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THERE IS A +-RAMSEY MAD FAMILY

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ABSTRACT

We answer an old question of Michael Hrušák by constructing a +-Ramsey MAD family without the need of any additional axioms beyond ZFC. We also prove that every Miller-indestructible MAD family is +-Ramsey; this improves a result of Michael Hrušák.

1. Introduction

A family $\mathcal{A} \subseteq [\omega]^\omega$ is **almost disjoint** (**AD**) if the intersection of any two different elements of $\mathcal{A}$ is finite, a MAD **family** is a maximal almost disjoint family. Almost disjoint families and MAD families have become very important in set theory, topology and functional analysis (see [7]). It is very easy to prove that the Axiom of Choice implies the existence of MAD families. However, constructing MAD families with special combinatorial or topological properties is a very difficult task without an additional hypothesis beyond ZFC. Constructing models of set theory for which certain kinds of MAD families do not exist is very difficult. We would like to mention some important examples regarding the existence or non-existence of special MAD families:

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(1) (Simon [21]) There is a MAD family which can be partitioned into two nowhere MAD families.
(2) (Mrówka [16]) There is a MAD family for which its Ψ-space has a unique compactification.
(3) (Raghavan [17]) There is a van Douwen MAD family.
(4) (Raghavan [18]) There is a model with no strongly separable MAD families.

If $\mathcal{I}$ is an ideal on ω, then by $\mathcal{I}^+$ we denote the set $\wp(\omega)\setminus\mathcal{I}$ and its elements are called $\mathcal{I}$-positive sets. If $\mathcal{A}$ is an AD family, by $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{A}$. In [12] Adrian Mathias proved that if $\mathcal{A}$ is a MAD family then $\mathcal{I}(\mathcal{A})^+$ is a happy family, which is a kind of Ramsey-like property. In [6] Michael Hrušák introduced a stronger Ramsey property:

Definition 1: (1) By $\mathcal{A}^\perp$ we denote the set of all $X \subseteq \omega$ such that $\mathcal{A} \cup \{X\}$ is almost disjoint.
(2) If $\mathcal{A}$ is an AD family, by $\mathcal{I}(\mathcal{A})^{++}$ we denote the set of all $X \subseteq \omega$ such that there is $\mathcal{B} \in [\mathcal{A}]^\omega$ such that $|X \cap B| = \omega$ for every $B \in \mathcal{B}$.
(3) Let $\mathcal{X} \subseteq [\omega]^\omega$, we say a tree $T \subseteq \omega^{<\omega}$ is a **$\mathcal{X}$-branching tree** if $\mathrm{suc}_T(s) \in \mathcal{X}$ for every $s \in T$ (where $\mathrm{suc}_T(s) = \{n \in \omega \mid s^\frown\langle n\rangle \in T\}$).
(4) An AD family $\mathcal{A}$ is **+-Ramsey** if for every $\mathcal{I}(\mathcal{A})^+$-branching tree T, there is $f \in [T]$ such that $\mathrm{im}(f) \in \mathcal{I}(\mathcal{A})^+$.

In [6] it is proved that there is a MAD family that is not +-Ramsey. On the other hand, +-Ramsey MAD families can be constructed under $\mathfrak{b} = \mathfrak{c}$, $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, $\mathfrak{a} < \mathsf{cov}(\mathcal{M})$ or $\diamondsuit(\mathfrak{b})$ (see [6] and [8]). Michael Hrušák asked the following:

Problem 2 (Hrušák [6]): Is there a +-Ramsey MAD family in ZFC?

In this note we will provide a positive answer to this question. In [20] (see also [7] and [15]) Saharon Shelah developed a novel and powerful method to construct MAD families. He used it to prove that there is a completely separable MAD family if $\mathfrak{s} \leq \mathfrak{a}$ or $\mathfrak{a} < \mathfrak{s}$ and a certain PCF-hypothesis holds. Our technique for constructing a +-Ramsey MAD is based on the technique of Shelah (however, in this case we were able to avoid the PCF-hypothesis). It is worth mentioning that the method of Shelah has been further developed in [19] and [15] where it is proved that weakly tight MAD families exist under $\mathfrak{s} \leq \mathfrak{b}$. Our notation is mostly standard; the definition and basic properties of the cardinal invariants of the continuum used in this note can be found in [2].

2. Preliminaries

A MAD family $\mathcal{A}$ is **completely separable** if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is $A \in \mathcal{A}$ such that $A \subseteq X$. This type of MAD families was introduced by Hechler in [5]. A year later, Shelah and Erdős asked the following question:

Problem 3 (Erdős–Shelah): Is there a completely separable MAD family?

It is easy to construct models where the previous question has a positive answer. It was shown by Balcar and Simon (see [1]) that such families exist assuming one of the following axioms: $\mathfrak{a} = \mathfrak{c}$, $\mathfrak{b} = \mathfrak{d}$, $\mathfrak{d} \leq \mathfrak{a}$ and $\mathfrak{s} = \omega_1$. In [20] (see also [7] and [15]) Shelah developed a novel and powerful method to construct completely separable MAD families. He used it to prove that there are such families if either $\mathfrak{s} \leq \mathfrak{a}$ or $\mathfrak{a} < \mathfrak{s}$ and a certain (so-called) PCF-hypothesis holds (which holds, for example, if the continuum is less than $\aleph_\omega$). Since the construction of Shelah of a completely separable MAD family under $\mathfrak{s} \leq \mathfrak{a}$ is key for our construction of a +-Ramsey MAD family, we will recall it on this section. This exposition is based on [15] and [7].

Definition 4:

1. (1) We say that S **splits** X if $S \cap X$ and $X \setminus S$ are both infinite.
2. (2) $\mathcal{S} \subseteq [\omega]^\omega$ is a **splitting family** if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X.
3. (3) Let $S \in [\omega]^\omega$ and $\mathcal{P} = \{P_n \mid n \in \omega\}$ be an interval partition. We say S **block-splits** $\mathcal{P}$ if both of the sets
$$\{n \mid P_n \subseteq S\} \quad \text{and} \quad \{n \mid P_n \cap S = \emptyset\}$$
are infinite.
4. (4) A family $\mathcal{S} \subseteq [\omega]^\omega$ is called a **block-splitting family** if every interval partition is block-split by some element of $\mathcal{S}$.

Recall that the **splitting number** $\mathfrak{s}$ is the smallest size of a splitting family. It is well known that $\mathfrak{s}$ has uncountable cofinality; it is below the dominating number $\mathfrak{d}$ and independent from the unbounding number $\mathfrak{b}$ (see [2]). Regarding the smallest size of a block splitting family we have the following result of Kamburelis and Węglorz:

Proposition 5 ([10]): *The smallest size of a block-splitting family is* $\max\{\mathfrak{b}, \mathfrak{s}\}$.

Some other notions of splitting are the following:

Definition 6: Let $S \in [\omega]^\omega$ and $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$.

(1) We say that S ω-**splits** $\overline{X}$ if S splits every X_n.
(2) We say that S (ω,ω)-**splits** $\overline{X}$ if both the sets $\{n \mid |X_n \cap S| = \omega\}$ and $\{n \mid |X_n \cap (\omega \backslash S)| = \omega\}$ are infinite.
(3) We say that $\mathcal{S} \subseteq [\omega]^\omega$ is an ω-**splitting family** if every countable collection of infinite subsets of ω is ω-split by some element of S.
(4) We say that $\mathcal{S} \subseteq [\omega]^\omega$ is an (ω,ω)-**splitting family** if every countable collection of infinite subsets of ω is (ω,ω)-split by some element of S.

It is easy to see that every block splitting family is an ω-splitting family. The following is a fundamental result of Mildenberger, Raghavan and Steprāns:

PROPOSITION 7 ([15]): *There is an (ω,ω)-splitting family of size $\mathfrak{s}$.*

The key combinatorial feature of (ω,ω)-splitting families is the following result of Raghavan and Steprāns:

PROPOSITION 8 ([19]): *If $\mathcal{S}$ is an (ω,ω)-splitting family, $\mathcal{A}$ an* AD *family and $X \in \mathcal{I}(\mathcal{A})^+$, then there is $S \in \mathcal{S}$ such that $X \cap S$, $X \cap (\omega \setminus S) \in \mathcal{I}(\mathcal{A})^+$.*

Given $X \subseteq \omega$ we denote $X^0 = X$ and $X^1 = \omega \setminus X$. By the previous result, if $\mathcal{A}$ is an AD family, $X \in \mathcal{I}(\mathcal{A})^+$ and $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ is an (ω,ω)-splitting family, then there are $\alpha < \mathfrak{s}$ and $\tau_X^{\mathcal{A}} \in 2^\alpha$ such that:

(1) If $\beta < \alpha$ then $X \cap S_\beta^{1-\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$.
(2) $X \cap S_\alpha$, $X \setminus S_\alpha \in \mathcal{I}(\mathcal{A})^+$.

Clearly $\tau_X^{\mathcal{A}} \in 2^{<\mathfrak{s}}$ is unique, and if $Y \in [X]^\omega \cap \mathcal{I}(\mathcal{A})^+$ then $\tau_Y^{\mathcal{A}}$ extends $\tau_X^{\mathcal{A}}$. We can now prove the main result of this section:

THEOREM 9 (Shelah [20]): *If $\mathfrak{s} \leq \mathfrak{a}$, then there is a completely separable* MAD *family.*

Proof. Let $[\omega]^\omega = \{X_\alpha \mid \alpha < \mathfrak{c}\}$. We will recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha \mid \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{s}}$ such that for every $\alpha < \mathfrak{c}$ the following holds (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$):

(1) $\mathcal{A}_\alpha$ is an AD family.
(2) If $X_\alpha \in \mathcal{I}(\mathcal{A}_\alpha)^+$ then $A_\alpha \subseteq X_\alpha$.
(3) If $\alpha \neq \beta$ then $\sigma_\alpha \neq \sigma_\beta$.
(4) If $\xi < \mathrm{dom}(\sigma_\alpha)$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.

It is clear that if we manage to do this, then we will have achieved to construct a completely separable MAD family. Assume $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ has already been constructed. Let $X = X_\delta$ if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$, and if $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)$ let X be any other element of $\mathcal{I}(\mathcal{A}_\delta)^+$. We recursively find $\{X_s \mid s \in 2^{<\omega}\} \subseteq \mathcal{I}(\mathcal{A}_\delta)^+$, $\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{s}}$ and $\{\alpha_s \mid s \in 2^{<\omega}\}$ as follows:

(1) $X_\emptyset = X$.
(2) $\eta_s = \tau^{\mathcal{A}_\delta}_{X_s}$ and $\alpha_s = \mathrm{dom}(\eta_s)$.
(3) $X_{s^\frown 0} = X_s \cap S_{\alpha_s}$ and $X_{s^\frown 1} = X_s \cap (\omega \setminus S_{\alpha_s})$.

Note that if $t \subseteq s$, then $X_s \subseteq X_t$ and $\eta_t \subseteq \eta_s$. On the other hand, if s is incompatible with t, then η_s and η_t are incompatible. For every $f \in 2^\omega$ let

$$\eta_f = \bigcup_{n\in\omega} \eta_{f\restriction n}.$$

Since $\mathfrak{s}$ has uncountable cofinality each η_f is an element of $2^{<\mathfrak{s}}$, and if $f \neq g$ then η_f and η_g are incompatible nodes of $2^{<\mathfrak{s}}$. Since δ is smaller than $\mathfrak{c}$, there is $f \in 2^\omega$ such that there is no $\alpha < \delta$ such that σ_α extends η_f. Since $\{X_{f\restriction n} \mid n \in \omega\}$ is a decreasing sequence of elements in $\mathcal{I}(\mathcal{A}_\delta)^+$, there is $Y \in \mathcal{I}(\mathcal{A}_\delta)^+$ such that $Y \subseteq^* X_{f\restriction n}$ for every $n \in \omega$ (see [12] proposition 0.7 or [7] proposition 2).

Letting $\beta = \mathrm{dom}(\eta_f)$, we claim that if $\xi < \beta$ then $Y \cap S^{1-\eta_f(\xi)}_\xi \in \mathcal{I}(\mathcal{A})$. To prove this, let n be the first natural number such that $\xi < \mathrm{dom}(\eta_{f\restriction n})$. By our construction we know that $X_{f\restriction n} \cap S^{1-\eta_f(\xi)}_\xi \in \mathcal{I}(\mathcal{A})$, and since $Y \subseteq^* X_{f\restriction n}$ the result follows.

For every $\xi < \beta$ let $F_\xi \in [\mathcal{A}]^{<\omega}$ be such that $Y \cap S^{1-\eta_f(\xi)}_\xi \subseteq^* \bigcup F_\xi$ and let $W = \{A_\alpha \mid \sigma_\alpha \subseteq \eta_f\}$. Let

$$\mathcal{D} = W \cup \bigcup_{\xi<\beta} F_\xi$$

and note that $\mathcal{D}$ has size less than $\mathfrak{s}$, hence it has size less than $\mathfrak{a}$. In this way we conclude that $Y \restriction \mathcal{D}$ is not a MAD family, so there is $A_\delta \in [Y]^\omega$ that is almost disjoint with every element of $\mathcal{D}$ and define $\sigma_\delta = \eta_f$. We claim that A_δ is almost disjoint with $\mathcal{A}_\delta$. To prove this, let $\alpha < \delta$; in case $A_\alpha \in W$ we already know $A_\alpha \cap A_\delta$ is finite so assume $A_\alpha \notin W$. Letting $\xi = \Delta(\sigma_\delta, \sigma_\alpha)$ we know that $A_\alpha \subseteq^* S^{1-\sigma_\delta(\xi)}_\xi$ so $A_\alpha \cap A_\delta \subseteq^* \bigcup F_\xi$, but since $F_\xi \subseteq \mathcal{D}$ we conclude that A_δ is almost disjoint with $\bigcup F_\xi$ and then $A_\alpha \cap A_\delta$ must be finite. ∎

Recall that an AD family $\mathcal{A}$ is **nowhere** MAD if for every $X \in \mathcal{I}(\mathcal{A})^+$ there is $Y \in [X]^\omega$ such that Y is almost disjoint with $\mathcal{A}$. A key feature in the previous proof is that each $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ is nowhere MAD.

The first step to construct a +-Ramsey MAD family is to prove that every Miller-indestructible MAD family has this property. If $\mathcal{A}$ is a MAD family and $\mathbb{P}$ is a partial order, then we say $\mathcal{A}$ is $\mathbb{P}$-**indestructible** if $\mathcal{A}$ is still a MAD family after forcing with $\mathbb{P}$. The destructibility of MAD families has become a very important area of research with many fundamental questions still open (the reader may consult [8], [9], or [4] to learn more about the indestructibility of MAD families and ideals). The following property of MAD families plays a fundamental role in the study of destructibility:

Definition 10: A MAD family $\mathcal{A}$ is **tight** if for every

$$\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$$

there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.

In [8] it is proved that every tight MAD family is Cohen-indestructible and that every tight MAD family is +-Ramsey. We will prove that every Miller-indestructible MAD family is +-Ramsey; this improves the previous result since Miller-indestructibility follows from Cohen-indestructibility (see [4]). First we need the following lemma:

LEMMA 11: *Let $\mathcal{A}$ be a MAD family and T an $\mathcal{I}(\mathcal{A})^+$-branching tree. Then there is a subtree $S \subseteq T$ with the following properties:*

(1) *If $s \in S$, there is $A_s \in \mathcal{A}$ such that $\mathrm{suc}_S(s) \in [A_s]^\omega$.*
(2) *If s and t are two different nodes of S, then $A_s \neq A_t$ and*

$$\mathrm{suc}_S(s) \cap \mathrm{suc}_S(t) = \emptyset.$$

Proof. Since T is an $\mathcal{I}(\mathcal{A})^+$-branching tree and $\mathcal{A}$ is MAD, $\mathrm{suc}_T(t)$ infinitely intersects many infinite elements of $\mathcal{A}$ for every $t \in T$. Recursively, for every $t \in T$ we choose $A_t \in \mathcal{A}$ and $B_t \in [A_t \cap \mathrm{suc}_T(t)]^\omega$ such that $B_t \cap B_s = \emptyset$ and $A_s \neq A_t$ whenever $t \neq s$. We then recursively construct $S \subseteq T$ such that, if $s \in S$, then $\mathrm{suc}_S(s) = B_s$. ∎

With the previous lemma we can now prove the following:

PROPOSITION 12: *If $\mathcal{A}$ is Miller-indestructible then $\mathcal{A}$ is +-Ramsey.*

Proof. Let $\mathcal{A}$ be a Miller-indestructible MAD family and T an $\mathcal{I}(\mathcal{A})^+$-branching tree. Let S be an $\mathcal{I}(\mathcal{A})$-branching subtree of T as in the previous lemma. We can then view S as a Miller tree. Let $\dot{r}_{\mathrm{gen}}$ be the name of the generic real and $\dot{X}$ the name of the image of $\dot{r}_{\mathrm{gen}}$.

We will first argue that $S \Vdash$ "$\dot{X} \notin \mathcal{I}(\mathcal{A})$". Assume this is not true, so there is $S_1 \leq S$ and $B \in \mathcal{I}(\mathcal{A})$ (B is an element of V) such that $S_1 \Vdash$ "$\dot{X} \subseteq B$". In this way, if t is a splitting node of S_1 then $\mathrm{suc}_{S_1}(t) \subseteq B$, but note that if $t_1 \neq t_2$ are two different splitting nodes of S_2 then $\mathrm{suc}_{S_1}(t_1)$ and $\mathrm{suc}_{S_1}(t_2)$ are two infinite sets contained in different elements of $\mathcal{A}$, so then $B \in \mathcal{I}(\mathcal{A})^+$ which is a contradiction.

In this way, $\dot{X}$ is forced by S to be an element of $\mathcal{I}(\mathcal{A})^+$, but since $\mathcal{A}$ is still MAD after performing a forcing extension of Miller forcing, we then conclude there are names $\{\dot{A}_n \mid n \in \omega\}$ for different elements of $\mathcal{A}$ such that S forces that $\dot{X} \cap \dot{A}_n$ is infinite. We then recursively build two sequences $\{S_n \mid n \in \omega\}$ and $\{B_n \mid n \in \omega\}$ such that for every $n \in \omega$ the following holds:

(1) S_n is a Miller tree and $B_n \in \mathcal{A}$.
(2) $S_0 \leq S$, and if $n < m$ then $S_m \leq S_n$.
(3) $S_n \Vdash$ "$\dot{A}_n = B_n$" (it then follows that $B_n \neq B_m$ if $n \neq m$).
(4) If $i \leq n$ then $\mathrm{stem}(S_n) \cap B_i$ has size at least n.

We then define $r = \bigcup_{n\in\omega} \mathrm{stem}(S_n)$, so clearly $r \in [S]$ and $\mathrm{im}(r) \in \mathcal{I}(\mathcal{A})^+$. ∎

The converse of the previous result is not true in general; this will be shown in Corollary 27. It is known that every MAD family of size less than $\mathfrak{d}$ is Miller-indestructible (see [4]). We can then conclude the following unpublished result of Michael Hrušák, which he proved by completely different means.

COROLLARY 13 (Hrušák): *Every MAD family of size less than $\mathfrak{d}$ is +-Ramsey. In particular, if $\mathfrak{a} < \mathfrak{d}$ then there is a +-Ramsey MAD family.*

3. The construction of a +-Ramsey MAD family

In this section we will construct a +-Ramsey MAD family without any extra hypothesis beyond ZFC. We will use the construction of Shelah of a completely separable MAD family; however, the previous result will help us avoid the need of a PCF-hypothesis for our construction. From now on, we will always assume that all Miller trees are formed by increasing sequences. If p is a Miller tree, we denote $\mathrm{Split}(p)$ the set of all splitting nodes of p.

Definition 14: Let p be a Miller tree. Given $f \in [p]$ we define

$$Sp(p, f) = \{f(n) \mid f \upharpoonright n \in \mathrm{Split}(p)\}$$

and

$$[p]_{\mathrm{split}} = \{Sp(p, f) \mid f \in [p]\}.$$

We will need the following definitions:

Definition 15: Let p be a Miller tree and $H : \mathrm{Split}(p) \longrightarrow \omega$. We then define:

(1) $\mathrm{Catch}_{\exists}(H)$ is the set

$$\{Sp(f,p) \mid f \in [p] \wedge \exists^{\infty} n (f \upharpoonright n \in \mathrm{Split}(p) \wedge f(n) < H(f \upharpoonright n))\}.$$

(2) $\mathrm{Catch}_{\forall}(H)$ is the set

$$\{Sp(f,p) \mid f \in [p] \wedge \forall^{\infty} n (f \upharpoonright n \in \mathrm{Split}(p) \wedge f(n) < H(f \upharpoonright n))\}.$$

(3) Define $\mathcal{K}(p)$ as the collection of all $A \subseteq [p]_{\mathrm{split}}$ for which there is $G : \mathrm{Split}(p) \longrightarrow \omega$ such that

$$A \subseteq \mathrm{Catch}_{\exists}(G).$$

Note that if $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ is an unbounded family of increasing functions, then for every infinite partial function $g \subseteq \omega \times \omega$ there is $\alpha < \mathfrak{b}$ such that the set $\{n \in \mathrm{dom}(g) \mid g(n) < f_\alpha(n)\}$ is infinite. With this simple observation we can prove the following lemma.

Lemma 16: *$\mathcal{K}(p)$ is a σ-ideal in $[p]_{\mathrm{split}}$ that contains all singletons and* $\mathfrak{b} = \mathsf{add}(\mathcal{K}(p)) = \mathsf{cov}(\mathcal{K}(p))$.

Proof. In order to prove that $\mathfrak{b} \leq \mathsf{add}(\mathcal{K}(p))$, it is enough to show that if $\kappa < \mathfrak{b}$ and $\{H_\alpha \mid \alpha < \kappa\} \subseteq \omega^{\mathrm{Split}(p)}$ then $\bigcup_{\alpha<\kappa} \mathrm{Catch}_{\exists}(H_\alpha) \in \mathcal{K}(p)$. Since κ is smaller than $\mathfrak{b}$, we can find $H : \mathrm{Split}(p) \longrightarrow \omega$ such that if $\alpha < \kappa$, then $H_\alpha(s) < H(s)$ for almost all $s \in \mathrm{Split}(p)$. Clearly

$$\bigcup_{\alpha<\kappa} \mathrm{Catch}_{\exists}(H_\alpha) \subseteq \mathrm{Catch}_{\exists}(H).$$

Now we must prove that $\mathsf{cov}(\mathcal{K}(p)) \leq \mathfrak{b}$. Let $\mathrm{Split}(p) = \{s_n \mid n < \omega\}$ and $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be an unbounded family of increasing functions. Given $\alpha < \mathfrak{b}$ define $H_\alpha : \mathrm{Split}(p) \longrightarrow \omega$, where

$$H_\alpha(s_n) = f_\alpha(n).$$

We will show that $\{\mathrm{Catch}_{\exists}(H_\alpha) \mid \alpha < \mathfrak{b}\}$ covers $[p]_{\mathrm{split}}$. Letting $f \in [p]$, define $A = \{n \mid s_n \sqsubseteq f\}$ and construct the function $g : A \longrightarrow \omega$, where

$$g(n) = f(|s_n|) + 1$$

for every $n \in A$. By the previous remark, there is $\alpha < \mathfrak{b}$ such that $f_\alpha \upharpoonright A$ is not dominated by $g \upharpoonright A$. It is then clear that $S_p(p,f) \in \mathrm{Catch}_{\exists}(H_\alpha)$. ∎

Letting p be a Miller tree and $S \in [\omega]^\omega$, we define the game $\mathcal{G}(p, S)$ as follows:

I	s_0		s_1		$\cdots$
II		r_0		r_1	

(1) Each s_i is a splitting node of p.
(2) $r_i \in \omega$.
(3) s_{i+1} extends s_i.
(4) $s_{i+1}(|s_i|) \in S$ and is bigger than r_i.

Player I wins the game if she can continue playing for infinitely many rounds. Given $S \in [\omega]^\omega$, we denote by $\mathrm{Hit}(S)$ the set of all subsets of ω that have infinite intersection with S.

Lemma 17: *Letting p be a Miller tree and $S \in [\omega]^\omega$, for the game $\mathcal{G}(p, S)$ we have the following:*

(1) *Player I has a winning strategy if and only if there is $q \leq p$ such that $[q]_{\mathrm{split}} \subseteq [S]^\omega$.*
(2) *Player II has a winning strategy if and only if there is $H : \mathrm{Split}(p) \longrightarrow \omega$ such that if $f \in [p]$, then the set $\{f \restriction n \in \mathrm{Split}(p) \mid f(n) \in S\}$ is almost contained in $\{f \restriction n \in \mathrm{Split}(p) \mid f(n) < H(f \restriction n)\}$ (in particular $[p]_{\mathrm{split}} \cap \mathrm{Hit}(S) \in \mathcal{K}(p)$).*

Proof. The first equivalence is easy so we leave it for the reader. Now assume there is a winning strategy π for II. We define $H : \mathrm{Split}(p) \longrightarrow \omega$ such that if $s \in \mathrm{Split}(p)$ then

$$\pi(\overline{x}) < H(s)$$

where $\overline{x}$ is any partial play in which player I has build s and II has played according to π (note there are only finitely many of those $\overline{x}$ so we can define $H(s)$). We want to prove that if $f \in [p]$, then $\{f \restriction n \in \mathrm{Split}(p) \mid f(n) \in S\}$ is almost contained in the set

$$\{f \restriction n \in \mathrm{Split}(p) \mid f(n) < H(f \restriction n)\}.$$

Assume this is not the case. Let B be the set of all $n \in \omega$ such that $f \restriction n \in \mathrm{Split}(p)$ with $f(n) \in S$ but $H(f \restriction n) \leq f(n)$. By our hypothesis B is infinite and then we enumerate it as $B = \{b_n \mid n \in \omega\}$ in increasing order. Consider the run of the game where I plays $f \restriction b_n$ at the n-th stage. This is possible since $f(b_n) \in S$ and $H(f \restriction b_n) \leq f(b_n)$ so I will win the game, which is a contradiction. The other implication is easy. ∎

Since $\mathcal{G}(p, S)$ is an open game for II by the Gale–Stewart theorem (see [11]) it is determined, so we conclude the following dichotomy:

Corollary 18: *If p is a Miller tree and $S \in [\omega]^\omega$ then one and only one of the following holds:*

(1) *There is $q \leq p$ such that $[q]_{\text{split}} \subseteq [S]^\omega$.*
(2) *There is $H : \text{Split}(p) \longrightarrow \omega$ such that if $f \in [p]$, then the set defined as $\{f \restriction n \in \text{Split}(p) \mid f(n) \in S\}$ is almost contained in the following set: $\{f \restriction n \in \text{Split}(p) \mid f(n) < H(f \restriction n)\}$ (and $[p]_{\text{split}} \cap \text{Hit}(S) \in \mathcal{K}(p)$).*

In particular, for every Miller tree p and $S \in [\omega]^\omega$ there is $q \leq p$ such that either $[q]_{\text{split}} \subseteq [S]^\omega$ or $[q]_{\text{split}} \subseteq [\omega \setminus S]^\omega$ (although this fact can be proved easier without the game).

Definition 19: Let p be a Miller tree and $S \in [\omega]^\omega$. We say S **tree-splits** p if there are Miller trees $q_0, q_1 \leq p$ such that

$$[q_0]_{\text{split}} \subseteq [S]^\omega \quad \text{and} \quad [q_1]_{\text{split}} \subseteq [\omega \setminus S]^\omega;$$

$\mathcal{S}$ is a **Miller tree-splitting family** if every Miller tree is tree-split by some element of $\mathcal{S}$.

It is easy to see that every Miller-tree splitting family is a splitting family and it is also easy to see that every ω-splitting family is a Miller-tree splitting family. We will now prove there is a Miller-tree splitting family of size $\mathfrak{s}$. I want to thank the referee for supplying the following argument which is simpler than the original one.

Proposition 20: *The smallest size of a Miller-tree splitting family is $\mathfrak{s}$.*

Proof. We will construct a Miller-tree splitting family of size $\mathfrak{s}$. In case $\mathfrak{b} \leq \mathfrak{s}$ there is an ω-splitting family of size $\mathfrak{s}$ (see Proposition 5 and the remark after Definition 6) and this is a Miller-tree splitting family as remarked above.

Now assume $\mathfrak{s} < \mathfrak{b}$. We will show that any splitting family of size $\mathfrak{s}$ is a Miller-tree splitting family. We argue by contradiction. Let

$$\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$$

be a splitting family which does not tree-split the Miller tree p. In this way, for every $\alpha < \mathfrak{s}$ there is $i(\alpha) < 2$ such that there is no $q \leq p$ for which $[q]_{\text{split}} \subseteq [S_\alpha^{i(\alpha)}]^\omega$. By Corollary 18, there is $H_\alpha : \text{Split}(p) \longrightarrow \omega$ such that for every $f \in [p]$ the following holds:

$$\{f \restriction n \in \mathrm{Split}(p) \mid f(n) \in S\} \subseteq^* \{f \restriction n \in \mathrm{Split}(p) \mid f(n) < H(f \restriction n)\}$$

or equivalently

$$\forall^\infty n(f \restriction n \in \mathrm{Split}(p) \wedge H_\alpha(f \restriction n) \leq f(n) \longrightarrow f(n) \in S_\alpha^{1-i(\alpha)}).$$

Since $\mathfrak{s} < \mathfrak{b}$ there exists $H : \mathrm{Split}(p) \longrightarrow \omega$ dominating each H_α. Take $f \in [p]$ such that for every $n \in \omega$, if $f \restriction n \in \mathrm{Split}(p)$ then $f(n) > H(f \restriction n)$. Then

$$\forall \alpha < \mathfrak{s} \forall^\infty n(f \restriction n \in \mathrm{Split}(p) \longrightarrow f(n) \in S_\alpha^{1-i(\alpha)}).$$

Let $X = \{f(n) \mid f \restriction n \in \mathrm{Split}(p)\}$; note that $X \subseteq^* S_\alpha^{1-i(\alpha)}$ for every $\alpha < \mathfrak{s}$. But this contradicts that $\mathcal{S}$ was a splitting family. ∎

The following lemma is probably well known:

Lemma 21: *Assume $\kappa < \mathfrak{d}$, and for every $\alpha < \kappa$ let $\mathcal{F}_\alpha \subseteq [\omega]^{<\omega}$ be an infinite set of disjoint finite subsets of ω and $g_\alpha : \bigcup \mathcal{F}_\alpha \longrightarrow \omega$. Then there is $f : \omega \longrightarrow \omega$ such that for every $\alpha < \kappa$ there are infinitely many $X \in \mathcal{F}_\alpha$ such that*

$$g_\alpha \restriction X < f \restriction X.$$

Proof. Given $\alpha < \kappa$, find an interval partition $\mathcal{P}_\alpha = \{P_\alpha(n) \mid n \in \omega\}$ such that for every $n \in \omega$ there is $X \in \mathcal{F}_\alpha$ such that $X \subseteq P_\alpha(n)$ (this is possible since $\mathcal{F}_\alpha$ is infinite and its elements are pairwise disjoint). Then define the function $\overline{g}_\alpha : \omega \longrightarrow \omega$ such that $\overline{g}_\alpha \restriction P_\alpha(n)$ is the constant function $\max\{g_\alpha[P_\alpha(n+1)]\}$. Since κ is smaller than $\mathfrak{d}$, we can then find an increasing function $f : \omega \longrightarrow \omega$ that is not dominated by any of the $\overline{g}_\alpha$. It is easy to prove that f has the desired property. ∎

Now we can prove the following lemma that will be useful:

Lemma 22: *Let q be a Miller tree and $\kappa < \mathfrak{d}$. If $\{H_\alpha \mid \alpha < \kappa\} \subseteq \omega^{\mathrm{Split}(q)}$ then there is $r \leq q$ such that $\mathrm{Split}(r) = \mathrm{Split}(q) \cap r$ and*

$$[r]_{\mathrm{split}} \cap \bigcup_{\alpha<\kappa} \mathrm{Catch}_\forall(H_\alpha) = \emptyset.$$

Proof. We will first prove there is $G : \mathrm{Split}(q) \longrightarrow \omega$ such that $\bigcup_{\alpha<\kappa} \mathrm{Catch}_\forall(H_\alpha)$ is a subset of $\mathrm{Catch}_\exists(G)$. Given $t \in \mathrm{Split}(q)$, let $T(t, \alpha)$ be the subtree of q such that if $f \in [T(t, \alpha)]$ then $t \sqsubseteq f$, and if $t \sqsubseteq f \restriction n$ and $f \restriction n \in \mathrm{Split}(q)$ then $f(n) \in H_\alpha(f \restriction n)$. Clearly $T(t, \alpha)$ is a finitely branching subtree of q. Then define $\mathcal{F}(t,\alpha) = \{\text{Split}_n(q) \cap T(t,\alpha) \mid n < \omega\}$ which is an infinite collection of pairwise disjoint finite sets, and let $g_{(t,\alpha)} : \bigcup \mathcal{F}(t,\alpha) \longrightarrow \omega$ given by

$$g_{(t,\alpha)}(s) = H_\alpha(s).$$

Since $\kappa < \mathfrak{d}$ by the previous lemma, we can find $G : \text{Split}(q) \longrightarrow \omega$ such that if $\alpha < \kappa$ and $t \in \text{Split}(q)$, then there are infinitely many $Y \in \mathcal{F}(t,\alpha)$ such that $g_{(t,\alpha)} \upharpoonright Y < G \upharpoonright Y$. We will now prove that $\bigcup_{\alpha<\kappa} \text{Catch}_\forall(H_\alpha) \subseteq \text{Catch}_\exists(G)$. Let $\alpha < \kappa$ and $f \in \text{Catch}_\forall(H_\alpha)$. Find $t \in \text{Split}(q)$ such that $t \sqsubseteq f$, and if $t \sqsubseteq f \upharpoonright m$ and $f \upharpoonright m \in \text{Split}(q)$ then $f(m) \in H_\alpha(f \upharpoonright m)$. Note that f is a branch through $T(t,\alpha)$. Let $Y \in \mathcal{F}(t,\alpha)$ such that $g_{(t,\alpha)} \upharpoonright Y < G \upharpoonright Y$ and, since $f \in [T(t,\alpha)]$, there is $n \in \omega$ such that $f \upharpoonright n \in Y$ so $f(n) < H_\alpha(f \upharpoonright n) < G(f \upharpoonright n)$.

Define $r \leq q$ such that $\text{Split}(r) = \text{Split}(q) \cap r$ and $\text{suc}_r(s) = \text{suc}_q(s) \setminus G(s)$. Clearly $[r]_{\text{split}}$ is disjoint from $\text{Catch}_\exists(G)$. ∎

We can then finally prove our main theorem.

THEOREM 23: *There is a +-Ramsey* MAD *family.*

Proof. If $\mathfrak{a} < \mathfrak{s}$ then $\mathfrak{a}$ is smaller than $\mathfrak{d}$ so then there is a +-Ramsey MAD family (in fact, there is a Miller-indestructible MAD family, see Corollary 13). So we assume $\mathfrak{s} \leq \mathfrak{a}$ for the rest of the proof. Fix an (ω,ω)-splitting family $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{s}\}$ that is also a Miller-tree splitting family. Let $\{L, R\}$ be a partition of the limit ordinals smaller than $\mathfrak{c}$ such that both L and R have size continuum. Enumerate by $\{X_\alpha \mid \alpha \in L\}$ all infinite subsets of ω and by $\{T_\alpha \mid \alpha \in R\}$ all subtrees of $\omega^{<\omega}$. We will recursively construct $\mathcal{A} = \{A_\xi \mid \xi < \mathfrak{c}\}$ and $\{\sigma_\xi \mid \xi < \mathfrak{c}\}$ as follows:

(1) $\mathcal{A}$ is an AD family and $\sigma_\alpha \in 2^{<\mathfrak{s}}$ for every $\alpha < \mathfrak{c}$.
(2) If $\sigma_\alpha \in 2^\beta$ and $\xi < \beta$, then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.
(3) If $\alpha \neq \beta$, then $\sigma_\alpha \neq \sigma_\beta$.
(4) If $\delta \in L$ and $X_\delta \in \mathcal{I}(\mathcal{A}_\delta)^+$, then $A_{\delta+n} \subseteq X_\delta$ for every $n \in \omega$ (where $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$).
(5) If $\delta \in R$ and T_δ is an $\mathcal{I}(\mathcal{A}_\delta)^+$-branching tree, then there is $f \in [T_\delta]$ such that $A_{\delta+n} \subseteq \text{im}(f)$ for every $n \in \omega$.

It is clear that if we manage to perform the construction, then $\mathcal{A}$ will be a +-Ramsey MAD family (and it will be completely separable too). Let δ be a limit ordinal and assume we have constructed A_ξ for every $\xi < \delta$. In case $\delta \in L$ we just proceed as in the case of the completely separable MAD family, so assume $\delta \in R$. Since $\mathcal{A}_\delta = \{A_\xi \mid \xi < \delta\}$ is nowhere-MAD (recall that $\mathcal{A}_\delta$ is nowhere-MAD by the proof of Theorem 9) we can find p, an $\mathcal{A}_\delta^\perp$-branching subtree of T_δ (recall that $\mathcal{A}_\delta^\perp$ is the set of all infinite sets that are almost disjoint with every element of $\mathcal{A}_\delta^\perp$).

First note that since $\mathcal{S}$ is a Miller-tree splitting family, for every Miller tree q there is $\alpha < \mathfrak{s}$ and $\tau_q \in 2^\alpha$ such that:

(1) S_α tree-splits q.
(2) If $\xi < \alpha$, then there is no $q' \leq q$ such that $[q']_{\text{split}} \subseteq [S_\xi^{1-\tau_q(\xi)}]^\omega$.

Note that if $q' \leq q$, then $\tau_{q'}$ extends τ_q. If $q \leq p$ and $\tau_q \in 2^\alpha$ we fix the following items:

(1) $W_0(q) = \{\xi < \alpha \mid \exists \beta < \delta(\sigma_\beta = \tau_q \restriction \xi)\}$

and

$W_1(q) = \{\xi < \alpha \mid \exists \beta < \delta(\Delta(\sigma_\beta, \tau_q) = \xi)\}$.[1]

(2) Let $\xi \in W_0(q)$. We then find β such that $\sigma_\beta = \tau_q \restriction \xi$ and define $G_{q,\xi} : \text{Split}(q) \longrightarrow \omega$ such that if $s \in \text{Split}(q)$ then

$$A_\beta \cap \text{suc}_q(s) \subseteq G_{q,\xi}(s)$$

(this is possible since q is $\mathcal{A}_\delta^\perp$-branching).

(3) Given $\xi \in W_1(q)$ we know there is no $q' \leq q$ such that

$$[q']_{\text{split}} \subseteq [S_\xi^{1-\tau_q(\xi)}]^\omega.$$

We know that there is $H_{q,\xi} : \text{Split}(q) \longrightarrow \omega$ such that if $f \in [q]$, the set defined as $\{f \restriction n \in \text{Split}(q) \mid f(n) \in S_\xi^{1-\tau_q(\xi)}\}$ is almost contained in the set

$$\{f \restriction n \in \text{Split}(q) \mid f(n) < H_{q,\xi}(f \restriction n)\}.$$

(4) If $U \in [W_0(q)]^{<\omega}$ and $V \in [W_1(q)]^{<\omega}$ choose any $J_{q,U,V} : \text{Split}(q) \longrightarrow \omega$ such that if $s \in \text{Split}(q)$, then

$$J_{q,U,V}(s) > \max\{G_{q,\xi}(s) \mid \xi \in U\}, \max\{H_{q,\xi}(s) \mid \xi \in V\}.$$

(5) $\mathcal{A}(q) = \{A_\xi \in \mathcal{A}_\delta \mid \tau_q \nsubseteq \sigma_\xi\}$.

Note that if $\xi \in W_0(q)$, then there is a unique $\beta < \delta$ such that $\sigma_\beta = \tau_q \restriction \xi$ (although the analogous remark is not true for the elements of $W_1(q)$). The following claim will play a fundamental role in the proof:

Claim 24: *If $q \leq p$, then there is $r \leq q$ such that $[r]_{\text{split}} \subseteq \mathcal{I}(\mathcal{A}(q))^+$.*

Proof of Claim 24. Let $\alpha < \mathfrak{s}$ such that $\tau_q \in 2^\alpha$. Since $\mathfrak{s} \leq \mathfrak{d}$, we know there is $r \leq q$ such that $[r]_{\text{split}}$ is disjoint from

$$\bigcup \{\text{Catch}_\forall(J_{q,U,V}) \mid U \in [W_0(q)]^{<\omega}, V \in [W_1(q)]^{<\omega}\}$$

and $\text{Split}(r) = \text{Split}(q) \cap r$. We will now prove $[r]_{\text{split}} \subseteq \mathcal{I}(\mathcal{A}(q))^+$ but assume this is not the case. Therefore, there is $f \in [r]$ and $F \in [\mathcal{A}(q)]^{<\omega}$ such that

$$X = Sp(r, f) \subseteq^* \bigcup F.$$

Let $F = F_1 \cup F_2$ and $U \in [W_0(q)]^{<\omega}$, $V \in [W_1(q)]^{<\omega}$ such that for every $A_\beta \in F_1$ there is $\xi_\beta \in U$ such that $\sigma_\beta = \tau_q \restriction \xi_\beta$, and for every $A_\gamma \in F_2$ there is $\eta_\gamma \in V$ such that $\Delta(\tau_q, \sigma_\gamma) = \eta_\gamma$. Let $D \subseteq \{n \mid f \restriction n \in \text{Split}(r)\}$ be the (infinite) set of all $n < \omega$ such that the following holds:

(1) $f \restriction n \in \text{Split}(r)$ and $f(n) \in \bigcup F$.
(2) If $\eta_\gamma \in V$ then $A_\gamma \setminus n \subseteq S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$.
(3) $f(n) > J_{q,U,V}(f \restriction n)$.
(4) If $\eta \in V$ and $f(n) \in S_\eta^{1-\tau_q(\eta)}$, then $f(n) < H_{q,\eta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ (recall that $\{f \restriction m \in \text{Split}(q) \mid f(m) \in S_\eta^{1-\tau_q(\eta)}\}$ is almost contained in $\{f \restriction m \in \text{Split}(q) \mid f(m) < H_{q,\eta}(f \restriction m)\}$).

We first claim that if $n \in D$, $\xi_\beta \in U$ and $\eta_\gamma \in V$, then $f(n) \notin A_\beta \cup S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$. On one hand, since $A_\beta \cap \text{suc}_q(f \restriction n) \subseteq G_{q,\xi_\beta}(f \restriction n) < J_{q,U,V}(f \restriction n)$ and $f(n) > J_{q,U,V}(f \restriction n)$ then $f(n) \notin A_\beta$. On the other hand, if it was the case that $f(n) \in S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$ so $f(n) < H_{q,\eta}(f \restriction n) < J_{q,U,V}(f \restriction n)$, but we already know that $f(n) > J_{q,U,V}(f \restriction n)$. Since $n \leq f(n)$ (recall every branch through p is increasing) $f(n) \notin A_\gamma$ for every $\eta_\gamma \in V$ because $A_\gamma \setminus n \subseteq S_{\eta_\gamma}^{1-\tau_q(\eta_\gamma)}$. This implies $f(n) \notin \bigcup F$ which is a contradiction and finishes the proof of the claim. ∎

Back to the proof of the theorem, we recursively build a tree of Miller trees $\{p(s) \mid s \in 2^{<\omega}\}$ with the following properties:

(1) $p(\emptyset) = p$.
(2) $p(s^\frown i) \leq p(s)$ and the stem of $p(s^\frown i)$ has length at least $|s|$.
(3) $\tau_{p(s^\frown 0)}$ and $\tau_{p(s^\frown 1)}$ are incompatible.
(4) $[p(s^\frown i)]_{\text{split}} \subseteq \mathcal{I}(\mathcal{A}(p(s)))^+$.

This is easy to do with the aid of the previous claim. For every $g \in 2^\omega$ let $\tau_g = \bigcup \tau_{p(g \restriction m)}$. Note that if $g_1 \neq g_2$, then τ_{g_1} and τ_{g_2} are two incompatible nodes of $2^{<\mathfrak{s}}$. Since $\mathcal{A}_\delta$ has size less than the continuum, there is $g \in 2^\omega$ such that there is no $\beta < \delta$ such that σ_β extends τ_g and then $\mathcal{A}_\delta = \bigcup_{m \in \omega} \mathcal{A}(p(g \restriction m))$.

Let f be the only element of $\bigcap_{m\in\omega}[p(g \restriction m)]$. Obviously, f is a branch through p and we claim that $\text{im}(f) \in \mathcal{I}(\mathcal{A}_\delta)^+$. This is easy since if $A_{\xi_1}, \ldots, A_{\xi_n} \in \mathcal{A}_\delta$, then we can find $m < \omega$ such that $A_{\xi_1}, \ldots, A_{\xi_n} \in \mathcal{A}(p(g \restriction m))$ and then we know that $Sp(p(g \restriction m+1), f) \nsubseteq^* A_{\xi_1} \cup \cdots \cup A_{\xi_n}$, and since $Sp(p(g \restriction m+1), f)$ is contained in $\text{im}(f)$ we conclude that $\text{im}(f) \in \mathcal{I}(\mathcal{A}_\delta)^+$.

Finally, find a partition $\{Z_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A}_\delta)^+$ of $\text{im}(f)$ and using the method of Shelah, construct $A_{\delta+n}$ such that $A_{\delta+n} \subseteq Z_n$. This finishes the proof. ∎

4. More constructions

In this last section, we will show the relationship between +-Ramsey and other properties of MAD families. Recall that an ideal $\mathcal{I}$ in ω is **tall** if for every $X \in [\omega]^\omega$ there is $Y \in \mathcal{I}$ such that $X \cap Y$ is infinite. Note that if $\mathcal{A}$ is an AD family, then $\mathcal{I}(\mathcal{A})$ is tall if and only if $\mathcal{A}$ is MAD. Note that the proof of Theorem 23 in fact gives the following result:

Corollary 25 ($\mathfrak{s} \leq \mathfrak{a}$): *If $\mathcal{I}$ is a tall ideal, then there is a +-Ramsey MAD family $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathcal{I}$.*

The following are properties of MAD families that have been studied in the literature:

Definition 26: Let $\mathcal{A}$ be a MAD family.

(1) $\mathcal{A}$ is $\mathbb{P}$-**indestructible** if $\mathcal{A}$ remains MAD after forcing with $\mathbb{P}$ (we are mainly interested where $\mathbb{P}$ is Cohen, random, Sacks or Miller forcing).
(2) $\mathcal{A}$ is **weakly tight** if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap X_n| = \omega$ for infinitely many $n \in \omega$.
(3) $\mathcal{A}$ is **tight** if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ there is $B \in \mathcal{I}(\mathcal{A})$ such that $B \cap X_n$ is infinite for every $n \in \omega$.
(4) $\mathcal{A}$ is **Laflamme** if $\mathcal{A}$ can not be extended to an F_σ-ideal.
(5) $\mathcal{A}$ is +-**Ramsey** if for every $\mathcal{I}(\mathcal{A})^+$-branching tree T, there is $f \in [T]$ such that $\text{im}(f) \in \mathcal{I}(\mathcal{A})^+$.

It is known that tightness implies both weak tightness and Cohen indestructibility (see [8]). It is also easy to see that Cohen indestructibility implies Miller indestructibility and Sacks indestructibility is weaker than both Miller indestructibility and random indestructibility (see [4]).

Corollary 27 ($\mathfrak{s} \leq \mathfrak{a}$): *There is a +-Ramsey* MAD *family that is not Sacks indestructible, Laflamme or weakly tight.*

Proof. The corollary follows by the previous result. In [9] it was proved that there is a tall ideal $\mathcal{I}$ such that every MAD family contained in $\mathcal{I}$ is Sacks destructible. A similar result for weak tightness was proved in [3]. ∎

The following is a very important definition:

Definition 28: We say $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\omega\}$ is a **lower semicontinuous submeasure** if the following hold:

(1) $\varphi(\omega) = \omega$.
(2) $\varphi(A) = 0$ if and only if $A = \emptyset$.
(3) $\varphi(A) \leq \varphi(B)$ whenever $A \subseteq B$.
(4) $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for every $A, B \subseteq X$.
(5) (lower semicontinuity) if $A \subseteq \omega$ then $\varphi(A) = \sup\{\varphi(A \cap n) \mid n \in \omega\}$.

Given a lower semicontinuous submeasure φ we define $\mathsf{Fin}(\varphi)$ as the family of those subsets of ω with finite submeasure. The following is a very interesting result of Mazur:

Proposition 29 (Mazur [13]): *$\mathcal{I}$ is an F_σ-ideal if and only if there is a lower semicontinuous submeasure such that $\mathcal{I} = \mathsf{Fin}(\varphi)$.*

If $a \subseteq \omega^{<\omega}$ we define

$$\pi(a) = \{f \in \omega^\omega \mid \exists^\infty n (f \restriction n \in a)\}.$$

Given $f \in \omega^\omega$, define

$$\widehat{f} = \{f \restriction n \mid n \in \omega\}$$

and let $\mathcal{BR} = \{\widehat{f} \mid f \in \omega^\omega\}$. By $\mathcal{J}$ we denote the ideal on $\omega^{<\omega}$ consisting of all sets $a \subseteq \omega^{<\omega}$ such that $\pi(a)$ is finite. Clearly $\mathcal{BR} \subseteq \mathcal{J}$. The next result follows easily from the results in [14], but we include a proof for the convenience of the reader:

Lemma 30: *$\mathcal{J}$ cannot be extended to an F_σ-ideal.*

Proof. Let $\varphi : \wp(\omega) \longrightarrow \omega \cup \{\omega\}$ be a lower semicontinous submeasure. We will prove that $\mathcal{J}$ is not a subset of $\mathsf{Fin}(\varphi)$. Given $s \in \omega^{<\omega}$, we denote

$$B_0(s) = \{t \in \omega^{<\omega} \mid s \subseteq t\} \quad \text{and} \quad B_1(s) = \{t \in \omega^{<\omega} \mid s \perp t\}$$

(where $s \perp t$ denotes that s and t are incompatible). Let $\omega^{<\omega} = \{s_n \mid n \in \omega\}$. We recursively construct two sequences $\langle i_n \mid n \in \omega\rangle$ and $\langle F_n \mid n \in \omega\rangle$ such that for every $n \in \omega$ the following holds:

(1) $i_n \in \{0, 1\}$.
(2) F_n is a finite subset of ω and $\varphi(F_n) \geq n + 1$.
(3) $\bigcap_{j \leq n} B_{i_j}(s_j) \in \mathsf{Fin}(\varphi)^+$.
(4) $F_n \subseteq \bigcap_{j \leq n} B_{i_j}(s_j)$.

The construction is very easy to perform. Let $G = \bigcup_{n\in\omega} F_n$. Note that $G \in \mathsf{Fin}(\varphi)^+$. Furthermore, for every $s \in \omega^{<\omega}$ either G is almost contained in $B_0(s)$ or is almost disjoint from it. It is easy to see that $G \in \mathcal{J}$ so $\mathcal{J}$ is not contained in $\mathsf{Fin}(\varphi)$. ∎

We can now prove the following:

Proposition 31 (CH): *There is a Laflamme* MAD *family that is not +-Ramsey.*

Proof. Let $\{\mathcal{I}_\alpha \mid \alpha \in \omega_1\}$ be the set of all F_σ-ideals in $\omega^{<\omega}$. We construct $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that the following holds:

(1) $\mathcal{A} \cup \mathcal{BR}$ is an AD family.
(2) If $s \in \omega^{<\omega}$, then $\mathcal{A}$ contains an infinite partition of $\{s^\frown n \mid n \in \omega\}$ into infinite sets.
(3) If $\mathcal{A}_\alpha \cup \mathcal{BR} \subseteq \mathcal{I}_\alpha$, then $A_\alpha \notin \mathcal{I}_\alpha$ (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$).
(4) $\mathcal{A}_\alpha$ is countable.

At step α assume that $\mathcal{BR} \cup \mathcal{A}_\alpha \subseteq \mathcal{I}_\alpha$. Since $\mathcal{I}_\alpha$ is an F_σ-ideal and it contains all branches, there is $a \in \mathcal{I}_\alpha^+ \cap \mathcal{J}$. Let $\pi(a) = \{f_1, \ldots, f_n\}$ and we now define $b = a \setminus (\widehat{f_1} \cup \cdots \cup \widehat{f_n})$. Note that $\pi(b) = \emptyset$ and $b \in \mathcal{I}_\alpha^+$. Let φ be a lower semicontinuous submeasure such that $\mathcal{I}_\alpha = \mathsf{Fin}(\varphi)$ and let $\mathcal{A}_\alpha = \{B_n \mid n \in \omega\}$. We recursively find $s_n \subseteq b \setminus (B_0 \cup \cdots \cup B_n)$ such that $\varphi(s_n) \geq n$ (this is possible since $b \in \mathcal{I}_\alpha^+$). Then $A_\alpha = \bigcup_{n\in\omega} s_n$ is the set we were looking for. It is easy to see that $\mathcal{A} \cup \mathcal{BR}$ is a Laflamme MAD family that is not +-Ramsey. ∎

We will now prove that weak tightness does not imply being +-Ramsey. Given $s \in \omega^{<\omega}$ we define

$$[s] = \{t \in \omega^{<\omega} \mid s \subseteq t\}.$$

Lemma 32: *If $A \subseteq \omega^{<\omega}$ does not have infinite antichains, then A can be covered with finitely many chains.*

Proof. Define S as the set of all unsplitting nodes of A, i.e., $s \in A$ if and only if every two extensions of s in A are compatible. Note that $S \subseteq A$ and every element of A can be extended to an element of S (otherwise A would contain a Sacks tree and hence an infinite antichain). Let $B \subseteq S$ be a maximal (finite) antichain. For every $s \in B$ let $b_s \in \omega^\omega$ be the unique branch such that $A \cap [s] \subseteq \widehat{b}_s$. Then (by the maximality of B) we conclude $A \subseteq \bigcup_{s \in B} \widehat{b}_s$. ∎

We need the following lemma:

Lemma 33: *If $A = \{A_n \mid n \in \omega\} \subseteq \wp(\omega^{<\omega})$ is a collection of infinite antichains, then there is an antichain B such that $B \cap A_n$ is infinite for infinitely many $n \in \omega$.*

Proof. We say $s \in \omega^{<\omega}$ **watches**A_n if s has infinitely many extensions in A_n. Define $T \subseteq \omega^{<\omega}$ such that $s \in T$ if and only if there are infinitely many $n \in \omega$ such that s watches A_n. Note that T is a tree. First assume there is $s \in T$ that is a maximal node. By shrinking A if needed, we may assume s watches every element of A. We now define the set $C = \{A_n \mid \exists^\infty m (A_n \cap [s^\frown m] \neq \emptyset)\}$. In case C is infinite, we can find an antichain B that has infinite intersection with every element of C. Now assume that C is finite; by shrinking A we may assume C is the empty set. In this way, for every A_n there is m_n such that $s^\frown m_n$ watches A_n. We can then find an infinite set $X \in [\omega]^\omega$ such that $m_n \neq m_r$ whenever $n \neq r$ and $n, r \in X$ (recall that s is maximal). Then $B = \bigcup_{n \in X} [s^\frown m_n] \cap A_n$ is the set we were looking for.

Now we may assume T does not have maximal nodes. If T contains a Sacks tree, then we can find an infinite antichain $Y \subseteq T$. For every $s \in Y$ we choose n_s such that s watches A_{n_s}, and if $s \neq t$ then $A_{n_s} \neq A_{n_t}$. Then $B = \bigcup_{s \in Y} [s] \cap A_{n_s}$ is the set we were looking for.

The only case left is that there is $s \in T$ that does not split in T and is not maximal. Let $f \in [T]$ be the only branch that extends s. We may assum+e s watches every element of A and every A_n is disjoint from $\widehat{f}$ (this is because A_n is an antichain and f is a branch). We say A_n is a **comb** with f if $\Delta(A_n \cap [s], \widehat{f})$ is infinite. We may assume that either every element of A is a comb with f or none is. In case all of them are combs we can easily find the desired antichain. So assume none of them are combs. In this way, for every $n \in \omega$ we can find t_n extending s but incompatible with f of minimal length such that t_n watches A_n. Since $t_n \notin T$ we can find $W \in [\omega]^\omega$ such that $t_n \neq t_m$ for all $n, m \in W$ where $n \neq m$. Then we recursively construct the desired antichain. ∎

We can then conclude the following:

PROPOSITION 34 (CH): *There is a weakly tight* MAD *family that is not +-Ramsey.*

Proof. Let $\{\overline{X}_\alpha \mid \omega \le \alpha < \omega_1\}$ enumerate all countable sequences of infinite subsets of $\omega^{<\omega}$. Let $\mathcal{BR} = \{\widehat{f} \mid f \in \omega^\omega\}$. We construct $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ such that the following holds:

(1) Every A_α is an antichain.
(2) $\mathcal{A} \cup \mathcal{BR}$ is an AD family.
(3) If $s \in \omega^{<\omega}$, then $\mathcal{A}$ contains a partition of $\mathrm{suc}(s) = \{s^\frown n \mid n \in \omega\}$.
(4) For every $\omega \le \alpha < \omega_1$, if $\overline{X}_\alpha = \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A}_\alpha \cup \mathcal{BR})^+$ then $A_\alpha \cap X_n$ is infinite for infinitely many $n \in \omega$ (where $\mathcal{A}_\alpha = \{A_\xi \mid \xi < \alpha\}$).

At step $\alpha = \{\alpha_n \mid n \in \omega\}$ assume $\overline{X}_\alpha = \{X_n \mid n \in \omega\} \subseteq (\mathcal{A}_\alpha \cup \mathcal{BR})^+$. We first claim that there is an infinite antichain $X'_n \subseteq X_n$ such that $X_n \in \mathcal{A}_\alpha^\perp$. Let $\Sigma = \{A \in \mathcal{A}_\alpha \mid |A \cap X_n| = \omega\}$. In case Σ is finite, by Lemma 32 we can find an infinite antichain $X'_n \subseteq X_n \setminus \bigcup \Sigma$. If Σ is infinite, then by Lemma 33 we can find an infinite $\Sigma' \subseteq \Sigma$ and $B_A \in [A \cap X_n]^\omega$ for $A \in \Sigma'$ such that $\bigcup\{B_A \mid A \in \Sigma'\}$ is an antichain. It is then easy to choose distinct $\{s_A \in B_A \mid A \in \Sigma'\}$ so that $X'_n = \{s_A \in B_A \mid A \in \Sigma'\} \in \mathcal{A}_\alpha^\perp$.

Let $Y_n = X'_n \setminus (A_{\alpha_0} \cup \cdots \cup A_{\alpha_n})$ which is an infinite antichain. By Lemma 33 we can find an antichain

$$A_\alpha \subseteq \bigcup_{n \in \omega} Y_n$$

such that $A_\alpha \cap Y_n$ is infinite for infinitely many $n \in \omega$.

Clearly $\mathcal{A} \cup \mathcal{BR}$ is not +-Ramsey (recall that weakly tight families are maximal). ∎

Recall that Miller indestructibility implies being +-Ramsey. We will now prove that (in particular) Sacks or random indestructibility are not enough to get +-Ramseyness. We will say a family $\mathcal{A}$ on $\omega^{<\omega}$ is a **standard $\mathcal{K}_\sigma$ family** if the following holds:

(1) $\mathcal{A}$ is an AD family.
(2) If $A \in \mathcal{A}$, either $\pi(A) = \emptyset$ or A is a finitely branching tree on $\omega^{<\omega}$.
(3) If $s \in \omega^{<\omega}$, then $\{s^\frown n \mid n \in \omega\} \in \mathcal{I}(\mathcal{A})^{++}$.

Recall that if $a \subseteq \omega^{<\omega}$, we denoted $\pi(a) = \{f \in \omega^\omega \mid \exists^\infty n (f \restriction n \in a)\}$. We now need the following lemma:

LEMMA 35: *Let $\mathbb{P}$ be an ω^ω-bounding forcing and $\mathcal{A}$ a countable standard $\mathcal{K}_\sigma$ family. If $p \in \mathbb{P}$ and $\dot{b}$ is a $\mathbb{P}$-name for an infinite subset of $\omega^{<\omega}$ such that $p \Vdash$ "$\dot{b} \in \mathcal{A}^\perp$", then there are $q \leq p$ and $\mathcal{B}$ a countable standard $\mathcal{K}_\sigma$ family such that $\mathcal{A} \subseteq \mathcal{B}$ and $q \Vdash$ "$\dot{b} \notin \mathcal{B}^\perp$".*

Proof. Let $\mathcal{A} = \{T_n \mid n \in \omega\} \cup \{a_n \mid n \in \omega\}$ where T_n is a finitely branching subtree of $\omega^{<\omega}$ and $\pi(a_n) = \emptyset$ for every $n \in \omega$. We may assume that p forces that $\pi(\dot{b})$ is either empty or a singleton. We first assume there is $\dot{r}$ such that $p \Vdash$ "$\pi(\dot{b}) = \{\dot{r}\}$". Since $\mathbb{P}$ is ω^ω-bounding, we may find $p_1 \leq p$ and $T \in V$ a finitely branching well pruned subtree of $\omega^{<\omega}$ such that $p_1 \Vdash$ "$\dot{r} \in [T]$". Once again, since $\mathbb{P}$ is ω^ω-bounding we may find $p_2 \leq p_1$ and $f \in \omega^\omega$ such that the following holds:

(1) f is an increasing function.
(2) $p_2 \Vdash$ "$(T_n \cup a_n) \cap \widehat{r} \subseteq \omega^{<f(n)}$".

For each $n \in \omega$, define

$$\widetilde{T}_n = \{s \in T_n \mid f(n) \leq |s|\}$$

and define the set $\widetilde{a}_n$ as $\{t \mid \exists s \in a_n (s \in a_n \wedge f(n) \leq f(n))\}$. Let

$$K = T \setminus \bigcup_{n\in\omega}(\widetilde{T}_n \cup \widetilde{a}_n).$$

It is easy to see that K is a finitely branching tree, $p_2 \Vdash$ "$\dot{r} \in [K]$" and $K \in \mathcal{A}^\perp$. We now simply define $\mathcal{B} = \mathcal{A} \cup \{K\}$.

Now we consider the case where $\pi(\dot{b})$ is forced to be empty. Let $\dot{S}$ be the tree of all $s \in \omega^{<\omega}$ such that s has infinitely many extensions in $\dot{b}$. We will first assume there are $p_1 \leq p$ and s such that p_1 forces that s is a maximal node of $\dot{S}$. Since $\mathbb{P}$ is ω^ω-bounding, we can find a ground model interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ and $p_2 \leq p_1$ such that if $n \in \omega$, then p_2 forces that there is $\dot{m}_n \in P_n$ such that $([s^\frown \dot{m}_n] \cap \dot{b}) \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n) \neq \emptyset$. Given $n, m \in \omega$ we define $K_{n,m} = \{s^\frown i^\frown t \mid i \in P_n \wedge t \in m^m\}$. Using once again that $\mathbb{P}$ is ω^ω-bounding, we may find $p_3 \leq p_2$ and an increasing function $f : \omega \longrightarrow \omega$ such that if $n \in \omega$ then p_3 forces $(K_{n,f(n)} \cap \dot{b}) \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n)$ is non-empty for every $n \in \omega$. We now define

$$a = \bigcup_{n\in\omega} K_{n,f(n)} \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n).$$

It is easy to see that $\pi(a) = \emptyset$, $a \in \mathcal{A}^\perp$, and p_3 forces that a and $\dot{b}$ have infinite intersection.

Now we assume that p forces that $\dot{S}$ does not have maximal nodes. Let $\dot{r}$ be a name for a branch of $\dot{S}$. First assume that $\dot{r}$ is forced to be a branch through some element of $\mathcal{A}$. We may assume that $p \Vdash$ "$\dot{r} \in [T_0]$". Since $\mathbb{P}$ is ω^ω-bounding, we may find $p_1 \leq p$ and an increasing ground model function $f : \omega \longrightarrow \omega$ such that if $n \in \omega$, then p_1 forces that all extentions of $\dot{r} \restriction f(n)$ to $\dot{b}$ are not in $T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n$. Once again, we may find $p_2 \leq p_1$ and $g : \omega \longrightarrow \omega$ such that if $n \in \omega$, then $\dot{b}$ has a non-empty intersection with the set $\{\dot{r} \restriction f(n)^\frown t \mid t \in g(n)^{g(n)}\} \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n)$. We now define

$$a = \bigcup_{s \in (T_0)_{f(n)}} (\{s^\frown t \mid t \in g(n)^{g(n)}\} \setminus (T_0 \cup \cdots \cup T_n \cup a_0 \cup \cdots \cup a_n)).$$

It is easy to see that a has the desired properties.

Finally, in case that $\dot{r}$ is not forced to be a branch through some element of $\mathcal{A}$, we find a finitely branching tree $T \in \mathcal{A}^\perp$ such that $p \Vdash$ "$\dot{r} \in [T]$" as we did at the beginning of the proof. If T has infinite intersection with $\dot{b}$ we are done, and if not then we apply the previous case. ∎

With a standard bookkeeping argument we can then conclude the following:

Proposition 36 (CH): *If $\mathbb{P}$ is a proper ω^ω-bounding forcing of size ω_1, then there is a* MAD *family $\mathcal{A}$ that is $\mathbb{P}$ indestructible but is not +-Ramsey.*

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