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Preservation theorems for Namba forcing ☆

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We study preservation properties of Namba forcing on κ. We prove that if $\mathcal{I}$ is an ideal with a Borel base on ω^ω and $\kappa > \omega_1$ is a regular cardinal less than the uniformity number or bigger than the covering number of $\mathcal{I}$, then the κ-Namba forcing preserves the covering of $\mathcal{I}$. The situation at $\kappa = \omega_1$, also treated here, is more complex.

1. Introduction

Namba forcing was introduced in [12] in order to show that it is possible to change the cofinality of ω_2 to ω while preserving ω_1. It also adds a new countable sequence of ordinals, yet it may not add new reals. In this paper, we prove additional preservation properties for Namba forcing. Their status may be different in different models of set theory; for example, it is consistent that Namba forcing adds Cohen reals, while it is also consistent that it has the Sacks property. We consider Namba forcing on various cardinals. The symbol $\mathbb{N}_\kappa$ will denote the Namba forcing on κ.

We consider preservation properties of the following sort: given an ideal $\mathcal{I}$ with a Borel base on a Polish space X, and a forcing notion $\mathbb{P}$, we say that $\mathbb{P}$ *preserves the covering of* $\mathcal{I}$ if $\mathbb{P} \Vdash \forall x \in X\ \exists B \in \mathcal{I} \cap V\ x \in B$. For example, if $\mathcal{M}$ denotes the ideal of meager sets and $\mathcal{N}$ the ideal of null sets on ω^ω, then preserving the

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covering of $\mathcal{M}$ means not adding Cohen reals and preserving the covering of $\mathcal{N}$ is not adding random reals. The following is an example of the results we will prove in this note:

Theorem 1. *Let $\kappa > \omega_1$ be a regular cardinal.*

(1) *If $\kappa < \mathsf{non}(\mathcal{M})$ or $\mathsf{cov}(\mathcal{M}) < \kappa$ then $\mathbb{N}_\kappa$ does not add Cohen reals.*
(2) *If $\kappa < \mathsf{non}(\mathcal{N})$ or $\mathsf{cov}(\mathcal{N}) < \kappa$ then $\mathbb{N}_\kappa$ does not add random reals.*
(3) *If $\mathsf{add}(\mathcal{N}) < \kappa$ or $\kappa < \mathsf{cof}(\mathcal{N})$ then $\mathbb{N}_\kappa$ has the Sacks property.*

The case $\kappa = \omega_1$ is more nuanced. We can prove theorems of this type with some additional assumptions, but in general, the following question remains open even for the σ-ideals of meager and null sets:

Problem 2. *Suppose that $\mathcal{I}$ is a σ-ideal of Borel sets on a Polish space and $\omega_1 < \mathrm{non}(\mathcal{I})$. Does $\mathbb{N}_{\omega_1}$ preserve the covering of $\mathcal{I}$?*

The methods of this paper do not seem to help to answer interesting questions of the following type.

Problem 3. *Can $\mathbb{N}_\kappa$ add Sacks, Laver, Mathias or Miller reals?*

In [7] Simon, Hrušák and Zindulka, in effect, asked if $\mathfrak{b}$ is the first regular, uncountable cardinal κ such that $\mathbb{N}_\kappa$ adds an unbounded real. We answer this question positively. The relationship between Namba forcing and weak partition properties will be further studied in [6].

Our notation is mostly standard. If X is a set, by $\wp(X)$ we denote the power set of X. An ideal $\mathcal{I} \subseteq \wp(X)$ on X is a collection of subsets of X closed under taking subsets and unions; for convenience, all our ideals will be proper (i.e. $X \notin \mathcal{I}$). A σ-ideal is an ideal closed under countable unions. If X is a topological space, we say $\mathcal{I}$ has a Borel base if every element of $\mathcal{I}$ is contained in a Borel set in $\mathcal{I}$. In this paper, the expression "for almost all" means for all except finitely many. Given a topological space X, we denote by $\mathsf{Borel}(X)$ the collection of all Borel subsets of X. Given a Borel set $B \subseteq \kappa^\omega$ and W a model of ZFC extending V, we may wish to reinterpret B in W. It is well known how to reinterpret Borel sets in the case where $\kappa = \omega$, but the general case presents some new difficulties. In [14] the third author developed a general framework for reinterpreting spaces and Borel sets on *interpretable spaces,* which are the open continuous images of a Čech complete space. In this paper, all interpretable spaces are in fact completely metrizable. The reader can consult [3] for the definition of the cardinal invariants used in this paper, and [5], [10] or [9] for more on Namba forcing.

2. Basic properties of Namba forcing and absoluteness results

We start the treatment of Namba forcing with a couple of basic definitions.

Definition 4. Let κ be an infinite cardinal.

(1) A tree $T \subseteq \kappa^{<\omega}$ is called a κ-*Namba tree* if there is $s \in T$ (called the *stem of* T) such that every $t \in T$ is comparable with s; furthermore if $t \sqsubset s$ then t has just one immediate successor and if $s \sqsubseteq t$ then t has κ many immediate successors.
(2) The Namba forcing $\mathbb{N}_\kappa$ is the set of all κ-Namba trees ordered by inclusion.

Note that $\mathbb{N}_\omega$ is the Laver forcing. If $G \subset \mathbb{N}_\kappa$ is a generic filter, then $\bigcup\bigcap G$ is an element of κ^ω, the name for which we denote $\dot{x}_{\mathrm{gen}}$.

Definition 5. Let κ be an infinite cardinal.

(1) If $F\colon \kappa^{<\omega} \longrightarrow [\kappa]^{<\kappa}$ is a function then $C(F)$ is the set $\{f \in \kappa^\omega \mid \exists^\infty n\ f(n) \in F(f \restriction n)\}$.
(2) The *κ-Namba ideal* $\mathcal{L}_\kappa$ is the ideal on κ^ω generated by the sets $C(F)$ as F varies over all functions from $\kappa^{<\omega}$ to $[\kappa]^{<\kappa}$.

In this way, $\mathcal{L}_\omega$ is the usual Laver ideal in ω^ω (see [13] page 44). Our first theorem establishes the basic relationship between $\mathbb{N}_\kappa$ and $\mathcal{L}_\kappa$ with a slight generalization of [13, Example 2.1.13]:

Theorem 6. (Quotient presentation) *Let κ be an infinite cardinal. For every Borel set $B \subset \kappa^\omega$, exactly one of the following occurs:*

(1) *$B \in \mathcal{L}_\kappa$;*
(2) *there is a Namba tree $T \in \mathbb{N}_\kappa$ such that $[T] \subset B$.*

As an immediate corollary, $\mathbb{N}_\kappa$ is naturally isomorphic to a dense subset of the quotient poset of Borel subsets of κ^ω modulo $\mathcal{L}_\kappa$.

Proof. Given a set $B \subseteq \kappa^\omega$ consider the following game.

I	X_0		X_1		X_2		X_3	$\cdots$
II		α_0, i_0		α_1, i_1		α_2, i_2		$\cdots$

where, $X_n \in [\kappa]^{<\kappa}$, $\alpha_n \in \kappa$ and $i_n \in 2$ for all $n \in \omega$. Player II wins if and only if the following conditions hold:

(1) $\langle \alpha_n \rangle_{n\in\omega} \in B$,
(2) there is $n \in \omega$ such that $i_n = 1$, and
(3) if $i_n = 1$ and $m \geq n$ then $\alpha_m \notin X_m$.

Note that if $i_n = 1$ and $m \geq n$ then i_m is irrelevant, so we may ignore it. Now, suppose that the set B is Borel. By Borel determinacy [8, Section 20] one of the players has a winning strategy. Thus, we can consider two complementary cases.

Case 1. Player I has a winning strategy σ. Note that for every $t \in \kappa^{<\omega}$ there are at most $|t|$ possible ways in which player II can reach t and player I was following σ; let $F(t) \in [\kappa]^{<\kappa}$ be the union of all possible answers by the strategy σ. It is then easy to see that $A \subseteq C(F)$.

Case 2. Player II has a winning strategy σ. We can then find $n \in \omega$ and $t \in \kappa^n$ such that i_n is the first such that $i_n = 1$ and player II reached t during a partial play with the strategy σ. It is now easy to see that there is $T \in \mathbb{N}_\kappa$ with stem t such that $[T] \subseteq A$.

The theorem follows. □

Theorem 7. (Continuous reading of names) *Let κ be an infinite cardinal, let $T \in \mathbb{N}_\kappa$ be a condition, let Y be a completely metrizable space, and let $\dot{y}$ be an $\mathbb{N}_\kappa$-name for an element of Y. There is $S \in \mathbb{N}_\kappa$ below T and a continuous function $f\colon [S] \to Y$ such that $S \Vdash F(\dot{x}_{gen}) = \dot{y}$.*

Proof. The proof is based on a claim with a game-theoretic proof.

Claim 8. *Suppose a tree $T \in \mathbb{N}_\kappa$ is a condition and let $D \subseteq \mathbb{N}_\kappa$ be an open dense set below T. Then there is a condition $S \leq T$ with the same trunk as T and a front $F \subset S$ (i.e. F is an antichain and every branch of S extends an element of F) such that for every $u \in F$, $S \restriction u \in D$.*

Proof. Let t be the trunk of T. Consider the following game:

I	X_0		X_1		X_2		X_3	$\cdots$
II		α_0		α_1		α_2		$\cdots$

in which $X_n \in [\kappa]^{<\kappa}$, $\alpha_n \in \kappa \setminus X_n$, and Player II wins if there is a number $n \in \omega$ such that the sequence $s = t^\frown\langle\alpha_i : i \in n\rangle$ is in T and there is a tree $U \subset T \restriction s$ such that U has trunk s and $U \in D$.

We claim that Player I has no winning strategy. Indeed, if σ was such a strategy, the tree T' of all nodes $s \in T$ which can be reached in a counterplay against σ is a Namba tree. Let $U \subset T'$ be a condition in D with trunk u longer than t. Then, Player II can beat the strategy σ by playing so that u is reached.

Therefore, Player II has a winning strategy σ. It is not difficult to build a Namba tree $T' \subset T$ with trunk t and a function p whose domain is the set of all nodes in T' extending the trunk of T', so that $p(u)$ is a play according to the strategy σ in which Player II played exactly the ordinals on the sequence $u \setminus t$, and such that $u \subset v$ implies $p(u) \subset p(v)$. Since σ is a winning strategy for Player II, the set

$$F = \{u \in T' : \text{ there is some } U_u \leq T \text{ such that the trunk of } U_u \text{ is } u \text{ and } U_u \in D\}$$

is a front of T'. Let S be the tree obtained from T' by replacing $T' \restriction u$ with U_u for each $u \in F$ and note that the tree S works. $\square$

We can now prove the theorem. Let d be a complete compatible metric for the space Y. For each $n \in \omega$ let D_n be the set of all conditions $U \leq T$ such that there is a basic open set $O \subset Y$ of d-radius $\leq 2^{-n}$ such that $U \Vdash \dot{y} \in \bar{O}$. Using the claim repeatedly, we can find a Namba tree $S \subset T$ and fronts F_n in the tree S such that for each $s \in F_n$, $S \restriction s \in D_n$. Let $f : [S] \to Y$ be the function defined by letting $f(x)$ equal the unique element of $\bigcap \bar{O}_n$ where $O_n \subset Y$ is the basic open set of radius $\leq 2^{-n}$ such that $S \restriction s_n \Vdash \dot{y} \in \bar{O}_n$, where s_n is the unique initial segment of x belonging to F_n. It is immediate that the tree S and the function f are as required. $\square$

Theorem 10 below uses a standard tool of descriptive set theory generalized to the setting of arbitrary completely metrizable spaces. We record the main properties of this tool in a separate proposition:

Proposition 9. *Let μ be an infinite cardinal. Let Y be a completely metrizable space of weight $\leq \mu$, and let $B \subset Y$ be a Borel set. Then there is a continuous function $f : \mu^\omega \to Y$ such that in all forcing extensions, the interpretation of B is equal to the range of the interpretation of f.*

Proof. We first argue that there is such a function in the case $B = Y$. To see this, fix a complete metric d on the space Y and use the weight assumption to construct basic open sets $O_t \subset Y$ for all $t \in \mu^{<\omega}$ such that $O_t = Y$, the closure of $O_{t^\frown\alpha} \subset O_t$, $O_t = \bigcup_{\alpha\in\mu} O_{t^\frown\alpha}$, and the d-diameter of O_t is smaller than $2^{-|t|}$ whenever $t \neq 0$. In the end, let $f(x)$ be the unique element of $\bigcap_n O_{x\restriction n}$ for every point $x \in \mu^\omega$. Note that the function f is well-defined by the completeness of the metric d. The function f is continuous, and the range of its interpretation will be the whole interpretation of Y, since interpretations of open sets preserve unions.

To prove the proposition, consider the collection of all Borel subsets $B \subset Y$ for which the conclusion of the theorem holds. We will show that this collection contains all open sets and all closed sets, and is closed under countable union and intersection. Since every Borel subset of a topological space can be built from the open sets and closed sets by repetition of the operations of countable unions and intersections, this will conclude the proof of the proposition.

Suppose first that $B \subset Y$ is either a closed or an open set. Then B is a completely metrizable space in the inherited topology, and its weight is still $\leq \mu$. Thus, we can use the first paragraph of the present proof with Y replaced by B to produce the desired function f. Suppose that $B = \bigcup_n B_n$ and the conclusion of the proposition is known for each B_n and exemplified by functions $f_n\colon \mu^\omega \to Y$. Consider the space $X = \omega \times \mu^\omega$, which is homeomorphic to μ^ω, and the function $f\colon X \to Y$ given by $f(n, x) = f_n(x)$. This function exemplifies the conclusion of the proposition for B as interpretations respect countable unions and homeomorphisms.

Finally, suppose that $B = \bigcap_n B_n$ and the conclusion of the proposition is known for each B_n and exemplified by $f_n\colon X \to Y$. Let $X = (\mu^\omega)^\omega$, and let $X' = \{\langle x_n : n \in \omega\rangle \in X : \forall n\, \forall m\; f_n(x_n) = f_m(x_m)\}$. It is not difficult to see that $X' \subset X$ is a closed set. Use the first paragraph of the present proof to find a continuous function $g\colon \mu^\omega \to X'$ such that in all forcing extensions the image of the interpretation of g is X'. Let $f\colon \mu^\omega \to Y$ be the function $f_0 \circ h \circ g$ where $h\colon X' \to \mu^\omega$ is the projection function into the first coordinate. The function f exemplifies the conclusion of the proposition for B as interpretations respect all operations used in the construction of f. □

Theorem 10. (σ-ideal preservation) *Let κ be an uncountable cardinal and Y be a completely metrizable space of weight $< \mathrm{cof}(\kappa)$. Let $\mathcal{I}$ be a family of Borel subsets of Y such that no countable subcollection of $\mathcal{I}$ covers Y. Then $\mathbb{N}_\kappa$ forces that no countable subcollection of $\mathcal{I}$ covers Y.*

Proof. Fix $\mu < \mathrm{cof}(\kappa)$ and assume that $Y = \mu^\omega$. Let $T \in \mathbb{N}_\kappa$ and for each $n \in \omega$ let $\dot{S}_n$ a $\mathbb{N}_\kappa$-name for a Borel set such that T forces that $\dot{S}_n \in \mathcal{I}$. We must find $T' \leq T$ and a continuous function $g\colon [T'] \to Y$ such that $T' \Vdash \dot{g}(\dot{x}_{\mathrm{gen}}) \notin \bigcup_n \dot{S}_n$. For every $n \in \omega$ let $\dot{f}_n$ be the name of a continuous surjective function in the ground model, from μ^ω to Y, such that $T \Vdash \dot{f}_n\colon \mu^\omega \longrightarrow \mu^\omega \setminus \dot{S}_n$; such a function has to exist by Proposition 9. By the continuous reading of names, we may assume that there is a sequence $\langle F_n\rangle_{n\in\omega}$ with the following properties:

(1) Each F_n is a front of T.
(2) Every element of F_{n+1} properly extends an element of F_n.
(3) If $t \in F_n$ then there is a continuous function $f_n^t\colon \mu^\omega \to Y$ such that $T_t \Vdash \dot{f}_n = f_n^t$.

For simplicity we assume that T has empty stem. Consider the following game:

I	X_0		X_1		X_2		X_3	$\cdots$
II		β_0		β_1		β_2		$\cdots$

where $X_n \in [\kappa]^{<\kappa}$ and $\beta_n \in \kappa$. Furthermore, through the game, Player II is required to build sequences (one element at a time) $L_n = \{s_n^i \mid i \in \omega\} \subseteq \mu^{<\omega}$ (she is allowed to wait any number of finite steps before playing an s_n^i). Player II wins the game if the following condition holds, writing $x = \langle \beta_n\rangle_{n\in\omega}$:

(1) $\beta_n \notin X_n$ for every $n \in \omega$,
(2) $x \in [T]$,
(3) $|s_n^i| = i$ for every $n, i \in \omega$,
(4) $s_n^i \subseteq s_n^{i+1}$ for every $n, i \in \omega$, and
(5) the value of $f_n^t(\bigcup_i s_n^i) \in Y$, where t is the unique initial segment of x in F_n, does not depend on n.

The game has Borel payoff. We claim that Player I does not have a winning strategy. Assume Player I has a winning strategy, since $\mu < \mathrm{cof}(\kappa)$ it is easy to see that she has a winning strategy σ that ignores the L_n. Let M be a countable elementary submodel of a large enough structure such that $T, \{(\dot{S}_n, \dot{f}_n) \mid n \in \omega\}, \sigma \in M$. Since M is countable then there is some point $y \in Y \setminus \bigcup(\mathcal{I} \cap M)$. Let $x = \langle \beta_n\rangle_{n\in\omega} \in [T]$ be any sequence in the model M resulting from a play against the strategy σ respecting item (1) above. For every $n \in \omega$ let $t_n \in F_n$ be such that $t_n \subseteq x$. Since $t_n \in M$ then $f_n^{t_n} \in M$ so we conclude that $y \in \bigcap_{n\in\omega} \mathrm{rng}\,(Z^{t_n})$. For every $n \in \omega$, let $x_n \in \mu^\omega$ be such that $f_n^{t_n}(x_n) = y$. Then if Player II plays x and $L_n = \{x_n \restriction i \mid i \in \omega\}$ (which is possible since the strategy σ ignores the L_n) she will win the game, which is a contradiction.

By Borel Determinacy, we conclude that Player II has a winning strategy. We can then build a tree $T' \leq T$ and a continuous function $h\colon [T'] \to \mu^\omega$ which records the sequence $\bigcup_i s_i^0 \in \mu^\omega$ as Player II builds a branch in the tree T' and the auxiliary objects s_i^0 for $i \in \omega$. Let $g\colon [T'] \to Y$ be the continuous function given by $g(x) = f_0^t(h(x))$ where t is the unique initial segment of x in the front F_n. Clearly, $T' \Vdash \dot{g}(\dot{x}_{\text{gen}}) \notin \bigcup_n \dot{S}_n$ as desired. □

To conclude this section, we record an absoluteness result which will come handy in several places in the paper.

Theorem 11. (Absoluteness) *Let κ be an infinite cardinal, $M \subseteq V$ be a transitive model of (a large portion of) ZFC such that $\kappa \in M$ and every countable subset of κ is a subset of a set countable in M. Then the membership of Borel sets in $\mathcal{L}_\kappa$ is absolute between M and V.*

In particular, if M is a transitive model of a large portion of ZFC which computes ω_1 correctly, then the membership of Borel sets in $\mathcal{L}_{\omega_1}$ is absolute between M and V.

Proof. We first argue that for any Borel sets $B, C \subset \kappa^\omega$, coded in M, if $M \models B \subseteq C$ then in fact $B \subseteq C$ holds in V. Note that in ZFC, $B \subseteq C$ is equivalent to "for cofinally many $a \in [\kappa]^{\aleph_0}$, $B \cap a^\omega \subseteq C \cap a^\omega$". Now, for a given set $a \subset \kappa$ which is in M and countable in M, the inclusion $B \cap a^\omega \subseteq C \cap a^\omega$ is calculated correctly by M by the Mostowski absoluteness between M and V. Moreover, the quantification over the sets a is also calculated correctly by M by the covering assumption on M.

Suppose that $B \subset \kappa^\omega$ is a Borel set coded in M. Suppose first that $M \models B \in \mathcal{L}_\kappa$. By the definitions, there is a function $F\colon \kappa^{<\omega} \to [\kappa]^{<\kappa}$ in the model M such that $M \models B \subset C(F)$. But then $B \subset C(F)$ holds also in V by the first paragraph, and so $B \in \mathcal{L}_\kappa$ holds. Suppose now that $M \models B \notin \mathcal{L}_\kappa$. Then by Theorem 6 there is a Namba tree T such that $M \models [T] \subset B$. By the first paragraph again, $[T] \subset B$ holds in V as well, and so $B \notin \mathcal{L}_\kappa$ holds. □

3. The case of $\kappa > \omega_1$ regular

It turns out that the treatment of preservation properties of Namba forcing $\mathbb{N}_\kappa$ is easiest in the case of $\kappa > \omega_1$ regular.

Theorem 12. *Let $\kappa > \omega_1$ be a regular cardinal. Let Y be a completely metrizable space and let $\mathcal{I}$ be a σ-ideal of Borel subsets of Y containing all singletons. Suppose that one of the following holds:*

(1) $\text{cov}(\mathcal{I}) < \kappa$;
(2) $\text{non}(\mathcal{I}) > \kappa$.

Then $\mathbb{N}_\kappa$ preserves covering by $\mathcal{I}$.

Proof. The proof in the case of (1) is easier. Let $\mathcal{J} \subset \mathcal{I}$ be a set of size $< \kappa$ such that $\bigcup \mathcal{J} = Y$; we will show that the equality $\bigcup \mathcal{J} = Y$ persists to the Namba extension. Suppose that $T \in \mathbb{N}_\kappa$ is a condition and τ is a Namba name for an element of the (interpretation of the) space y. Using the continuous reading of names–Theorem 7, thin out the tree T if necessary to find a continuous function $f\colon [T] \to Y$ such that $T \Vdash \tau = \dot{f}(\dot{x}_{\text{gen}})$. For each set $B \in J$ consider the preimage $f^{-1}B$. It is impossible for all of these sets to belong to the ideal $\mathcal{L}_\kappa$, since their union is the $\mathcal{L}_\kappa$-positive set $[T]$ and the ideal $\mathcal{L}_\kappa$ is $< \kappa$-additive. Thus, there must be a set $B \in \mathcal{J}$ such that the set $f^{-1}B$ is $\mathcal{L}_\kappa$-positive. By the quotient presentation theorem, there is a Namba tree $S \subset T$ such that $[S] \subset f^{-1}B$. Then $S \Vdash \tau = \dot{f}(\dot{x}_{\mathrm{gen}}) \in B$ as desired.

The proof in the case of (2) makes use of the following key claim.

Claim 13. $\mathrm{non}(\mathcal{L}_\kappa) = \kappa$.

Proof. Let $S \subset \kappa$ be the set of all limit ordinals of cofinality ω. By a result of Shelah (see e.g. [1, Theorem 2.17]), there is a set $\{c_\alpha \colon \alpha \in S\}$ such that for every ordinal $\alpha \in S$, the set c_α is a cofinal subset of α of ordertype ω, and for every closed unbounded set $C \subset \kappa$ there is $\alpha \in S$ such that $c_\alpha \subset C$. Let $D \subset \kappa^\omega$ be the set of all increasing enumerations of the sets c_α for $\alpha \in S$. Clearly, $|D| = \kappa$, and it will be enough to show that $D \notin \mathcal{L}_\kappa$.

To this end, suppose $f \colon \kappa^{<\omega} \to \kappa$ be a function; we must produce a sequence $d \in D$ such that for every number $n \in \omega$, $d(n) > f(d \restriction n)$. Let $C \subset \kappa$ be the closed unbounded set of all ordinals closed under the function f. Let $\alpha \in S$ be an ordinal such that $c_\alpha \subset C$. It is immediate that $d =$the increasing enumeration of the set c_α works as required. $\square$

Now, suppose that $T \in \mathbb{N}_\kappa$ is a condition and τ is a name for an element of the (interpretation of the) space Y. Using the continuous reading of names, thinning out the tree T if necessary we may find a continuous function $f \colon [T] \to Y$ such that $T \Vdash \tau = \dot{f}(\dot{x}_{\mathrm{gen}})$. Since there is a $\mathcal{L}_\kappa$-preserving injection from κ^ω to $[T]$, the claim shows that there is a $\mathcal{L}_\kappa$-positive set $D \subset [T]$ of size κ. Since $f''D \subset Y$ is a set of size $\leq \kappa$, the initial assumptions show that there is a Borel set $B \in \mathcal{I}$ such that $f''D \subset B$. Then $f^{-1}B \subset [T]$ is a Borel $\mathcal{L}_\kappa$-positive set, and by Theorem 6, it contains all branches of some Namba tree $S \subset T$. Then $S \Vdash \tau = \dot{f}(\dot{x}_{\mathrm{gen}}) \in \dot{B}$ as required. $\square$

Let $\kappa > \omega_1$ be a regular cardinal. Applying Theorem 12 to certain standard ideals one can conclude the following (see [12] for the first and second item).

(1) $\mathbb{N}_\kappa$ does not collapse ω_1;
(2) If $\mathfrak{c} < \kappa$ then $\mathbb{N}_\kappa$ does not add new reals;
(3) If $\mathrm{cov}(\mathcal{M}) < \kappa$ or $\kappa < \mathrm{non}(\mathcal{M})$ then $\mathbb{N}_\kappa$ does not add Cohen reals;
(4) If $\mathrm{cov}(\mathcal{N}) < \kappa$ or $\kappa < \mathrm{non}(\mathcal{N})$ then $\mathbb{N}_\kappa$ does not add random reals;
(5) If $\mathfrak{d} < \kappa$ or $\kappa < \mathfrak{b}$ then $\mathbb{N}_\kappa$ does not add unbounded reals;
(6) $\mathbb{N}_\kappa$ adds a dominating real if and only if $\mathfrak{b} = \mathfrak{d} = \kappa$;
(7) If $\mathfrak{r} < \kappa$ or $\kappa < \mathfrak{s}$ then $\mathbb{N}_\kappa$ does not add splitting reals;
(8) If $\mathrm{non}(\mathcal{M}) < \kappa$ or $\kappa < \mathrm{cov}(\mathcal{M})$ then $\mathbb{N}_\kappa$ preserves category;
(9) If $\mathrm{add}(\mathcal{N}) < \kappa$ or $\kappa < \mathrm{cof}(\mathcal{N})$ then $\mathbb{N}_\kappa$ has the Sacks property.

The proofs of the above items use the standard characterizations of the cardinal invariants involved; we only point out the proof of (1) and (2). For (1), consider the space $Y = \omega_1^\omega$ and the σ-ideal $\mathcal{I}$ generated by the closed sets $B_\alpha = \{y \in Y \colon \mathrm{rng}(y) \subset \alpha\} \subset Y$ for $\alpha \in \omega_1$. Clearly $\mathrm{cov}(\mathcal{I}) = \omega_1 < \kappa$ and so (1) of Theorem 12 applies. For the second item, let $\mathcal{I}$ be the collection of countable sets of reals and apply (1) of Theorem 12 to prove the left-to-right direction. For the right-to-left direction, if $\mathfrak{c} \geq \kappa$ then fix an injection $h \colon \kappa \to 2^\omega$ and consider the function $f \colon \kappa^\omega \to (2^\omega)^\omega$ given by $f(x) = h \circ x$. It is immediate that f is a continuous injection, and so $\mathbb{N}_\kappa \Vdash \dot{f}(\dot{x}_{\mathrm{gen}}) \notin V$.

We now give a condition under which $\mathbb{N}_\kappa$ does not preserve the covering of an ideal:

Proposition 14. *Let κ be a cardinal, let Y be a completely metrizable space of weight $< \kappa$, and let I be a σ-ideal of Borel sets in Y. If $\mathrm{add}(I) = \mathrm{cof}(I) = \kappa$ then $\mathbb{N}_\kappa$ adds an element of Y which belongs to no elements of I coded in the ground model.*

Proof. The cardinal assumptions show that there is an inclusion-increasing sequence $\langle B_\alpha\colon \alpha\in\kappa\rangle$ of Borel sets in I such that every set in I is a subset of some set on the sequence. By Theorem 10, I still generates a proper σ-ideal in the $\mathbb{N}_\kappa$-extension on the (interpretation of the) space Y. In particular, if $x\in\kappa^\omega$ is the $\mathbb{N}_\kappa$-generic sequence, then in the model $V[x]$ there is a point $y\in Y$ which belongs to none of the sets $B_{x(n)}$ for any $n\in\omega$. Since $\mathrm{rng}(x)\subset\kappa$ is a cofinal set, this means that this point y belongs to no ground model coded elements of the σ-ideal I. □

4. The case $\kappa>\omega_1$ singular

It may appear that in the case of a singular cardinal κ of uncountable cofinality, the poset $\mathbb{N}_\kappa$ is very close to $\mathbb{N}_{\mathrm{cof}(\kappa)}$. Indeed, in the case of preservation properties considered in this paper, there is a close relationship:

Theorem 15. *Suppose that κ is an uncountable cardinal, Y is a completely metrizable space, and $\mathcal{I}$ is an ideal on Y with a Borel base. If $\mathbb{N}_\kappa$ preserves covering of $\mathcal{I}$ then $\mathbb{N}_{\mathrm{cof}(\kappa)}$ preserves covering of $\mathcal{I}$.*

Proof. Argue in contrapositive. Write $\mu=\mathrm{cof}(\kappa)$. Suppose that $\mathbb{N}_\mu$ does not preserve covering of $\mathcal{I}$. By the continuous reading of names there must be a continuous function $H\colon\mu^\omega\to Y$ such that H-preimage of any Borel set in $\mathcal{I}$ belongs to $\mathcal{L}_\mu$. Express $\kappa=\bigcup_{\alpha\in\mu}b_\alpha$ as a union of pairwise disjoint pieces of cardinality less than κ. Consider the function $G\colon\kappa^\omega\to\mu^\omega$ defined by $G(x)(n)=\alpha$ if $x(n)\in b_\alpha$ and the continuous function $H\circ G\colon\kappa^\omega\to Y$. It is not difficult to check that $H\circ G$-preimages of Borel sets in $\mathcal{I}$ belong to $\mathcal{L}_\kappa$. Therefore, the $\mathbb{N}_\kappa$-name given by the function $G\circ H$ shows that $\mathbb{N}_\kappa$ destroys covering by $\mathcal{I}$. □

Yet, the forcings $\mathbb{N}_\kappa$ and $\mathbb{N}_{\mathrm{cof}(\kappa)}$ may be quite different as the following theorem shows[1]:

Theorem 16. *The following statement is consistent with ZFC. There is a cardinal κ of uncountable cofinality such that $\mathbb{N}_\kappa$ does not add generic sequences for $\mathbb{N}_{\mathrm{cof}(\kappa)}$.*

Proof. We start with a model of GCH and let $\kappa=\omega_{\omega_1}$. Let $\mathbb{P}$ be a ccc forcing notion that forces $\kappa^+<\mathfrak{p}$ and let $G\subset\mathbb{P}$ be a generic filter. We claim that in the resulting model $V[G]$, the poset $\mathbb{N}_\kappa$ does not add generic sequences for $\mathbb{N}_{\omega_1}$.

To see this, first note that by the c.c.c. of the poset $\mathbb{P}$, $\mathrm{cof}[\kappa]^{<\kappa}=\kappa^+<\mathfrak{p}$. In this context, by a result of Miller [11], $\mathbb{N}_{\omega_1}$ and $\mathbb{N}_\kappa$ both have minimal real degree of constructibility, and also the generic extension for both of them is given by a real. In such a situation, to show that $\mathbb{N}_\kappa$ does not add a generic sequence for $\mathbb{N}_{\omega_1}$ is equivalent to showing that $\mathbb{N}_{\omega_1}$ does not add a generic sequence for $\mathbb{N}_\kappa$.

To prove the latter statement, it is enough to show that $\mathbb{N}_{\omega_1}\Vdash$ every countable set of ordinals is covered by a set of size $\aleph_1$ in the ground model. To prove this, suppose that $T\in\mathbb{N}_\kappa\Vdash\dot a$ is a countable set of ordinals. By the continuous reading of names, thinning out the tree if necessary, we may find a continuous function $f\colon[T]\to\mu^\omega$ for a suitable ordinal μ such that $T\Vdash\dot a=\mathrm{rng}(f(\dot x_{\mathrm{gen}}))$. Let $\{O_\beta\colon\beta\in\omega_1\}$ be an enumeration of a basis of the topology of $[T]$ and for each natural number $n\in\omega$ and each ordinal $\beta\in\omega_1$ let $g(\beta,n)=$the unique $\alpha\in\mu$ if it exists such that for all $x\in O_\beta$, $f(x)(n)=\alpha$. By the continuity of the function f, the set $\mathrm{rng}(f(x))$ is a subset of $\mathrm{rng}(g)$ for every point $x\in[T]$. Thus, $T\Vdash\dot a\subset\mathrm{rng}(g)$; at the same time $|\mathrm{rng}(g)|\le\aleph_1$ and the proof is complete. □

[1] Recall that a forcing notion $\mathbb{P}$ has *minimal real degree of constructibility* if for every generic filter $G\subseteq\mathbb{P}$ if $x\in V[G]\cap 2^\omega$ then either $x\in V$ or $G\in V[x]$.

5. The case $\kappa = \omega_1$

The case $\kappa = \omega_1$ is more challenging than the case of a regular cardinal $\kappa > \omega_1$. The main reason is that the equality $\mathrm{non}(\mathcal{L}_\kappa) = \kappa$ instrumental in the proof of Theorem 12 can fail at $\kappa = \omega_1$, and the failure is in fact implied by (a very small portion of) the Proper Forcing Axiom.

Theorem 17.
(1) $\mathrm{non}(\mathcal{L}_{\omega_1}) \leq \mathfrak{d}$.
(2) *The Proper Forcing Axiom implies* $\mathrm{non}(\mathcal{L}_{\omega_1}) > \omega_1$.

Proof. For (1), let *Part* denote the set of all interval partitions (partitions in finite sets) of ω. We may define an order in *Part* as follows, given $\mathcal{P}, \mathcal{Q} \in Part$ we say $\mathcal{P} \leq \mathcal{Q}$ if for all $Q \in \mathcal{Q}$ there is $P \in \mathcal{P}$ such that $P \subseteq Q$. In [3] it is proved that the smallest size of a dominating family of interval partitions is precisely $\mathfrak{d}$.

Let $\mathcal{P} = \{\mathcal{P}_\gamma \mid \gamma \in \mathfrak{d}\}$ be a dominating family of interval partitions where $\mathcal{P}_\gamma = \{[P_\gamma(n), P_\gamma(n+1)) \mid n \in \omega\}$. For every limit ordinal $\alpha < \omega_1$, choose $C_\alpha = \langle \alpha_n \rangle_{n \in \omega}$ an increasing sequence cofinal in α. For every $\alpha < \omega_1$ and $\gamma < \mathfrak{d}$ we define $g^\gamma_\alpha \colon \omega \longrightarrow \omega_1$ given by $g^\gamma_\alpha(n) = \alpha_{P_\gamma(n+1)}$. We claim that the set $X = \{g^\gamma_\alpha \mid \alpha$ is a countable limit ordinal and $\gamma \in \mathfrak{d}\}$ does not belong to $\mathcal{L}_{\omega_1}$.

Let $F \colon \omega_1^{<\omega} \longrightarrow \omega_1$ and as before, let $D \subseteq \omega_1$ be a club such that if $\alpha \in D$ and $s \in \alpha^{<\omega}$ then $F(s) < \alpha$. Choose any $\alpha \in D$ which is also a limit point of D. Now we define an interval partition $\mathcal{Q} = \{[Q(n), Q(n+1)) \mid n \in \omega\}$ such that $[\alpha_{Q(n)}, \alpha_{Q(n+1)}) \cap D \neq \emptyset$ for every $n \in \omega$. Since $\mathcal{P}$ is a dominating family of interval partitions, then there is $\gamma < \mathfrak{d}$ such that $\mathcal{Q} \leq \mathcal{P}_\gamma$. It is then easy to see that $g^\gamma_\alpha \notin C(F)$.

For (2), let P denote Baumgartner's forcing for adding a club with finite conditions. A condition $p \in P$ is a pair $\langle a_p, b_p \rangle$ where $a \subset \omega_1$ is a finite set and b_p is a finite set of closed intervals in ω_1 disjoint from the set a_p. The ordering is that of coordinatewise reverse inclusion. It is well known that P is a proper forcing and the union of the first coordinates of conditions in the generic filter is a closed unbounded subset of ω_1 consisting of indecomposable ordinals only. Let $\dot{F} \colon \omega_1^{<\omega} \to \omega_1$ be a P-name for the function which assigns to each sequence t the first element of this generic club larger than all ordinals listed by t. By a standard genericity argument, it will be enough to show that $P \Vdash \check{x} \in C(\dot{F})$ for every sequence $x \in \omega_1^\omega$.

To this end, let $p \in P$ and $n \in \omega$ be given; we must find $q \leq p$ and $m > n$ such that $q \Vdash \check{x}(m) \in \dot{F}(\check{x} \restriction m)$. If there is $m > n$ such that $x(m) \leq x(m-1)$ then $p \Vdash \check{x}(m) \in \dot{F}(\check{x} \restriction m)$ as required. Otherwise, the sequence x is increasing beyond n and so there must be a number $m > n$ such that the interval $[x(m), x(m+1)]$ contains no elements of a_p. Then, $q = \langle a_p, b_p \cup \{[x(m), x(m+1)]\} \rangle$ is a condition in P stronger than p and $q \Vdash \check{x}(m) \in \dot{F}(\check{x} \restriction m)$ as required. □

The upshot is that we cannot answer the central preservation question for $\mathbb{N}_{\omega_1}$ in general:

Problem 18. *If $\mathcal{I}$ is an ideal generated by Borel sets in ω^ω and $\omega_1 <$ $\mathsf{non}(\mathcal{I})$, is it true that $\mathbb{N}_{\omega_1}$ preserves covering of $\mathcal{I}$?*

Nevertheless, for certain specific ideals the question does have a positive answer. This section contains the partial results of this kind that we were able to prove.

Theorem 19. *If $\omega_1 <$ $\mathsf{cov}(\mathcal{M})$ then $\mathbb{N}_{\omega_1}$ does not destroy category.*

Proof. We need to prove that for every continuous function $H \colon \omega_1^\omega \longrightarrow \omega^\omega$ there is $h \in \omega^\omega$ such that the preimage of the set $\{f \in \omega^\omega \mid |f \cap h| = \omega\}$ is not in $\mathcal{L}_{\omega_1}$.

Let M be an elementary submodel of a large structure such that $H \in M$, $\omega_1 \subseteq M$ and $|M| = \omega_1$. Since $\omega_1 <$ $\mathsf{cov}(\mathcal{M})$, there is $c : \omega \longrightarrow \omega$ which is Cohen over M. Let $B = \{f \in \omega^\omega \mid |f \cap c| = \omega\}$, clearly B

is a Borel set and $B \in M[c]$. Let $A = H^{-1}(B)$ we now claim $M[c] \models A \notin \mathcal{L}_{\omega_1}$. We argue in $M[c]$, let $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in M[c]$. Since $M[c]$ is a ccc extension of M there is $g \in (\omega_1^\omega \cap M) \setminus C(F)$. In this way, $H(g) \in M$ and since c is Cohen over M, then $H(g) \cap c \neq \emptyset$ so $g \in A \setminus C(F)$. So, $M[c] \models A \notin \mathcal{L}_{\omega_1}$, hence $A \notin \mathcal{L}_{\omega_1}$ by Theorem 11. □

Theorem 20. *If $\omega_1 < \mathfrak{b}$ then $\mathbb{N}_{\omega_1}$ does not add unbounded reals.*

Proof. We need to prove that for every continuous function $H : \omega_1^\omega \to \omega^\omega$ there is a function $f \in \omega^\omega$ such that the set $H^{-1}\{h \in \omega^\omega : h \leq^* f\}$ does not belong to $\mathcal{L}_{\omega_1}$ Let M be an elementary submodel of a large structure containing the function H, containing ω_1 as a subset, and of size $\aleph_1$. By the cardinal invariant assumption, there is an increasing function f which modulo finite dominates every function in M. We claim that the function f works.

To that end, we consider the *Hechler forcing* $\mathbb{H}$: elements of $\mathbb{H}$ are pairs of the form (s, g) where $s \in \omega^{<\omega}$ and $g \in \omega^\omega$, and the order is given by $(s, g) \leq (z, h)$ if $z \subseteq s$, $h \leq g$ and if $i \in \mathrm{dom}(s) \setminus \mathrm{dom}(z)$ then $s(i) \geq h(i)$.

Claim 21. *There is a c.c.c. partial order $\mathbb{Q} \in V$ that adds a $g : \omega \longrightarrow \omega$ such that $g \leq f$ and g is Hechler over M.*

Proof. Write $h <_n g$ to mean that $h(m) < g(m)$ for every $m \geq n$. Let $\mathbb{Q}$ be the suborder of $\mathbb{H}$ consisting of all pairs $(s, h) \in \mathbb{H} \cap M$ such that $s \leq f$ and $h \leq_{|s|} f$. Clearly $\mathbb{Q}$ adds a function $g : \omega \longrightarrow \omega$ and $g \leq f$. We will show that g is Hechler over M (note that $\mathbb{Q}$ is not in M). It is enough to show that if $D \in M$ and $D \subseteq \mathbb{H}$ is open dense then $D \cap \mathbb{Q}$ is dense for $\mathbb{Q}$.

Pick any $(s, h) \in \mathbb{Q}$ with $|s| = n_0$, for every $i > n_0$ let $s^i = s^\frown h \restriction [n_0, i]$. Note that $\left(s^i, h\right) \in \mathbb{Q}$ and it extends (s, h). Inside M, we recursively construct two sequences $\{(s_i, h_i) \mid i \in \omega\} \subseteq \mathbb{H}$ and $\{n_i \mid i \in \omega\} \subseteq \omega$ so that $(s_0, h_0) = (s, h)$, and for every $i \in \omega$, $(s_{i+1}, h_{i+1}) \in D$, $|s_i| = n_i$, $(s_{i+1}, h_{i+1}) \leq (s^{n_i}, h)$, and $h_i \leq_{n_{i+1}} h_{i+1}$.

We then define $l = s^\frown (s_1 + h_1) \restriction [n_0, n_1)^\frown (s_2 + h_1 + h_2) \restriction [n_2, n_3)^\frown \ldots$ and note that $l \in M$, therefore, there is $i \in \omega$ such that $l <_{n_i} f$. This entails that $(s_{i+1}, h_i) \in \mathbb{Q}$. □

Let $g \in \omega^\omega$ be a function generic for the poset $\mathbb{Q}$. Let $B = \{h \in \omega^\omega \mid h \leq^* g\}$, and let $A = H^{-1}(B)$ which is a Borel set in $M[g]$. We will prove that $M[g] \models A \notin \mathcal{L}_{\omega_1}$. Let $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in M[g]$ and since $M[g]$ is a ccc extension of M then $(\omega_1^\omega \cap M) \setminus C(F) \neq \emptyset$. Let $x \in (\omega_1^\omega \cap M) \setminus C(F)$ then $H(x) \in M$ and since g is Hechler over M we conclude that $H(x) \leq^* g$ so $x \in A \setminus C(F)$.

Thus, $M[g] \models A \notin \mathcal{L}_{\omega_1}$. By Theorem 11, we conclude that $V[g] \models A \notin \mathcal{L}_{\omega_1}$. Since $g \leq f$, it must be the case that $V[g] \models H^{-1}\{h \in \omega^\omega : h \leq^* f\} \notin \mathcal{L}_{\omega_1}$. By Theorem 11 again, $V \models H^{-1}\{h \in \omega^\omega : h \leq^* f\} \notin \mathcal{L}_{\omega_1}$ as desired. □

As a consequence, we can answer a question of [7]:

Corollary 22. *$\mathfrak{b}$ is the first uncountable regular cardinal κ such that $\mathbb{N}_\kappa$ adds an unbounded real.*

Using a similar method, we can prove the following:

Theorem 23. *If $\omega_1 < \mathfrak{d}$ then $\mathbb{N}_{\omega_1}$ does not add dominating reals. Thus, $\mathbb{N}_{\omega_1}$ adds a dominating real if and only if $\mathfrak{d} = \omega_1$.*

Proof. We need to prove that for every continuous function $H : \omega_1^\omega \longrightarrow \omega^\omega$ there is $f \in \omega^\omega$ such that the preimage of $\{h \in \omega^\omega \mid f \not\leq^* h\}$ is not in $\mathcal{L}_{\omega_1}$. Let M be an elementary submodel of size ω_1 such that $\omega_1 \subseteq M$ and $H \in M$. Since $\omega_1 < \mathfrak{d}$ there is a function $f \in \omega^\omega$ that is unbounded over M.

Claim 24. *There is a c.c.c. partial order $\mathbb{Q} \in V$ that adds a function $g: \omega \longrightarrow \omega$ such that $g \leq f$ and g is Cohen over M.*

Proof. Let $\mathbb{Q}$ be the suborder of $\omega^{<\omega}$ given by $\mathbb{Q} = \{s \in \omega^{<\omega} | s \leq f\}$, clearly $\mathbb{Q}$ adds a function $g : \omega \longrightarrow \omega$ and $g \leq f$. We will show that g is Cohen over M and it is enough to show that if $D \in M$ and $D \subseteq \omega^{<\omega}$ is open dense then $D \cap \mathbb{Q}$ is dense for $\mathbb{Q}$.

Let $s \in \mathbb{Q}$ with $|s| = n_0$ and for every $i > n_0$ define $s^i = s^\frown \overline{0} \upharpoonright [n_0, i]$ where $\overline{0}$ is the constant 0 function. Note that $s^i \in \mathbb{Q}$ and it extends s. Inside M, we recursively construct two sequences $\{s_i \mid i \in \omega\} \subseteq \mathbb{C}$ and $\{n_i \mid i \in \omega\} \subseteq \omega$ so that $s_0 = s$, and for every $i \in \omega$, $s_{i+1} \in D$, $|s_i| = n_i$, $n_i < n_{i+1}$, and $s_{i+1} \leq s^{n_i}$.

We now define $l : \omega \longrightarrow \omega$ where $l\,(i)$ is the largest value in the range of s_{i+1}, and note that $l \in M$, therefore, there is $i \in \omega$ such that $l\,(i) < c\,(i)$, and since $i \leq n_i$ there is a condition $s_i \in D$ extending s. □

Let $g \in \omega^\omega$ be a function generic over the poset $\mathbb{Q}$. Let $B = \{h \in \omega^\omega \mid g \not\leq^* h\}$ and $A = H^{-1}(B)$ which is a Borel set in $M\,[g]$. We will prove that $M[g] \models A \notin \mathcal{L}$. Let $F : \omega_1^{<\omega} \longrightarrow \omega_1 \in M[g]$. We know that $M[g]$ is a ccc extension of M, so $(\omega_1^\omega \cap M) \setminus C(F) \neq \emptyset$. Let $h \in (\omega_1^\omega \cap M) \setminus C\,(F)$. Clearly $H(h) \in M$, and since g is Cohen over M then $g \not\leq^* H(h)$ so $h \in A \setminus C(F)$. The last part of the argument is similar to the previous theorem. □

Theorem 25. *If $\omega_1 < \mathsf{add}(\mathcal{N})$ then $\mathbb{N}_{\omega_1}$ has the Sacks property.*

Proof. We need to prove that for every continuous function $F : \omega_1^\omega \longrightarrow \omega^\omega$ there is a *slalom* S, i.e. $S : \omega \to [\omega]^{<\omega}$ such that $\sum n \in \omega \frac{S(n)}{2^n} < \infty$, such that the preimage of $\{f \in \omega^\omega \mid f \sqsubseteq^* S\}$ is not in $\mathcal{L}_{\omega_1}$. Here, $f \sqsubseteq^* S$ means that $\forall^\infty n \in \omega\ f(n) \in S(n)$.

To that end we consider the *n-Amoeba forcing* $\mathbb{A}_n$ defined as the set of all open subsets of 2^ω with Lebesgue measure less than $\frac{1}{n}$. If $U_1, U_2 \in \mathbb{A}$ then $U_1 \leq U_2$ if $U_1 \subseteq U_2$. It can be proved that $\mathbb{A}_n$ and $\mathbb{A}_m$ are forcing equivalent for every $n, m \in \omega$–see [2, Lemma 3.1.11]. In this way, forcing with $\mathbb{A}_2$ adds a null set containing every ground model null set. It is well known that $\mathbb{A}_2$ is ccc and Judah and Repický proved that the Martin number of $\mathbb{A}_2$ is $\mathsf{add}(\mathcal{N})$–see [2, Theorem 3.4.17].

Let M be an elementary submodel of size ω_1 such that $F \in M$ and $\omega_1 \subseteq M$. Since $\omega_1 < \mathsf{add}(\mathcal{N})$ then there is a filter $G \subseteq \mathbb{A}_2$ that is $(M, \mathbb{A}_2)$-generic. In this way, in $M\,[G]$ there is a null set containing every null set from M so then there is a slalom S such that $f \sqsubseteq^* S$ for every $f \in M$. Let $B = \{f \in \omega^\omega \mid f \sqsubseteq^* S\}$ and $A = F^{-1}\,(B)$ which is a Borel set. We claim that $M\,[G] \models A \notin \mathcal{L}_{\omega_1}$, let $H : \omega_1 \longrightarrow \omega_1 \in M$ and since $M\,[G]$ is a ccc extension of M, then there is $x \in M \cap (\omega_1^\omega \setminus C_H)$. But then $F\,(x) \in M$ so $F\,(x) \sqsubseteq^* S$ hence $x \in A$ which implies that A is not contained in C_H so $M\,[G] \models A \notin \mathcal{L}_{\omega_1}$ and then $A \notin \mathcal{L}_{\omega_1}$ by Theorem 11. □

We remark that neither $\mathbb{N}_{\mathrm{add}(\mathcal{N})}$ nor $\mathbb{N}_{\mathrm{cof}(\mathcal{N})}$ have the Sacks property. This will be proved in [6] (which answers another question of [7]).

6. Bukovský forcing

In this section we will briefly consider a forcing very similar to $\mathbb{N}_\kappa$ introduced in [4].

Definition 26. Let κ be a regular cardinal.

(1) A tree $T \subseteq \kappa^{<\omega}$ is a *κ-Bukovský tree* if the following conditions hold:
  (a) if $s \in T$ then either $|\mathrm{suc}_T\,(s)| = 1$ or $|\mathrm{suc}_T\,(s)| = \kappa$;
  (b) for every $s \in T$ there is $t \in T$ extending s such that $|\mathrm{suc}_T\,(t)| = \kappa$.

(2) $\mathbb{M}_\kappa$ denotes the set of all κ-Bukovský trees ordered by inclusion.

In this way, $\mathbb{M}_\omega$ is the usual Miller forcing. For every $F\colon \kappa^{<\omega} \longrightarrow \kappa$ define $D_F = \{x \in \kappa^\omega \mid \forall^\infty n \in \omega\,(x(n) < F(x \restriction n))\}$ and let $\mathcal{K}_\kappa$ be the ideal generated by $\{D_F \mid F\colon \kappa^{<\omega} \longrightarrow \kappa\}$. It is easy to see that if $\kappa > \omega$ is a regular cardinal then $\mathcal{K}(\kappa)$ is a σ-ideal. For a set $B \subseteq \kappa^\omega$ consider the following game $\mathcal{G}(B)$:

I	s_0		s_1		s_2		s_3	$\cdots$	$\bigcup s_n \in B$
II		α_0		α_1		α_2		$\cdots$	

such that $s_n \in \kappa^{<\omega}$, $s_n \subseteq s_{n+1}$, $\alpha_n \in \kappa$ and $\alpha_n < s_{n+1}(|s_n|)$ for every $n \in \omega$. Player I wins the game if $\bigcup s_n \in B$. The following proposition is easy and left to the reader:

Proposition 27. *Let $\kappa > \omega$ be a regular cardinal and $B \subseteq \kappa^\omega$.*

(1) *Player I has a winning strategy in $\mathcal{G}(B)$ if and only if there is $T \in \mathbb{M}_\kappa$ such that $[T] \subseteq B$.*
(2) *Player II has a winning strategy in $\mathcal{G}(B)$ if and only if $B \in \mathcal{K}_\kappa$.*
(3) *Every Borel set of κ^ω either contains the branches of a κ-Bukovský tree or belongs to $\mathcal{K}_\kappa$.*
(4) *$\mathbb{M}_\kappa$ is forcing equivalent to Borel(κ^ω) modulo $\mathcal{K}_\kappa$.*

We have the following result, the proof of which is left to the reader:

Proposition 28. *Let κ, μ be cardinals such that $\kappa > \omega$ is a regular cardinal. Let $\mathcal{I} \subseteq \mu^\omega$ be an ideal with a Borel base. If $\mathbb{N}_\kappa$ preserves covering of $\mathcal{I}$ then $\mathbb{M}_\kappa$ preserves covering of $\mathcal{I}$.*

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