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On Pospíšil ideals ☆

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Dedicated to the memory of William Wistar Comfort

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We study a class of ideals introduced by Pospíšil. We answer a question of the second author by proving that there is an $F_{\sigma\delta\sigma}$ ideal $\mathcal{I}$ such that every filter of character less than $\mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. We also prove that this statement is consistently false for $F_{\sigma\delta}$-ideals.

1. Introduction

Ultrafilters and independent families occupy one of the central places in Wis Comfort's research (see e.g. [4–8]). We revisit an old construction of B. Pospíšil involving both concepts. In his 1939 paper [17], Pospíšil proved that there is an ultrafilter on ω of character $\mathfrak{c}$. He did it by defining a certain filter from an independent family of size $\mathfrak{c}$, and then proved that any ultrafilter extending the filter has character $\mathfrak{c}$ (see [4, 2.6 and 2.7]). It is this filter of his, or the dual ideal, which is the main object of study here.

Recall that a family $\mathcal{X} = \{X_\alpha \mid \alpha \in \kappa\} \subseteq [\omega]^\omega$ is *independent* if for any two disjoint $F, G \in [\kappa]^{<\omega}$ the set

$$\Big(\bigcap_{\alpha\in F} X_\alpha\Big) \cap \bigcap_{\beta\in G} (\omega \setminus X_\beta)$$

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is infinite. The fact that there are independent families of size $\mathfrak{c}$ was probably first proved by G. Fichtenholz and L. Kantorovitch [9]. In retrospect, their construction provides an independent family which is *perfect* as a subspace of the Cantor set.

Given a perfect independent family P define the *Pospíšil ideal of* P (denoted by $\mathsf{Pos}(P)$) as the ideal generated by the finite sets and

$$\{\omega \setminus x \mid x \in P\} \cup \{\bigcap C \mid C \in [P]^{\omega}\}.$$

Given an ideal $\mathcal{I}$ on a set X, J. Baumgartner [1] introduced the notion of an $\mathcal{I}$-ultrafilter as follows: An ultrafilter $\mathcal{U}$ on ω is an $\mathcal{I}$-*ultrafilter* if for every function $f : \omega \to X$ there is a $U \in \mathcal{U}$ such that $f[U] \in \mathcal{I}$. The notion of an $\mathcal{I}$-ultrafilter is closely tied with the *Katětov order*[1] as an ultrafilter $\mathcal{U}$ is an $\mathcal{I}$-ultrafilter if and only if $\mathcal{I} \nleq_{\mathsf{K}} \mathcal{U}^*$. Most of the standard properties of ultrafilters can then be characterized in this way, using Borel ideals[2] of low complexity:

(1) $\mathcal{U}$ is a Ramsey ultrafilter if and only if $\mathcal{U}$ is an $\mathcal{ED}$-ultrafilter.
(2) $\mathcal{U}$ is a P-point if and only if $\mathcal{U}$ is a $\mathsf{Fin}\times\mathsf{Fin}$-ultrafilter if and only if $\mathcal{U}$ is a conv-ultrafilter.
(3) $\mathcal{U}$ is a Q-point if and only if $\mathcal{ED}_{\mathsf{Fin}} \nleq_{\mathsf{KB}} \mathcal{U}^*$.
(4) $\mathcal{U}$ is a nowhere dense ultrafilter if and only if $\mathcal{U}$ is a nwd-ultrafilter.
(5) $\mathcal{U}$ is rapid if and only if $\mathcal{J} \nleq_{\mathsf{KB}} \mathcal{U}^*$ for any analytic P-ideal $\mathcal{J}$.

The reader may consult [1,2,11,12] for more information. In other words, the Katětov order naturally stratifies ultrafilters by "upward cones" of Borel ideals. Ultrafilters satisfying any of the above properties cannot be constructed in ZFC alone, so one has to wonder whether this stratification may consistently be vacuous. On the other hand, extending Pospíšil's argumentation slightly, one can show that $\mathsf{Pos}(P)$-ultrafilters do exist in ZFC, in fact, they *exist generically*, i.e. any filter of character $< \mathfrak{c}$ can be extended to a $\mathsf{Pos}(P)$-ultrafilter. However, $\mathsf{Pos}(P)$ is analytic, and appears not to be Borel. This led the second author to ask:

Problem 1 *([12]).* Is there a Borel ideal $\mathcal{I}$ such that $\mathcal{I}$-ultrafilters exist in ZFC?

We shall answer this question in the positive by defining a Borel (in fact $F_{\sigma\delta\sigma}$) version of the Pospíšil ideal. Moreover, we will define an $F_{\sigma\delta\sigma}$ ideal $\mathcal{J}$ for which $\mathcal{J}$-ultrafilters exist generically. Then we shall show that the complexity cannot be lowered, i.e. consistently $\mathcal{I}$-ultrafilters do not exist generically for any $F_{\sigma\delta}$ ideal $\mathcal{I}$.

We conclude this introduction by fixing some notation. Given $A, B \subseteq \omega$ we will say that A *is an almost subset of* B (or B *almost covers* A) if $A \setminus B$ is finite, and this will be denoted by $A \subseteq^* B$. If $A \subseteq \omega$ we denote by A^* the complement of A and if $\mathcal{X} \subseteq \wp(\omega)$[3] we define $\mathcal{X}^* = \{A^* \mid A \in \mathcal{X}\}$. If $\mathcal{I}$ is an ideal, $\mathcal{I}^+$ denotes the family of all subsets of ω that are not in $\mathcal{I}$. Given $A \in \mathcal{I}^+$, the *restriction of* $\mathcal{I}$ *to* A is defined as $\mathcal{I} \cap \wp(A)$. We say a family $\mathcal{B} \subseteq \mathcal{I}$ is *cofinal in* $\mathcal{I}$ if for every $A \in \mathcal{I}$ there is $B \in \mathcal{B}$ such that $A \subseteq B$. By

[1] Recall that given two ideals $\mathcal{I}$ and $\mathcal{J}$ on sets X and Y respectively we say that $\mathcal{I}$ *is Katětov below* $\mathcal{J}$ and denote by $\mathcal{I} \leq_{\mathsf{K}} \mathcal{J}$, if there is a function $f : Y \to X$ such that $f^{-1}[I] \in \mathcal{J}$ for every $I \in \mathcal{I}$ (see [14]) We say that $\mathcal{I}$ *is Katětov-Blass below* $\mathcal{J}$ and denote by $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{J}$ if, moreover, the witnessing function is finite-to-one.

[2] For every $n \in \omega$ we define $C_n = \{(n, m) \mid m \in \omega\}$ and if $f : \omega \longrightarrow \omega$ let $D(f) = \{(n, m) \mid m \leq f(n)\}$
Recall that:

(1) Fin is the ideal of all finite subsets of ω.
(2) $\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\}$ and (the graphs of) functions from ω to ω.
(3) $\mathcal{ED}_{\mathsf{Fin}}$ is the restriction of $\mathcal{ED}$ to $\Delta = \{(n, m) \mid m \leq n\}$.
(4) $\mathsf{Fin}\times\mathsf{Fin}$ is the ideal on $\omega \times \omega$ generated by $\{C_n \mid n \in \omega\} \cup \{D(f) \mid f \in \omega^{\omega}\}$.
(5) conv is the ideal on $[0, 1] \cap \mathbb{Q}$ generated by all sequences converging to a real number.
(6) nwd is the ideal on $\mathbb{Q}$ generated by all nowhere dense sets.

[3] If X is a set, we denote its power set by $\wp(X)$.

$\mathsf{cof}(\mathcal{I})$ we denote the smallest size of a cofinal family of $\mathcal{I}$. In this note, a *tree* T will be a subset of $\omega^{<\omega}$ closed under taking initial segments. If T is a tree and $n \in \omega$, we denote $T_n = T \cap \omega^n$. The *set of branches of* T is defined as $[T] = \{f \mid \forall n\,(f \restriction n \in T)\}$.

2. Pospíšil ideals

We will say that a perfect tree $T \subseteq 2^{<\omega}$ is *independent* if the set of its *branches* $[T]$ is independent.[4] Abusing notation a bit, given an independent tree $T \subseteq 2^{<\omega}$ we shall denote the Pospíšil ideal $\mathsf{Pos}([T])$ simply by $\mathsf{Pos}(T)$, i.e. the ideal generated by the finite sets and $\{x^* \mid x \in [T]\} \cup \{\bigcap C \mid C \in [[T]]^{\omega}\}$.

We now present the following mild extension of Pospíšil's argument crucial for our considerations.

Lemma 2. *If $T \subseteq 2^{<\omega}$ is an independent tree then $\mathsf{Pos}(T)$ is a proper ideal and $\mathsf{cof}\,(\mathcal{J}) = \mathfrak{c}$ for every ideal $\mathcal{J}$ extending $\mathsf{Pos}(T)$.*

Proof. First we will show that $\mathsf{Pos}(T)$ is a proper ideal, i.e. that $\omega \notin \mathsf{Pos}(T)$. Let $x_0, ..., x_n \in [T]$ and $C_0, ..., C_m$ be countable subsets of $[T]$. For each $i \leq m$ we choose $y_i \in C_i$ such that $y_i \notin \{x_0, ..., x_n\}$. Clearly $(\bigcap C_0) \cup ... \cup (\bigcap C_m)$ is a subset of $y_0 \cup ... \cup y_m$. Since T is independent, $x_0^* \cup ... \cup x_n^*$ does not almost cover $\omega \setminus (y_0 \cup ... \cup y_m)$, hence $\omega \notin \mathsf{Pos}(T)$.

Now, aiming toward a contradiction assume that there is an ideal $\mathcal{J}$ such that $\mathsf{Pos}(T) \subseteq \mathcal{J}$ and $\mathsf{cof}(\mathcal{J}) < \mathfrak{c}$. Let $\mathcal{B} \subseteq \mathcal{J}$ be a cofinal family of size less than $\mathfrak{c}$. Since $\{x^* \mid x \in [T]\} \subseteq \mathcal{J}$, there is $C \in [[T]]^{\omega}$ and $B \in \mathcal{B}$ such that $x^* \subseteq B$ for every $x \in C$. On one hand $\bigcup\limits_{x \in C} x^* \subseteq B$ and on the other hand

$$\bigcap C = \left(\bigcup_{x \in C} x^*\right)^*$$

belongs to $\mathcal{J}$ so $\omega \in \mathcal{J}$ which is a contradiction. □

Recall that an ideal $\mathcal{I}$ is P^- (see [13]) if for every $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}^*$ there is $Y \in \mathcal{I}^+$ such that $Y \subseteq^* X_n$ for every $n \in \omega$. We shall need the following proposition in the next section.

Proposition 3. *If T is an independent tree then $\mathsf{Pos}(T)$ is not* P^-.

Proof. Let $D \subseteq [T]$ be a countable dense set. Clearly $D \subseteq \mathcal{I}^*$, (where $\mathcal{I} = \mathsf{Pos}(T)$) and we will show that every pseudo-intersection of D is in $\mathsf{Pos}(T)$. Letting A be a pseudo-intersection of D, we must show that $A \in \mathsf{Pos}(T)$. Let $f : D \longrightarrow \omega$ such that $A \setminus d \subseteq f(d)$ for every $d \in D$. We may assume that there are two $x, y \in D$ such that $A \subseteq x \cap y$. Choose two subsets $\{x_n \mid n \in \omega\}$, $\{y_n \mid n \in \omega\}$ of $D \setminus \{x, y\}$ such that $x \restriction n = x_n \restriction n$ and $y \restriction n = y_n \restriction n$ for every $n \in \omega$. We recursively define two increasing sequences of natural numbers $\langle n_i \rangle_{i \in \omega}$ and $\langle m_i \rangle_{i \in \omega}$ as follows:

(1) $n_0 = 0$.
(2) $m_i > \max\{f(x_{n_i}), n_i\}$.
(3) $n_{i+1} > \max\{f(y_{m_i}), m_i\}$.

This is very easy to do. Let $X = \{x_{n_i} \mid i \in \omega\} \cup \{x\}$ and $Y = \{y_{m_i} \mid i \in \omega\} \cup \{y\}$. It is easy to see that $A \subseteq \bigcap X \cup \bigcap Y$ so $A \in \mathsf{Pos}(T)$. □

[4] We are identifying a set with its characteristic function.

Next we shall show that not all Pospíšil ideals are isomorphic. In fact, they can have rather dramatically different combinatorial properties. Recall that an ideal $\mathcal{I}$ is *ω-hitting* if for every $\{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ there is $B \in \mathcal{I}$ such that $|B \cap X_n| = \omega$ for every $n \in \omega$. We will see that there are examples of Pospíšil ideals that are ω-hitting, but there are also some that are not.[5] In order to prove this, we need the following definitions: Let $T \subseteq 2^{<\omega}$ be an independent tree.

(1) We say that T is a *hitting tree* if whenever $s \in T$, for almost all $n \in \omega$ there is $t \in T_{n+1}$ extending s such that $t(n) = 1$.
(2) We say that T has the *generic property* if for every $n \in \omega$, $X \subseteq T_n$ and $c : X \longrightarrow 2$ there are infinitely many $m > n$ such that for every $s \in T_{m+1}$ if $s \restriction n \in X$ then $s(m) = c(s \restriction n)$.

Note that the above properties are mutually exclusive.

Proposition 4. *Let $T \subseteq 2^{<\omega}$ be an independent tree.*

(1) *If T has the generic property then $\mathsf{Pos}(T)$ is not ω-hitting.*
(2) *If T is a hitting tree then $\mathsf{Pos}(T^*)$ is ω-hitting.*

Proof. Let T be an independent tree with the generic property. Let $\mathcal{W}$ be the family of all pairs $p = (X, Y)$ such that there is an $n \in \omega$ such that X and Y are two disjoint non empty subsets of T_n. Given $p = (X, Y) \in \mathcal{W}$ such that $X, Y \subset T_n$, let $W_p \in [\omega]^\omega$ such that for every $m \in W_p$ and for every $s \in T_{m+1}$ if $s \restriction n \in X$ then $s(m) = 1$ and if $s \restriction n \in Y$ then $s(m) = 0$. Since $\mathcal{W}$ is a countable family, $\{W_p \mid p \in \mathcal{W}\}$ is a countable family of infinite sets.

We claim that no element of $\mathsf{Pos}(T)$ has infinite intersection with each W_p. Letting $A \in \mathsf{Pos}(T)$, we may assume there are $x_0, ..., x_n \in [T]$ and $C_0, ..., C_m$ countable subsets of T such that $A = \bigcup\limits_{i \leq n} x_i^* \cup \bigcup\limits_{j \leq m} (\bigcap C_j)$. For every $j \leq m$ we choose $y_j \in C_j$ such that $y_j \notin \{x_0, ..., x_n\}$ and $y_j \neq y_k$ if $j \neq k$. We may then find $l \in \omega$ such any two different elements of $\{x_i \mid i \leq n\} \cup \{y_j \mid j \leq m\}$ differ before l. We now define $p = (X, Y)$ where $X = \{x_i \restriction l \mid i \leq n\}$ and $Y = \{y_j \restriction l \mid j \leq m\}$. Note that if $k \in W_p$ then $k \in x_0 \cap ... \cap x_n$ and $k \notin y_0 \cup ... \cup y_m$ so $k \notin A$.

Finally, it is easy to see that if T is a hitting tree then $[T] \subseteq \mathsf{Pos}(T^*)$ is already ω-hitting. □

It should be noted here that both kinds of trees actually exist:

Proposition 5. *There are independent trees $T, S \subseteq 2^{<\omega}$ such that T has the generic property and S is a hitting tree.*

Proof. By $\mathbb{T}$ we denote the set of all finite trees $p \subseteq 2^{<\omega}$ such that all maximal nodes of p have the same height. This common value will be denoted by $ht(p)$. Given $p, q \in \mathbb{T}$ we define the following:

(1) $p \leq_0 q$ if $p \cap 2^{ht(q)} = q$ (hence $q \subseteq p$).
(2) $p \leq_1 q$ if $p \leq_0 q$ and for every $n \in \omega$ and $s \in q$ if $ht(q) \leq n < ht(p)$ then there is $t \in p$ extending s such that $t(n) = 1$.

By $\mathsf{max}(p)$ we denote the set of maximal nodes of p. We define the following sets:

[5] There is in general no relation between being ω-hitting and P^- even for tall Borel ideals. The ideals $\mathcal{ED}_{\mathsf{Fin}}$, $\mathsf{Fin} \times \mathsf{Fin}$, nwd and $\mathsf{Fin} \times \mathcal{ED}_{\mathsf{Fin}}$ cover all the possibilities.

(1) For every $n \in \omega$ we define $D_0(n)$ as the set of all $p \in \mathbb{T}$ such that $\left|s^{-1}(0)\right|, \left|s^{-1}(1)\right| \geq n$ for every $s \in \max(p)$.
(2) For every $n \in \omega$ we define $D_1(n)$ as the set of all $p \in \mathbb{T}$ such that there is $k \in \omega$ such that $n < k < ht(p)$ and every node in p_k is a splitting node.
(3) Given $n \in \omega$, let $D_2(n)$ be the set of all $p \in \mathbb{T}$ for which there is k such that $n < k < ht(p)$ with the property that for every $c : p_k \longrightarrow 2$ and for every $\{s_t \mid t \in p_k\} \subseteq \max(p)$ such that $t \subseteq s_t$, it is the case that $\left|\bigcap_{t \in p_k} s_t^{-1}(c(s_t \restriction k))\right| \geq n$.
(4) For every $n, m \in \omega$ we define $B_{n,m}$ as the set of all $p \in \mathbb{T}$ such that for every $X \subseteq p_n$ and $c : X \longrightarrow 2$ there are $j_0 < ... < j_m < ht(p)$ such that for every $i \leq m$ and for every $s \in p_{j_i+1}$ if $s \restriction n \in X$ then $s(j_i) = c(s \restriction n)$.

Let $\mathcal{D} = \{D_i(n) \mid i < 3 \wedge n \in \omega\}$ and $\mathcal{B} = \mathcal{D} \cup \{B_{n,m} \mid n, m \in \omega\}$. We claim that each $D_i(n)$ is $\leq_1$-dense (i.e. for every $p \in \mathbb{T}$ there is $q \in D_i(n)$ such that $q \leq_1 p$) and each $B_{n,m}$ is $\leq_0$-dense. The only non trivial part is that the sets $D_2(n)$ are $\leq_1$-dense, which we will prove now.

Let $p \in \mathbb{T}$ and $k = ht(p)$, we may assume that $k > n$. Let A be the set of all functions $g\colon \max(p) \longrightarrow \{0,1\}^2$ such that if $s \in p_k$, then either $g_0(s) = 1$ or $g_1(s) = 0$ (where $g(s) = (g_0(s), g_1(s))$). Fix an enumeration $A = \left\{g^i \mid i < l\right\}$ for some $l \in \omega$. Given $s \in \max(p)$ we will recursively define $\{(s_i(0), s_i(1)) \mid i \leq l\}$ as follows:

(1) $s_0(0) = s^\frown \langle 0\rangle$ and $s_0(1) = s^\frown \langle 1\rangle$.
(2) $s_{i+1}(0) = s_i(0)^\frown \left\langle g_0^i(s)\right\rangle^n$ and $s_{i+1}(1) = s_i(1)^\frown \left\langle g_1^i(s)\right\rangle^n$ (recall that $g^i\colon \max(p) \longrightarrow \{0,1\}^2$ and $g^i(s) = \left(g_0^i(s), g_1^i(s)\right)$. If $j \in \{0,1\}$, by $\langle j\rangle^n$ we denote the constant sequence of length n with constant value j).

Let q be the smallest tree such that $s_{l+1}(0), s_{l+1}(1) \in q$ for every $s \in \max(p)$. It is easy to see that $q \in \mathbb{T}$ and $q \leq_1 p$. We claim that $q \in D_2(n)$. To see this, let $c : q_k \longrightarrow 2$ (note that $q_k = \max(p)$) and $\{z_s \mid s \in \max(p)\} \subseteq \max(q)$ such that $s \subseteq z_s$. Note that for every $s \in \max(p)$, there is $m_s \in \{0,1\}$ such that $s_0(m_s) \subseteq z_s$. We can find $g^i \in A$ such that $g^i_{m_s}(s) = c(s)$ and $g^i_{1-m_s}(s) = 1$. It is easy to see that $\left|\bigcap_{s \in q_k} (s_{i+1}(m_s))^{-1}(c(s))\right| \geq n$, since $s_{i+1}(m_s) \subseteq z_s$, the result follows.

By the Rasiowa–Sikorski lemma (see [15]) there are $G_0, G_1 \subseteq \mathbb{T}$ with the following properties:

(1) G_0 is a filter in $(\mathbb{T}, \leq_0)$.
(2) $G_0 \cap W \neq \emptyset$ for every $W \in \mathcal{B}$.
(3) G_1 is a filter in $(\mathbb{T}, \leq_1)$.
(4) $G_1 \cap W \neq \emptyset$ for every $W \in \mathcal{D}$.

It is then easy to see that $T = \bigcup G_0$ has the generic property and $S = \bigcup G_1$ is a hitting tree. □

3. $\mathcal{I}$-ultrafilters

We will now provide a positive answer to Problem 1 here. Given an ideal $\mathcal{I}$, we say that *$\mathcal{I}$-ultrafilters exist generically* if every filter of character less than $\mathfrak{c}$ can be extended to an $\mathcal{I}$-ultrafilter. Generic existence of $\mathcal{I}$-ultrafilters can be conveniently characterized by the *generic existence number* or *exterior cofinality*

$\mathsf{cof}^*(\mathcal{I})$ defined as the smallest cofinality of an ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$, introduced and studied by Brendle and Flašková in [2] and, independently, by Hong and Zhang in [10][6]:

Lemma 6 *([2], [10]). If $\mathcal{I}$ is an ideal on ω then $\mathcal{I}$-ultrafilters exist generically if and only if $\mathsf{cof}^*(\mathcal{I}) = \mathfrak{c}$.*

In particular, by Lemma 2 if T is an independent tree, then $\mathsf{Pos}(T)$-ultrafilters exist generically.

The problem is that $\mathsf{Pos}(T)$ does not seem to be Borel. However, we will now prove that every ideal $\mathsf{Pos}(T)$ can be extended to a Borel ideal, and as cof^* is increasingly monotone, $\mathcal{I}$-ultrafilters exist generically for this new, Borel, ideal $\mathcal{I}$ as well. Now, the existence of such a Borel ideal can be deduced directly from a theorem of H. Sakai [19] who showed that every analytic ideal can be extended to a Borel one. However, Sakai's proof does not give any bound on the complexity. We shall give an explicit definition of an $F_{\sigma\delta\sigma}$ ideal extending $\mathsf{Pos}(T)$ here.

Given a set A and $m \in \omega$ we define $Z_m(A)$ as the set of all $\overline{y} = (\overline{y}(i))_{i<m} \in A^m$ such that $\overline{y}(i) \neq \overline{y}(j)$ whenever $i \neq j$. Let $T \subseteq 2^{<\omega}$ be an independent tree.

(1) For every $\overline{x} \in [T]^n$ let $C(\overline{x}) = \bigcup_{i<n} \left(\overline{x}(i)^*\right)$ and $D(\overline{x}) = \bigcap_{i<n} x(i)$.
(2) For every $\overline{x} \in [T]^n$ and $\overline{y}_1, ..., \overline{y}_k \in Z_m([T])$ we define $H(\overline{x}, \overline{y}_1, ..., \overline{y}_k) = C(\overline{x}) \cup \bigcup_{j\leq k} D(\overline{y}_j)$.
(3) For every $n > 0$ we define $\mathcal{H}(n)$ as the set of all $A \subseteq \omega$ such that for every $m > n$ there are $k \geq 1$, $\overline{x} \in [T]^n$ and $\overline{y}_1, ..., \overline{y}_k \in Z_m([T])$ such that $A \subseteq H(\overline{x}, \overline{y}_1, ..., \overline{y}_k)$.

It is easy to see that $\mathcal{H}(n) \subseteq \mathcal{H}(n+1)$ for every $n \in \omega$. We now introduce the following definition:

Definition 7. If $T \subseteq 2^{<\omega}$ is an independent tree, we define $\mathsf{Pos}_{\mathsf{B}}(T) = \bigcup_{n\in\omega} \mathcal{H}(n)$.

We will need the following lemma:

Lemma 8. *Let $n > 0$. If $A, B \in \mathcal{H}(n)$ then $A \cup B \in \mathcal{H}(2n)$.*

Proof. Let $A, B \in \mathcal{H}(n)$ and $m > 2n$. Since $A, B \in \mathcal{H}(n)$ there are $\overline{x}, \overline{a} \in [T]^n$, $\overline{y}_1, ..., \overline{y}_{k_1} \in Z_m([T])$ and $\overline{b}_1, ..., \overline{b}_{k_2} \in Z_m([T])$ such that $A \subseteq H(\overline{x}, \overline{y}_1, ..., \overline{y}_{k_1})$ and $B \subseteq H\left(\overline{a}, \overline{b}_1, ..., \overline{b}_{k_2}\right)$. It follows that

$$A \cup B \subseteq H(\overline{x}^\frown\overline{a}, \overline{y}_1, ..., \overline{y}_{k_1}, \overline{b}_1, ..., \overline{b}_{k_2}). \quad \square$$

We now have the following result:

Proposition 9. *Let $T \subseteq 2^{<\omega}$ be an independent tree.*

(1) *$\mathsf{Pos}_{\mathsf{B}}(T)$ is an $F_{\sigma\delta\sigma}$-ideal extending $\mathsf{Pos}(T)$.*
(2) *$\mathsf{Pos}_{\mathsf{B}}(T)$ can not be extended to an $F_{\sigma\delta}$-ideal.*

Proof. We will first prove $\mathsf{Pos}_{\mathsf{B}}(T)$ is an ideal. It is closed under unions by the previous lemma, so it is enough to prove that $H(\overline{x}, \overline{y}_1, ..., \overline{y}_k) \neq \omega$ for every $\overline{x} \in [T]^n$ and $\overline{y}_1, ..., \overline{y}_k \in Z_m([T])$ with $n < m$. Since $n < m$, there is $i < m$ such that $z = \overline{y}_1(i) \neq \overline{x}(j)$ for every $j < n$. Note that if $z \in im(\overline{y}_j)$ (for any $j \leq k$) then $D(\overline{y}_j) \subseteq z$. Since T is independent, we know that z^* is not almost contained in $C(\overline{x}) \cup \bigcup\{D(\overline{y}_j) \mid z \notin im(\overline{y}_j)\}$ so $H(\overline{x}, \overline{y}_1, ..., \overline{y}_m)$ does not almost contain z^*.

[6] In [2] the cardinal invariant is denoted by $\mathfrak{ge}(\mathcal{I})$, and in [10] by $\mathsf{non}^{**}(\mathcal{I})$.

We will now prove that $\mathsf{Pos}(T) \subseteq \mathsf{Pos}_{\mathsf{B}}(T)$. Let $x \in [T]$ and $C = \{y_i \mid i \in \omega\} \in [[T]]^{\omega}$. Since $x^* \cup \bigcap C \subseteq x^* \cup (y_0 \cap ... \cap y_m)$ for every $m \in \omega$, $x^* \cup \bigcap C \in \mathcal{H}(1)$, hence $\mathsf{Pos}(T) \subseteq \mathsf{Pos}_{\mathsf{B}}(T)$.

Next we shall prove that $\mathsf{Pos}_{\mathsf{B}}(T)$ is an $F_{\sigma\delta\sigma}$ ideal. Let $a_m(T)$ be the set of finite sequences $\overline{s} = (s_1, ..., s_m) \in T^m$ such that s_i is incompatible with s_j whenever $i \neq j$. For every $\overline{s} = (s_1, ..., s_m) \in a_m(T)$ we define $\langle \overline{s} \rangle = \{(y_1, ..., y_m) \mid \forall i \leq m\, (y_i \in [T_{s_i}])\}$. For every $0 < n < m$, $k > 0$ and $\overline{s}_1, ..., \overline{s}_k \in a_m(T)$ we define

$$\mathcal{H}(n, m, \overline{s}_1, ..., \overline{s}_k) = \{H(\overline{x}, \overline{y}_1, ..., \overline{y}_k) \mid \overline{x} \in [T]^n \wedge \forall i \leq k\, (\overline{y}_i \in \langle \overline{s}_i \rangle)\}.$$

Note that $\mathcal{H}(n, m, \overline{s}_1, ..., \overline{s}_k)$ is a closed set since it is a continuous image of $[T]^n \times \prod_{i \leq k} \langle \overline{s}_i \rangle$. In this way, the subset closure of $\mathcal{H}(n, m, \overline{s}_1, ..., \overline{s}_k)$ (denoted by $\mathcal{H}_{\downarrow}(n, m, \overline{s}_1, ..., \overline{s}_k)$) is closed as well.

Let $\mathcal{B}(n, m) = \bigcup_{k \in \omega} \{\mathcal{H}_{\downarrow}(n, m, \overline{s}_1, ..., \overline{s}_k) \mid \overline{s}_1, ..., \overline{s}_k \in a_m(T)\}$ and note that $\mathcal{B}(n, m)$ is an F_{σ}-set. Clearly $\mathcal{H}(n) = \bigcap_{m > n} \mathcal{B}(n, m)$ so each $\mathcal{H}(n)$ is an $F_{\sigma\delta}$-set and then $\mathsf{Pos}_{\mathsf{B}}(T) = \bigcup_{n > 0} \mathcal{H}(n)$ is an $F_{\sigma\delta\sigma}$-set. Finally, by results of Solecki, Laczkovich and Recław (see [21] and [16]), no $F_{\sigma\delta}$ ideal is Katětov above $\mathsf{Fin} \times \mathsf{Fin}$. Since $\mathsf{Fin} \times \mathsf{Fin} \leq_{\mathsf{K}} \mathsf{Pos}_{\mathsf{B}}(T)$ by Proposition 3, and the fact (see [13]) that an ideal $\mathcal{I}$ is P^- if and only if $\mathsf{Fin} \times \mathsf{Fin} \not\leq_{\mathsf{K}} \mathcal{I}$, the result follows. □

We can then conclude the desired result:

Corollary 10. *There is an $F_{\sigma\delta\sigma}$-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist generically.*

The previous result is optimal in the sense that it is no longer true for $F_{\sigma\delta}$-ideals, as we will prove now.

Definition 11 *(see [3,18,20]).* An ideal $\mathcal{I}$ is *Shelah-Steprāns* if for every sequence $\{s_n \mid n \in \omega\} \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$ one of the following holds:

(1) There is $A \in \mathcal{I}$ such that $A \cap s_n \neq \emptyset$ for every $n \in \omega$.
(2) There is $B \in \mathcal{I}$ such that $s_n \subseteq B$ for infinitely many $n \in \omega$.

Let $\mathcal{I}$ be an ideal and $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$. We say that X witness that $\mathcal{I}$ is not Shelah-Steprāns if neither of the two possibilities above hold for X.

Lemma 12. *Let $\mathcal{I}$ be an analytic ideal, and let $X = \{s_n \mid n \in \omega\}$ witnesses that $\mathcal{I}$ is not Shelah-Steprāns. If $\mathbb{P}$ is a forcing notion, then X still witnesses that $\mathcal{I}$ is not Shelah-Steprāns after forcing with $\mathbb{P}$.*

Proof. Let $\mathcal{B} = \{A \mid \forall n\, (A \cap s_n \neq \emptyset)\}$ and $\mathcal{D} = \{B \mid \exists^{\infty} n\, (s_n \subseteq B)\}$. Both $\mathcal{B}$ and $\mathcal{D}$ are Borel sets, and not being Shelah-Steprāns simply means that $\mathcal{I} \cap (\mathcal{B} \cup \mathcal{D}) = \emptyset$. Since this is a coanalytic statement, it holds in any forcing extension. □

We say that an ideal $\mathcal{I}$ is *nowhere Shelah-Steprāns* if $\mathcal{I} \restriction A$ is not Shelah-Steprāns for every $A \in \mathcal{I}^+$. We now have the following absoluteness result:

Lemma 13. *Let W be a forcing extension of V and $\mathcal{I} \in V$ an analytic ideal. If $W \models$ "$\mathcal{I}$ is nowhere Shelah-Steprāns" then $V \models$ "$\mathcal{I}$ is nowhere Shelah-Steprāns".*

Proof. Note that $\mathcal{I}$ is nowhere Shelah-Steprāns if the following statement holds:

$$\forall A \exists X \subseteq [A]^{<\omega} \forall x\, ((x \notin \mathcal{B}_X \cup \mathcal{D}_X) \vee x \notin \mathcal{I})$$

where $\mathcal{B}_X = \{x \mid \forall s \in X\,(x \cap s \neq \emptyset)\}$ and $\mathcal{D}_X = \{x \mid \exists^\infty s \in X\,(s \subseteq x)\}$. This statement is a $\prod_1^3$ statement and therefore it is downward absolute. □

Recall that given an ideal $\mathcal{I}$ on ω (or on any countable set), the *Mathias forcing* $\mathbb{M}(\mathcal{I})$ associated with $\mathcal{I}$ is the set of all pairs (s, A) where $s \in [\omega]^{<\omega}$ and $A \in \mathcal{I}$. If $(s, A), (t, B) \in \mathbb{M}(\mathcal{I})$ then $(s, A) \leq (t, B)$ if the following conditions hold:

(1) t is an initial segment of s.
(2) $B \subseteq A$.
(3) $(s \setminus t) \cap B = \emptyset$.

If $G \subseteq \mathbb{M}(\mathcal{I})$ is a generic filter, we define *the generic real* as $r_{gen} = \bigcup\{s \mid \exists A\,((s, A) \in G)\}$.

Lemma 14. *Let $\mathcal{I}$ be a nowhere Shelah-Steprāns analytic ideal. If $G \subseteq \mathbb{M}(\mathcal{I})$ is a generic filter then the following holds in $V[G]$:*

(1) $r_{gen} \in \mathcal{I}^+$.
(2) *If $A \in V \cap \mathcal{I}^+$ then $r_{gen} \cap A \in \mathcal{I}^+$.*
(3) *If $A \in V \cap \mathcal{I}$ then $r_{gen} \cap A$ is finite.*

Proof. Note that the first item follows from the second by taking $A = \omega$. Letting $A \in V \cap \mathcal{I}^+$, we will prove that $r_{gen} \cap A \in \mathcal{I}^+$. Since $\mathcal{I}$ is nowhere Shelah-Steprāns and $A \in \mathcal{I}^+$, there is $X = \{s_n \mid n \in \omega\} \subseteq [A]^{<\omega} \setminus \{\emptyset\}$ with the following properties:

(1) For every $B \in \mathcal{I}$ there is $n \in \omega$ such that $s_n \cap B = \emptyset$.
(2) If $W \in [\omega]^\omega$ then $\bigcup_{n \in W} s_n \in \mathcal{I}^+$.

Furthermore, since $\mathcal{I}$ is analytic the two previous properties hold in a forcing extension of V. By a simple genericity argument and the first property, we can conclude that there are infinitely many $n \in \omega$ such that $s_n \subseteq r_{gen} \cap A$ and then $r_{gen} \cap A \in \mathcal{I}^+$ by the second property.

The third item is easy and holds for every ideal. □

The following result was proved in [3]:

Proposition 15. *If $\mathcal{I}$ is a Borel ideal, then $\mathcal{I}$ is Shelah-Steprāns if and only if $\mathsf{Fin}\times\mathsf{Fin} \leq_{\mathsf{K}} \mathcal{I}$.*

As was mentioned earlier, no $F_{\sigma\delta}$-ideal is Katětov above $\mathsf{Fin}\times\mathsf{Fin}$, so $F_{\sigma\delta}$-ideals are nowhere Shelah-Steprāns. The following result is based on the results of [2].

Proposition 16. *It is consistent that $\mathcal{I}$-ultrafilters do not exist generically for every analytic nowhere Shelah-Steprāns ideal $\mathcal{I}$ (in particular for $F_{\sigma\delta}$-ideals).*

Proof. Given a model of set theory W, we define $\mathbb{P}(W)$ as the finite support iteration of the Mathias forcing of all analytic nowhere Shelah-Steprāns ideals. Let V be a model where $\mathfrak{c} = \omega_2$. We perform a finite support iteration $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \omega_1\}$ where $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{P}}(V_\alpha)$", where V_α is the model obtained after forcing with $\mathbb{P}_\alpha$. We will argue that V_{ω_1} is the desired model. Let $\mathcal{I} \in V_{\omega_1}$ be an analytic nowhere Shelah-Steprāns ideal. Since $\mathcal{I}$ can be coded with a real, there is $\alpha < \omega_1$ such that $\mathcal{I} \in V_\alpha$, and by downwards absoluteness, we know that $V_\alpha \models$"$\mathcal{I}$ is nowhere Shelah-Steprāns". Given $\beta > \alpha$ let $r_\beta^{\mathcal{I}}$ be the $\mathbb{M}(\mathcal{I})$ generic real added by $\mathbb{P}_{\beta+1}$. Let $x^{\mathcal{I}}_{\beta} = \omega \setminus r^{\mathcal{I}}_{\beta}$ and define $\mathcal{J}$ as the ideal generated by $\left\{ x^{\mathcal{I}}_{\beta} \mid \alpha < \beta < \omega_1 \right\}$. By the previous result, it follows that $\mathcal{J}$ is a proper ideal and $\mathcal{I} \subseteq \mathcal{J}$ so $\mathsf{cof}^* (\mathcal{I}) \leq \mathsf{cof}(\mathcal{J}) = \omega_1$. □

In [2] Brendle and Flašková proved that if $\mathcal{I}$ is an F_σ-ideal then $\mathsf{cof}^* (\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$ (where $\mathcal{N}$ denotes the ideal of all null sets). This can actually be deduced directly using some results that can be currently found in the literature: In [19] Sakai proved that there is an analytic P-ideal $\mathcal{P}_{\mathsf{max}}$ such that $\mathcal{I} \leq_{\mathsf{KB}} \mathcal{P}_{\mathsf{max}}$ where $\mathcal{I}$ is either an F_σ-ideal or an analytic P-ideal. In particular, $\mathsf{cof}^* (\mathcal{I}) \leq \mathsf{cof}(\mathcal{P}_{\mathsf{max}})$ for every F_σ-ideal $\mathcal{I}$. In [22] Todorcevic showed that the cofinality of every analytic P-ideal is at most $\mathsf{cof}(\mathcal{N})$. Therefore, we conclude that if $\mathcal{I}$ is an F_σ-ideal then $\mathsf{cof}^* (\mathcal{I}) \leq \mathsf{cof}(\mathcal{N})$. The following questions remain open:

Problem 17. Is there an F_σ-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist?

Problem 18. Is there an $F_{\sigma\delta}$-ideal $\mathcal{I}$ for which $\mathcal{I}$-ultrafilters exist? What about the density zero ideal or $\mathcal{P}_{\mathsf{max}}$?

Note that by the aforementioned result of Sakai [19] the non-existence of a $\mathcal{P}_{\mathsf{max}}$-ultrafilter would imply a negative answer to Problem 17, i.e. consistency of $\mathcal{I} \leq_{\mathsf{K}} \mathcal{U}^*$ for every ultrafilter $\mathcal{U}$ and every F_σ ideal $\mathcal{I}$.

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