space). This solves a 25 years old question of V. I. Malykhin and the first author.

1. Introduction

Given a MAD family $\mathcal{A}$, its *Mrówka-Isbell space* $\Psi(\mathcal{A})$ is defined as follows: the underlying space is $\omega \cup \mathcal{A}$, the points of ω are isolated, and if $A \in \mathcal{A}$, it has a local base of the form $\{(A \setminus F) \cup \{A\} \mid F \in [\omega]^{<\omega}\}$. It is easy to see that $\Psi(\mathcal{A})$ is a Hausdorff, first countable, separable, locally compact, zero dimensional space. These spaces are important not only in set theory and topology, but also in other areas like functional analysis (in the study of Banach spaces and Banach algebras). The reader can learn more on Mrówka-Isbell spaces in [11] or [13].

Since $\Psi(\mathcal{A})$ is locally compact, we can take its Alexandroff (one-point) compactification (see [17]). This space is denoted as $\mathsf{Fr}(\mathcal{A})$ and is called the *Franklin space of* $\mathcal{A}$. We need to recall the following notion:

Definition 1. Let X be a topological space. We say that X is *Fréchet–Urysohn* if for every $a \in X$ and $A \subseteq X$, if $x \in \overline{A}$, then there is a sequence $\langle x_n \rangle_{n<\omega}$ in A such that $\lim_{n\to\infty} x_n = x$.

If $\mathcal{A}$ is a MAD family, it is easy to see that $\mathsf{Fr}(\mathcal{A})$ is not a Fréchet–Urysohn space (the maximality of $\mathcal{A}$ implies that the point at infinity cannot be the limit point of a convergent sequence of natural numbers).

☆ Research of the first-named author was supported by the PAPIIT grant no. IN-100122. The second author gratefully acknowledges support from a CONACyT grant A1-S-16164 and a PAPIIT grant IN 104220.

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https://doi.org/10.1016/j.topol.2021.107871
0166-8641/

This suggests to look at some weakening of the Fréchet–Urysohn property that may hold for the Franklin space of a MAD family.

Definition 2 *(Folklore).* Let $\mathcal{U}$ be an ultrafilter on ω and X a topological space.

1. Let $x \in X$ and $\langle x_n \rangle_{n\in\omega}$ be a sequence of elements of X. We say that x is the *$\mathcal{U}$-limit of* $\langle x_n \rangle_{n\in\omega}$ (denoted by $x = \lim_{\mathcal{U}} \langle x_n \rangle_{n\in\omega}$) if for every open neighborhood $V \subseteq X$ of x, the set $\{n < \omega \mid x_n \in V\}$ belongs to $\mathcal{U}$.
2. We say that X is *Fréchet–Urysohn with respect to* $\mathcal{U}$ (for short, X is $\mathsf{FU}(\mathcal{U})$) if for every $x \in X$ and $A \subseteq X$, if $x \in \overline{A}$, then there is a sequence $\langle x_n \rangle_{n\in\omega}$ in A such that $x = \lim_{\mathcal{U}} \langle x_n \rangle_{n\in\omega}$.

Given a MAD family $\mathcal{A}$ we may wonder if $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$ for some an ultrafilter $\mathcal{U}$ on ω. In [4] Boldjiev and V. I. Malykhin proved the following:

Proposition 3 *([4]). If $\mathcal{A}$ is a MAD family and $\mathcal{U}$ is not a P-point, then $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$.*

In fact, by combining results of [4] and [8], in [9] it was pointed out the following:

Proposition 4 *(CH). Let $\mathcal{U}$ be an ultrafilter. The following are equivalent:*

1. *$\mathcal{U}$ is a P-point.*
2. *There is a MAD family $\mathcal{A}$ such that $\mathsf{Fr}(\mathcal{A})$ is not $\mathsf{FU}(\mathcal{U})$.*

By the above results, the question whether $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$ is only interesting in the case where $\mathcal{U}$ is a P-point. A famous theorem of S. Shelah establishes that it is consistent that there are no P-points (see [2], see also [7] for a different model). In this way, it is consistent that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$ for every ultrafilter. The following problem was suggested by V. I. Malykhin and S. Garcia-Ferreira:

Problem 5 *([8]). Does CH imply that for every MAD family $\mathcal{A}$, there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$?*

In the paper [9], the authors made an important advanced on this problem:

Proposition 6 *([9]). Let $V \models$ CH. If $\mathcal{A}$ is a K-uniform MAD family, then there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$.*

The definition of a K-uniform MAD family will be reviewed in the next section.

In this paper, we will provide a complete answer to the Problem 5. Concretely, we will prove that the Continuum Hypothesis implies that for every MAD family $\mathcal{A}$, there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$.

2. Preliminaries

An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite, a MAD *family* is a maximal almost disjoint family. For $A, B \in [\omega]^\omega$, by $A \subseteq^* B$ we mean that $A \setminus B$ is finite and we say that A is an *almost subset of* B. An *ultrafilter* is a maximal filter on ω. In this note, all ultrafilters are assumed to be non-principal. An ultrafilter $\mathcal{U}$ on ω is a *P-point* if for every $\mathcal{H} = \{A_n \mid n < \omega\} \subseteq \mathcal{U}$ there is $B \in \mathcal{U}$ such that $B \subseteq^* A_n$ for every $n < \omega$ (in this case, we say that B is a *pseudointersection of* $\mathcal{H}$).

We assume that ideals on ω contain all finite sets, but not ω (i.e. all ideals are non-trivial). If $\mathcal{Y} \subseteq [\omega]^\omega$ by $\langle \mathcal{Y} \rangle$ we denote the ideal generated by $\mathcal{Y}$ (and finite sets). For an AD family $\mathcal{A}$, by $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{A}$ and by $\mathcal{I}(\mathcal{A})^*$ we denote the dual filter. If $\mathcal{I}$ is an ideal, we define $\mathcal{I}^+ = \wp(\omega) \diagup \mathcal{I}$.

Definition 7. Let $\mathcal{U}$ and $\mathcal{V}$ ultrafilters on ω.

1. We say $f : \omega \longrightarrow \omega$ is a *Rudin-Keisler function from* $(\omega, \mathcal{U})$ to $(\omega, \mathcal{V})$ if for every $A \subseteq \omega$ the following holds:
$$A \in \mathcal{V} \text{ if and only if } f^{-1}(A) \in \mathcal{U}$$
2. We say $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ if there is a Rudin-Keisler function from $(\omega, \mathcal{U})$ to $(\omega, \mathcal{V})$.

Given a function $f : \omega \longrightarrow \omega$ and an ultrafilter $\mathcal{U}$, define $f(\mathcal{U}) = \{A \mid f^{-1}(A) \in \mathcal{U}\}$. It is not hard to see that $f(\mathcal{U})$ is an ultrafilter and it is generated by the family $\{f[B] \mid B \in \mathcal{U}\}$ (see [17]). Note that if $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters, then $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ if and only if there is a function $f : \omega \longrightarrow \omega$ such that $\mathcal{V} = f(\mathcal{U})$. The Rudin-Keisler order can also be thought as follows: Given a function $f : \omega \longrightarrow \omega$, we may view it as a continuous function $f : \omega \longrightarrow \beta\omega$, so we can find a (unique) continuous extension $\overline{f} : \beta\omega \longrightarrow \beta\omega$. In this way, it is not hard to see that $f : \omega \longrightarrow \omega$ is a Rudin-Keisler function from $(\omega, \mathcal{U})$ to $(\omega, \mathcal{V})$ if and only if $\mathcal{V} = f(\mathcal{U})$.

Definition 8. Let A and B be two countable sets, $\mathcal{I}, \mathcal{J}$ ideals on X and Y respectively and $f : Y \longrightarrow X$ a function.

1. We say f is a *Katětov function from* $(Y, \mathcal{J})$ *to* $(X, \mathcal{I})$ if $f^{-1}(A) \in \mathcal{J}$ for every $A \in \mathcal{I}$.
2. We define $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ ($\mathcal{I}$ *is Katětov smaller than* $\mathcal{J}$ or $\mathcal{J}$ *is Katětov above* $\mathcal{I}$) if there is a Katětov function from $(Y, \mathcal{J})$ to $(X, \mathcal{I})$.

A MAD family A is called *K-uniform* if $\mathcal{I}(\mathcal{A}) \restriction X \leq_{\mathrm{K}} \mathcal{I}(\mathcal{A})$ for every $X \in \mathcal{I}(\mathcal{A})^+$. In order to learn more about the Katětov order, the reader is referred to [15], [14], [1] or [12]. As mentioned before, P. Szeptycki and S. Garcia-Ferreira proved (under the Continuum Hypothesis), that if $\mathcal{A}$ is a K-uniform MAD family, then there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$. This was an important advance in solving the Problem 5. Nevertheless, it does not fully answer the problem since it is known that there are MAD families that are not K-uniform. However, it is interesting to note that it is unknown if K-uniform MAD families exist in ZFC (they exist under certain hypothesis like CH or $\mathfrak{p} = \mathfrak{c}$).

The main tool in order to decide if a Franklin space is $\mathsf{FU}(\mathcal{U})$ (for an ultrafilter $\mathcal{U}$) is the following result from [8]:

Proposition 9 *([8]). Let $\mathcal{A}$ be a MAD family and $\mathcal{U}$ an ultrafilter. The following are equivalent:*

1. *The space $\mathsf{Fr}(\mathcal{A})$ is not $\mathsf{FU}(\mathcal{U})$.*
2. *There is $C \in \mathcal{I}(\mathcal{A})^+$ such that for every ultrafilter $\mathcal{V}$, if $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ and $C \in \mathcal{V}$, then $\mathcal{I}(\mathcal{A})^* \nsubseteq \mathcal{V}$.*

We rewrite this result in a more convenient way for us:

Proposition 10 *([8]). Let $\mathcal{A}$ be a MAD family and $\mathcal{U}$ an ultrafilter. The following are equivalent:*

1. *The space $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$.*
2. *For every $C \in \mathcal{I}(\mathcal{A})^+$ there is $\mathcal{V} \leq_{\mathrm{RK}} \mathcal{U}$ such that $C \in \mathcal{V}$ and $\mathcal{I}(\mathcal{A})^* \subseteq \mathcal{V}$.*

Using this result, we will prove that the Continuum Hypothesis implies that for every MAD family $\mathcal{A}$, there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$. For the convenience of the reader we follow the following conventions: $\mathcal{A}$ and $\mathcal{B}$ we will denote AD families, $\mathcal{U}$ and $\mathcal{V}$ will denote ultrafilters and p, q, r will denote conditions in the forcing that we will define in the next section.

3. MAD families and P-points

In this section we solve Problem 5. In fact, we will prove that for every MAD family $\mathcal{A}$ there is a σ-closed forcing notion $\mathbb{P}_{\mathcal{A}}$ such that $\mathbb{P}_{\mathcal{A}}$ adds a P-point $\mathcal{U}_{gen}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U}_{gen})$. Furthermore, it will be clear by the construction that in order to build an ultrafilter with those properties, we only need to intersect $\mathfrak{c}$ many open dense subsets of $\mathbb{P}_{\mathcal{A}}$ (where $\mathfrak{c}$ denotes the size of the continuum). It follows that the construction can be carried out under the Continuum Hypothesis.

Definition 11. Let $\mathcal{R}$ be a family of ideals on ω. We say that $\mathcal{R}$ is a *compatible set of ideals* if $\bigcup\{\mathcal{I} \mid \mathcal{I} \in \mathcal{R}\}$ generates an ideal (equivalently, for every $\mathcal{I}_1, ..., \mathcal{I}_n \in \mathcal{R}$, the family $\mathcal{I}_1 \cup ... \cup \mathcal{I}_n$ generates an ideal).

We will say that two ideals $\mathcal{I}$ and $\mathcal{J}$ are compatible if $\{\mathcal{I}, \mathcal{J}\}$ is a set of compatible ideals. If $\mathcal{R}$ is a collection of AD families, we will say that $\mathcal{R}$ is a compatible set of AD families if the family of its respective ideals is compatible. We now introduce the following forcing notion:

Definition 12. Define $\mathbb{P}$ as the set of all $p = (S, \mathcal{Y})$ with the following properties:

1. $S \in [\omega]^{\omega}$.
2. $\mathcal{Y} = \{\mathcal{A}_n \mid n \in \omega\}$ is a countable collection of compatible AD families. Let $\mathcal{I}(\mathcal{Y})$ be the ideal generated by $\bigcup\limits_{n\in\omega} \mathcal{I}(\mathcal{A}_n)$.
3. $S \in \mathcal{I}^+_{\mathcal{Y}}$.

If $p = (S_p, \mathcal{Y}_p)$ and $q = (S_q, \mathcal{Y}_q)$ are in $\mathbb{P}$, define $p \leq q$ if the following holds:

1. $\mathcal{Y}_q \subseteq \mathcal{Y}_p$.
2. $S_p \setminus S_q \in \mathcal{I}(\mathcal{Y}_p)$.

If we want to force with $\mathbb{P}$, we must first prove the following:

Lemma 13. $(\mathbb{P}, \leq)$ *is a preorder.*

Proof. We only need to see that the relation is transitive. Let $p = (S_p, \mathcal{Y}_p)$, $q = (S_q, \mathcal{Y}_q)$ and $r = (S_r, \mathcal{Y}_r)$ be elements in $\mathbb{P}$ such that $p \leq q \leq r$. We need to prove that $p \leq r$. We know the following:

1. $\mathcal{Y}_r \subseteq \mathcal{Y}_q$ and $\mathcal{Y}_q \subseteq \mathcal{Y}_p$.
2. $S_q \setminus S_r \in \mathcal{I}(\mathcal{Y}_q)$ and $S_p \setminus S_q \in \mathcal{I}(\mathcal{Y}_p)$.

From the first item it follows that $\mathcal{Y}_r \subseteq \mathcal{Y}_p$. Now, note that $S_p \setminus S_r \subseteq (S_p \setminus S_q) \cup (S_q \setminus S_r)$. Furthermore, since $\mathcal{I}(\mathcal{Y}_q) \subseteq \mathcal{I}(\mathcal{Y}_p)$ we get that $S_p \setminus S_r \in \mathcal{I}(\mathcal{Y}_p)$ by the item 2 above. $\square$

The following is a very important notion in the theory of ideals on countable sets:

Definition 14. Let $\mathcal{I}$ be an ideal. We say that $\mathcal{I}$ is a P^+-ideal if for every $\subseteq$-decreasing sequence $\mathcal{H} = \{B_n \mid n \in \omega\} \subseteq \mathcal{I}^+$, there is $A \in \mathcal{I}^+$ that is a pseudointersection of $\mathcal{H}$.

The following is a very well known result. The reader may consult [13] for a proof.

Proposition 15. *If $\mathcal{A}$ is an AD family, then $\mathcal{I}(\mathcal{A})$ is a P^+-ideal.*

We will now prove the following:

Proposition 16. *Let $\mathcal{I}$ be an ideal and $\mathcal{B}$ an AD family such that $\mathcal{I}$ and $\mathcal{I}(\mathcal{B})$ are compatible. If $\mathcal{I}$ is P^+, then $\mathcal{J} = \langle \mathcal{I} \cup \mathcal{I}(\mathcal{B}) \rangle$ is P^+.*

Proof. Let $\mathcal{X} = \{X_n \mid n \in \omega\} \subseteq \mathcal{J}^+$ be an $\subseteq$-decreasing sequence. We need to prove that $\mathcal{X}$ has a pseudointersection in $\mathcal{J}^+$. The following is a trivial remark, but we need to keep it in mind:

If $Y \in \mathcal{I}^+$ is a pseudointersection of $\mathcal{X}$ with $Y \in \mathcal{J}$, then
there are $B \in \mathcal{B}$ and $C \subseteq B \cap Y$ such that $C \in \mathcal{I}^+$.

We will proceed by contradiction, so assume that $\mathcal{X}$ does not have a pseudointersection in $\mathcal{J}^+$.

Given $A \in \mathcal{I}(\mathcal{B})$ and $m \in \omega$, define $\mathcal{X}_m(A) = \{X_n \setminus A \mid n > m\}$. It is clear that $\mathcal{X}_m(A)$ is a countable subset of $\mathcal{J}^+$. We will now recursively find $\langle Y_n, B_n, C_n \rangle_{n\in\omega}$ such that for every $n \in \omega$, the following conditions hold:

1. $Y_n \in \mathcal{I}^+$ and $Y_n \subseteq X_n$.
2. Y_n is a pseudointersection of $\mathcal{X}_n(\bigcup_{i<n} B_i)$.
3. $B_n \in \mathcal{B}$ and $B_n \neq B_m$ whenever $n \neq m$.
4. $C_n \subseteq B_n \cap Y_n$ and $C_n \in \mathcal{I}^+$.

The construction is essentially trivial. First, we take $Y_0 \subseteq X_0$ a pseudointersection of $\mathcal{X}$ such that $Y_0 \in \mathcal{I}^+$. Since $Y_0 \in \mathcal{J}$, we choose $B_0 \in \mathcal{B}$ and $C_0 \subseteq B_0 \cap Y_0$ with $C_0 \in \mathcal{I}^+$ (recall the remark at the beginning of the proof). Assume we have already defined Y_n, B_n and C_n. Let $Y_{n+1} \subseteq X_{n+1}$ be a pseudointersection of $\mathcal{X}_{n+1}(\bigcup_{i\leq n} B_i)$ such that $Y_{n+1} \in \mathcal{I}^+$. Since $Y_{n+1} \in \mathcal{J}$, we choose $B_{n+1} \in \mathcal{B}$ and $C_{n+1} \subseteq B_{n+1} \cap Y_{n+1}$ with $C_{n+1} \in \mathcal{I}^+$.

Define $W = \bigcup_{n\in\omega} C_n$ and note that it is a pseudointersection of $\mathcal{X}$. We claim that $W \in \mathcal{J}^+$. For this, it is enough to prove that if $D \in \mathcal{I}$ and $E \in \mathcal{I}(\mathcal{B})$, then W is not contained in $D \cup E$. Since $E \in \mathcal{I}(\mathcal{B})$, we can find $m \in \omega$ such that $E \cap C_m$ is finite. Since $C_m \in \mathcal{I}^+$, we know that C_m is not almost cover by D, so W is not covered by $D \cup E$. In this way, $W \in \mathcal{J}^+$ and is a pseudointersection of $\mathcal{X}$, which is a contradiction. □

We can now prove the following lemma:

Lemma 17. *Let $\mathcal{R} = \{\mathcal{A}_n \mid n \in \omega\}$ be a compatible collection of AD families. The ideal $\mathcal{J} = \langle \bigcup_{n\in\omega} \mathcal{I}(\mathcal{A}_n) \rangle$ is a P^+-ideal.*

Proof. For every $n \in \omega$, let $\mathcal{I}_n = \langle \bigcup_{i\leq n} \mathcal{I}(\mathcal{A}_i) \rangle$ and $\mathcal{K} = \{\mathcal{I}_n \mid n \in \omega\}$. Clearly $\mathcal{K}$ is a compatible set of ideals. Furthermore, by the Proposition 16, every $\mathcal{I}_n$ is a P^+-ideal. Let $\{B_n \mid n \in \omega\}$ be a decreasing sequence of $\mathcal{J}^+$-sets. Since each $\mathcal{I}_n$ is a P^+-ideal, for every $m \in \omega$, we can find A_m with the following properties:

1. $A_m \in \mathcal{I}_m^+$.
2. A_m is a pseudointersection of $\{B_n \mid n \in \omega\}$.
3. $A_m \subseteq B_m$.

Let $A = \bigcup_{m\in\omega} A_m$, it is easy to see that A is a pseudointersection of $\{B_n \mid n \in \omega\}$ and $A \in \mathcal{J}^+$. $\square$

By identifying $\wp(\omega)$ with 2^ω and providing this latter with the product topology, we can talk about the topological properties of ideals, such as being Borel or being meager. Understanding the topological nature of ideals is fundamental in order to study them.

Definition 18. Let $\mathcal{I}$ be an ideal on ω and $\mathcal{P} = \{P_n \mid n \in \omega\}$ a partition of ω into finite sets. We say that $\mathcal{P}$ is a *Talagrand partition for* $\mathcal{I}$ if for every $X \subseteq \omega$, if X contains infinitely many elements of $\mathcal{P}$, then $X \in \mathcal{I}^+$.

The following is a classical theorem in the theory of ideals. The reader may consult [2] for a proof:

Theorem 19 *(Talagrand, Jalali-Naini). Let $\mathcal{I}$ be an ideal on ω. The following are equivalent:*

1. *$\mathcal{I}$ is meager.*
2. *$\mathcal{I}$ has a Talagrand partition.*

The following result is well known, we prove it here for the sake of completeness:

Lemma 20. *If $\mathcal{I}$ is a meager ideal, then there is an infinite partition of ω in $\mathcal{I}^+$ pieces.*

Proof. Let $\mathcal{P} = \{P_n \mid n \in \omega\}$ be a Talagrand partition for $\mathcal{I}$. Choose any $\{B_n \mid n \in \omega\}$ partition of ω such that each B_n is infinite. For every $n \in \omega$, define $A_n = \bigcup_{i\in B_n} P_i$. It is clear that $\{A_n \mid n \in \omega\}$ has the desired properties. $\square$

By the results above, it follows that if $p = (S, \mathcal{Y}) \in \mathbb{P}$, then $\mathcal{I}(\mathcal{Y})$ is a P^+-ideal. Furthermore, since the ideal of every AD family is meager, it follows that $\mathcal{I}(\mathcal{Y})$ is meager as well.

Let $\mathbb{Q}$ be a partial preorder. Recall that $\mathbb{Q}$ is *σ-closed* if every countable decreasing sequence of conditions in $\mathbb{Q}$ has a lower bound. Armed with the previous results, we can prove the following:

Lemma 21. $\mathbb{P}$ *is σ-closed.*

Proof. Let $\langle (S_n, \mathcal{Y}_n)\rangle_{n\in\omega}$ be a decreasing sequence of conditions of $\mathbb{P}$. Define $\mathcal{Y} = \bigcup \mathcal{Y}_n$. We know that $\mathcal{Y}$ is a countable collection of AD families. Note that $\mathcal{I}(\mathcal{Y})$ is an ideal. We now have the following:

Claim 22. $S_n \in \mathcal{I}(\mathcal{Y})^+$ *for every $n \in \omega$.*

Let $n \in \omega$, it is enough to argue that $S_n \in \mathcal{I}(\mathcal{Y}_m)^+$ for every $m \in \omega$. If $m \leq n$, by definition we know that $S_n \in \mathcal{I}(\mathcal{Y}_n)^+$. Since $\mathcal{I}(\mathcal{Y}_m) \subseteq \mathcal{I}(\mathcal{Y}_n)$, we are done in this case. Now, assume that $n < m$. We know that $S_m \setminus S_n \in \mathcal{I}(\mathcal{Y}_m)$. If $S_n \in \mathcal{I}(\mathcal{Y}_m)$ we would have that $S_m \in \mathcal{I}(\mathcal{Y}_m)$, but by definition we know that this is not the case.

By Lemma 17, $\mathcal{I}(\mathcal{Y})$ is a P^+-ideal, so there is $Z \in \mathcal{I}(\mathcal{Y})^+$ a pseudointersection of $\langle S_n\rangle_{n\in\omega}$. Let $p = (Z, \mathcal{Y})$, it is clear that p is a lower bound for $\langle (S_n, \mathcal{Y}_n)\rangle_{n\in\omega}$. $\square$

The following notion will be very important for the rest of the paper:

Definition 23. Let $G \subseteq \mathbb{P}$ be a generic filter. In the extension $V[G]$, define $\mathcal{U}_{gen} = \{S \mid \exists \mathcal{Y}((S, \mathcal{Y}) \in G)\}$.

We will refer to $\mathcal{U}_{gen}$ as the *generic ultrafilter*. Of course, we need to prove first that $\dot{\mathcal{U}}_{gen}$ will be an ultrafilter (in fact a P-point) in the extension. The following result will help us to achieve that:

Lemma 24. *Let $p = (S, \mathcal{Y}) \in \mathbb{P}$ and $W \in [\omega]^\omega$. The following two statements are equivalent:*

1. $p \Vdash$ "$W \in \dot{\mathcal{U}}_{gen}$".
2. $S \setminus W \in \mathcal{I}(\mathcal{Y})$.

Proof. We will first prove that 1 implies 2. We argue by contradiction, assume that $p \Vdash$"$W \in \dot{\mathcal{U}}_{gen}$", but $S \setminus W \notin \mathcal{I}(\mathcal{Y})$, so $S \setminus W \in \mathcal{I}(\mathcal{Y})^+$. Let $q = (S \setminus W, \mathcal{Y})$, this is a condition extending p. Let $G \subseteq \mathbb{P}$ be a generic filter such that $q \in G$. Since $r \Vdash$"$W \in \dot{\mathcal{U}}_{gen}$", we know that $W \in \dot{\mathcal{U}}_{gen}[G]$. In this way, there must be a condition $r = (W, \mathcal{X}) \in G$.

Since $q, r \in G$ and G is a filter, there must be $\overline{q} = (R, \mathcal{Z}) \in G$ extending both q and r. Since $\overline{q} \leq q$, we know that $R \setminus (S \setminus W) \in \mathcal{I}(\mathcal{Z})$. Furthermore, since $R \cap W \subseteq R \setminus (S \setminus W)$, we know that $R \cap W$ is in $\mathcal{I}(\mathcal{Z})$ as well. Since $\overline{q} \leq r$, we have that $R \setminus W \in \mathcal{I}(\mathcal{Z})$. In this way, we get that $R \in \mathcal{I}(\mathcal{Z})$, which is a contradiction (since $\overline{q}$ is a condition).

We will now prove that 2 implies 1. Once again, we argue by contradiction. Assume that $S \setminus W \in \mathcal{I}(\mathcal{Y})$ but unfortunately p does not force that "$W \in \dot{\mathcal{U}}_{gen}$". In this way, we can find $q = (Z, \mathcal{X}) \leq p$ such that $q \Vdash$"$W \notin \dot{\mathcal{U}}_{gen}$". Since q extend p, we know that $Z \setminus S \in \mathcal{I}(\mathcal{X})$. Now we have the following:

1. $Z \in \mathcal{I}(\mathcal{X})^+$.
2. $Z \subseteq (S \setminus W) \cup (Z \setminus S) \cup (Z \cap W)$.
3. $S \setminus W$, $Z \setminus S \in \mathcal{I}(\mathcal{X})$.

It follows that $Z \cap W \in \mathcal{I}(\mathcal{X})^+$. In this way, $r = (Z \cap W, \mathcal{X})$ is a condition extending q. However, r also extends $(W, \mathcal{X})$, so r is an extension of q forcing that W is in $\dot{\mathcal{U}}_{gen}$, which is a contradiction. □

We say that a family $\mathcal{R} \subseteq [\omega]^\omega$ is *upwards closed* if whenever $A \in \mathcal{R}$ and $A \subseteq B$, we have that $B \in \mathcal{R}$. We can now prove the following:

Proposition 25.

1. *$\dot{\mathcal{U}}_{gen}$ is forced to be upwards closed.*
2. *If $p = (S, \mathcal{Y}) \in \mathbb{P}$, then $p \Vdash$ "$\{S\} \cup \mathcal{I}(\mathcal{Y})^* \subseteq \dot{\mathcal{U}}_{gen}$".*
3. *$\dot{\mathcal{U}}_{gen}$ is forced to be a P-point.*

Proof. We will first prove that $\dot{\mathcal{U}}_{gen}$ is forced to be upwards closed. It is enough to show that if $p = (S, \mathcal{Y}) \in \mathbb{P}$, $A, B \in [\omega]^\omega$ are such that $A \subseteq B$ and $p \Vdash$"$A \in \dot{\mathcal{U}}_{gen}$", then $p \Vdash$"$B \in \dot{\mathcal{U}}_{gen}$". Since $p \Vdash$"$A \in \dot{\mathcal{U}}_{gen}$", by Lemma 24, we know that $S \setminus A \in \mathcal{I}(\mathcal{Y})$. Since $A \subseteq B$, we have that $S \setminus B \in \mathcal{I}(\mathcal{Y})$ and again by Lemma 24, we get that $p \Vdash$"$B \in \dot{\mathcal{U}}_{gen}$".

We will now show that if $p = (S, \mathcal{Y}) \in \mathbb{P}$, then $p \Vdash$ "$\{S\} \cup \mathcal{I}(\mathcal{Y})^* \subseteq \dot{\mathcal{U}}_{gen}$". By definition, we know that $p \Vdash$"$S \in \dot{\mathcal{U}}_{gen}$". Now, let $B \in \mathcal{I}(\mathcal{Y})^*$. Clearly we have that $S \setminus B \in \mathcal{I}(\mathcal{Y})$, so by the Lemma 24 we get that $p \Vdash$"$B \in \dot{\mathcal{U}}_{gen}$".

Now we need to prove that $\dot{\mathcal{U}}_{gen}$ is forced to be a P-point. We already know that $\dot{\mathcal{U}}_{gen}$ is forced to be closed upwards. We will now prove that it will be closed under intersections. It is enough to show that if $p = (S, \mathcal{Y}) \in \mathbb{P}$, $A, B \in [\omega]^\omega$ are such that $p \Vdash$"$A, B \in \dot{\mathcal{U}}_{gen}$", then $p \Vdash$"$A \cap B \in \dot{\mathcal{U}}_{gen}$". By the Lemma 24 we know that $S \setminus A$, $S \setminus B \in \mathcal{I}(\mathcal{Y})$. In this way, we get that $S \setminus (A \cap B) \in \mathcal{I}(\mathcal{Y})$, which we know it implies that $p \Vdash$"$A \cap B \in \dot{\mathcal{U}}_{gen}$".

Now we will prove that $\dot{\mathcal{U}}_{gen}$ is forced to be an ultrafilter. Note that by the Lemma 24, $\dot{\mathcal{U}}_{gen}$ will not contain any finite set. Since $\mathbb{P}$ is σ-closed, we only need to prove that for every $A \subseteq \omega$ (with A in the ground model) and $p = (S, \mathcal{Y}) \in \mathbb{P}$, there is $q \leq p$ such that either $q \Vdash$"$A \in \dot{\mathcal{U}}_{gen}$" or $q \Vdash$"$\omega \setminus A \in \dot{\mathcal{U}}_{gen}$". Since

$S \in \mathcal{I}(\mathcal{Y})^+$ we know that one of $q_1 = (A \cap S, \mathcal{I}(\mathcal{Y}))$ or $q_1 = (S \setminus A, \mathcal{I}(\mathcal{Y}))$ is a condition (maybe both). Since $\dot{\mathcal{U}}_{gen}$ is forced to be upward closed, we are done.

Finally, we will prove that $\dot{\mathcal{U}}_{gen}$ will be a P-point. For this (since $\mathbb{P}$ is σ-closed), it is enough to prove that if $p = (S, \mathcal{I}(\mathcal{Y})) \in \mathbb{P}$ and $\{B_n \mid n \in \omega\}$ is a decreasing sequence such that $p \Vdash$"$B_n \in \dot{\mathcal{U}}_{gen}$", then there are A a pseudointersection of $\{B_n \mid n \in \omega\}$ and $q \leq p$ such that $q \Vdash$"$A \in \dot{\mathcal{U}}_{gen}$". Since $p \Vdash$"$B_n \in \dot{\mathcal{U}}_{gen}$" for every $n \in \omega$, it follows that $S \cap B_n \in \mathcal{I}(\mathcal{Y})^+$ for every $n \in \omega$. We know that $\mathcal{I}(\mathcal{Y})$ is a P^+-ideal, so we can find $Z \subseteq S$ such that $Z \in \mathcal{I}(\mathcal{Y})^+$ and $Z \subseteq^* B_n$ for every $n \in \omega$. It follows that $q = (Z, \mathcal{Y})$ is the desired condition. □

If $\mathcal{A}$ is a MAD family, define $\mathbb{P}_{\mathcal{A}} = \{p \in \mathbb{P} \mid p \leq (\omega, \{\mathcal{A}\})\}$. Note that forcing with $\mathbb{P}_{\mathcal{A}}$ adds a P-point extending $\mathcal{I}^*(\mathcal{A})$. We now have the following:

Theorem 26. *If $\mathcal{A}$ is a MAD family, then $\mathbb{P}_{\mathcal{A}} \Vdash$"$\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\dot{\mathcal{U}}_{gen})$".*

Proof. Let $p = (S, \mathcal{Y}) \in \mathbb{P}_{\mathcal{A}}$ and $C \in \mathcal{I}(\mathcal{A})^+$. By extending p if necessary, we may assume that either $p \Vdash$"$C \in \dot{\mathcal{U}}_{gen}$" or $p \Vdash$"$C \notin \dot{\mathcal{U}}_{gen}$". In case that $p \Vdash$"$C \in \dot{\mathcal{U}}_{gen}$", it follows that $\dot{\mathcal{U}}_{gen}$ is forced to have the desired properties. Assume that $p \Vdash$"$C \notin \dot{\mathcal{U}}_{gen}$", by extending if necessary, we may assume that $S \cap C = \emptyset$.

Claim 27. *There is a bijection $f : S \longrightarrow C$ such that $\mathcal{I}(\mathcal{Y})$ and the ideal $\mathcal{J} = \left\{f^{-1}(A) \mid A \in \mathcal{I}(\mathcal{A})\right\}$ are compatible. Furthermore, S is not in the ideal generated by $\mathcal{I}(\mathcal{Y})$ and $\mathcal{J}$.*

Let $\{A_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})$ be a partition of C such that if $n \neq m$, then A_n and A_m are almost contained in different elements of $\mathcal{A}$ (remember that $C \in \mathcal{I}(\mathcal{A})^+$). Since $\mathcal{I}_{\mathcal{Y}} \restriction S$ is a meager ideal, by Lemma 20, we can find $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{Y})^+$ a partition of S. Choose $f : S \longrightarrow C$ a bijection such that $f[X_n] = A_n$. We claim that f is as desired. Let $E \in \mathcal{I}(\mathcal{A})$ and $D \in \mathcal{I}(\mathcal{Y})$, we need to prove that $S \nsubseteq f^{-1}(E) \cap D$. Since $E \in \mathcal{I}(\mathcal{A})$, we know that there is $n \in \omega$ such that $E \cap A_n$ is finite, so $f^{-1}(E) \cap X_n$ is finite too. Since $X_n \in \mathcal{I}(\mathcal{Y})^+$, it can not be almost covered by D, so S can not be almost covered by $f^{-1}(E) \cap D$.

Since f is a bijection, we can find $\mathcal{B}$ an AD family such that $\mathcal{I}(\mathcal{B}) = \mathcal{J}$. By the above claim, we know that $q = (S, \mathcal{Y} \cup \{\mathcal{B}\})$ is a condition extending p. Let $g : \omega \longrightarrow \omega$ a bijection extending f. Let $\dot{\mathcal{V}}$ be a $\mathbb{P}_{\mathcal{A}}$-name for $g(\dot{\mathcal{U}}_{gen})$. In other words, $\dot{\mathcal{V}}$ is the name for $\{X \mid g^{-1}(X) \in \dot{\mathcal{U}}_{gen}\}$. We claim that q forces that $\dot{\mathcal{V}}$ has the desired properties.

Since $q \Vdash$"$S \in \dot{\mathcal{U}}_{gen}$" and $g^{-1}(C) = S$, it follows that $q \Vdash$"$C \in \dot{\mathcal{V}}$". It is clear that q also forces that $\dot{\mathcal{V}}$ is Rudin-Keisler below $\dot{\mathcal{U}}_{gen}$. Now, since $q \Vdash$"$\mathcal{I}(\mathcal{B})^* \subseteq \dot{\mathcal{U}}_{gen}$", it follows that $q \Vdash$"$\mathcal{I}(\mathcal{A})^* \subseteq \dot{\mathcal{V}}$".

Note that $\mathcal{A}$ remains a MAD family since $\mathbb{P}_{\mathcal{A}}$ does not add new reals.

By the Proposition 10, we get the desired conclusion. □

We can now answer the problem of Malykhin and the first author:

Theorem 28 *(CH). For every MAD family $\mathcal{A}$, there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is a $\mathsf{FU}(\mathcal{U})$-space.*

Proof. Let $\mathcal{A}$ be a MAD family. By the Theorem 26, we know that $\mathbb{P}_{\mathcal{A}}$ forces that $\mathsf{Fr}(\mathcal{A})$ is a $\mathsf{FU}(\dot{\mathcal{U}}_{gen})$-space. However, it is clear by the proof that in order to construct $\dot{\mathcal{U}}_{gen}$, we only need to meet $\mathfrak{c}$ many dense sets. Since $\mathbb{P}_{\mathcal{A}}$ is σ-closed, we can construct such ultrafilter using the Continuum Hypothesis. □

4. Open questions

In this last section, we will state some problems that we do not know how to solve. The following question was motivated by Lemma 17:

Problem 29. *Let $\mathcal{R} = \{\mathcal{I}_n \mid n \in \omega\}$ be a family of compatible P^+-ideals. Is the ideal $\mathcal{J} = \langle \bigcup_{n\in\omega} \mathcal{I}_n \rangle$ a P^+-ideal?*

We proved this is indeed the case where the ideals are generated by AD families, but we were unable to prove the general case.

We may wonder what happens when we are no longer in the realm of the Continuum Hypothesis. Of course, the question is only interesting in models where there are P-points. Recall that the *dominating number* (denoted by $\mathfrak{d}$) is the smallest size of a family $\mathcal{D} \subseteq \omega^\omega$ that is cofinal under the eventual domination (the reader may consult [3] to learn more about $\mathfrak{d}$ and other cardinal invariants of the continuum). The following is a classical result of Ketonen: ([16], see also [2])

Theorem 30 *([16]). The following statements are equivalent:*

1. $\mathfrak{d} = \mathfrak{c}$.
2. *P-points exist generically.*[1]

In particular, there are many P-points if $\mathfrak{d}$ is equal to $\mathfrak{c}$. It is then natural to ask the following:

Problem 31. *Does $\mathfrak{d} = \mathfrak{c}$ imply that for every MAD family $\mathcal{A}$, there is a P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$?*

We conjecture that the previous problem has a positive answer. The following question was suggested to us by the referee, which we think is very interesting:

Problem 32. *Is there a model of ZFC in which there are P-points, yet there is a MAD family $\mathcal{A}$ for which there is no P-point $\mathcal{U}$ such that $\mathsf{Fr}(\mathcal{A})$ is $\mathsf{FU}(\mathcal{U})$?*

Acknowledgement

The authors would like to thank Michael Hrušák for several conversations on the topic of the paper. We would also like to thank the referee for his/her comments and corrections.

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[1] We say that a class of ultrafilters *exists generically* if every filter of character less than $\mathfrak{c}$ can be extended to an ultrafilter of that class. The reader may consult [6], [5] and [10] to learn more about this interesting topic.

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